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GENERAL

★Séminaire Henri Cartan, 14ième année: 1961/62. 2953
Topologie différentielle.
École Normale Supérieure.
Secrétariat mathématique, Paris, 1964. ii + 116 *pp.*

This volume contains the substance of three lectures of A. Douady, four by C. Morlet, and one each by J. Cerf and J.-P. Serre. The individual titles are listed in the Differential Topology section of this issue.

★Séminaire Henri Cartan, 15ième année: 1962/63. 2954
Topologie différentielle.
École Normale Supérieure.
Secrétariat mathématique, Paris, 1964. ii + 156 *pp.*

This volume contains the substance of four lectures each by J. Cerf and B. Malgrange and one each by H. Cartan and I. Tamura. The individual titles are listed in the Differential Topology section of this issue.

★Philosophy of Science: The Delaware Seminar. Vol. 2: 1962–1963. 2955
Edited by Bernard Baumrin.
Interscience Publishers [*John Wiley & Sons*], *New York-London-Sydney,* 1963. xviii + 551 *pp.* $14.50.

Vol. 1 (1961/62 [Interscience, New York, 1963]) was reviewed in MR **28** #15.

This further instalment of the Delaware colloquium is exciting if only because it gives the distinct impression that philosophy of science as here interpreted through a host of distinguished contributors is involved in a continual debate whereby old postures and results are constantly questioned, thereby fulfilling the ancient demand of all philosophy, which is to be a kind of eternal dialogue between contending partners rather than the accomplishment of a set body of results. This volume is specifically devoted to problems in the field of the physical sciences. Part II discusses topics under the heading of space, time and relativity; Part III is on particles, fields, and quantum mechanics; Part IV on cosmology. On the other hand, there are also included some of the more general matters that form part of philosophy of science. Here Part I heavily scores with exciting demolitions of the philosophical shibboleths of yesteryear such as 'logical empiricism', 'hypothetico-deductive theories', 'ultimate' requirements for explanation in terms of precision and completeness, or the old exclusion of probabilistic approaches from any final viewpoint. All these assumptions are questioned vigorously. Feyerabend's chapter, though partly going over ground covered by him elsewhere, considerably clarifies his position, which has apparently arisen out of an attempt to make the anti-Copenhagen spirit of physicists like Bohm and Vigier at least methodologically respectable. To the implied logical criteria of the older empiricism of consistency and meaning invariance Feyerabend opposes an approach which seeks to test existing theories not just by pitting them against given neutral 'facts' but rather against alternative theories, which must first be formulated so that some of the critical 'facts', observed or inferred, shall be forthcoming at all. Ultimately, such alternative theories need not be mutually consistent, nor need the meanings of the terms which they contain (theoretical or empirical) all have a fixed connotation.

Sellars's chapter goes into the problem on the reality of theoretical entities via the elegant method of a discussion of the nature of the correspondence rules of various kinds which coordinate the 'abstract' with the 'empirical' expressions of some given theory, or even the theoretical properties of two different theories, e.g., physical and chemical, thus involving incidentally a clarification of the notion of the 'reduction' of one to the other.

The paper on 'Space, Time and Language' by Shapere investigates the relation of the ordinary meanings of the space-time concepts to their scientific counterparts, in order to investigate the origin of the seemingly paradoxical air of some of the scientific statements involving space and time. Gruenbaum, by means of an example from relativity, shows something of the importance of the philosophy of science for solving questions in its history, e.g., the genesis of relativity ideas; and Hilary Putnam in turn criticises extensively Gruenbaum's own views of geometry. In Part II, Hill, Pais and Suppes consider such matters as 'the status and applicability of mathematics in physical theories, the plethora of elementary particles, and the nonstandard character of the mathematical equipment of quantum mechanics'.

Part IV includes an important paper on the theory of measurement by Brian Ellis, and in the last part Hanson and Wheeler contribute discussions on philosophical aspects of contemporary cosmologies ('continuous creation' vs. 'big bang', and a theory of creation of matter).

Space forbids reference to all the papers and names in this collection, let alone adequate discussion, but it is obvious that although here and there it seems as though old wine was being poured into new bottles, to go by the experience of these seminars at last an intimation of fulfilment of the old dream of close cooperation between scientist and philosopher of science is coming into view.

G. Buchdahl (Cambridge, England)

HISTORY AND BIOGRAPHY

See also 2959.

Aaboe, Asger 2956

★Episodes from the early history of mathematics.

Random House, New York; The L. W. Singer Co., Syracuse, N.Y., 1964. x + 133 *pp.*

In vier Kapiteln, welche (1) die babylonische Mathematik (u.a. Positionssystem, quadratische Gleichung, $\sqrt{2}$), (2) die der Griechen vor und bei Euklid (besonders Lunulae, Irrationalität von $\sqrt{2}$, Elemente, Primzahl, Fünfeck), (3) Archimedes (u.a. Spirale, Winkeldreiteilung, Siebeneck, die Kugel in der Methodenlehre) und (4) Ptolemäos (Almagest, Sehnentafel, Epizykelmodell) zum Inhalt haben, die aber auch Ausblicke auf die weitere Entwicklung geben (nichteuklidische Geometrie, Konstruierbarkeit der Polygone, Trigonometrie in späterer Zeit), schildert der Verfasser die Leistungen der Antike für die Grundlegung und den Ausbau mathematischer Erkenntnisse an ausgewählten, die einzelnen Perioden besonders kennzeichnenden und klar dargestellten Beispielen, die der Schulalgebra und Geometrie zugänglich sind. Das Buch, das mit seinen zahlreichen Abbildungen auch manch Neues bringt (z.B. der mesopotamische Schulraum (S. 13), die "Tomahawk"-Winkeldreiteilung (S. 87)), ist der 13. Band der von der School Mathematics Study Group veranlaßten "Neuen mathematischen Bibliothek" und durchaus geeignet, das Interesse auch an der Geschichte der Mathematik zu wecken und dem Schüler den grundlegenden Unterschied in der Einstellung zu mathematischen Dingen zu vermitteln. Dieser Absicht dienen auch eine Reihe von Aufgaben (nebst Lösungen) und Hinweise auf weitere Literatur. Der Titel des Buches selbst ist vielleicht zu anspruchslos gewählt: Es sind keine Einschiebsel, die der Verfasser darbietet, keine "Epeisodia", die so nebenher "in den Weg" kommen, sondern sowohl die anonymen Babylonier, wie Euklid, Archimedes und Ptolemäos gehören zu den Hauptakteuren des mathematischen Geschehens. *K. Vogel* (Munich)

Gundlach, Karl-Bernhard; von Soden, Wolfram 2957

Einige altbabylonische Texte zur Lösung "quadratischer Gleichungen".

Abh. Math. Sem. Univ. Hamburg **26** (1963/64), 248–263.

Ergänzt nach der philologischen und inhaltlichen Seite die bisherige Interpretation von vier Aufgaben in zwei babylonischen Texten aus Tell Harmal [T. Baqir, Sumer **6** (1950), 130–148] und Susa [E. M. Bruins und M. Rutten, *Textes mathématiques de Suse*, Texte XIII, 82–83, Mém. Mission Archéologique en Iran, Tome XXXIV, Librairie Orientaliste Paul Geuthner, Paris, 1961; MR **23** #A1487], wozu der verbesserte Text nebst neuer Übersetzung gegeben wird. Die Deutung einer Aufgabe aus Tell Harmal unterscheidet sich grundlegend von der bisherigen. Während Bruins [Rev. Assyriologie et Archéologie Orientale **47** (1953), 185–188] einen Lehrtext zur babylonisch-griechischen Näherungsformel der quadratischen Gleichung annimmt, sehen hier die Verfasser das Rezept zur Berechnung der Seiten eines Vierecks aus Seitensumme und Fläche. Dies würde ganz zu den anderen Aufgaben passen, in denen allen die einzelnen Schritte der Lösungsmethode des quadratischen Gleichungssystems der "Normalform" ($x+y=a$; $xy=b$) vorgerechnet wird. Einleitend werden die babylonischen Methoden der quadratischen Gleichungslösung dargelegt. *K. Vogel* (Munich)

Rychlík, Karel 2958

★Theorie der reellen Zahlen in Bolzanos handschriftlichem Nachlasse.

Verlag der Tschechoslowakischen Akademie der Wissenschaften, Prague, 1962. 103 *pp.* (1 *plate*) *Kčs.* 8.00.

There is as yet no complete edition of the mathematical works of Bernard Bolzano (1781–1848), mathematician and philosopher at Prague. From an unpublished manuscript preserved in the Vienna National Library, K. Rychlík (who also edited Bolzano's *Functionenlehre* [Königliche Böhmische Ges. Wiss., Prague, 1930]) now offers Bolzano's treatise on real numbers. To quote from the editor's preface:

"In der Theorie der reellen Zahlen versucht Bolzano eine Arithmetisierung der Theorie der reellen Zahlen durchzuführen, die viel später auf drei verschiedene Weisen von Weierstrass (1860), Méray (1869) und G. Cantor (1872) und endlich von Dedekind (1872) entwickelt wurde. Bolzano kann mit vollem Recht als Vorläufer dieser Mathematiker betrachtet werden: Der Gedanke der rein arithmetischen Begründung der reellen Zahlen tritt nämlich bei ihm ganz klar hervor, obwohl seine Ausführungen nicht als ganz stichhaltig betrachtet werden können. Dann bringt Bolzano die Entwicklung der reellen Zahlen in die sogenannten 'Cantorschen Reihen' und beweist weitere Sätze aus der Theorie der reellen Zahlen: die Trichotomie der Beziehungen 'grösser als' und 'kleiner als', den Satz von Archimedes, den Satz, dass die Menge der reellen Zahlen überall dicht ist, den Satz von Cauchy-Bolzano, den Satz von Bolzano-Weierstrass und endlich einen Satz, der an den Satz von Dedekind erinnert. Diese Entwicklungen könnten ohne wesentliche Veränderungen zu der heute verlangten Schärfe ausgefeilt werden. Tatsächlich hätte diese Handschrift, wäre sie selbst so wie sie ist veröffentlicht worden, den Fortschritt der Mathematik beschleunigen können."

The edition is equipped with an introduction, notes, an index and a bibliography. *C. J. Scriba* (Hamburg)

LOGIC AND FOUNDATIONS

See also 2955, 3886, 3902, 3903.

Scholz, Heinrich 2959

★Concise history of logic.

Translated by Kurt F. Leidecker. Philosophical Library, New York, 1961. xiv + 140 *pp.* $3.75.

This is a translation of a work (*Geschichte der Logik*) which first appeared in 1931 [Junker und Dünnhaupt, Berlin, 1931], and which in its time pioneered the view that ancient and medieval logic was not something totally different (for better or for worse) from what modern logicians are doing by mathematical means. The author was one of the first to see clearly that there is no better aid than modern logic to make clear what Aristotle, the Stoics, the Schoolmen and also a few post-Renaissance figures like Leibniz, were really after. This small book, however, is

now itself of mainly historical interest, so much more having been done along the lines which it helped to open up. *A. N. Prior* (Manchester)

da Costa, Newton C. A. 2960

Calculs des prédicats pour les systèmes formels inconsistants.

C. R. Acad. Sci. Paris **258** (1964), 27–29.

This paper is an extension to predicate calculi of the author's earlier paper [same C. R. **257** (1963), 3790–3792; MR **28** #1123]. *E. J. Cogan* (Bronxville, N.Y.)

Čavčanidze, V. V. 2961

Fundamental relations of the analytic theory of propositional algebra. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **33** (1964), 27–34.

The author presents a scheme for designating conditions for the variables in a formula of sentential calculus so that the formula is true. For formulas involving m variables, the designation is a column vector of 2^m entries. In case $m=1$ and the variable is x, $\binom{\sigma}{\sigma}$ represents formulas that are logically true, $\binom{\Lambda}{\Lambda}$ represents formulas that are logically false, $\binom{\sigma}{\Lambda}$ represents formulas that are true if x is true, and $\binom{\Lambda}{\sigma}$ represents formulas that are true if x is false. The scheme is carried thus to cases for $m>1$, and an algebra of vectors based on $+$ for disjunction and $\cdot$ for conjunction is defined. The equivalence of the algebra to operations with truth tables is quite evident.

E. J. Cogan (Bronxville, N.Y.)

Cohen, Paul J. 2962

The independence of the continuum hypothesis. II.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 105–110.

In this paper which is a continuation of the previous communication [same Proc. **50** (1963), 1143–1148; MR **28** #1118] the author concludes his proof that the continuum hypothesis is independent of the axioms of Zermelo-Fraenkel set-theory with the axiom of choice included. In the previous note the author has shown how starting from a denumerable model $\mathcal{M}$ of Zermelo-Fraenkel axioms in which the axiom of constructibility is valid he can construct a new family of sets $\mathcal{N}$ by adjunction of generic sets $a_\delta\subset\omega$ where δ ranges over a set of power $\aleph_\tau$ (in the sense of $\mathcal{M}$). He now proves that all the axioms of Zermelo-Fraenkel are valid in $\mathcal{N}$. There are two non-trivial steps in this proof: the verification of the axiom of replacement and of the power-set axiom. The verification of the axiom (schema) of replacement requires a formalization of the notion of forcing. This, however, is not possible in the full extent since the binary relation which a finite set of conditions P bears to an unlimited statement $\mathfrak{a}$ if and only if P forces $\mathfrak{a}$ is not definable just as the relation of satisfaction in a model is not definable in the model. Thus the relevant lemma says merely that if $\mathfrak{a}(x,y)$ is a fixed unlimited statement, then the relation "P forces $\mathfrak{a}(F_\alpha, F_\beta)$" is definable in $\mathcal{M}$. This is sufficient to verify that particular case of the axiom of replacement for the statement $\mathfrak{a}$ is valid in $\mathcal{N}$.

The power-set axiom is satisfied in $\mathcal{N}$ since—as the author shows—if $F_\beta\subset F_\alpha$ then there is a γ such that $F_\beta=F_\gamma$ and γ is limited from above as F_β ranges over all subsets of F_α. The bound is given by card $\gamma\leqq$ card $(\alpha)+\aleph_\tau$ (where the cardinals are meant in the sense of $\mathcal{M}$). The proof uses a device similar to that used in the main lemma of Gödel in his consistency proof of the generalized continuum hypothesis as well as the following curious lemma: If W has as elements mutually inconsistent finite sets P of conditions each having the form $n\in a_\delta$ or $\sim(n\in a_\delta)$, then W is at most denumerable in $\mathcal{M}$.

The verification of the axiom of choice in $\mathcal{N}$ is easy.

In order to show that the continuum hypothesis is not valid in $\mathcal{N}$ the author shows (using the lemma mentioned above) that the ordinals which are of different power in $\mathcal{M}$ continue to have a different cardinality in $\mathcal{N}$. Since the set of all a_δ's has the cardinality $\aleph_\tau$ in $\mathcal{M}$, the set of all subsets of ω has in $\mathcal{N}$ at least the cardinality $\aleph_\tau$. An additional discussion shows that if τ is cofinal with ω (in $\mathcal{M}$), then $2^{\aleph_0}=\aleph_{\tau+1}$ in $\mathcal{N}$ and that otherwise $2^{\aleph_0}=\aleph_\tau$. It follows in particular that taking $\tau=2$ we obtain a model $\mathcal{N}$ in which $2^{\aleph_0}=2^{\aleph_1}$ is true. This settles a hypothesis put forward by Lusin [Fund. Math. **25** (1935), 109–131].

The author adds a remark that his method can be used to show the existence of models in which constructible reals form a denumerable set and of models in which the union of denumerably many denumerable sets is not always denumerable.

In the final section the author discusses the problem of how to formalize his independence proof and sketches an argument which shows that one can prove in arithmetic the implication Con(Z-F)$\to$Con$(\text{Z-F})_1$ where $(\text{Z-F})_1$ is a system obtained from the Zermelo-Fraenkel system Z-F by adding to it, e.g., the axiom $2^{\aleph_0}=\aleph_2$ or the axiom $2^{\aleph_0}=\aleph_{\omega^2+1}$ or generally $2^{\aleph_0}=\aleph_\tau$ with an arithmetically definable τ. *A. Mostowski* (Warsaw)

Mostowski, Andrzej 2963

The Hilbert epsilon function in many-valued logics.

Acta Philos. Fenn. Fasc. 16 (1963), 169–188.

This paper deals with the generalization to the many-valued case of two fundamental properties of predicate logic: the compactness theorem, and the axiomatizability of the set of valid formulas. The many-valued logic J has individual and predicate variables, p_ξ-ary propositional connectives $\mathfrak{F}_\xi$ $(\xi<a)$, and quantifiers $\mathfrak{Q}_\eta$ $(\eta<b)$, where a and b are arbitrary ordinals. Formulas Φ of J are constructed in the usual way, with quantifiers applied only to individual variables. Let J have an interpretation consisting of a compact Hausdorff space V of truth values, a closed subset $V^+\subset V$ of distinguished truth values, continuous functions F_ξ on V^{p_ξ} into V, and functions Q_η on the power set of V into V which are uniformly continuous in the sense that if $S_0, S_1\subset V$ and every point of S_i is close to some point of S_{1-i} $(i=0,1)$, then $Q_\eta(S_0)$ is close to $Q_\eta(S_1)$. A model of J over a set I is a function μ which correlates with each k-ary predicate variable a function on I^k into V. The truth value $\mathrm{Val}(\Phi,\mu)\in V$ of a closed formula Φ in the model μ is defined in the natural way.

The author extends J to a language J_∞ which has a functor $f_{\Phi qCs}$ for each formula Φ of J_∞, natural number q, a finite covering C of V by basic open sets, and set $s\in C$. He then considers the set $\mathfrak{M}_r$ of regular special models; these are models μ of J_∞, over the set $\mathbf{T}^*$ of all terms of J_∞ which

contain no variables, in which the functors $f_{\Phi q C_1}$ play a role analogous to that of the Hilbert ε-operator in two-valued logic. A topology is introduced on the set $\mathfrak{M}_r$ and the following lemmas are proved. (A) $\mathfrak{M}_r$ is a compact Hausdorff space. (B) For every closed formula Φ in J_∞, the function $f(\mu) = \mathrm{Val}(\Phi, \mu)$ is continuous on $\mathfrak{M}_r$. (C) For every model ν of J, there is a model $\mu \in \mathfrak{M}_r$ such that $\mathrm{Val}(\Phi, \mu) = \mathrm{Val}(\Phi, \nu)$ for all closed formulas Φ of J.

Using (A), (B), (C), a new proof of the following compactness theorem of Chang and the reviewer [Bull. Amer. Math. Soc. **68** (1962), 107–109; MR **25** #12] is given. Let $S(\Phi)$ be a closed subset of V for each closed formula Φ of J, and suppose that for every finite set **F** of closed formulas there is a model $\mu_{\mathbf{F}}$ of J such that $\mathrm{Val}(\Phi, \mu_{\mathbf{F}}) \in S(\Phi)$ for all $\Phi \in \mathbf{F}$. Then there is a model μ for J such that $\mathrm{Val}(\Phi, \mu) \in S(\Phi)$ for all Φ. A "Lowenheim-Skolem type" result estimating the power of μ also follows from the proof. The author mentions that the idea of his proof came from the two-valued proof of Beth [Nederl. Akad. Wetensch. Proc. Ser. A **54** (1951), 436–444; MR **13**, 614].

A closed formula Φ of J is said to be satisfiable if there is a model μ such that $\mathrm{Val}(\Phi, \mu) \in V^+$. Using J_∞ and regular special models, two characterizations of satisfiable formulas are obtained, and then a sufficient condition is given for the set of all satisfiable formulas to be the complement of a recursively enumerable set. As an example, the author shows that his sufficient condition holds when $V = 2^\omega$, where ω is the set of all integers, V has the usual product topology, $V^+ = \{X : 0 \in X\}$, $F_0(X) = \{x+1 : x \in X\}$, $F_1, \cdots, F_{a-1}$ are finitely many Boolean functions, $b = 2$, and Q_0, Q_1 are the operations of union and intersection. The paper concludes with a description of two many-valued logics, arising from modal and intuitionistic logics, for which the compactness and axiomatizability problems are still open. *H. J. Keisler* (Madison, Wis.)

Gandy, R. O.; Kreisel, G.; Tait, W. W. 2964a
Set existence.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **8** (1960), 577–582.

Gandy, R. O.; Kreisel, G.; Tait, W. W. 2964b
Set existence. II.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **9** (1961), 881–882.

$\mathscr{A}$ is a system of axioms in the notation of finite type theory which includes a relation symbol S (for successor) with appropriate axioms for S, and is such that every model M of $\mathscr{A}$ has (not necessarily distinct) individuals $\gamma_0^M, \gamma_1^M, \cdots$, the natural numbers in M (the set of which is denoted N_M). If M has no other individuals, it is an ω-model; otherwise it is a general model. The first paper gives some results about the kind of sets whose existence is established by certain such $\mathscr{A}$. Two senses are distinguished in which $\mathscr{A}$ establishes the existence of a set. In one of these, a set is assured by $\mathscr{A}$ for ω-models (for general models) if and only if X belongs to all ω-models (all general models) of $\mathscr{A}$, where, for type 1, X^1 belongs to M if and only if for some Y of M, (i) $Y \subseteq N_M$, (ii) $X^1 = \{n \mid \gamma_n^M \in Y\}$. In the other sense, a set is said to be represented in all ω-models (all general models) of $\mathscr{A}$, where, for type 1, X is represented by Y in M provided (ii) above holds, and for higher types [denoting by χ^α the set of all sets of type α which are represented in every ω-model (every general model) of $\mathscr{A}$], $X \subseteq \chi^\alpha$, Y is in M and $(X')_{\chi^\alpha}(Y')(Y'')$ [$(Y'$ represents X' & Y'' represents $X') \rightarrow (Y' \in Y \leftrightarrow Y'' \in Y)$] and $(X')_{\chi^\alpha}[X' \in X \leftrightarrow (EY')(Y'$ represents X' & $Y' \in Y)]$. Theorem 1: If $\mathscr{A}$ is Π_1^1 and set X of natural numbers is assured by $\mathscr{A}$ for ω-models, then X is hyperarithmetic. This converse of a result by Grzegorczyk, Mostowski and Ryll-Nardzewski [J. Symbolic Logic **23** (1958), 188–206; MR **21** #4908] is obtained by the use of (part (a) of) a lemma of independent interest: If R is recursive and the functions $q_0, q_1, \cdots$ are (a) not HA or (b) bounded by 1 and not recursive, then $(E\alpha)(x)R(\alpha(x)) \rightarrow (E\alpha^*)[R'(\bar{\alpha}^*(x))$ & (i) (q_i is not primitive recursive in α^*)], where under (a), R' is R and under (b), $R'(x) \equiv [\alpha(x) \leq 1$ & $R(x)]$. As a corollary of Theorem 1, the intersection of all ω-models of second-order arithmetic is not a model for it. Theorem 2: (i) If X is assured by $\mathscr{A}$ for general models, then X is hereditarily finite. (ii) If $\mathscr{A}$ is Π_1^1 and includes all instances of the comprehensive axiom $(EX^1)(x)[x \in X^1 \leftrightarrow P(x)]$, P having no bound variables of higher type, then if X^2 is assured by $\mathscr{A}$ for ω-models, $(X)\{X \in X^2 \rightarrow (Ea)[a \in O$ &, $|a| < \omega_1$ & X is recursive in $H_a]\}$. Theorem 2 is cited as illustrating the analogy between finite sets in general models of arbitrary $\mathscr{A}$ and HA sets in ω-models of $\Pi_1^1\mathscr{A}$. Using part (b) of the lemma above, one obtains Theorem 3: If $\mathscr{A}$ is Σ_1^0 and X is represented in all general models of $\mathscr{A}$, then X is recursive.

The second paper contains a strengthening of Theorem 2 (as given above) and some minor corrections.
P. Axt (E. Lansing, Mich.)

Turquette, Atwell R. 2965
Modality, minimality, and many-valuedness.
Acta Philos. Fenn. Fasc. 16 (1963), 261–276.

The author considers the problem of relating many-valued and modal logics. He offers a number of suggestions for considering a special choice of modalities together with a special choice for m-valued logic. These relate mainly to an identification of the logic-values 1, 2, 3 with the modal values 'necessary', 'contingent', 'impossible', respectively, and to the use of modal operators O_n, O_t, O_i which are special cases of the operators J_1, J_2, J_3 of Rosser and Turquette [J. Symbolic Logic **10** (1945), 61–82; MR **7**, 185]. Concepts of 'minimal sets of operators' and 'minimal sets of axioms' are introduced, primarily in relation to elegance and economy, but applications of these concepts to modality are discussed also. *A. Rose* (Nottingham)

Chang, C. C. 2966
Logic with positive and negative truth values.
Acta Philos. Fenn. Fasc. 16 (1963), 19–39.

The author considers infinite-valued propositional and predicate calculi with truth-values in the closed interval $[-1, 1]$. He states, without proof, a number of results concerning formulae whose truth-values belong to certain subsets of this interval. Proofs are given of the axiomatizability of two sets of formulae. These proofs depend on properties of certain algebras, called MV^*-algebras, which are extensions of the MV-algebras (or Chang algebras) developed previously by the author [Trans. Amer. Math. Soc. **88** (1958), 467–490; MR **20** #821].
A. Rose (Nottingham)

Rose, Alan 2967
Formalisations de calculs propositionnels polyvalents à foncteurs variables.
C. R. Acad. Sci. Paris **258** (1964), 1951–1953.
This paper gives a method for introducing the Leśniewski-Łukasiewicz concept of a "variable functor" into many-valued logic. As with Łukasiewicz [see Proc. Roy. Irish Acad. Sect. A **54** (1951), 25–35, pp. 26–28; MR **13**, 3], δ denotes a variable functor, but its use is extended to M-valued propositional calculi with S designated values. Let I denote "implication" and $J_1, \cdots, J_M$ "assertion" operators which satisfy "standard conditions" in the sense of Rosser and Turquette. Consider an M-valued propositional calculus with S designated values in which $I, J_1, \cdots, J_M$ can be defined and which can be formalized axiomatically. The author shows that such a calculus can be extended to include variable functors by adding the following M highly intuitive axioms:

$$IJ_iPIJ_iQI\delta P\delta Q \qquad (i = 1, \cdots, M).$$

The author's result naturally suggests the following question: Does there exist for many-valued logic a single shortest axiom, involving variable functors which is comparable in elegance to Meredith's six-letter axiom for 2-valued logic $C\delta\delta O\delta P$ [ibid. **54** (1951), 37–47; MR **13**, 3]?
A. Turquette (Champaign, Ill.)

Anderson, Alan Ross 2968
Some open problems concerning the system E of entailment.
Acta Philos. Fenn. Fasc. 16 (1963), 7–18.
The author presents a list of problems concerning the system E of entailment [the author and Belnap, J. Symbolic Logic **27** (1962), 19–52; MR **27** #4739]. The two most important of these problems are the following. (I) Do there exist formulae A, B of E such that A, $\bar{A} \vee B$ are provable but B is unprovable? (II) The problem of finding a decision procedure for E.
A. Rose (Nottingham)

Moisil, Gr. C. 2969
Les logiques non-chrysippiennes et leurs applications.
Acta Philos. Fenn. Fasc. 16 (1963), 137–152.
An expository account of three-valued propositional calculus. Among the topics discussed are the calculi of Łukasiewicz and of Łukasiewicz-Słupecki, the representation theory of Łukasiewicz trivalent algebras, their connection with Boolean algebras, and applications.
H. J. Keisler (Madison, Wis.)

Cleave, J. P. 2970
A hierarchy of primitive recursive functions.
Z. Math. Logik Grundlagen Math. **9** (1963), 331–346.
The hierarchies considered are based on a measure of complexity of computation. Ritchie ["Classes of recursive functions of predictable complexity", Ph.D. Diss., Princeton Univ., Princeton, N.J., 1961; see also Trans. Amer. Math. Soc. **106** (1963), 139–173; MR **28** #2045] gave a chain of type ω, $F^0 \subset F^1 \subset F^2 \subset \cdots$ of classes of functions such that F^0 is the set of functions computed by finite automata, F^{i+1} is the class of functions computed by Turing machines whose computations use a length of tape which is a function in F^i of the argument, and such that the union of the F^i is the class K of Kalmar elementary functions. The author considers here computations by a more general type of machine similar to those used by Shepherdson and Sturgis [J. Assoc. Comput. Mach. **10** (1963), 217–255; MR **27** #1359]. His machines have a finite number N of registers holding natural numbers $R_1, \cdots, R_N$ and instructions $J_i(\alpha, \beta)$ $(i = 1, \cdots, n)$: jump to line α of the program if $R_i = 1$, to line β if $R_i \neq 1$; together with, for each function f in a given set Σ, the instructions: place $f(R_1, \cdots, R_n)$ in R_m. The complexity of the computation is measured by the number of jump orders followed, the $(r+1)$st class E_{r+1} of the hierarchy consisting of those functions $\phi(x_1, \cdots, x_k)$ which are computable by a Σ-program which makes $\leqq \psi(x_1, \cdots, x_n)$ jumps, where $\psi \in \Sigma$. The starting class E_0 (this differs, but only trivially, from the author's notation) is taken as the class of constant functions. This gives an ω-sequence $E_0 \subset E_1 \cdots, E_\omega = \bigcup_{i=1}^{\omega} E_i$ which is now extended to $E_{\omega+1}, E_{\omega+2}, \cdots$ by replacing Σ by E_ω in the basic instructions. Continuing this procedure an ω^2-sequence is obtained. The author investigates in detail the closure properties of the sequence under various operations: composition, simultaneous iteration, limited primitive recursion, limited summation and multiplication. Σ is always supposed to contain $\{+, \times, \delta\}$ ($\delta(x, y) = 0, 1$ according as $x \neq y$ or $x = y$), so that if no bound on the number of jumps were imposed, all partial recursive functions would be computable by a Σ-program. Most of the results require also that there should be a function h of one variable obtainable by composition from elements of Σ and the projection functions whose iterates in a certain sense dominate all other such functions (of one or more variables). This condition is satisfied if Σ consists solely of $\{+, \times, \delta\}$, and in this particular case it turns out there is a simple relation with Grzegorczyk's hierarchy $\mathscr{E}^0 \subset \mathscr{E}^1 \cdots$ (for which $\mathscr{E}^3 = K$ and $\bigcup \mathscr{E}^i = P$, the class of all primitive recursive functions), viz., $E_{\omega s} = \mathscr{E}^{s+2}$ for $s \geqq 1$, hence $E_\omega = K$, $E_{\omega^2} = P$.
J. C. Shepherdson (Bristol)

Enderton, H. B. 2971
Hierarchies in recursive function theory.
Trans. Amer. Math. Soc. **111** (1964), 457–471.
The author constructs a hierarchy of subsets of N (the set of natural numbers) using partial well-orderings in place of ordinals, and particular sets in place of degrees. A partial well-ordering $<_t$ is a transitive binary relation such that any non-empty set has a minimal element for $<_t$. The field D_t of $<_t$ is the set $\{x : (\exists y)(x <_t y \text{ or } y <_t x)\}$. A function $|x|_t$ from D_t into ordinals is defined so that $|x|_t$ is the least ordinal β such that $(\forall y <_t x)|y|_t < \beta$. A system of notations is a partial well-ordering $<_t$ such that (1) $b \in D_t$ & $|b|_t = 0 \rightarrow b = 1$; (2) each $b \in D_t$ has a unique successor 2^b; (3) if $|b|_t$ is a limit ordinal, $b \neq 2^{(b)_0}$. Writing x^* for 2^x $(x \neq 0)$, we have for $x \in D_t$, $|x^*|_t = |x|_t + 1$. If the representative function of A is recursive in B, with Gödel number e, the author writes $A \leqq [e]B$.

A function j from a subset to a subset of N is called a jump if there is an integer e and a partial recursive function J such that for any sets A, B

$$A \leqq [n]B \rightarrow j(A) \leqq [J(n)]j(B),$$
$$A' \leqq [e]j(A),$$

where

$$A' = \{x : (\exists y)T_1{}^A(x, x, y)\}.$$

A hierarchy is defined as an ordered pair $(j, <_t)$. For the hierarchy $(j, <_t)$ a function H_t is defined so that $H_t(1) = N$, $H_t(b^*) = j(H_t(b))$, $b \in D_t$, and if $|b|_t$ is a limit ordinal,

$$H_t(b) = \{z : y <_t b \ \& \ x \in H_t(y)\},\ z = 2^x \cdot 3^y.$$

A system of notations $<_1$ is said to be canonical for a jump j if and only if for any system of notations $<_2$ there is a partial recursive ϕ such that

$$|a|_1 \leq |s|_2 \to H_1(a) \leq |\phi(a, s)| H_2(s).$$

The central theorem is that a system $<_1$ is canonical if there are partial recursive functions F, G such that if $|b|_1 \neq 0$ then $F(b) <_1 b$ and the set $\{y : y <_1 b\}$ is recursive in $H_1(F(b))$ with Gödel number $G(b)$.

R. L. Goodstein (Leicester)

McLaughlin, T. G. 2972

Some observations on quasicohesive sets.

Michigan Math. J. **11** (1964), 83–87.

Let N be the natural numbers. If $\alpha, \beta \subseteq N$, say that β splits α if $(\alpha \cap \beta)$ and $(\alpha \cap (N - \beta))$ are both infinite. Call α cohesive [r-cohesive] if α is infinite and α is not split by any recursively enumerable [recursive] set. Call α quasicohesive if α is the union of a finite number of cohesive sets. Call α decomposable if there exist disjoint recursively enumerable sets W, W' such that $\alpha \subseteq W \cup W'$ and $\alpha \cap W$, $\alpha \cap W'$ are both infinite. Theorem: There exist cohesive sets K_1, K_2 such that there exists a recursively enumerable set containing K_1 and disjoint from K_2, but there exists no recursively enumerable set containing K_2 and disjoint from K_1. Theorem: There are r-cohesive, non-quasicohesive sets. Theorem: There exists a recursive sequence $\{W_{\phi(i)}\}$ of pairwise disjoint recursively enumerable sets and a sequence $\{K_i\}$ of cohesive sets such that each $K_i \subseteq W_{\phi(i)}$ and $\bigcup_i K_i$ is r-cohesive. (From this it follows that decomposable r-cohesive sets exist.) There are other results as well. The methods are elementary.

A. Nerode (Ithaca, N.Y.)

Guillaume, Marcel 2973

Sur les structures hilbertiennes polyadiques.

C. R. Acad. Sci. Paris **258** (1964), 1957–1960.

A polyadic Hilbert ring is intended to be the counterpart in algebraic logic of the first-order functional calculus with the Hilbert symbol; the author announces the counterpart of the theorem that that symbol, if not present, can be adjoined with no fear of contradiction. The proof, not given, is asserted to be difficult. {The style, as is frequent with articles in the Comptes Rendus, is telegraphic to the point of being cryptic; the article is more an announcement of intentions than inventions.}

P. R. Halmos (Ann Arbor, Mich.)

Engeler, Erwin 2974

A reduction-principle for infinite formulas.

Math. Ann. **151** (1963), 296–303.

Let T be a first-order theory. Let ϕ be a "formula" built up by allowing countable conjunctions and disjunctions as well as the usual operations for building up first-order formulas. The author introduces a system of logic for such "formulas" which yields an analysis of the question "when is ϕ true in all models of T?" in first-order terms. The analysis is elegant and depends on devices of Henkin and Schütte.

A. Nerode (Ithaca, N.Y.)

Kripke, Saul A. 2975

The undecidability of monadic modal quantification theory.

Z. Math. Logik Grundlagen Math. **8** (1962), 113–116.

This is an important by-product of the author's earlier results [J. Symbolic Logic **24** (1959), 1–14; MR **22** #1514]. Using the models in that paper, he shows that the modal system $S5$ (and consequently weaker modal systems) with first-order monadic quantification theory is decidable if and only if unmodalized first-order dyadic quantification theory is, and so that it is undecidable. Basically, he shows that a formula in the latter calculus containing a dyadic predicate $R(x, y)$ is provable if and only if the corresponding formula with $R(x, y)$ replaced by $\Diamond(P(x) \wedge Q(y))$ is provable in the former. A contrary result of M. J. Poliferno [Logique et Analyse (N.S.) **4** (1961), 138–153] is shown to involve certain errors.

The author adds an "intuitive comment" which is not entirely happy. He says that "if we grant only that, for any positive integer n, it is possible that there are n planets, and that it is necessary that the number of planets is unique", and if, "given any relation $R(x, y)$ between positive integers, we take $P(x)$ as 'there is a z such that there are exactly z planets and $R(x, z)$', and $Q(y)$ as 'there are exactly y planets'", we can prove the equivalence of $R(x, y)$ and $\Diamond(P(x) \wedge Q(y))$. But if we let our $R(x, y)$ be "$x = x$ and there are exactly y planets", and take the case in which the number of planets is not y, $R(x, y)$ will be false and its alleged equivalent true. However, if we confine our R to strictly arithmetical relations, the author's contention is correct, and in any case the error does not affect his main result.

A. N. Prior (Manchester)

Starke, Peter H. 2976

Über die Darstellbarkeit von Ereignissen in nichtinitialen Automaten.

Z. Math. Logik Grundlagen Math. **9** (1963), 315–319.

A finite automaton is an ordered triple (Z, X, f) consisting of finite non-empty sets Z, X, and a function $f: Z \times X \to Z$. Here Z is the set of internal states, X is the alphabet, $f(z, x)$ is the new state given that the old state was z and symbol x was observed. If $F(X)$ is the set of words in the alphabet X and $e \in F(X)$ is the empty word, extend f to $Z \times F(X)$ by the recursive requirements that $f(z, e) = z$, $f(z, px) = f(f(z, p), x)$ for $p \in F(X)$, $x \in X$. The author calls a subset E of $F(X)$ an n-representable event if (1) every $p \in E$ is of length $\geqq n$, (2) there exists a finite automaton (Z, X, f) and a set $M \subseteq Z$ such that for every $p \in F(X)$ of length $\geqq n$, and every $z \in Z$, we have $p \in E$ if and only if $f(z, p) \in M$. This notion is very close to Kleene's notion of a definite event, and a straightforward examination of analysis and synthesis is given.

A. Nerode (Ithaca, N.Y.)

SET THEORY

See 2962, 3383.

COMBINATORIAL ANALYSIS

See also 3879.

Novák, Jiří 2977

Contribution to the theory of combinations. (Czech. Russian and German summaries)

Časopis Pěst. Mat. **88** (1963), 129–141.

Es seien $n \geqq k > r \geqq 0$ ganze Zahlen. Unter einem Kombinat $[n, k, r]$ wird ein System von Kombinationen k-ter Klasse aus n Elementen verstanden, wobei (1) je zwei Kombinationen aus $[n, k, r]$ höchstens r Elemente gemeinsam haben, (2) jede nicht in $[n, k, r]$ liegende Kombination k-ter Klasse aus den n Elementen mit einigen Kombinationen aus $[n, k, r]$ wenigstens $r+1$ gemeinsame Elemente besitzt [dieselben Časopis **84** (1959), 257–282; MR **24** #A447]. In der vorliegenden Arbeit werden Eigenschaften der sogenannten dichten Kombinate untersucht. Ein Kombinat $[n, k, r]$ heisst dicht, wenn jede Kombination r-ter Klasse aus den n Elementen in $m = [(n-r):(k-r)]$ Kombinationen von $[n, k, r]$ vorkommt. Einige Resultate: Ein Kombinat $[n, k, r]$ ist dicht dann und nur dann, wenn es von

$$[(n-r):(k-r)] \cdot \binom{n}{r} : \binom{k}{r}$$

Kombinationen gebildet wird. Dichte Kombinate $[n, 3, 1]$ bezw. $[n, 4, 2]$ gibt es genau für $n \equiv 0, 1, 2, 3 \pmod 6$ bezw. $n \equiv 1, 2, 3, 4 \pmod 6$. *O. Borůvka* (Brno)

Burau, W. 2978

Über die zur Kummerkonfiguration analogen Schemata von 16 Punkten und 16 Blöcken und ihre Gruppen.

Abh. Math. Sem. Univ. Hamburg **26** (1963/64), 129–144.

The author determines all symmetric balanced incomplete block designs with $\lambda = 2$, $k = 6$, and $v = 16$. Apart from the classical self-dual Kummer configuration, there are exactly two other such designs. These are likewise self-dual, but unlike the Kummer configuration, they cannot be represented by points and planes in complex 3-space. The author also shows that the automorphism groups of these designs are transitive and have orders $4! \cdot 2^4$ and $4! \cdot 2^5$.

P. Dembowski (Frankfurt a.M.)

Ajamzin, A. I. 2979

An example of a pair of orthogonal Latin squares of order ten. (Russian)

Uspehi Mat. Nauk **18** (1963), *no.* 5 (113), 173–174.

A pair of orthogonal latin squares of order 10 is exhibited. The system is non-isotopic to the system published by E. T. Parker [Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 859–862; MR **21** #3344]. Its symmetries are of different type than the original Parker squares. (Parker has since obtained a large number of non-isotopic pairs.) The author has not tested his pair to see if it admits a third orthogonal. (None of Parker's do.) *J. D. Swift* (Los Angeles, Calif.)

Johnson, Selmer M. 2980

Generation of permutations by adjacent transposition.

Math. Comp. **17** (1963), 282–285.

The author gives a method for generating all $n!$ permutations on the marks $1, 2, \cdots, n$ in which each permutation is obtained from its predecessor by a single interchange of two adjacent marks. Thus if the cost of moving a mark is an increasing function of the distance moved, this method minimizes the total cost of making all permutations. The method can be explained inductively as follows. The trivial case of $n = 2$ gives the two permutations 12, 21. Having generated and listed all permutations of the marks $1, 2, \cdots, n-1$, we expand this list by replacing each item by n copies of itself. We now adjoin the new mark n at the extreme right of the first entry in the expanded list. In succeeding permutations we interchange n with its left neighbor until n arrives in $n-1$ steps at the extreme left where it remains for one step while the old marks undergo their first interchange. The mark n now infiltrates from left to right during the next $n-1$ steps. The process continues back and forth until all $n!$ permutations have been generated and the process ends with $2, 1, 3, 4, 5, \cdots, n$. A rather complicated formula for the interchange position at the Nth step is given in terms of "digits" d_k in the unique representation

$$N = n!\left[\frac{d_2}{2!} + \frac{d_3}{3!} + \cdots + \frac{d_n}{n!}\right]$$

with $0 \leqq d_k < k$.

The reviewer has coded this procedure for the IBM 7090 and has found it excellent.

D. H. Lehmer (Berkeley, Calif.)

ORDER, LATTICES

See also 2987, 2988, 3108, 3902.

Wolk, E. S. 2981

On decompositions of partially ordered sets.

Proc. Amer. Math. Soc. **15** (1964), 197–199.

Let k be an infinite cardinal (defined as an initial ordinal). The author shows that if $Q(k) = k \times k$ is partially ordered by means of the natural ordering of the components, then every totally ordered subset of $Q(k)$ is finite and every decomposition of $Q(k)$ into disjoint chains is of power k. This generalizes an example due to Sierpiński. $Q(k)$ is then used to construct, for every pair of cardinals l, m for which $l \leqq m$, an example of a graph G in which l is the l.u.b. of the cardinalities of the complete subgraphs of G and m is the minimum of the cardinalities of the decompositions of G into independent sets of vertices.

R. P. Dilworth (Pasadena, Calif.)

Devidé, V. 2982

On monotonous mappings of complete lattices.

Fund. Math. **53** (1963/64), 147–154.

The author generalizes a theorem of A. Tarski [Pacific J. Math. **5** (1955), 285–309; MR **17**, 574]. Let $S_1, S_2, \cdots, S_n$ be complete lattices, and let $f_i: S_i \to S_{(i+1) \bmod n}$, $i = 1, 2, \cdots, n$, be order-preserving functions. For $S = S_1 \times S_2 \times \cdots \times S_n$ define $f\colon S \to S$ by $f(x_1, x_2, \cdots, x_n) = (f_n(x_n), f_1(x_1), \cdots, f_{n-1}(x_{n-1}))$. The principal theorem of this paper states that if $\xi \in S$, then there exists an $\eta \in S$ such that $\eta = \xi \vee f(\eta)$ [dually, $\eta = \xi \wedge f(\eta)$], and if $\lambda \in S$ satisfies $\xi \vee f(\lambda) \leqq \lambda$ [$\lambda \leqq \xi \wedge f(\lambda)$], then $\eta \leqq \lambda$ [$\lambda \leqq \eta$], where the operations and order are those of the cardinal product. Several corollaries

are stated and used, as an illustration of their applicability, to prove the Cantor-Bernstein theorem.

A. R. Bednarek (Gainesville, Fla.)

Molnárová, Katarína 2983

Congruence relations in the free lattice $FL(2+2)$. (Slovak. Russian summary)

Mat.-Fyz. Časopis Sloven. Akad. Vied **12** (1962), 108–122.

Es sei S ein Verband, M eine Menge von Intervallen des Verbandes S, $\langle a, b\rangle$ ein Intervall von S. Wenn $a=b$, oder wenn es solche Elemente $a=a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n = b$ gibt, dass $\langle a_{i-1}, a_i\rangle \in M$ für $i=1, 2, \cdots, n$ gilt, heisst das Intervall $\langle a, b\rangle$ zusammengesetzt aus den Intervallen der Menge M. Ist das Intervall $\langle a \cap b, a \cup b\rangle$ aus den Intervallen der Menge M zusammengesetzt, so wird $aR(M)b$ gesetzt. Für die Menge M werden Bedingungen formuliert, unter welchen $R(M)$ eine Kongruenzrelation auf S ist. Ist A eine Menge von Intervallen, $v=\langle c, d\rangle$ ein Intervall von A, so gibt es eine solche Menge $B \subseteq A$, dass $R(B)$ mit der kleinsten Kongruenzrelation identisch ist, für welche $c \equiv d$ gilt; die letztere wird mit $R(v)$ bezeichnet. Für $v \in A$ und $B \subseteq A$ werden Bedingungen formuliert, unter welchen $R(B) = R(v)$ ist.

Eine spezielle Menge A von Intervallen des Verbandes $S = FL(2+2)$, was ein durch die Menge $2+2$ erzeugter freier Verband ist, wird untersucht. Als Anwendung der obigen Ergebnisse wird zu jedem $v \in A$ ein solches System $B(v) \subseteq A$ gefunden, dass $R[B(v)] = R(v)$ gilt. Jede Kongruenzrelation auf S kann als Supremum einer geeigneten Teilmenge von A geschrieben werden. Diese Teilmenge ist unter gewissen Bedingungen, welche hier formuliert werden, endlich.

M. Novotný (Brno)

Varlet, Jules 2984

Congruences dans les treillis pseudo-complémentes.

Bull. Soc. Roy. Sci. Liège **32** (1963), 623–635.

Let L be a pseudo-complemented lattice and define on L the binary relation R: $a \equiv b(R)$ if and only if $a^* = b^*$. E.g., if L is distributive, R is a congruence relation. Stone lattices are characterized in terms of R. Certain equivalence classes modulo R are considered. Some results on elements of the form $a^* \cup a^{**}$ are also given.

G. Gratzer (University Park, Pa.)

Fort, Jacques 2985

Éléments injectifs (ou compléments) dans les treillis modulaires.

C. R. Acad. Sci. Paris **258** (1964), 1377–1379.

The author investigates the notion of a complement in a complete, modular, meet-continuous lattice L. An element a of L is called a complement if there is an element b of L such that a is maximal in the set of all x for which $x \wedge b = 0$. If a and b are elements of L such that $b \geq a$, then b is called an essential extension of a if $0 \leq x \leq b$ and $x \wedge a = 0$ imply $x=0$. An element is called injective if it has no proper essential extension. An element is a complement if and only if it is injective. An injective envelope of a is an element which is maximal in the set of essential extensions of a. Each element has an injective envelope.

A nonzero element a is called co-irreducible if 0 is meet-irreducible in $a/0$. Co-irreducibility of an element is characterized in terms of its injective envelopes. Relationships between meet-irreducible, maximal co-irreducible, and minimal and maximal injective elements are established.

Proofs and details will be published shortly.

J. Kist (University Park, Pa.)

Fort, Jacques 2986

Éléments Γ-isotypiques dans les $\mathscr{T}$-algèbres modulaires.

C. R. Acad. Sci. Paris **258** (1964), 1676–1678.

In a complete, modular, $\cap$-continuous lattice L, satisfying either the ascending or the descending chain condition, two projective quotients provided with the associated isomorphism form by definition a similitude. Denote by Σ the class of similitudes of quotients of L, by T the class of lattice isomorphisms acting on the quotients of L, and let Γ be an intermediate class ($\Sigma \subseteq \Gamma \subseteq T$) closed under composition, taking inverses and taking restrictions to quotients. In the set C of co-irreducible quotients A/B of L (i.e., $A \neq B$ and B is $\cap$-irreducible in the lattice A/B), A_1/B_1 and A_2/B_2 are partially Γ-isomorphic when there exist non-trivial sub-quotients C_1/B_1 and C_2/B_2 of A_1/B_1 and A_2/B_2 respectively such that the lattices C_1/B_1 and C_2/B_2 are isomorphic under an isomorphism of the class Γ; this may be used to define a Γ-equivalence in C, and each Γ-equivalence class defines a Γ-type of co-irreducible quotient.

For $X \in L$, $X \neq U$, where U is the universal element of L, and a decomposition $X = X_1 \cap X_2 \cap \cdots \cap X_n$ of X into an irredundant finite intersection of $\cap$-irreducible elements of L, the Γ-types of the co-irreducible quotients U/X_1, $U/X_2, \cdots, U/X_n$ are independent of the particular decomposition considered; these Γ-types are called the Γ-types associated with X, and X is defined to be "Γ-isotypique" if all the Γ-types associated with X are the same.

The author states (details and proofs are to be published later), that: (1) Every element except U of L is an irredundant intersection of a finite number of Γ-isotypique elements. (2) Any two such representations of the same element have the same number of components, and the sets of Γ-types are the same in each.

He then compares the concept of a Γ-isotypique element with that of a tertiary element [see Lesieur and Croisot, same C. R. **243** (1956), 1988–1991; MR **18**, 637].

R. McFadden (Belfast)

GENERAL MATHEMATICAL SYSTEMS

See also 3071.

Schmidt, E. T. 2987

Universale Algebren mit gegebenen Automorphismengruppen und Unteralgebrenverbänden.

Acta Sci. Math. (*Szeged*) **24** (1963), 251–254.

Let G be a group and L a compactly generated lattice. It is known that there exist universal algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ such that the automorphism group of $\mathfrak{A}_1$ and the subalgebra lattice of $\mathfrak{A}_2$ are isomorphic to G and L respectively [see G. Birkhoff, Rev. Un. Mat. Argentina **11** (1946), 155–157; MR **7**, 411; G. Birkhoff and O. Frink, Jr., Trans. Amer. Math. Soc. **64** (1948), 299–316; MR **10**, 279]. The author proves the existence of a universal algebra $\mathfrak{A}$ satisfying the two requirements simultaneously.

G. Gratzer (University Park, Pa.)

Šul'geĭfer, E. G. 2988

The lattice of ideals of an object of a category. II. (Russian)

Mat. Sb. (*N.S.*) **62 (104)** (1963), 335–344.

In Part I [same Sb. (N.S.) **54 (96)** (1961), 209–224; MR **27** #2452], the author investigated categories K satisfying certain axioms, (I) to (VI), and has shown that, in such a category, the lattice $I(a)$ of all ideals of a given object a is modular. Here, condition (VI) is slightly strengthened as the existence of not only finite but also infinite direct products is required {in the paper, direct products and direct sums are respectively called direct unions and free unions; in the review of Part I, the reviewer has erroneously translated "direct union" by "direct sum"}, and the question is raised as to whether (1) For every object a in K, $I(a)$ is fully modular (Dedekind), this meaning that for any two sets $\{A_i\}$, $\{B_i\}$ $(i \in I)$ of ideals of a, in 1-1 correspondence, and such that $A_j \leq B_i$ whenever $j \neq i$, one has

$$(\bigcup_i A_i) \cap (\bigcap_i B_i) = \bigcup_i (A_i \cap B_i).$$

For a given category K, condition (1) is shown to be equivalent with (2) Every object generated by an orthogonal system of ideals (i.e., a set of ideals each of which intersects trivially the union of the others) is a "special subdirect sum" of these ideals, in the sense of M. S. Čalenko [ibid. (N.S.) **57 (99)** (1962), 75–94; MR **25** #1123]. In an abelian category with infinite direct sums and direct products, an equivalent expression of (2) is (2′) The canonical mapping of the direct sum of any set of objects into their direct product is injective. Finally, it is remarked that the dual of the category of abelian groups provides an example of an abelian category where (2′), and therefore also (1), is not valid. *J. L. Tits* (Bonn)

THEORY OF NUMBERS

See also 3039, 3134, 3277, 3523.

Dedekind, Richard 2989

★Essays on the theory of numbers. I: Continuity and irrational numbers. II: The nature and meaning of numbers.

Authorized translation by Wooster Woodruff Beman.

Dover Publications, Inc., New York, 1963. iii + 115 *pp.* $1.00.

This is an unabridged and unaltered republication of an earlier English translation [Open Court, Chicago, Ill., 1901].

Gillies, Donald B. 2990

Three new Mersenne primes and a statistical theory.

Math. Comp. **18** (1964), 93–97.

The ILLIAC II computer has been used to test the primality of all Mersenne numbers $2^p - 1$ for $p < 12000$ except those cases in which a small prime factor of $2^p - 1$ is already known. Three new cases of primality were discovered, and these constitute the largest primes so far identified. The number of Mersenne primes now stands at 23. They are $2^p - 1$ for the following list of p's: 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213. A list of final remainders in the Lucas test taken modulo 2^{15} is given for $7000 < p < 12000$. A complete list of all prime factors q with $2^{34} < q < 2^{36}$ dividing $2^p - 1$ with $5000 < p < 17000$ is also given.

The author makes a conjecture from which it would follow that the number of Mersenne primes $\leq x$ is asymptotic to $(2/\log 2) \log\log x$ so that one could expect two Mersenne primes between two consecutive Fermat numbers. The heuristic argument is based on the prime number theorem and the fact that the prime factors of $2^p - 1$ are of the form $px + 1$.

From the data given it can be deduced that the time required to test $2^p - 1$ by the ILLIAC II is only $.006p^3$ microseconds. *D. H. Lehmer* (Berkeley, Calif.)

Selfridge, J. L.; Hurwitz, Alexander 2991

Fermat numbers and Mersenne numbers.

Math. Comp. **18** (1964), 146–148.

This note discusses the results of primality tests for Fermat and Mersenne numbers. The main results are that $2^p - 1$ is composite for $5000 < p < 6000$, and that $F_{14} = 2^{2^{14}} + 1$ is composite. For checking these results against possible future reruns a table of final remainders taken with respect to various checking moduli, such as 2^{15} and 2^{36}, is given. Complete remainders on punched cards are available from the authors. The machine used was the IBM 7090. Incorporated in the routine was a checking subroutine modulo $2^{35} - 1$. It is disconcerting to read that "several times" this checking subroutine caught the machine in error [see also #2990].

The least Fermat number beyond F_4 that can be a prime is now F_{17}, the testing of which, the authors say, would require 128 weeks of machine time.

D. H. Lehmer (Berkeley, Calif.)

Brillhart, John 2992

On the factors of certain Mersenne numbers. II.

Math. Comp. **18** (1964), 87–92.

This is a second list of new prime factors of Mersenne numbers [see the author and G. D. Johnson, Math. Comp. **14** (1960), 365–369; MR **23** #A832]. These 899 factors and previous results in the literature constitute all primes $q < 2^{35}$ of Mp with $103 \leq p \leq 257$ and all $q < 2^{34}$ of Mp with $257 < p < 20000$. These results are indirectly of great use in discovering Mersenne primes.

D. H. Lehmer (Berkeley, Calif.)

Kurtz, Harold C. 2993

The linear cubic p-adic recurrence and its value function.

Illinois J. Math. **8** (1964), 125–131.

Let $p > 3$ be a prime and let $\phi(W)$ be the p-adic value of W in the p-adic field. The author considers the values $\phi(W_n)$, where W_0, W_1, W_2 are not all zero and $W_{n+3} = PW_{n+2} - QW_{n+1} + RW_n$. Here P, Q, R, W_0, W_1, W_2 are p-adic rationals. There are three cases. The case in which Q and R are non-units is easy. The case in which P and Q are units and R a non-unit falls back on the results of M. Ward for quadratic recurrences [Proc. Amer. Math. Soc. **13** (1962), 102–106; MR **25** #2025]. Finally, if P, Q, R are units, the results of Ward for the rational integers are extended to the p-adic case [Trans. Amer. Math. Soc. **79** (1955), 72–90; MR **16**, 906].

D. H. Lehmer (Berkeley, Calif.)

Narkiewicz, W. 2994

On a class of arithmetical convolutions.

Colloq. Math. **10** (1963), 81–94.

Let A_n be a set of divisors of n. The set of all complex-valued arithmetical functions with ordinary addition and the convolution $f * g = \sum_{d \in A_n} f(d)g(n/d)$ as multiplication is a ring R_A. In this paper, conditions are examined under which R_A is commutative, associative, and has a unit. E.g., R_A is commutative if and only if $d \in A_n$ implies $n/d \in A_n$. Furthermore, sums over certain arithmetical functions, obtained by convolution, are evaluated.

G. J. Rieger (Munich)

de Bruijn, N. G.; van Lint, J. H. 2995

On the number of integers $\leq x$ whose prime factors divide n.

Acta Arith. **8** (1962/63), 349–356.

Let $f(n, x)$ be the number of integers not exceeding x which are products of powers of prime factors of n. In an earlier paper [Illinois J. Math. **6** (1962), 137–141; MR **26** #4977], de Bruijn showed that if

$$F(x) = \sum_{n \leq x} f(n, x), \qquad G(x) = \sum_{n \leq x} f(n, n),$$

then

$$\log(x^{-1}F(x)) \sim \log(x^{-1}G(x)) \sim (8 \log x)^{1/2} (\log\log x)^{-1/2}.$$

In the present paper it is shown that $F(x) \sim G(x)$. The proof quickly reduces to showing that

$$\sum_{k \leq x} \frac{k}{\alpha(k)} = o\left(x \sum_{k \leq x} \frac{1}{\alpha(k)}\right)$$

as $x \to \infty$, where $\alpha(k) = \prod_{p|k} p$ is the maximal square-free factor of k; the verification of this is somewhat intricate.

W. J. LeVeque (Boulder, Colo.)

Pearson, Erna H. 2996

On the congruences $(p-1)! \equiv -1$ and $2^{p-1} \equiv 1 \pmod{p^2}$.

Math. Comp. **17** (1963), 194–195.

No primes p with $30000 < p \leq 200183$ satisfy either of the congruences in the title. Thus the only known solutions are $p = 5$, 13, 563 for the first congruence and $p = 1093$, 3511 for the second one.

D. H. Lehmer (Berkeley, Calif.)

Muskat, J. B. 2997

On the solvability of $x^e \equiv \varepsilon \pmod p$.

Pacific J. Math. **14** (1964), 257–260.

Let e be an integer greater than 1, and let p be a prime $\equiv 1 \pmod e$. What conditions must p satisfy if e is an eth power residue modulo p? Let g be a fixed primitive root of p. If $p \nmid a$, define ind a as the least non-negative integer t such that $g^t \equiv a \pmod p$. For fixed h, k, $0 \leq h, k \leq e-1$, define the cyclotomic number (h, k) as the number of solutions of ind $n \equiv h \pmod e$, ind $(n+1) \equiv k \pmod e$, $1 \leq n \leq p-2$. For e odd put

$$S = \sum_{i=1}^{(e-1)/2} i \sum_{v=1}^{e-2} [B(e-i, v) - B(i, v)],$$

where $B(i, v) = \sum_{h=0}^{e-1} (h, i-vh)$. The author first proves the following two theorems. Theorem 1:

$$\text{ind}\, e \equiv \tfrac{1}{2}(p-1) - \sum_{h=1}^{e-1} (h, 0)h \pmod e.$$

Theorem 2: If e is an odd prime, then e ind $e \equiv S \pmod{e^2}$. He then deduces from Theorem 2 the criteria of C. E. Bickmore and L. Tanner [Bickmore, Messenger of Math. **26** (1896), 1–38] in the cases $e = 5$ and $e = 7$.

A. L. Whiteman (Los Angeles, Calif.)

Chowla, Paromita 2998

The problem of "four twos".

Norske Vid. Selsk. Forh. (Trondheim) **36** (1963), 142–144.

Put $S(x) = \sqrt{x}$, $G(x) \equiv [x]$, $C(x) = x^3$. The author proves that if t is an integer $\geqq 1$ then non-negative integers a, b exist such that $GS^bC^a(3) = t$. It follows from this that, for given t, non-negative integers a_n and b_n $(1 \leqq n \leqq m)$ exist such that $GS^{b_m}C^{a_m} \cdots GS^{b_1}C^{a_1}(3) = t$. The proof makes use of the fact that the numbers $a \log 3 - b \log 2$ are everywhere dense in $(-\infty, \infty)$.

L. Carlitz (Durham, N.C.)

Stevens, Harlan 2999

Some congruence properties of the Hermite polynomials.

Arch. Math. **14** (1963), 391–398.

Let $H_n(x)$ denote the Hermite polynomial defined by

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)t^n/n!.$$

The reviewer [same Arch. **10** (1959), 460–465; MR **22** #689] showed that

$$(*) \qquad \sum_{s=0}^{r} (-1)^s \binom{r}{s} (2x)^{(r-s)m} H_{n+sm} \equiv 0 \pmod{m^{r_1}},$$

where $m \geqq 1$, $n \geqq 0$, $r \geqq 1$ and $r_1 = [(r+1)/2]$. The author shows, making use of some general results on congruences [Math. Z. **79** (1962), 180–192], that the congruence in (*) holds

$$(\text{mod}\ 2^{e_0 r} p_1^{(e_1-1)r+r_1} \cdots p_k^{(e_k-1)r+r_1}),$$

where $m = 2^{e_0} p_1^{e_1} \cdots p_k^{e_k}$. This result is now improved as follows. The author proves that (*) holds (mod $2^{\tau_0} p_1^{\tau_1} \cdots p_k^{\tau_k}$), where

$$\tau_0 = e_0 r + [\tfrac{1}{2}n], \quad \tau_i = (e_i - 1)r + r_1 + \mu_i(r_1) \quad (i = 1, \cdots, k),$$

and $p_i^{\mu_i(r_1)}$ is the highest power of p_i dividing $r_1!$.

Finally, the author considers the question of when this result is best possible. A complete answer is not obtained. However, he shows in particular that

$$[D^p - (2x)^p]^r \sum_{n=0}^{\infty} H_n(t)t^n/n! \not\equiv 0 \pmod{p^{r_1+\mu(r_1)+1}},$$

where $D = d/dt$ and r satisfies $\mu(r) = 2\mu([r/2])$.

L. Carlitz (Durham, N.C.)

Mullin, Albert A. 3000

On a final multiplicative formulation of the fundamental theorem of arithmetic.

Z. Math. Logik Grundlagen Math. **10** (1964), 159–161.

From the author's introduction: "The purpose of this note is to prove various characteristics of certain integer-valued number-theoretic functions all derivable, by simple

means, from one particular integer-valued number-theoretic function, ψ. The discussion will proceed from the Fundamental Theorem of Arithmetic (the unicity of prime decomposition) and it will terminate in that part of analytic number theory dealing with the limiting distribution for square-free natural numbers."

Möller, Kurt 3001

Über mehrgradige Gleichungen und Quadrate.

Math. Nachr. **27** (1963), 67–75.

Ayant rappelé la définition d'un système multigrade, système diophantien de la forme $\sum_{\nu=1}^{s} a_\nu^k = \sum_{\nu=s+1}^{2s} a_\nu^k$, k étant un entier naturel prenant successivement les valeurs $k_1, k_2, \cdots, k_r$, l'auteur définit un carré multigrade comme la matrice carrée $a_{\mu,\nu}$ ($\mu, \nu = 1, 2, \cdots, s, s+1, \cdots, 2s$) telle qu'on ait simultanément $\sum_{\nu=1}^{s} a_{\mu,\nu}^k = \sum_{\nu=s+1}^{2s} a_{\mu,\nu}^k$, et $\sum_{\mu=1}^{s} a_{\mu,\nu}^k = \sum_{\mu=s+1}^{2s} a_{\mu,\nu}^k$ ($k = k_1, k_2, \cdots, k_r$). À l'instar d'un système multigrade d'ordre r ($k = 1, 2, \cdots, r$), et d'un système multigrade normal d'ordre r ($s = r+1$), il définit les carrés multigrades d'ordre r et les carrés multigrades normaux d'ordre r.

Il démontre ensuite le théorème: Si des fonctions linéaires $l_\nu(x, y) = \alpha_\nu x + \beta_\nu y$ ($\nu = 1, 2, \cdots, s, s+1, \cdots, 2s$) vérifient la relation $\sum_{\nu=1}^{s} l_\nu^k(x, y) \equiv \sum_{\nu=s+1}^{2s} l_\nu^k(x, y)$, dans laquelle k est un entier naturel, et si l'on pose $a_{\mu,\nu} = \alpha_\mu\beta_\nu - \alpha_\nu\beta_\mu$, on a $\sum_{\nu=1}^{s} a_{\mu,\nu}^k = \sum_{\nu=s+1}^{2s} a_{\mu,\nu}^k$ et $\sum_{\mu=1}^{s} a_{\mu,\nu}^k = \sum_{\mu=s+1}^{2s} a_{\mu,\nu}^k$. L'auteur applique ce théorème pour obtenir, à partir de théorèmes sur les multigrades donnés par M. Frolov, G. Tarry, A. Gloden, des théorèmes analogues pour les carrés multigrades. En particulier, il démontre qu'il existe des carrés multigrades de tout ordre.

En construisant des systèmes multigrades normaux d'ordre 2, 3, 4, 5, dont les termes sont des fonctions linéaires, il en déduit respectivement des carrés multigrades normaux du même ordre. *A. Gloden* (Luxembourg)

Davenport, H. 3002

★Analytic methods for Diophantine equations and Diophantine inequalities.

The University of Michigan, Fall Semester, 1962.

Ann Arbor Publishers, Ann Arbor, Mich., 1963. viii + 168 *pp.* $4.00.

The first ten sections of this course of lectures deal with the analytic theory of Diophantine equations of additive type; the other ten deal with the more recent application of the Hardy-Littlewood method to general Diophantine equations. It is convenient to describe these two halves separately.

After a couple of introductory sections, the author gives in §§ 3–6 the simplest available solution of Waring's problem: He proves that every large enough integer is a sum of (2^k+1) kth powers by using Hua's inequality. In §§ 7–8, asymptotic formulae are obtained for the number of solutions of equations of type $\sum_{i=1}^{s} c_i x_i^k = N$ for $s > 2^k$. In § 9, Vinogradov's result that for $k > k_0(\delta)$ every large enough integer is a sum of at most $(6+\delta)k \log k$ kth powers is proved, following the account given by H. Heilbronn [Acta Arith. **1** (1936), 212–221]. In § 10, the author returns to the equation $\sum c_i x_i^k = 0$. Throughout this first part, the treatment of the singular series is unusually meticulous. These lectures were more or less contemporary with the proof by the author and D. J. Lewis [Proc. Roy. Soc. Ser. A **274** (1963), 443–460; MR **27** #3617] that for $k > 16$ every diagonal equation $\sum c_i x_i^k = 0$ with $s > k^2$ is soluble in integers; in this problem, the singular series caused the principal difficulty.

The author then turns to general homogeneous equations. First, in § 11, he proves a theorem of the reviewer [Mathematika **4** (1957), 102–105; MR **20** #3828] that a rational equation of odd degree has a zero so long as it involves enough variables. §§ 12–18 contain the main meat of the lectures; the author gives a complete exposition of his theorem that every rational cubic in s variables has a rational zero for $s \geqq 17$. This is in fact much the clearest presentation of this important work. The proof given here contains an extra idea and so is simpler than the author's original proof [Philos. Trans. Roy. Soc. London Ser. A **251** (1959), 193–232; MR **21** #4136] valid for $s \geqq 32$; on the other hand, one avoids the extra difficulties of detail in the author's proof valid for $s \geqq 16$ [Proc. Roy. Soc. Ser. A **272** (1963), 285–303; MR **27** #5734]. § 19 describes related work of the reviewer. § 20 contains a result of a different character, the theorem of the author and H. Heilbronn [J. London Math. Soc. **21** (1946), 185–193; MR **8**, 565] that for every real $\lambda_1, \cdots, \lambda_5$, not all of the same sign, one can find integers $x_1, \cdots, x_5$ so that $|\sum_{i=1}^{5} \lambda_i x_i^2| < 1$.

These notes are almost completely self-contained; there is even a section proving relevant lemmas from the Geometry of Numbers. The author normally chooses the simplest available proofs—these are often his own, and often simpler than those available elsewhere; but he refers to alternative lines of attack when they exist. Plenty of references are given, both historical and to current work. These notes should be very valuable both to those starting research and to more experienced workers.
B. J. Birch (Manchester)

Heilbronn, H. 3003

On the representation of a rational as a sum of four squares by means of regular functions.

J. London Math. Soc. **39** (1964), 72–76.

It has been known since Lagrange, if not Diophantus, that every positive rational can be expressed as a sum of four squares of rational numbers. The author obtains the following beautiful extension. Theorem 1: There exist four integral functions $f_j(x)$ which are positive for $x > 0$, satisfy $\sum_{j=1}^{4} f_j^2(x) = 1$, and have the property that $x^{1/2} f_j(x)$ is rational for positive rational x.

Next, let $k \geqq 2$ be a rational integer, and let $s = s(k)$ be the smallest integer such that for each rational $x > 0$ the rational points are everywhere dense on the surface $\sum_{\sigma=1}^{s} y_\sigma^k = x$ ($y_\sigma > 0$). Also let L denote the complex plane exclusive of the negative real axis $x \leqq 0$. The author sketches a proof of Theorem 2: There exist s functions $g_\sigma(x)$, $1 \leqq \sigma \leqq s$, regular in L, such that (1) $g_\sigma(x)$ is rational for rational $x > 0$ and (2) $\sum_{\sigma=1}^{s} g_\sigma^k(x) = x$ in L.

He also remarks that $s = s(k)$ satisfies $s = O(k \log k)$, by the methods of Vinogradoff. On the other hand, it seems hard to prove that $s > 3$ for a single odd k.
S. Chowla (University Park, Pa.)

Graham, R. L. 3004

On finite sums of reciprocals of distinct nth powers.

Pacific J. Math. **14** (1964), 85–92.

A classical result states that every positive rational number can be written as a finite sum of reciprocals of distinct positive integers. In the present paper, those

rational numbers are characterized that can be written as finite sums of reciprocals of the nth powers of distinct positive integers, where n is a fixed positive integer. Let t be the largest integer k such that $k^{-n} > \sum_{j=1}^{\infty} (k+j)^{-n}$. Then a rational number r can be written as a finite sum of reciprocals of the nth powers of distinct positive integers if and only if it lies in a half-open interval of the form $[\pi_S, \pi_S + \sum_{j=1}^{\infty} (t+j)^{-n})$, where S is a subset of the set of the first t positive integers and $\pi_S = \sum_{j\in S} j^{-n}$. In particular, the rational number r can be expressed as the finite sum of reciprocals of distinct squares if and only if it belongs to one of the sets $[0, -1+\frac{1}{6}\pi^2)$ and $[1, \frac{1}{6}\pi^2)$.

W. H. Mills (Princeton, N.J.)

Bombieri, E. 3005

Sul teorema di Tschebotarev. (English summary)

Acta Arith. **8** (1962/63), 273–281.

Let Λ be an n-dimensional lattice with determinant $d(\Lambda) \neq 0$. Write $G(\mathbf{x}) = |x_1 \cdots x_n|$, and define $\mu(\Lambda) = \sup_{x_0} \inf_{x \equiv x_0(\Lambda)} G(\mathbf{x})$ and $\mathscr{L}_1 = \sup_\Lambda \mu(\Lambda)/d(\Lambda)$. Minkowski's conjecture on the inhomogeneous minimum of the product of n linear forms asserts that $\mathscr{L}_1 = 2^{-n}$. Tchebotarev proved that $\mathscr{L}_1 \leq 2^{-n/2}$, and this has been improved by various authors. The author improves the previous record by a factor of about 3 by showing that for large enough n, $\mathscr{L}_1 \leq \gamma_n 2^{-n/2}$ with $\gamma_n < \frac{1}{3}(2e-1)^{-1}$; his main new idea is an improvement of a theorem of Siegel's (see the author [Boll. Un. Mat. Ital. (3) **17** (1962), 283–288; MR **26** #4972]).

B. J. Birch (Manchester)

Siegel, Carl Ludwig 3006

★**Lectures on the analytical theory of quadratic forms.**

Notes by Morgan Ward. Third revised edition.

Buchhandlung Robert Peppmüller, Göttingen, 1963. iii+243 *pp. DM* 15.00.

This is a revision of the second edition [Inst. Advanced Study, Princeton, N.J., 1949]. A modern bibliography on the theory of modular functions and forms, abelian functions and abelian varieties has been included, with suitable references in the text, and a number of small errors and misprints have been eliminated. Otherwise the content, as before, is roughly this: the basic theorem on the density of representations of one definite quadratic form by the genus of a second; modular functions and forms, and the connection between the modular field and the moduli of an algebraic curve; extension of the basic theorem to indefinite forms; further extension to forms over totally real algebraic number fields.

W. J. LeVeque (Boulder, Colo.)

Makai, E. 3007

On a minimum problem. II.

Acta Math. Acad. Sci. Hungar. **15** (1964), 63–66.

Let $z_0 = 1, z_1, z_2, \cdots, z_n$ be complex numbers with $|z_j| \leq |z_k|$ for $j < k$, m an arbitrary positive integer and $b_0, b_1, \cdots, b_n$ arbitrary complex numbers. V. T. Sós and P. Turán [same Acta **6** (1955), 241–255; MR **17**, 1061] proved the inequality

$$(*) \quad \max_{\nu=m,m+1,\cdots,m+n} \left| \sum_{j=0}^{n} b_j z_j^{\nu} \right| \geq \left(\frac{n}{8e(m+n)} \right)^{n+1} \min_{k=0,1,\cdots,n} \left| \sum_{j=0}^{k} b_j \right|.$$

The present author [Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **3–4** (1960/61), 177–182; MR **24** #A2006] showed that the exponent $n+1$ can be replaced by n.

Let A be the lim inf of those constants that can replace $8e$ in (*). Erdős (see first reference above) showed that $A > 1.32$; this result was improved by S. Uchiyama [Acta Math. Acad. Sci. Hungar. **9** (1958), 379–380; MR **21** #1294] and the author [ibid. **10** (1959), 405–411; MR **22** #3715]. In the present paper the author shows that $A \geq 4e$.

L. Carlitz (Durham, N.C.)

Yin, Wen-lin 3008

The lattice-points in a circle.

Sci. Sinica **11** (1962), 10–15.

Let $R(x)$ denote the number of lattice points in and on the circle $u^2+v^2 \leq x$. It is well known that $R(x) = x + O(x^a)$ for some $a < 1$. Let b be the greatest lower bound of a. The author outlines an estimation of the trigonometric sums involved to obtain the upper bound $b \leq 12/37$ thus adding another link to the chain of known upper bounds, namely, $b \leq 1/2$, $1/3$, $(1/3)-\epsilon$, $37/112$, $163/494$, $27/82$, $15/46$ and $13/40$, due to Gauss [*Werke*, Bd. II, pp. 269–291, Ges. Wiss., Göttingen, 1863], Sierpiński [Prace Mat.-Fiz. **17** (1906), 77–118], van der Corput [Math. Ann. 87 (1922), 39–65], Littlewood and Walfisz [Proc. Roy. Soc. London Ser. A **106** (1924), 478–488], Walfisz [Math. Z. **26** (1927), 66–88], Nieland [Math. Ann. **98** (1928), 717–736], Titchmarsh [Proc. London Math. Soc. (2) **38** (1934), 96–115] and Hua [Quart. J. Math. Oxford Ser. **13** (1942), 18–29; MR **4**, 190], respectively. The best lower bound known is apparently $b \geq 1/4$, due to Hardy and Landau [Landau, *Vorlesungen über Zahlentheorie*, Bd. II, Achter Teil, Kap. 6, pp. 233–239, Hirzel, Leipzig, 1927] in 1915.

A. C. Woods (Columbus, Ohio)

Novak, B. 3009

Integral points in hyperellipsoids. (Russian)

Dokl. Akad. Nauk SSSR **153** (1963), 762–764.

The author considers exponential sums $A(x) = \sum \exp[2\pi i \sum_{l=1}^{r} \alpha_l \mu_l]$. The summation is extended over the sets $(\mu_1, \cdots, \mu_r)$ satisfying $Q(\mu_1, \cdots, \mu_r) \leq x$, where Q is a positive definite quadratic form with integral coefficients, $r > 4$, and in addition, $\mu_l \equiv b_l \pmod{M_l}$ with given b_l, M_l for all $l = 1, \cdots, r$. For $\alpha_l = b_l = 0$ and $M_l = 1$ for all i, $A(x)$ reduces to the number of lattice points in the hyperellipsoid ("integral volume"). $A(x)$ is then compared with the "generalized integral volume" and various estimates for the error term are given. The results include and improve earlier results by Val'fiš [Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze **20** (1954), 1–20; MR **17**, 133; ibid. **21** (1955), 3–64; MR **18**, 115] and earlier work by Jarník [Časopis Pěst. Mat. Fys. **69** (1940), 57–60; MR **1**, 294].

V. Linis (Ottawa, Ont.)

Karacuba, A. A. 3010

Trigonometric sums of a special type and their applications. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 237–248.

The author proves that if χ is a primitive character modulo D, where $D = p^n$ and (for $0 < \theta < 1$) $c_1 n^{1+2/\theta} \geq$

$\log p \geqq c_2 n^\theta$, then the Dirichlet L-series $\sum_1^\infty \chi(n)/n^s$ does not vanish in the region

$$|s| < c_3, \quad \sigma > 1 - \frac{1}{\log^{\theta/(\theta+1)} D},$$

where c_1, c_2, c_3 are certain positive absolute constants.

The argument is based on I. M. Vinogradov's estimates of exponential sums. *S. Chowla* (University Park, Pa.)

Mori, Mitsuya 3011
Über die rationale Darstellbarkeit der Heckeschen Operatoren.
J. Math. Soc. Japan **15** (1963), 256–267.

Let **Q** and **C** stand for the fields of rational and complex numbers, respectively; $\mathfrak{H}$ the upper half-plane, Γ a Fuchsian group such that $\mathfrak{H}/\Gamma$ has finite measure, $\mathfrak{K}$ is the field of functions (algebraic, of one variable over **C**) automorphic under Γ and $\mathfrak{S}_{2\kappa}(\Gamma)$ the vector space of cusp forms of dimension -2κ. ϕ stands for an indefinite algebra of quaternions over **Q** of discriminant $d(\phi)$, $\mathfrak{o}$ a maximal order of ϕ, $\mathfrak{a}$ a (two-sided) ideal of $\mathfrak{o}$, Γ_Ξ the group of units of $\mathfrak{o}$ of norm 1; $\Gamma_\mathfrak{a} = \{\gamma \in \Gamma_\Xi | \gamma \equiv 1 \pmod{\mathfrak{a}}\}$. It may be shown that $\Gamma_\mathfrak{a}$ is a Fuchsian group. If $\mathfrak{a}$ and $d(\phi)$ are coprime, then $\mathfrak{a} = N\mathfrak{o}$ (N a positive integer), the group $\Gamma_\mathfrak{a}$ is denoted by $\Gamma_N(\mathfrak{o})$ and generalizes the classical principal congruence subgroup $\Gamma(N)$ of level ("Stufe") N. Indeed, if ϕ contains any divisors of zero, then $\phi = M_2(\mathbf{Q})$ (ring of 2×2 matrices over **Q**) and $\Gamma_N(\mathfrak{o})$ is isomorphic to $\Gamma(N)$. The purpose of the paper is to show that the linear transformations of $\mathfrak{S}_{2\kappa}(\Gamma_N(\mathfrak{o}))$, induced by the (generalized) Hecke operators $T(n; N\mathfrak{o})$, may be rationally represented. In the classical case $\phi = M_2(\mathbf{Q})$; this result can be obtained by combining Eichler's trace formula for the Hecke operators T_n [Eichler, Math. Z. **67** (1957), 267–298; MR **19**, 740] with theorems of Petersson [Math. Ann. **116** (1939), 401–412; ibid. **117** (1939), 39–64; MR **1**, 294; ibid. **117** (1940), 277–300; MR **2**, 151]. Here the result is obtained in the full generality stated above. The proof is rather long and somewhat involved, and will not be described here. *E. Grosswald* (Philadelphia, Pa.)

Shanks, Daniel; Wrench, John W., Jr. 3012
The calculation of certain Dirichlet series.
Math. Comp. **17** (1963), 136–154.

The Dirichlet series under consideration are $L_a(s) = \sum(-a/n)n^{-s}$ summed over all positive odd integers n. The symbol is that of Jacobi. The first part of the paper discusses the known methods available for expressing $L_a(s)$ in closed form or as a Fourier series. The cases considered are $a = \pm q$, where q is a square-free integer < 16. The fact that in some cases $L_a(1)$ can be computed from the class number formula is brought out. There is a discussion of the exact evaluation of $L_a(s)$, where s is an integer < 1. Finally, a table of $L_a(s)$ to $30D$ for $s = 1(1)10$ and for a equal to the positive and negative divisors of 18. These tables suffice to present the Dirichlet $L(\chi, s)$ series for all the characters χ mod k, for $k = 8$, 12 and 24. *D. H. Lehmer* (Berkeley, Calif.)

Shanks, Daniel 3013
Supplementary data and remarks concerning a Hardy-Littlewood conjecture.
Math. Comp. **17** (1963), 188–193.

Let $P_a(N)$ be the number of primes of the form $n^2 + a$ for $1 \leqq n \leqq N$. Let $\Pi_a(N)$ be the number of primes $\leqq N$ having $-a$ as a non-residue. Let h_a be the infinite product

$$h_a = \prod_{p>2}\left(1 - \left(\frac{-a}{p}\right)\frac{1}{p-1}\right).$$

According to a conjecture of Hardy and Littlewood, $P_a(N)/\Pi_a(N) \sim h_a$. Previous numerical data on P_a and Π_a were given for $a = 1$, ± 2, ± 3, and 4 [the author, Math. Comp. **14** (1960), 320–332; MR **22** #10960]. This note supplies further data for $a = \pm 5$, ± 6, and ± 7. More precisely, the two functions $P_a(N)$ and $\Pi_a(N)$ and their ratio are tabulated for $N = 10^4(10^4)18\cdot 10^4$. Agreement when $a = \pm 6$ is good. For $a = \pm 5$ and ± 7 no accurate values of h_a are known but the ratios P_a/Π are well-behaved. The most popular form is $n^2 + 7$. There is a discussion of the computation of the constant h_a, and an irregular graph for $-20 \leqq a \leqq 9$ is given. *D. H. Lehmer* (Berkeley, Calif.)

Rosser, J. Barkley 3014
Unexpected dividends in the theory of prime numbers.
Proc. Sympos. Appl. Math., Vol. XV, pp. 259–268. *Amer. Math. Soc., Providence, R.I.*, 1963.

This paper gives some account of numerical investigations of the distribution of primes $< 10^8$. In spite of the results of Littlewood that show that $\pi(x) = \mathrm{li}(x) + O(\mathrm{li}(x^{1/2}))$ and $\theta(x) = x + O(x^{1/2})$ are both false, the author studies the functions

$$\mathrm{PI}(x) = x^{-1/2}(\mathrm{li}(x) - \pi(x))\log x,$$

$$\mathrm{TH}(x) = x^{-1/2}(x - \theta(x)),$$

$$\mathrm{TL}(x) = x^{2/3}\int_x^\infty (\mathrm{TH}(y) - 1)y^{-3/2}\,dy.$$

This last function is remarkably constant for $x < 10^8$, as is shown by a table of its maxima and minima in this range. In fact, the function lies between 1.33 and 2.04. The author considers the combination

$$(1) \quad x^{1/6}[\mathrm{PI}(x) - \mathrm{TH}(x) + 1 - \tfrac{1}{2}x^{-1/2}\,\mathrm{li}(x^{1/2})\log x]\log x = x^{-1/3}\log^2 x\int_c^x (\mathrm{TH}(y) - 1)y^{-1/2}\log^{-2} y\,dy,$$

where $c = 1.653$.

A table of its maxima and minima shows it also to be remarkably constant, fluctuating between 4.17 and 6.88. The author is puzzled by the fact that $\mathrm{TH}(y) - 1$ should be approximately $y^{1/6}$, and when this average value is substituted on the right-hand side of (1), he gets a function that is "approximately constant, and equal to 3". The reviewer can explain some of this difficulty by pointing out that the approximate value 3 results from cancelling $\log^2 x$ against $\log^{-2} y$. If one avoids this simplification, one gets a larger nearly constant function which for $x = 10^8$, for example, is equal to 4.97091. *D. H. Lehmer* (Berkeley, Calif.)

Gel'fond, A. O.; Linnik, Ju. V.; Čudakov, N. G.; Jakubovič, V. A. 3015
Letter to the editor: On an erroneous paper of N. I. Gavrilov. (Russian)
Uspehi Mat. Nauk **17** (1962), *no.* 1 (103), 265–267.

The paper in question appeared in Odess. Gos. Univ. Naučn. Ežegodnik **1961**, no. 2, 7–40, and was reviewed earlier [MR **23** #A867]. Gavrilov has since published an extended version of the paper (see #3016 below).

Gavrilov, N. I. [Гаврилов, Н. И.] 3016

★An asymptotic law for the distribution of prime numbers [Асимптотический закон распределения простых чисел].

Odess. Gosudarstv. Univ., Odessa, 1962. 79 pp. 0.75 r.

This brochure consists of two chapters. In the first, the author expounds some elements of the theory of the Riemann zeta-function (the analytic continuation, the functional equation, the theorems of Hadamard and de la Vallée Poussin) and gives a routine proof of the prime number theorem. At the very end of this Chapter 1 (p. 34), the author is incorrect when saying that the Riemann hypothesis would imply the relation $\pi(x)-x\log^{-1}x=O(x^{1/2})$. In fact, it is known without any hypothesis that $\liminf_{x\to\infty}|\pi(x)-x\log^{-1}x|x^{-1}\log^2 x>0$. The second chapter contains an alleged proof of the Riemann hypothesis. Its material coincides, except for some slight developments, with the author's previous work published in 1961 [Odess. Gos. Univ. Naučn. Ežegodnik **1961**, no. 2, 7–40; MR **23** #A867]. He supposes that the Riemann hypothesis breaks down, i.e., that there exists a zero $\alpha+i\theta_0$ ($\alpha>0$, $\theta_0>0$) of the Riemann Ξ-function, and on this conjecture he investigates two differential equations bound up with $\Xi(s)$ and θ_0. He studies their solutions and derives a number of facts which eventually lead him to a contradiction. The author's arguments are rather disorganized but it is fairly obvious that he never makes any serious use of his main assumption: $\theta_0>0$. Hence, his method cannot be correct, because otherwise it should also function (with $\theta_0=0$) in a proof that $\zeta(\sigma+it)$ does not vanish in $0<\sigma<1$, $-\infty<t<+\infty$, except at a few numerical zeros used in the computations in § 2.

S. Knapowski (Poznań)

Lavrik, A. F. 3017

On the distribution of k-twin primes. (Russian)

Dokl. Akad. Nauk SSSR **132** (1960), 1258–1260.

Die Arbeit beschäftigt sich mit Primzahl-k-Zwillingen, d.h. mit Zahlenpaaren p, q für die p und q Primzahlen sind und $p-q=2k$ gilt. Es bezeichne p_{ki} die i-te Primzahl, für die $p_{ki}+2k$ auch eine Primzahl ist. Dann wird der Beweis folgender dreier Sätze skizziert. Satz 1. Es sei $f(t)$ eine Funktion, für die die Relation $f(t)\to\infty$ $(t\to\infty)$ gilt und für die $(\log^2 t)/f(t)$ monoton zunimmt. Dann gilt für fast alle k $p_{ki}-p_{k(i-1)}>(\log^2 p_{ik})/f(p_{ik})$. Satz 2. Man setze $\alpha_k=\prod_{p|k,p>5}(p-2)/(p-4)$. Dann gilt $p_{ki}-p_{k(i-1)}>\log^2 p_{ki}/\alpha_k f(p_{ki})$ für alle k mit $1\leq k\leq x/2\log x$ bis auf eine Ausnahmemenge mit höchstens $cx/(\log x)^M$ Gliedern, wobei $M>1$, aber sonst beliebig ist und c nur von M abhängt. Satz 3. Man setze $E(x)=(\log^2 x)/f(x)$, $D(x)\ll E(x)$ [d.h. $D(x)=O(E(\varphi))$; es wird vorausgesetzt, daß $E(x)$ monoton gegen unendlich strebt]. Dann existieren für fast alle k im Intervall $1\leq k\leq x/2\log x$ Zahlen N und α' mit $N=[\alpha_k'(\log^2 x)/D(x)]$, $\alpha_k'=\beta\prod_{p|k,p>3}(p-4)/(p-2)$ (β konst.) derart, daß die Ungleichungen $p_{k(i+s)}-p_{k(i+s-1)}>D(x)$ $(s=1,2,\cdots,N)$ gelten. Die nur skizzierten Beweise beruhen auf einem ohne Beweis wiedergegebenen Satz des Verfassers, der sich auf die Anzahl der k-Prim-Zwillinge unterhalb x bezieht. *P. Szüsz* (Zbl **105**, 33)

Schmidt, Wolfgang M. 3018

Metrical theorems on fractional parts of sequences.

Trans. Amer. Math. Soc. **110** (1964), 493–518.

Let $\{x\}$ be the fractional part of x and $\|x\|$ the distance from x to the nearest integer. Let $I_{j,q}$ $(j=1,\cdots,n$; $q=1,2,\cdots)$ be intervals in $[0,1)$ of length $L_{j,q}$. (For present purposes, besides ordinary intervals, also sets of the form $[1-a,1)\cup[0,b)$ will be called intervals of length $a+b$ if $a+b<1$.) The following metric theorem is the main result of the paper. Let $P_1,\cdots,P_n$ be n non-constant polynomials with integer coefficients and $\psi(h)=\sum_{q=1}^h\prod_{j=1}^n L_{j,q}$. Denote by $N(h;\alpha_1,\cdots,\alpha_n)$ the number of integers q, $1\leq q\leq h$ with $\{\alpha_jP_j(q)\}\in I_{j,q}$. If the intervals are decreasing, i.e., $I_{j,1}\supset I_{j,2}\supset\cdots$, then, for each $\varepsilon>0$,

$$N(h;\alpha_1,\cdots,\alpha_n)=\psi(h)+O(\psi(h)^{1/2+\varepsilon})$$

for almost all real n-tuples $\alpha_1,\cdots,\alpha_n$.

If $\theta_j\in\bigcap_{q=1}^\infty I_{j,q}$ one can look on the problem as one concerning the simultaneous approximation of $\theta_1,\cdots,\theta_n$ by $\alpha_1P_1,\cdots,\alpha_nP_n$. The above theorem unifies and generalizes a large number of special cases which had been proved by Khintchine [Math. Ann. **92** (1924), 115–125], Cassels [Proc. Cambridge Philos. Soc. **46** (1950), 209–218; MR **12**, 162], Szüsz [Acta Math. Acad. Sci. Hungar. **9** (1958), 177–193; MR **20** #1671], LeVeque [J. Reine Angew. Math. **202** (1959), 215–220; MR **22** #12090], Erdős [Acta Arith. **5** (1959), 359–369; MR **22** #12091] and the author [Canad. J. Math. **12** (1960), 619–631; MR **22** #9482].

The proof is based on an application of Chebyshev's inequality, and careful analysis is needed for the estimate of the variance of an approximation to $N(h;\alpha_1,\cdots,\alpha_n)$.

In the special case $P_j(q)=q$, $I_{j,q}=I_j$, $q=1,2,\cdots,j=1,\cdots,n$ the error term $\psi(h)^{1/2+\varepsilon}$ may be replaced by $(\log h)^{n+1+\varepsilon}$. This is derived from an estimate for

$$\sum_{1\leq q_i\leq h}\left(q_1,\cdots,q_n\left\|\sum_{i=1}^n\alpha_ia_i(q)+\theta\right\|\right)^{-1}$$

for arbitrary fixed increasing sequences of integers $a_i(q)$ $q=1,2,\cdots$ and fixed θ.

There is also a discussion of the case where the restrictions $\{\alpha_jP_j(q)\}\in I_{j,q}$ are replaced by restrictions of the form $\{\sum_{j=1}^n\alpha_jP_j(q_j)\}\in I_{q_1\cdots q_n}$ for intervals $I_{q_1\cdots q_n}$.

H. Kesten (Ithaca, N.Y.)

Stemmler, Rosemarie M. 3019

The ideal Waring theorem for exponents 401–200,000.

Math. Comp. **18** (1964), 144–146.

The so-called ideal Waring theorem states that for each integer k every positive integer is the sum of $2^k+[(3/2)^k]-2$ non-negative kth powers, provided the fractional part of $(3/2)^k$ is less than $1-(3/4)^k$. Whether there is a k for which this proviso fails is an unsolved problem. The author shows that such a k must exceed 200000. A table gives the distribution of the fractional parts of $(3/2)^n$ into eighths for $n\leq k$ and for $k=100(100)\ 1000(1000)\ 10000(10000)200000$. The author feels that it is highly unlikely that the above proviso ever fails.

D. H. Lehmer (Berkeley, Calif.)

Siraždinov, S. H.; Azlarov, T. A. 3020

On a uniform local theorem. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1963**, no. 2, 32–37.

Let s be a positive integer and let R denote the number of representations of the integer N in the form $N=x_1^s+x_2^s+\cdots+x_n^s$, where $x_k=0, 1, 2, \cdots, p_k$. It is proved that there is an $n_0(s)$ such that as $n\geqq n_0(s)$ uniformly with respect to N

$$R=\frac{\prod_{k=1}^{n}(p_k+1)}{B_n}\left[\varphi\left(\frac{N-A_n}{B_n}\right)+\theta(s)\frac{1}{\lambda_n}\right].$$

Here, $A_n=a_1+\cdots+a_n$, $a_k=\sum_{m=0}^{p_k} m^s/p_k+1$, $B_n{}^2=\sigma_1{}^2+\cdots+\sigma_n{}^2$,

$$\sigma_k{}^2=\sum_{m=0}^{p_k}(m^s-a_k)^2/p_k+1,\ \varphi(x)=(2\pi)^{-1/2}\exp\{-\tfrac{1}{2}x^2\},$$

$\lambda_n=B_n\cdot\min_{1\leq k\leq n}(1/p_k{}^s)$ and $|\theta(s)|\leqq c(s)$, a constant depending on s only.

This generalizes and improves former results of A. G. Postnikov [Izv. Akad. Nauk SSSR Ser. Mat. **20** (1956), 751–764; MR **19**, 252] where the cases $s=1$, and $s=2$, $p_1=p_2=\cdots=p_k=\cdots$ had been considered.

H.-E. Richert (Syracuse, N.Y.)

Weiss, Edwin 3021

★Algebraic number theory.

McGraw-Hill Book Co., Inc., New York-San Francisco-Toronto-London, 1963. xii+275 *pp.* $9.95.

This book provides an introduction to the theory of algebraic numbers along the lines of Artin's often quoted (but now alas unobtainable) lecture notes *Algebraic numbers and algebraic functions*, I [Inst. Math. Mech., New York Univ., New York, 1951; MR **13**, 628], or of his smaller *Theory of algebraic numbers* [G. Striker, Göttingen, 1959; MR **24** #A1884]. As such, it is a useful addition to the literature and provides a good introduction for those who wish to learn something of modern algebraic number theory. Moreover, there is a large collection of exercises, some as "theorem examples" in the text and many at the ends of the chapters, which should prove to be of value to anyone seeking to gain practice and understanding in the ideas of the subject and familiarity with new definitions and techniques.

The author maintains that "the reader who has completed a full year van der Waerden type course in algebra should encounter no difficulty". Certainly he will see the power of the methods of abstract algebra in use in a highly interesting concrete situation and may well appreciate their significance for the first time. He will also find himself driven to consult works on topological groups and multilinear algebra, but such is the subject matter of the book that he may well do so with enthusiasm.

The book differs from Artin's in at least one particular; though the development is axiomatic and applies to a large extent to both number fields and to fields of algebraic functions in one variable over finite constant fields, the parallel development is not always carried out systematically. In consequence, opportunities for deepening insight are sometimes missed. For example, it is helpful (provided that the reader is familiar with fields of analytic functions of a complex variable) to think of the concepts of valuation ring and place in terms of regular functions and the value taken by a function, or of the ramification index in terms of the winding number (minus 1). In many instances the analogy is made explicitly and there are illuminating examples throughout. Some idea of the scope of the book will appear from the following summary.

Chapter 1 (Elementary valuation theory) begins with the theory of valuations, the Hausdorff topology determined by a valuation, the approximation theorem and independence of finite collections of valuations. Archimedean and non-archimedean prime divisors (defined as equivalence classes with respect to equivalence of valuations), valuation ring and residue class field follow. The rational numbers are discussed in the light of these concepts, including the product formula for canonical valuations, and the rational function field is set as an exercise. The problem of extending a valuation of a given field, F, to a valuation of an extension field, E, is introduced together with the related concepts of ramification index and residue class degree. The chapter concludes with an account of the theory of completions for archimedean and non-archimedean valuations, the former by means of Ostrowski's theorem.

Chapter 2 (Extension of valuations) begins with the problem of the existence and uniqueness of extensions of valuations for complete fields and is settled in the archimedean case by Ostrowski's theorem and in the non-archimedean case by Hensel's lemma. The general case is discussed by appealing to the complete case. The remainder of the chapter is devoted to showing how the extensions of a local field "fit together". If P is a prime divisor of F and F_P the completion of F at P, and if Q_i $(1\leqq i\leqq r)$ are the extensions of P to E, with corresponding completions E_{Q_i}, then the degree of E/F is related to the degrees of the local extensions E_{Q_i}/F_P. This is proved without assuming separability; in the case of a separable extension (for example $F=\mathbf{Q}$) the global degree is the sum of the local degrees.

Chapter 3 (Local fields) is concerned with a detailed study of the local field F_P and its finite extensions E_{Q_i}, and begins with Newton's polygon and Newton's method, illustrated by examples of factorization in p-adic fields. Unramified extensions are now defined and the concept of the inertia field (largest unramified subfield) introduced. Tamely ramified extensions, wildly ramified extensions and ramification field (largest tamely ramified field) follow. The inertia and ramification groups are defined, and the Frobenius automorph makes its appearance as a generator of the Galois group of an unramified extension. The different, discriminant and differential exponent are introduced, and the chapter concludes with Hilbert's formula for the differential exponent. (Hilbert's formula expresses the differential exponent in terms of the orders of the ramification groups.)

Chapter 4 (Ordinary arithmetic fields) takes up the questions of unique factorization and ideal theory (from the valuation-theoretic point of view). An ordinary arithmetic field (OAF) is a pair $\{F, \mathscr{S}\}$ where F is a field, $\mathscr{S}$ a (non-empty) collection of discrete prime divisors of F such that: (1) for $a\in F$, $\nu_P(a)\geqq 0$ for almost all $P\in\mathscr{S}$ ($\nu_P(a)$ is the order of a at P); (2) (strong approximation or Chinese remainder theorem) for P_1, $P_2\in\mathscr{S}$, $\exists a\in F$: $\nu_{P_1}(a-1)\geqq 1$, $\nu_{P_2}(a)\geqq 1$, $\nu_P(a)\geqq 0$ for all other $P\in\mathscr{S}$. The ring $O(F, \mathscr{S})=\{a\in F: \nu_P(a)\geqq 0, \forall P\in\mathscr{S}\}$ is the ring of integers of $\{F, \mathscr{S}\}$. (For example, we may take F to be a finite extension of $\mathbf{Q}$ and $\mathscr{S}$ the set of non-archimedean prime divisors of F, O is then the ring of algebraic integers of F. Again, take F to be a finite extension of the

transcendental field $k(x)$ and $\mathscr{S}$ the set of primes of F other than those which lie above x^{-1}, then O is the ring of integral algebraic functions of F which have no pole "outside infinity".) Ideals and divisors are defined, the connection between the two established and unique decomposition proved. It is shown that the concept of an OAF is essentially that of a Dedekind ring. For OAF's the isomorphism between the group of divisor classes and the group of ideal classes is established, the common order is the class number. The different and discriminant for a separable extension E/F are defined and the connection between the global theory and the local theory of the previous chapter is made. A prime ideal of F is ramified in E if and only if it divides the discriminant, and results on the factorization of prime ideals are obtained. Finally, the decomposition group (the subgroup of the Galois group of E/F which leaves a given prime ideal $\mathfrak{Q}$ fixed) of $\mathfrak{Q}$ over F is defined and the fundamental connection between this group, the Galois group of the related extension $E_{\mathfrak{Q}}/F_{\mathfrak{P}}$ and the ramification groups established. The Frobenius automorph corresponding to this prime ideal is defined and the Artin symbol introduced.

Chapter 5 (Global fields). A global field is either a number field of finite degree over $\mathbf{Q}$ or a function field in one variable over a finite constant field. In order to study the units of a number field or to find a formula for the class number (and to investigate the wider issues of class field theory and analytic questions) one must look at the field in its global setting, that is, at all the local fields together. To this end the adèle ring $\mathbf{F}$ (valuation vectors, repartitions) of such a field is constructed and the idèles are the invertible elements of the adèle ring. Principal adèles, and so on, are defined and the relation between idèles and ideals explained. The structure of the group of units is obtained and the finiteness of the class number proved. The class number formula is derived in the case of number fields. The topological structure of $\mathbf{F}$ is discussed.

In Chapters 6 and 7, the examples of quadratic fields and cyclotomic fields are presented as illustrations of the foregoing. Quadratic fields are studied in some detail (integral bases, discriminant, ideals, units, class number and norm residue symbol) and the reader may well find it helpful to turn to this chapter whilst reading the preceding ones. Chapter 7 introduces the reader to the theory of cyclotomic fields and, one hopes, whets his appetite for more. *J. V. Armitage* (Durham City)

Nečaev, V. I. 3022

A best-possible estimate of trigonometric sums for recurrent functions with non-constant coefficients. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 520–522.

The investigation of general recurrences over an algebraic number field, K, initiated by the author in an earlier paper [same Dokl. **152** (1963), 275–277; MR **27** #5748] is continued. Such a recurrence is defined by $\psi(x)=a_1(x)\psi(x-n)+\cdots+a_n(x)\psi(x-1)$, where the a_i are periodic modulo a prime ideal, ρ. With certain restrictions on the characteristic polynomial of ψ, sums of the form $\sum_{y=1}^{y}\exp(2\pi i S(\psi(x)\cdot\omega))$ are considered when S is the trace and $\omega=\lambda/\sigma'\rho$, σ' being the different and λ an appropriate ideal. Let q be the minimal common period of the a_i, and τ the primitive period of ψ; select ψ with maximal τ. Then the modulus of the sum will not exceed $qN(\rho)^{n/2}$ if $y=\tau$ or this value multiplied by $(1+\log q+n\log N(\rho))$ fo[r] $1\leq y\leq\tau$. (N is the norm in K.)

Modifications of the sum are also considered and th[e] best-possible character of the result is exhibited. Th[e] proofs are omitted. *J. D. Swift* (Los Angeles, Calif[.])

Rieger, G. J. 302[3]

Über die Folge der Zahlen der Gestalt p_1+p_2.

Arch. Math. **15** (1964), 33–41.

A famous theorem of Schnirelmann (1930) asserts that

$$A(n:n\leqq x,\ n=p_1+p_2)>Cx.$$

Here p_1,p_2 are primes, $A(*,\cdots)$ the number of $*$ wit[h] the properties $\cdots$, and n is a natural number. The autho[r] generalizes the above theorem to Theorem 1: For every [σ] with $0<\sigma\leqq\frac{1}{2}$, every α with $\frac{1}{2}<\alpha<1$ and (recall Hoheisel' theorem)

$$\pi(x+x^\alpha)-\pi(x)>\frac{x^\alpha}{2\log x}\,[x>C_7(\alpha)>1]$$

and every x such that

$$x>C_{10}(\sigma,\alpha)=\mathrm{Max}[(2/\sigma)^{1/(1-\alpha)},8C_7(\alpha)];$$

we have

$$A(n:n\leqq x,\ n=p_1+p_2,\ |p_1-\sigma n|<n^\alpha)\geqq C_{11}\sigma^2x,$$

where C_{11} is an absolute positive constant.

This is generalized in different ways, in particular, t[o] primes in arithmetic progressions. The author also sharpen[s] a well-known theorem of Rényi, which asserts that ever[y] large even number is a sum of a prime and an "almost" prime. *S. Chowla* (University Park, Pa.)

Rieger, G. J. 302[4]

Verallgemeinerung der Siebmethode von A. Selberg au[f] algebraische Zahlkörper. III.

J. Reine Angew. Math. **208** (1961), 79–90.

In Parts I and II [same J. **199** (1958), 208–214; MR **2**[0] #3115; ibid. **201** (1959), 157–171; MR **24** #A1903], the author concerned himself with the distribution of ideal[s] in an algebraic number field K having certain divisibility properties. Now he turns to the distribution of the elements themselves. The general sieve-type inequality for ideals, which was Theorem 1 of I, extends immediately to give an upper bound for the number of integers of a subset of K not divisible by any of a specified set o[f] prime ideals of K. In case $|\xi^{(i)}|<|\zeta^{(i)}|$ for $i=1,\cdots,n$, he writes $\xi\prec\zeta$. Oversimplifying considerably, the basic theorem permits him to estimate from above (Satz 15) the number of integers ξ such that $\xi\prec\zeta$, the principal ideal $\langle\xi\rangle$ is prime, and $\xi\equiv\gamma\pmod f$; (Satz 16) the number of solutions of $\xi+\eta=\zeta$ with $\langle\xi\rangle$ and $\langle\eta\rangle$ prime and $\xi,\eta\prec\zeta$; (Satz 17) the number of ξ with $\xi\prec\zeta$ such that both $\langle\xi\rangle$ and $\langle\xi+\zeta\rangle$ are prime for fixed ζ; (Satz 21) the number of ξ with $\xi\prec\zeta$, $\xi\equiv\gamma\pmod f$ such that $\mathfrak{p}\nmid\xi$ for all prime ideals $\mathfrak{p}$ with $N\mathfrak{p}\leqq z$.

W. J. LeVeque (Boulder, Colo.)

Vinogradov, A. I. 3025

Bounds from below by the sieve method in algebraic number fields. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 13–15.

The "sieve method" is generalized to problems of algebraic number theory. Amongst the generalizations discussed are those of Goldbach's conjecture (every even number $\geqq 4$

is a sum of two primes), and the zero-free regions of the Dedekind zeta function. A full account will presumably appear later, and is awaited with interest.

S. Chowla (University Park, Pa.)

Masuda, Katsuhiko 3026

Note on characters of the groups of units of algebraic number fields.

Tôhoku Math. J. (2) **13** (1961), 248–252.

Soient k un corps (de degré fini) de nombres algébriques, I le groupe des idèles de k, C le groupe des classes d'idèles, D la composante connexe de l'élément neutre de C. On identifie, comme d'habitude, le groupe P des idèles principaux avec le groupe multiplicatif k^* de k et, $\mathfrak{p}$ étant un diviseur premier de k, on identifie le groupe $I_\mathfrak{p}$ des idèles a, dont la composante $(a)_\mathfrak{q}$ est $=1$ pour tout diviseur premier $\mathfrak{q}=\mathfrak{p}$ de k, avec le groupe multiplicatif $k_\mathfrak{p}^*$ du corps local $k_\mathfrak{p}$ de k. Si Y est un sous-groupe de I organisé par la topologie induite, un caractère du groupe topologique Y est dit G-caractère ou D-caractère s'il est la restriction à Y d'un caractère de C respectivement d'un caractère de C/D (considérés comme caractères de I).

Soit E_0 le groupe des idèles a de k tels que: (1) $(a)_\mathfrak{p}=1$ pour tout diviseur archimédien $\mathfrak{p}$ de k; (2) $(a)_\mathfrak{p}$ est une unité pour tout autre diviseur $\mathfrak{p}$ de k. Soit T un ensemble arbitraire de diviseurs non-archimédiens de k, et soit T^* l'endomorphisme de I tel que $(T^*\cdot a)_\mathfrak{p}=(a)_\mathfrak{p}$ ou $=1$ selon que $\mathfrak{p}\in T$ ou $\mathfrak{p}\notin T$. Soit $E_T=E_0\cap T^*\cdot I$. Si χ est un caractère de E_0 et si $\mathfrak{p}$ est un diviseur non-archimédien de k, $\mathfrak{p}$ sera dit ramifié par χ si la restriction de χ à $E_{(\mathfrak{p})}$ n'est pas triviale. Soit $V(\chi)$ l'ensemble (fini) des $\mathfrak{p}$ ramifiés par χ. Soit A l'extension abélienne maximale de k, et soit G le groupe de Galois de A/k. La théorie des corps de classes définit un homomorphisme canonique σ de I sur G.

Soit, pour un $a\in I$, $Z(a)=\{\mathfrak{p};\ (a)_\mathfrak{p}\neq 1\}$, et soit J_0 l'ensemble (qui est un groupe) des aI tels que $Z(a)$ ne contienne aucun diviseur archimédien et ait la densité de Kronecker nulle. L'auteur a prouvé antérieurement que l'application canonique de J_0 dans C/D est injective. Ceci entraîne, si T est un ensemble de diviseurs non-archimédiens, dont la densité de Kronecker est nulle, que σ induit sur E_T un isomorphisme, d'où resulte que tout caractère de E_T en est un D-caractère. Il n'est pas, toutefois, toujours possible d'étendre un caractère χ de E_T jusqu'à un D-caractère χ' de E_0 tel que $V(\chi')\subseteq T$. L'auteur montre que ceci est possible si, et seulement si, le noyau de χ contient $F^*=V(\chi)^*\cdot F$, où F est le groupe des unités totalement positives de k (considéré comme un sous-groupe de $P\subset I$).

Soient M_T le composé des corps d'inertie de A/k pour les $\mathfrak{p}\in T$ et K_T/k la plus grande sous-extension de A/k ramifiée seulement en $\mathfrak{p}\in T$. Il résulte alors, du résultat précédent que le groupe de Galois de $A/\overline{K_TM_T}$ est canoniquement isomorphe avec la fermêture $\overline{T^*\cdot F}$ de $T^*\cdot F$ dans E_T.

M. Krasner (Paris)

FIELDS AND POLYNOMIALS

See also 3021, 3055, 3104, 3106, 3133, 3187.

Carlitz, L. 3027

Note on a problem of Dickson.

Proc. Amer. Math. Soc. **14** (1963), 98–100.

Let q be a power of an odd prime and $F(x)=a_0x^k+\cdots+a_k$ ($a_i\in GF(q)$, $a_0\neq 0$) be a polynomial such that $F(\alpha)$ is a (non-zero) square of $GF(q)$ for all $\alpha\in GF(q)$. The author has proved the existence of a number N_k such that if $q>N_k$, then $F(x)=H(x)^2$ ($H(x)\in GF[q,x]$) [Duke Math. J. **14** (1947), 1139–1140; MR **9**, 337; ibid. **19** (1952), 471–474; MR **14**, 539]. The author proves in this paper the existence of a polynomial $F(x)=x^{(q-1)/2}+c$ ($c\in GF(q)$) (for $q>17$) such that $F(\alpha)$ is a non-zero square of $GF(q)$ for all α in $GF(q)$, and claims that this implies $N_k>2k+1$. {But the reviewer thinks that what he proves is $N_{(q-1)/2}\geqq q$.} The author indicates that the case in which $q=4m+1$ and $F(x)=x^m+c$ can be treated similarly.

T. Hayashida (Yokohama)

Carlitz, L. 3028

The Euler-Fermat property in a polynomial domain.

Rev. Mat. Hisp.-Amer. (4) **23** (1963), 194–195.

If F is an arbitrary field, and if for any two relatively prime polynomials A, M in the polynomial domain $F[x]$ the equation (1) $A^e\equiv 1 \pmod M$ always holds, where e is a positive integer, $F[x]$ is said to possess the Euler-Fermat property. If (1) holds with e independent of A, we shall say that $F[x]$ has the strong Euler-Fermat property. Further, let F_p be the prime field of characteristic p and Ω_p the algebraic closure of F_p. The author proves the following two theorems. Theorem 1: The polynomial domain $F[x]$ has the Euler-Fermat property if and only if F is a subfield of some Ω_p. Theorem 2: The polynomial domain $F[x]$ has the strong Euler-Fermat property if and only if F is finite.

J. W. Andrushkiw (S. Orange, N.J.)

Mills, W. H. 3029

Polynomials with minimal value sets.

Pacific J. Math. **14** (1964), 225–241.

Let K be a finite field of characteristic p that contains exactly q elements. Let $F(x)$ be a polynomial over K of degree $f>0$ and let $r+1$ denote the number of distinct values $F(\tau)$ as τ ranges over K. Then $r\geqq[(q-1)/f]$. The question discussed in the present paper is the following: For what polynomials $F(x)$ does one have (*) $r=[(q-1)/f]\geqq 2$; the cases $r=0$ and $r=1$ are special. Carlitz, Lewis, Mills and Straus [Mathematika **8** (1961), 121–130; MR **25** #3038] determined all polynomials with $f<2p+2$ for which (*) holds. In the present paper all polynomials with $f\leqq\sqrt{q}$ for which (*) holds are determined. They are of the form $F(x)=\alpha L^v+\gamma$, where L is a polynomial that factors into distinct linear factors over K and that is of the form $L=\beta+\sum_i\phi_ix^{p^{ki}}$, where $v\mid p^k-1$ and q is a power of p^k. In particular, the case $q=p^2$ is now disposed of.

L. Carlitz (Durham, N.C.)

Horáková, Kornélia; Schwarz, Štefan 3030

Cyclic matrices and algebraic equations over a finite field. (Russian. English summary)

Mat.-Fyz. Časopis Sloven. Akad. Vied **12** (1962), 36–46.

Let $Z(a_0,a_1,\cdots,a_{r-1};n)$ be the $n\times n$ cyclic matrix

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_{r-2} & a_{r-1} & 0 & 0 & \cdots & 0\\ 0 & a_0 & \cdots & a_{r-3} & a_{r-2} & a_{r-1} & 0 & \cdots & 0\\ \vdots \\ a_1 & a_2 & \cdots & a_{r-1} & 0 & 0 & 0 & \cdots & a_1 \end{bmatrix}$$

with $a_i \in GF(q)$, $q = p^s$ ($s \geqq 1$, p prime). Given the polynomial $f(x) = a_0 + a_1 x + \cdots + a_{q-2} x^{q-2}$ ($a_0 \neq 0$) over the finite field $GF(q)$, the problem is to find the number σ_i of different irreducible factors of $f(x)$ of degree i. The following generalization of the König-Rados theorem [Rados, J. Reine Angew. Math. **99** (1886), 258–260] is proved:

$$\sigma_i = (1/i) \sum_{k|i} \mu(i/k)(q^k - 1 - h_k),$$

where μ is the Möbius function and $h_k = \text{rank } Z(a_0, a_1, \cdots, a_{q-2}; q^k - 1)$. The paper contains also the generalizations of Hurwitz's formula [Dickson, *History of the theory of numbers*, Vol. I, p. 232, Stechert, New York, 1934] and of a theorem by Rédei and Turán [Acta Arith. **5** (1959), 223–225; MR **21** #4148]. {In the formulation of Theorem 1 the whole line is missed.} *D. Ž. Djoković* (Belgrade)

Reufel, Manfred 3031

Spezialisierungen in Polynomringen.

Bonn. Math. Schr. No. 19 (1963), xxiii + 165 *pp.*

If σ is a homomorphism from the integral domain R to the integral domain S, then it is extendable (in the natural way) to a mapping σ^* of the set of all ideals of the polynomial domain $K[x_0, \cdots, x_n]$ to the set of all ideals of $L[y_0, \cdots, y_n]$, where K and L are the quotient fields of R and S, respectively. (σ^* is called a specialization in $K[x_0, \cdots, x_n]$.) In this work (a dissertation under the direction of W. Krull) the author investigates the relation between the ideal-theoretic structure of an ideal A in $K[x_0, \cdots, x_n]$ and that of its image, $\sigma^*(A)$, when R is a valuation ring of K and σ is a place (Stelle) of R. For handling homogeneous polynomials the concepts of "canonical algorithm" (of which Kapferer's elimination method is an example) and "H-invariant property" are developed, and the algorithm is used to show that the "most important" ideal properties, such as perfectness, are H-invariant. These concepts are, however, too complex to define here. Specializations in a graded module, the specialization of homogeneous ideals, and that of inhomogeneous ideals are studied, and it is proved, for example, that if $\sigma^*(A)$ is a prime (radical) ideal then so is A. *W. E. Deskins* (E. Lansing, Mich.)

Schinzel, A. 3032

Reducibility of polynomials in several variables.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 633–638.

Soient $X_1, \cdots, X_n$ des ensembles finis et disjoints d'indéterminées, et $F_i(X_i)$ ($i = 1, \cdots, n$) et $P(v_1, \cdots, v_n)$ des polynômes sur un corps K. On suppose que P est de degré > 0 par rapport à chaque v_i et qu'au moins deux des F_i sont non-constants. Alors le polynôme $P(F_1(X_1), \cdots, F_n(X_n))$ est réductible sur K si et seulement si il existe des polynômes G_i, H_i sur K tels que $F_i(X_i) = G_i(H_i(X_i))$ ($i = 1, \cdots, n$) et que $P(G_1(u_1), \cdots, G_n(u_n))$ soit réductible sur K. La démonstration utilise les fonctions symétriques, le théorème de Lüroth, et la méthode de substitution de Kronecker. *P. Samuel* (Paris)

Keller, Ott-Heinrich 3033

Zur Berechnung des Galois'schen Körpers und der Galois'schen Gruppe einer Gleichung in erträglich vielen Schritten.

Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Natur. Kl. **104**, *no.* 5, 23 *pp.* (1961).

L'auteur se propose de simplifier et de raccourcir le calcul du groupe de Galois d'une équation. Il fait la remarque (probablement inédite) que, si K est un corps de caractéristique 0 (bien que l'auteur dit simplement "corps") ne contenant pas les racines p-ièmes de l'unité (où p est premier), et si $\{\alpha_1, \alpha_2, \cdots, \alpha_q\}$ est un ensemble arbitraire de zéros d'un polynôme irréductible $f(x) \in K[x]$ de degré p, toute combinaison linéaire $\vartheta = \sum_{i=1}^{q} u_i \alpha_i$ des α_i avec les coefficients $u_i \in K$ inégaux deux à deux est un élément primitif de l'extension $K(\alpha_1, \alpha_2, \cdots, \alpha_q)/K$. Le reste du travail est une exposition de certains paragraphes de la "Moderne Algebra" de van der Waerden et de la "Théorie de Galois" de Tchebotarew, émaillée de quelques remarques triviales. Ces remarques ne permettent pas, en général (c'est-à-dire sauf quand le polynôme $f(x)$ est choisi ad hoc), de simplifier ou raccourcir le calcul du groupe de Galois, car, si un polynôme $\in K[x]$ est donné arbitrairement, l'auteur n'indique aucun procédé pour voir si l'on se trouve dans le cas, où une quelconque de ces remarques s'applique. Ainsi, sauf le premier résultat cité (qui ne semble, non plus, apporter grand chose pour le calcul du groupe de Galois), le niveau du travail diffère peu de celui d'un exercice scolaire. Le travail finit par quelques exemples de calcul (polynômes de degré 3 et 5) permettant l'application des remarques de l'auteur. *M. Krasner* (Paris)

Kreimer, H. F. 3034

The foundations for an extension of differential algebra.

Trans. Amer. Math. Soc. **111** (1964), 482–492.

A number of recent papers have dealt with a notion of "ring with operators", at once generalizing the notion of "differential ring" and that of "difference ring". In the present paper, the author defines the notion of "M-ring". In effect, an M-ring is a commutative ring R on which the elements of a set M operate so that $m(a+b) = ma + mb$ and $m(ab) = \sum_{n,r \in M} c_{mnr}(na)(rb)$ ($a, b \in R$, $m \in M$), the c_{mnr} being elements of R satisfying certain conditions. When suitably specialized, this yields differential rings, rings with higher derivations, difference rings, and mixed differential-difference rings, with operators which need not commute; an example shows that even these are not all. The appropriate notions of M-homomorphism, M-ideal, M-subring, etc., are defined. Conditions are given under which two M-extensions of an M-field R can be embedded in a single M-extension of R, and under which an M-isomorphism of R into an M-extension S of R can be extended to an M-isomorphism of S into an M-extension of S. *E. R. Kolchin* (New York)

ABSTRACT ALGEBRAIC GEOMETRY

Blackburn, K. 3035

An alternative approach to multiplicity theory.

Proc. London Math. Soc. (3) **14** (1964), 115–136.

The author proposes a new method of dealing with multiplicities in general noetherian rings, which uses neither the Hilbert function nor homological methods. Generally speaking, the idea is to start with one-dimensional

multiplicities and to use the associative law to define them in the general case. The starting point is the study of the function $e_R(x|M)=L_R(M/xM)-L_R(0:x)$, where R is a noetherian ring with unit, M is a finitely-generated (unitary) R-module, $L_R(A)$ is the length of an R-module A in the usual sense, and where it is assumed that $L_R(R/(x))<\infty$. If $0\to M'\to M\to M''\to 0$ is an exact sequence of finitely-generated R-modules, then $e(x|M)=e(x|M')+e(x|M'')$. For any positive integer k, $e(x^k|M)=ke(x|M)$. Let $P_1,\cdots,P_s$ be the minimal prime ideals of the zero submodule of the finitely-generated R-module M and let $N_1,\cdots,N_s$ be the corresponding primary components. Then $e(x|M)=\sum_{i=1}^s l_i e(x|R/P_i)$, where l_i is the length of a P_i-primary composition series from N_i to M. Now let $x\in R$ and let P, P' be prime ideals such that $P\subset P+(x)\subseteq P'$ and P' is a minimal prime ideal of $P+(x)$. Under these circumstances, define $\lambda(P|x|P')$ to be the length of the P'-primary component of $P+(x)$. This function does the trick for the one-dimensional case and behaves decently on transition to residue class rings and rings of quotients. For the general case, let $P\subset P'\subset R$ and let $x_1,\cdots,x_s$ be elements of R. Suppose P' is a minimal prime ideal of $P+(x_1,\cdots,x_s)$ and that rank $(P'/P)=s$. Then the multiplicity symbol $\mu(P|x_1,\cdots,x_s|P')$ is defined as $\sum_{(P_i)}\lambda(P_0|x_1|P_1)\lambda(P_1|x_2|P_2)\cdots\lambda(P_{s-1}|x_s|P_s)$, where (P_i) denotes a set of prime ideals $P_0,P_1,\cdots,P_s$ such that $P=P_0\subset P_1\subset\cdots\subset P_s=P'$ and $(x_1,\cdots,x_i)\subseteq P_i$. The remainder of the paper is devoted to the basic properties of the symbol μ, including two forms of the associative law.

W. E. Jenner (Chapel Hill, N.C.)

Lascu, Alexandru T. 3036

The order of a rational function at a subvariety of an algebraic variety.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 378–384.

Soient V une variété algébrique abstraite (au sens de Weil), $K(V)$ le corps des fonctions rationnelles sur V, U une sous-variété de V, et $\mathfrak{o}$ l'anneau local de U sur V. On suppose que l'anneau gradué associé gr($\mathfrak{o}$) est intègre, de sorte que la fonction d'ordre de $\mathfrak{o}$ est induite par une valuation v_U de $K(V)$. L'entier $v_U(x)$ s'appelle l'ordre de la fonction rationnelle $x\in K(V)$ en U. Alors, sur la transformée monoïdale V' de V de centre U, l'image réciproque U' de U est une sous-variété simple de codimension 1, et v_U est la valuation correspondante. De plus, si l'anneau $\mathfrak{o}$ est factoriel, et si $(x)=\sum_i a_iH_i$ est la décomposition du diviseur de $x\in K(V)$ en composantes irréductibles H_i, on a $v_U(x)=\sum_i a_i m(U,H_i)/m(U,V)$, où $m(A,B)$ désigne la multiplicité de la sous-variété A sur la variété B (on pose $m(A,B)=0$ si $A\not\subset B$).

P. Samuel (Paris)

Abhyankar, Shreeram 3037

Uniformization in p-cyclic extensions of algebraic surfaces over ground fields of characteristic p.

Math. Ann. **153** (1964), 81–96.

This paper consists of a proof of the theorem: Let K be a two-dimensional algebraic function field over an algebraically closed field k of characteristic $p\neq 0$, let K^* be a Galois extension of K of degree p, let w be a rational nondiscrete (or irrational, as well) valuation of K/k having only one extension w^* to K^*, and assume that w can be uniformized. Then w^* can be uniformized. This theorem was originally proved by the author as the most delicate step in his Local Uniformization Theorem for algebraic surfaces of characteristic p [cf. Ann. of Math. (2) **63** (1956), 491–526; MR **17**, 1134]. The present proof is similar to the original one in nature, but is presented in a more transparent fashion. The essential modification consists in a better formulation of induction with respect to suitably selected exponents which appear in a certain canonical form of the equation for the extension K^*/K. A proof of the same statement for a purely inseparable extension K^*/K is also included in the paper.

H. Hironaka (Waltham, Mass.)

Otsuka, Kayo 3038

On existence of a resolved surface of a singular surface.

J. Math. Kyoto Univ. **3** (1963), 81–84.

Dwork, Bernard 3039

On the zeta function of a hypersurface.

Inst. Hautes Études Sci. Publ. Math. No. 12 (1962), 5–68.

The author continues his strikingly original work on the zeta function of an algebraic variety defined over a finite field. Here he considers a nonsingular hypersurface H of degree d in projective n-space over the field $k=GF(q)$. The Weil conjectures predict that its zeta function has the form

$$(1)\qquad \zeta(H,t)=P(t)^{(-1)^n}/\prod_{0}^{n-1}(1-q^it),$$

where $P(t)$ is a polynomial of predicted degree. This paper gives a proof covering all cases except when char$(k)=2$ and d is even. The object studied is a certain linear operator α acting on a space of p-adic power series, whose connection with the ζ-function is taken over from an earlier paper (cited below), with the addition of (2) below. A partial spectral theory is developed for α here, and a key role is also played by a Koszul-type complex associated with a set of differential operators on the space of power series. Some theorems in p-adic analysis are included.

Let Q_p be the rational p-adic field, and Ω the completion of the algebraic closure of Q_p. We consider first an operator α of the general type needed for the problem. This α can be defined on the space of all power series $\Omega(X_0,\cdots,X_{n+1})$, but in order that its spectrum not be all of Ω^*, its action is restricted to a subspace $L(\kappa)$ of series whose coefficients grow in ordinal at least at rate κ (a real number). Fix a suitable series F in $L(\kappa)$, and define an operator α on $L(q\kappa)$ by $\alpha(G)=\Psi\cdot(F\cdot G)$, where Ψ is the qth root operator on $\Omega(X)$, sending monomials of the form $X^{(qu)}$ into $X^{(u)}$ and the others into 0. A "characteristic series" χ_F is introduced as the limit of the characteristic polynomials of α acting on truncations of $L(q\kappa)$. The spectral theory for α is now done assuming that the coefficients of F lie in a finite extension K_0 of Q_p. To each zero λ^{-1} of χ_F of multiplicity s_λ, it associates the primary subspace W_λ, of dim s_λ, which is the kernel of $(I-\lambda^{-1})^s$ for all $s\geqq s_\lambda$. If we restrict the coefficient field to K_0, then if λ^{-1} is not a zero of χ_F, then $I-\lambda^{-1}\alpha$ is surjective, with the obvious generalization using s_λ. {This spectral theory was subsequently generalized and simplified by Serre [same Publ. No. 12 (1962), 69–85; MR **26** #1733].}

For the application to $\zeta(H,t)$, various choices for F may be made. If $\bar{f}(X)$ is the defining polynomial for H over k, let $f(X)$ be the unique polynomial over Ω reducing mod p to $\bar{f}$ and whose coefficients are $(q-1)$st roots of unity. Put $H=\gamma X_0 f$, where $\gamma^{q-1}=1$, and then we fix $F=(\exp H)^\delta$, where in general $G(X)^\delta$ is defined to be $G(X)/G(X^q)$. Actually, other F's are usable and later one must switch from one to another.

From the author's earlier paper [Amer. J. Math. **82** (1960), 631–648; MR **25** #3914] the basic connection between $\zeta(H,t)$ and α is (letting H' be the hypersurface $X_1X_2\cdots X_{n+1}=0$)

$$\zeta(H-H', qt) = \chi_F^{-(-\delta)^{n+1}}(1-t)^{-(-\delta)^n}.$$

Let A be a nonempty subset of $S=\{1,\cdots,n+1\}$ and H_A be the hyperplane obtained by intersecting H with the hyperplanes $X_i=0$, $i\in A$. We can assume that H_A is nonsingular for all A. Letting the equation (1) for H_A define the rational function $P_A(t)$, one can deduce formally that

$$(2)\qquad \chi_F^{\delta^{n+1}} = (1-t)\prod_A P_A(qt).$$

Says the author: "We believe this equation is quite significant since χ_F is entire even if H is singular."

To show $P_S(qt)$ is a polynomial, the essential first step is to show that $\chi_F^{\delta^{n+1}}$ is a polynomial of degree d^n. This is true because there is a finite-dimensional quotient space $\mathfrak{W}$ of $L(q\kappa)$ on which α acts (as $\bar{\alpha}$, say) and

$$(3)\qquad \chi_F^{\delta^{n+1}} = \det(I-t\bar{\alpha}).$$

The idea of the proof of this is to introduce differential operators on $L(q\kappa)$: $D_iG=X_i\,\partial G/\partial X_i+HG$ $(i=1,\cdots,n+1)$. We have easily $\alpha\circ D_i=qD_i\circ\alpha$, showing that if λ^{-1} is an eigenvalue of α, so is $q\lambda^{-1}$, and in fact $D_i(W_\lambda)\subset W_{\lambda/q}$; thus α acts (as $\bar{\alpha}$) on $\mathfrak{W}=L(q\kappa)/\sum D_iL(q\kappa)$. The natural projection $L(q\kappa)\to\mathfrak{W}$ carries a primary subspace W_λ onto the eigenspace $\mathfrak{W}_\lambda$ if λ^{-1} is also an eigenvalue of $\bar{\alpha}$ (otherwise onto 0), and all eigenspaces of $\bar{\alpha}$ arise this way. Moreover, it induces an isomorphism $W_\lambda/\sum D_iW_{\lambda/q}\to\mathfrak{W}_\lambda$. All this follows from the spectral theory. Letting $\dim\mathfrak{W}_\lambda=b_\lambda$, what must be proved is therefore the first equality of

$$(3)\qquad \chi_F^{\delta^{n+1}} = \prod_\lambda (1-\lambda^{-1}t)^{b_\lambda} = \det(I-t\bar{\alpha})$$

the product being taken over the spectrum of α. To do this the author uses a complex which is a modification of the exterior algebra complex (here apparently invented ab ovo)

$$0\to F_{n+1}\to F_n\to\cdots\to F_0\to W_\lambda\to 0,$$

where $F_r=W_{\lambda/q^r}\otimes\Lambda^rE$. The differentiations in the complex are the usual ones, employing the $n+1$ commuting endomorphisms D_i. If one knows the sequence is exact, the equality (3) follows trivially; but by the usual formalism of these complexes, exactness follows if one knows that for all k, $D_k\beta=\sum D_i\beta_i\Rightarrow\beta=\sum D_i\beta_i'$, where $\beta_i,\beta_i'\in W_{\lambda/q}$ and $\beta\in W_\lambda$. This last statement is the crux of the matter; it is proved first when β,β_i,β_i' are simply in $L(q\kappa)$, using elementary but long computations, then the spectral theory is used to put the elements in the right primary subspaces.

It still must be shown that $P_S(qt)$ is a polynomial. For this purpose a decomposition $W=\sum W_A{}^A$ is given so that $\bar{\alpha}$ induces an $\bar{\alpha}_A{}^A$ on each summand, and

$$(4)\qquad \det(I-t\bar{\alpha}) = \prod_{A\subset S}\det(I-t\alpha_A{}^A) = \prod P_A(qt),$$

where the second equality follows from the first, (2), and (3). Now (4) is still valid if S is replaced by any subset $B\subset S$; the resulting system of relations shows easily that $P_S(qt)=\det(I-t\bar{\alpha}_S{}^S)$, which completes the proof, the degree being calculated via the Koszul resolution. The $W_A{}^A$ are obtained as natural quotients (via the D_i) of spaces $L_A{}^A(q\kappa)$ obtained by taking the power series in $L(q\kappa)$, setting $X_i=0$ for all $i\notin A$, and then taking just the series divisible by X_i, for all $i\in A$. This last condition causes technical complications in the rather involved (in algebra and convergence) calculations in $L(q\kappa)$ which establish both the decomposition of W and (4) above. The case $p|d$ is particularly troublesome and is excluded entirely when $p=2$.

Subsequent work by the author has established the missing case (p and d even) mentioned above, and proved the conjectured functional equation for $\zeta(H,t)$. The location of the zeros of $P(t)$ remains open.

A. Mattuck (Cambridge, Mass.)

Greenberg, Marvin J. 3040

Algebraic rings.

Trans. Amer. Math. Soc. **111** (1964), 472–481.

An algebraic ring is, roughly speaking, an associative algebra which is an algebraic variety. The author proves that (1) an algebraic ring is Artinian, (2) an algebraic ring is, under its addition, a unipotent algebraic group, (3) the unit group of an algebraic ring is a Zariski open subset. Furthermore, some structure theorems for algebraic rings are proved. For instance, (i) a commutative local ring variety of positive characteristic is a finite algebra over a ring of Witt vectors, and (ii) a simple ring variety is a matric ring (over an algebraically closed field). {Reviewer's note: (1) and (3) above were obtained independently by H. Yanagihara [J. Math. Kyoto Univ. **3** (1963), 103–110; MR **28** #96].}

M. Nagata (Kyoto)

Matsumura, Hideyuki 3041

On algebraic groups of birational transformations.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 151–155.

L'auteur prouve les résultats suivants. Soient V une variété algébrique et G un groupe algébrique connexe de transformations birationnelles de V en elle-même. (i) Si V est complète et non singulière, alors toute forme différentielle, partout régulière, est invariante par G. Si un multiple du système linéaire canonique est ample, le groupe G est réduit à l'identité. (ii) Soit A la variété d'Albanese de V, et soit f l'application canonique de V dans A. L'ensemble des g dans G tels que $f(g\cdot v)=f(v)$ pour v dans V générique est un sous-groupe algébrique linéaire de G.

P. Cartier (Strasbourg)

Schwarzenberger, R. L. E. 3042

The secant bundle of a projective variety.

Proc. London Math. Soc. (3) **14** (1964), 369–384.

Let Y be a scheme, and H an invertible sheaf of

O_Y-modules. (O_Y denotes the structure sheaf of Y.) For instance, take the case in which Y is given in a projective space and H is the one associated with hyperplane sections. Consider the following commutative diagram of schemes and morphisms,

$$\begin{array}{ccc} W \xrightarrow{i} & X \times Y & \\ {}^{f}\searrow \swarrow^{p} & & \searrow^{q} \\ X & & Y, \end{array}$$

where p and q are projections. The author calls such a diagram "a product scheme for Y" if f is a covering map, i.e., a projective finite-to-one morphism. Define a sheaf E of O_X-modules: $E = f_* i^* q^*(H)$. He applies the word "secant bundle" to $P(E) \rightarrow X$, the projective fibre space associated with E in the sense of Grothendieck. For example (Case (1)): $X = X_n$, the n-times cartesian product of Y, $W = W_n$, the reduced subscheme of $X_n \times Y$ whose points are of the form $(y_1, y_2, \cdots, y_n) \times y_t$ for $1 \leqq t \leqq n$, and i is the inclusion; (Case (2)): $X = X_n'$, the n-times symmetric product of Y, W is the canonical image of W_n in $X_n' \times Y$, and i is the inclusion. In Case (1), $d^*(E)$ with the diagonal map $d: Y \rightarrow X_n$ becomes related to the tangent bundle of Y. In this paper, only elementary (immediate from definitions and assumptions) properties of E are proved, either in the general case or in the above special cases. A short discussion on projective characters of Y with H (or with an imbedding in a projective space) is added in an attempt to show the usefulness of the sheaf E. It concerns the Chern classes of E.

H. Hironaka (Waltham, Mass.)

Gherardelli, Francesco 3043

Un teorema di Lefschetz sulle intersezioni complete.

Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur. (8) **28** (1960), 610–614.

Let V_n be a non-singular projective variety of complex dimension n ($n \geqq 3$) over the field of complex numbers. Let W_k be a non-singular algebraic subvariety of V_n, of complex dimension $k \geqq 2$. If the natural map $H^p(V_n, Z) \rightarrow H^p(W_k, Z)$ is bijective for $p \leqq k-1$ and injective for $p = k$, then the homomorphism of the additive group of (the equivalence classes of) the holomorphic line bundles on V_n into the analogous group on W_k is bijective for $k \geqq 3$ and injective for $k = 2$. Therefore, if $k \geqq 3$, every divisor on W_k is linearly equivalent to a divisor cut on W_k by a divisor on V_n, which is defined up to a linear equivalence. Letting $V_n = P_n(\mathbf{C})$, the last statement yields the theorem of Lefschetz, according to which, if W_k ($k \geqq 3$) is a complete intersection of $n-k$ projective hypersurfaces of $P_n(\mathbf{C})$, then every holomorphic divisor on W_k is cut on W_k by a projective hypersurface of $P_n(\mathbf{C})$. If the non-singular projective variety $W_k \subset P_n(\mathbf{C})$ ($k \geqq 1$) is such that $H^1(W_k, \mathbf{Z}) = 0$, if its base number is equal to 1 and if W_k does not have algebraic torsion, then a suitable multiple of any holomorphic divisor on W_k is cut on W_k by a hypersurface of $P_n(\mathbf{C})$. Let V_n be an abelian variety with divisors $\delta_1 \leqq \delta_2 \leqq \cdots \leqq \delta_n$ ($n \geqq 4$) and let the non-singular submanifold $W_k \subset V_n$ ($k \geqq 3$) be the complete intersection of the prime divisors of $n-k$ holomorphic theta functions of order $\geqq 3\delta_n$ belonging to the same ring of intermediate functions. Then every holomorphic divisor on W_k is the complete intersection of W_k with the zero-variety of a theta function.

E. Vesentini (Pisa)

LINEAR ALGEBRA

See also 3116, 3133, 3532.

Lefort, G. 3044

★Algèbre et analyse: Exercices. Illustration du Cours de Mathématiques Générales.

Collection Universitaire de Mathématiques, VII.

Dunod, Paris, 1961. xi+517 *pp*. 44 *NF*.

The book means more than its title and is purposefully written in the light of a modified syllabus as a companion volume to the work by C. Pisot and M. Zamansky [*Mathématiques générales*, Dunod, Paris, 1959] which covers the usual course prescribed for the Certificat in Mathématiques Générales of the French Lycées. It will greatly help those who desire to have a thorough grounding and working knowledge of the various topics contained therein, even by self-study.

The book begins with the algebra of sets, concept of function, complex numbers, polynomials, rational fractions, etc. The book is then arranged to cover linear and multilinear algebra (determinants and matrices), the theory of functions (continuity, derivation, integration, curvilinear integrals, functional series, differential equations, etc.) and is bound to be a valuable introduction to the higher studies of Modern Algebra and Analysis.

Aung Hla (Rangoon)

Kowalsky, Hans-Joachim 3045

★Lineare Algebra.

Walter de Gruyter & Co., Berlin, 1963. 340 *pp*. *DM* 48.00.

The author is primarily interested in linear mappings of vector spaces, using modern terminology. After eight chapters of introductory nature follow three chapters with the main results, one on the Jordan normal form, one on the dual space and finally one on multilinear algebra. Much information is woven into exercises whose solutions are also included.

O. Taussky-Todd (Pasadena, Calif.)

Carlson, David 3046

A note on M-matrix equations.

J. Soc. Indust. Appl. Math. **11** (1963), 1027–1033.

This paper generalizes to arbitrary M-matrices the method and the results of H. Schneider [Proc. Edinburgh Math. Soc. (2) **10** (1956), 108–122; MR **17**, 935] concerning singular ones. Assume that the matrix has been transformed to standard form (A_{ik}) with irreducible A_{ii}. The main point is to achieve results concerning the possible non-negative solutions of $Ax = c$ by using only the location of zero and non-zero elements in A and c and of the singular A_{ii}. For irreducible M-matrices this equation has been studied previously mainly by A. Ostrowski [Comment. Math. Helv. **10** (1937), 69–96].

O. Taussky-Todd (Pasadena, Calif.)

Gerstenhaber, Murray 3047

On semicommuting matrices.

Math. Z. **83** (1964), 250–260.

An ordered set $\mathfrak{a}_1, \cdots, \mathfrak{a}_t$ of n-by-n matrices is said to semicommute if there exists a scalar λ such that $\mathfrak{a}_i\mathfrak{a}_j = \lambda\mathfrak{a}_j\mathfrak{a}_i$ when $i < j$. Extending results of Williamson [Proc.

Edinburgh Math. Soc. (2) **3** (1933), 179–188; ibid. (2) **3** (1933), 231–240], the author gives canonical forms for such sets under the assumption that all the matrices are nonsingular. Without this assumption, he determines the components and the generic elements of the algebraic set of pairs $(\mathfrak{a}_1, \mathfrak{a}_2)$ of n-by-n matrices with $\mathfrak{a}_1\mathfrak{a}_2 = \lambda\mathfrak{a}_2\mathfrak{a}_1$, λ fixed. *D. Zelinsky* (Evanston, Ill.)

Lavoie, Jean L. 3048

On inverses of finite segments of the generalized Hilbert matrix.

Math. Comp. **18** (1964), 141–143.

R. B. Smith [MTAC **13** (1959), 41–43; MR **21** #679] has proved certain sum relations for the elements of the inverse of the matrix $H_n = [(p+i+j-1)^{-1}]$, $i, j = 1, 2, \cdots, n$. The author shows that these sum relations can be established from the (known) explicit expressions of the elements of H_n^{-1} by means of certain results on generalized hypergeometric series with unit argument.

A. Erdélyi (Pasadena, Calif.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 3031, 3035, 3040, 3062, 3075, 3076, 3104.

Kegel, Otto H. 3049

Zur Nilpotenz gewisser assoziativer Ringe.

Math. Ann. **149** (1962/63), 258–260.

The author proves the following analogue of his theorem on the product of two finite nilpotent groups [Arch. Math. **12** (1961), 90–93; MR **24** #A3199]. If the associative ring R is the sum of two nilpotent subrings, then R is nilpotent. The nilpotency class of R is bounded by a function of the nilpotency classes of the subrings. The proof is based on the following interesting lemma: If A is a subset of R, let $W_n(A)$ be the ideal generated by all n-fold monomials with factors in A. Suppose $R^+ = A + B$, where A and B are additive subgroups; then for each natural number n, there is a natural number $k(n)$ such that if $r \in R$ is a monomial of length $k(n)$, then $r = s + t$, where $s \in W_n(A)$ and $t \in W_n(B)$. *J. McLaughlin* (Ann Arbor, Mich.)

van der Walt, A. P. J. 3050

Contributions to ideal theory in general rings.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 68–77.

In an arbitrary ring R call an ideal P s-prime in case its complement contains a multiplicatively closed system which meets every principal ideal not contained in P. This paper shows that s-primeness can be used as the basis for a well-behaved general structure theory of the standard type. For example, the intersection of the s-prime ideals in R specifies an s-radical T such that R/T has zero radical, and such a ring is a subdirect sum of s-prime rings.

W. G. Lister (Stony Brook, N.Y.)

Gilmer, Robert; Ohm, Jack 3051

Integral domains with quotient overrings.

Math. Ann. **153** (1964), 97–103.

An integral domain D with field of quotients k is said to have the QR-property if every ring D' between D and k is a ring of quotients of D. The article studies integral domains with the QR-property. Starting with some simple properties of integral domains with the QR-property, the authors prove the following theorem. Let D be an integral domain with the QR-property, and let P be its arbitrary prime ideal. Then D_P is a valuation ring. If A is a finitely generated ideal of D, then A is invertible and some power of A is contained in a principal ideal.

As for the converse, the authors prove that if every valuation ring of k containing D is a ring of quotients of D and if every prime ideal of D is the radical of a principal ideal, then D has the QR-property. As a particular case, it follows that a Noetherian integral domain D has the QR-property if and only if it is a Dedekind domain in which the ideal class group is a torsion group. Some properties of intermediate rings for such rings are observed.

{Reviewer's note: Though the authors added that the result on Noetherian integral domains with the QR-property was obtained independently also by E. D. Davis, the reviewer is afraid that this special case may be known by some other people. Substantially the same result is also contained in the paper of Goldman reviewed below [#3052].} *M. Nagata* (Kyoto)

Goldman, Oscar 3052

On a special class of Dedekind domains.

Topology **3** (1964), *suppl.* 1, 113–118.

The main result is the following. Let R be a Dedekind domain which has the following two properties: (F1) R/P is a finite field for every maximal ideal P, and (F2) the group $U(R)$ of units of R is finitely generated. Let K be the field of quotients of R and let X be a transcendental element over K. Then there is a Dedekind domain S such that (1) $R[X] \subseteq S \subseteq K(X)$, (2) S satisfies (F1), and (3) $U(R) = U(S)$.

Some consequences of this theorem are also proved.

M. Nagata (Kyoto)

Muhly, H. T.; Sakuma, M. 3053

Asymptotic factorization of ideals.

J. London Math. Soc. **38** (1963), 341–350.

Let (D, M) be a local domain with quotient field F and let w be an integer-valued valuation of F such that the valuation ring $R_w \geqq D$ and $P \cap D = M$, where P is the maximal value of R_w; w is then called a divisor of the second kind relative to D if R_w/P is a finitely generated extension of D/M of transcendence degree $(r-1)$, where r is the dimension of D. The following sequence of ideals is associated with $w: A_0 > A_1 > \cdots$, where $A_0 = D$, $A_i = \{x | x \in D, w(x) > w(A_{i-1})\}$, $i = 1, 2, \cdots$. The above sequence is a filtration and w is said to be a Noetherian divisor if this filtration has a Noetherian subfiltration [see Samuel, An. Acad. Brasil. Ci. **30** (1958), 447–450; MR **21** #4967] for relevant definitions. A normal local domain (D, M) is said to satisfy condition (N) if each prime divisor of the second kind relative to D is Noetherian. The following main theorem is proved. Theorem: If (D, M) is a normal, analytically irreducible local domain of dimension 2 satisfying condition (N) and if A is an M-primary ideal, then there exist asymptotically irreducible ideals $W_1, \cdots, W_s$ and positive integers $f_1, \cdots, f_s$ such that $A \mathrel{\overline{\wedge}} W_1^{f_1} \cdots W_s^{f_s}$. The ideals $W_1, \cdots, W_s$ are unique to within projective

asymptotic equivalence and, when they are fixed, the integers $f_1, \cdots, f_s$ are unique to within a factor of proportionality. Note $A \barwedge B$ means A is projectively asymptotically equivalent to B [Samuel, op. cit.]. The authors use this result to obtain the result of Zariski [Zariski and Samuel, *Commutative algebra*, Vol. II, p. 386, Van Nostrand, Princeton, N.J., 1960; MR **22** #11006], which states that if Γ is the set of complete ideals in a two-dimensional regular local ring, then Γ is a Gaussian semi-group under multiplication. A number of very interesting connections between factorizations of ideals and pseudo-valuations (in the sense of Rees) are obtained. {The authors have pointed out a misprint. The first line of p. 342 should be the last line of p. 343.}

E. H. Batho (Durham, N.H.)

Jouanolou, Jean-Pierre 3054

Étude différentielle des anneaux locaux réguliers: Applications.

C. R. Acad. Sci. Paris **257** (1963), 2948–2950.

The author treats a differential characterization of a regular local ring and its application to the criterion of analytical unramifiedness. The main result is as follows: Let A be an equi-characteristic regular local ring with maximal ideal M such that $D(A)$ is flat and $\mathfrak{a}$ is an ideal of A such that $A/\mathfrak{a}$ is reduced. Then the following conditions are equivalent: (a) $A/\mathfrak{a}$ is regular;

(b) $$\mathrm{Tor}^i_{A/\mathfrak{a}}(A/M, D(A/\mathfrak{a})) = 0 \quad (i \geqq 1).$$

Moreover, if $D(A)$ is free, (a) is equivalent to (c) $D(A/\mathfrak{a})$ is free.

Y. Nakai (Hiroshima)

Masuda, Katsuhiko 3055

Automorphisms of algebras and a theorem concerning norms.

Math. J. Okayama Univ. **12** (1963/64), 39–47.

The following theorem, a related form of which is due to R. Bortfeld [J. Reine Angew. Math. **201** (1959), 196–206; MR **22** #12124], is proved: "Let K be a field, A a normal simple algebra over K of finite rank, σ a ring-automorphism of A, B the set of elements that σ keeps unchanged, and k the intersection $B \cap K$ of B and K. Suppose that the rank $[K:k]$ of K over k is finite and BK is simple. Then, there exists, for each simple subring H of A containing B, a ring-automorphism τ of A such that H coincides with the set of elements of A that τ keeps unchanged. Conversely, let μ be an arbitrary ring-automorphism of A that keeps each element of B unchanged, then the set M of elements of A that μ keeps unchanged is a simple ring." The proof uses the theory of commutator algebras, but not the general Galois theory of simple rings, and leads to a reduction to a theorem of field theory: "Let L be a field, M, K subfields, and k the intersection of M and K, respectively. Suppose that L is equal to MK and is of finite rank over k, Kk/k is separable and Galois, and, moreover, M/k is a simple extension. Then there exists for each non-zero element c of M a non-zero element d of L such that $cN_{L/M}d$ generates M over k; $M = k(cN_{L/M}d)$."

O. F. G. Schilling (Lafayette, Ind.)

Nagahara, Takasi; Tominaga, Hisao 3056

Some theorems on Galois theory of simple rings.

J. Fac. Sci. Hokkaido Univ. Ser. I **17** (1963), 1–13.

This paper is devoted to weakening the hypotheses of some theorems in a previous paper by the authors [Math. J. Okayama Univ. **11** (1962/63), 79–117; MR **28** #4007a]. For example, in proving that the composite correspondence ring→group→ring is the identity (loc. cit., Theorem 2.1), it is possible to drop the hypothesis that the overall ring extension is locally finite.

There is a forthcoming paper by Nagahara which improves these results further and should represent a reasonably final version of this Galois theory of simple Artinian rings.

D. Zelinsky (Evanston, Ill.)

Laxton, R. R. 3057

Prime ideals and the ideal-radical of a distributively generated near-ring.

Math. Z. **83** (1964), 8–17.

Each near-ring R is assumed to be distributively generated, i.e., to contain a multiplicative semigroup S which generates R^+ and is such that $s(x+y) = sx + sy$ for all $s \in S$, $x, y \in R$. An ideal P of R is called primitively prime if R/P has a faithful representation on some cyclic irreducible R-group. The intersection I of all primitively prime ideals of R is called the ideal-radical of R. It is shown that I contains all nilpotent ideals of R. If R is (left) Artinian, then I itself is nilpotent. If R is Artinian and N is the quasi-radical of R (i.e., N is the intersection of all maximal left ideals of R), then each irreducible R-group is isomorphic to one of the finite number of irreducible direct summands of $R^+ - N$. It is shown that the Jacobson radical (i.e., $\bigcap 0{:}G$, G an irreducible R-group) of an Artinian near-ring R is nilpotent if and only if every proper prime ideal of R is maximal.

R. E. Johnson (Rochester, N.Y.)

Herstein, I. N. 3058

Rings admitting certain automorphisms. (Turkish summary)

İstanbul Üniv. Fen Fak. Mec. Ser. A **26** (1961), 45–59.

Let R be a ring and let ϕ be an automorphism of R which is of prime period and has its fixed-point set in the center of R. The author shows that if R is semi-simple with descending chain condition, then R is commutative. For general R and ϕ of period two or three he shows that the commutator ideal of R must be nil. The proof for period three is not elementary.

J. McLaughlin (Ann Arbor, Mich.)

Belickiĭ, G. R. 3059

On chains of matrix norms. (Russian)

Dokl. Akad. Nauk SSSR **151** (1963), 9–10.

The author proves the hypothesis suggested by Ju. I. Ljubič: All maximal chains of matrix norms in the ring $\mathfrak{M}_n$ of all real $n \times n$ matrices partially ordered by the relation $\|A\|_1 \leqq \|A\|_2$, $A \in \mathfrak{M}_n$, are similar to the half-axis $[0, \infty)$.

M. Altman (Warsaw)

Talintyre, T. D. 3060

Quotient rings of rings with maximum condition for right ideals.

J. London Math. Soc. **38** (1963), 439–450.

A ring Q is called a (right) quotient ring of ring R if $Q \supset R$,

each regular element of R has an inverse in Q, and $Q = \{ab^{-1} | a, b \in R, b \text{ regular}\}$. Goldie proved [Proc. London Math. Soc. (3) **8** (1958), 589–608; MR **21** #1988] that a semiprime ring satisfying certain finiteness conditions has a simple Artinian ring as a quotient ring. It is assumed in the present paper that the ring R has a Wedderburn radical W, but that b is a regular element of R if and only if $b + W^n$ is regular in R/W^n for every positive integer n. If such a ring R is (right) Noetherian, then it is proved that R has a quotient ring Q which is (right) Artinian. Furthermore, Q is completely primary if and only if R/W is an integral domain, and primary if and only if R/W is prime. This work is closely related to that of Feller and Swokowski on reflective rings [Trans. Amer. Math. Soc. **99** (1961), 264–271; erratum, ibid. **101** (1961), 555; MR **22** #11011].

R. E. Johnson (Rochester, N.Y.)

Lesieur, L.; Croisot, R. 3061

Cœur d'un module.

J. Math. Pures Appl. (9) **42** (1963), 367–407.

Details and proofs of the results previously announced [C. R. Acad. Sci. Paris **252** (1961), 52–54; MR **26** #159].

D. Zelinsky (Evanston, Ill.)

NON-ASSOCIATIVE ALGEBRA

Barnes, Donald W. 3062

Conditions for nilpotency of Lie rings. II.

Math. Z. **81** (1963), 416–418.

The author presents a correction to his paper [same Z. **79** (1962), 289–296; MR **27** #181], as well as some new results. If we define $D_p(L) = \{x | x \in L, p^\mu x = 0 \text{ for some } \mu\}$ then Lemma 4.5 [op. cit.] can now be proved by a modification of the original methods. The author establishes the following two results. Theorem 1: Let L be a finite-dimensional Lie algebra over the Noetherian ring R. If every maximal subalgebra of L is an ideal, then L is ω-nilpotent. Theorem 2: Let A be a torsion-free associative algebra over the integral domain R. If A is generated as an R-module by a finite set of nilpotent elements, then A is nilpotent.

As the author points out, Theorem 2 generalizes the well-known theorem of Wedderburn, which states that if a finite-dimensional associative algebra A over a field has a basis of nilpotent elements, then A is nilpotent. The author derives some consequences from Theorem 2, one of the most interesting being the following. Theorem: If A is an associative algebra over the Noetherian ring R and if A can be generated as an A-module by a finite set of nilpotent elements, then A is ω-nilpotent (i.e., $\bigcap_{i=1}^{\infty} A^i = (0)$).

E. H. Batho (Durham, N.H.)

Bokut', L. A. 3063

A basis for free polynilpotent Lie algebras. (Russian)

Algebra i Logika Sem. **2** (1963), *no.* 4, 13–19.

Let L be the free Lie algebra (over a field F) on an ordered set of free generators, and denote by $L^{(k)}$ the terms of the derived series of L. The author constructs a basis of L adapted to the subspaces $L^{(k)}$ as follows: Any Lie product of the generators is called an R_k-word if it belongs to $L^{(k)}$, and these words are ordered as follows: If $u \in L^{(k)}$ and $v \notin L^{(k)}$ for some k, then $u > v$; if u and v are R_k-words for the same values of k but u is longer, then $u > v$, while for $u = u_1u_2$ and $v = v_1v_2$ of the same length, if $u_1 > v_1$ or $u_2 > v_2$, then $u > v$. Now regular words are defined by induction on the length as in Širšov [Mat. Sb. (N.S.) **45 (87)** (1958), 113–122; MR **20** #5796] (using the above ordering). Then the regular R_k-words form a basis of $L^{(k)}$. A similar method leads to a basis adapted to the subalgebra $(\cdots((L^{(n_1)})^{(n_2)})^{(n_3)}\cdots)^{(n_k)}$. By forming quotient algebras one obtains bases for the free soluble and the free polynilpotent Lie algebras, respectively.

P. M. Cohn (London)

Leger, G. 3064

Derivations of Lie algebras. III.

Duke Math. J. **30** (1963), 637–645.

Part II appeared in Proc. Amer. Math. Soc. **10** (1959), 10–11 [MR **21** #2679]. The contents of this paper concern two aspects of the relation between a Lie algebra L and its derivation algebra $D(L)$ in the characteristic 0 case: (1) the classification of nilpotent L's according to the structure of $D(L)$, (2) the consequences for L of the non-existence of outer derivations. In the area (1) the results concern the unique dimension $t(L)$ of a maximal abelian completely reducible subalgebra of $D(L)$. These can generally be interpreted as showing that $t(L)$ is a measure of the "commutativity" of L. The extremes correspond to L abelian ($t(L) = \dim(L)$) and L characteristically nilpotent ($t(L) = 0$). There are some results and conjectures for intermediate values.

The results concerning (2) are somewhat inconclusive and not easily summarized. Their line is that if L has no outer derivations, then the radical of L must satisfy some substantial structural conditions.

W. G. Lister (Stony Brook, N.Y.)

Jacobson, N. 3065

Some groups of transformations defined by Jordan algebras. II. Groups of type F_4.

J. Reine Angew. Math. **204** (1960), 74–98.

Part I appeared in same J. **201** (1959), 178–195 [MR **21** #5666]. In this paper, the author investigates the group $G(\mathfrak{J})$ of automorphisms of a reduced (non-division) exceptional simple Jordan algebra $\mathfrak{J}$ over a field Φ of characteristic $\neq 2, 3$ ($G(\mathfrak{J})$ is known to be the group of rational points of an algebraic group of type F_4; in particular, if $\mathfrak{J}$ is split, $G(\mathfrak{J})$ is the Chevalley group of type F_4 [G. B. Seligman, Trans. Amer. Math. Soc. **97** (1960), 286–316; MR **23** #A969]). The main results are that $G(\mathfrak{J})$ operates irreducibly on the 26-dimensional space of elements of trace 0 in $\mathfrak{J}$, and that if $\mathfrak{J}$ contains non-zero nilpotent elements, $G(\mathfrak{J})$ is a simple group. The paper also contains much valuable data on Cayley algebras (among other things, a statement and proof of the triality principle for a general Cayley algebra), on the primitive idempotents of $\mathfrak{J}$, and on certain subgroups of $G(\mathfrak{J})$, which are an essential tool in the proof of the main theorem. We state hereafter some of the most important auxiliary results.

Every primitive idempotent in $\mathfrak{J}$ belongs to a set of three mutually orthogonal idempotents (which are then the three diagonal units in a suitable representation of $\mathfrak{J}$ as a matrix algebra over a Cayley algebra $\mathfrak{C}$). A primitive idempotent is called an s-idempotent if its annihilator

contains non-zero nilpotent elements; any two s-idempotents are conjugate (i.e., the s-idempotents form a single orbit of $G(\mathfrak{J})$). {Reviewer's note: Assume that the Cayley algebra $\mathfrak{C}$, over which $\mathfrak{J}$ is constructed, is a division algebra. Then, the rays Φa determined by the elements $a \in \mathfrak{J}$ which are either primitive idempotents or of zero-square, are represented by the points of the projective plane on $\mathfrak{C}$ [see #3066 below], in which a certain hermitian polarity π is given; the zero-square elements correspond to the points of the "hermitian conic" Γ associated with π, and the s-primitive idempotents correspond to the points "outside" Γ, that is, the points whose polar line meets Γ in more than one point. The first result above asserts that a point not on Γ is a vertex of at least one autopolar triangle, while the second one is an immediate consequence of the fact that $G(\mathfrak{J})$ is doubly transitive on Γ. A similar interpretation of the two results can be given when $\mathfrak{C}$ is split; one must then consider the "plane" on $\mathfrak{C}$ introduced by the reviewer [cf., e.g., Séminaire Bourbaki, 1957/58, Exp. 162, Secrétariat mathématique, Paris, 1958; MR **21** #4981].}

Let $\mathfrak{D}$ be a subalgebra of $\mathfrak{J}$ and consider the "Galois group" $G(\mathfrak{J}/\mathfrak{D})$ of all automorphisms of $\mathfrak{J}$ which leave $\mathfrak{D}$ pointwise fixed. The author determines the structure of this group in the following cases: $\mathfrak{D} = \Phi e_1 + \Phi e_2 + \Phi e_3$, where the e_i's are three mutually orthogonal idempotents; $\mathfrak{D}$ is central simple of degree three; $\mathfrak{D} = \Phi e$ where e is a primitive idempotent. The last case leads to the following "triality principle": Let $\mathfrak{J} = \mathfrak{J}_0 + \mathfrak{J}_{1/2} + \mathfrak{J}_1$ be the Peirce decomposition of $\mathfrak{J}$ relative to e, let $\mathfrak{M}$ be the 9-dimensional space of all elements of trace 0 in $\mathfrak{J}_0$. In $\mathfrak{J}$, and therefore also in all the subspaces under consideration, there is a naturally defined symmetric bilinear form $\mathrm{Tr}(x \cdot y)$. Then, $G(\mathfrak{J}/\Phi e)$ is canonically isomorphic with the reduced Clifford group of $\mathfrak{M}$, its restriction to $\mathfrak{M}$ and $\mathfrak{J}_{1/2}$ being, respectively, the natural (orthogonal) and the spin representation of this group; furthermore, if C and B are orthogonal transformations of $\mathfrak{M}$ and $\mathfrak{J}_{1/2}$, respectively, such that $(a \cdot x)C = (aC)(xB)$ for all $a \in \mathfrak{J}_{1/2}$, $x \in \mathfrak{M}$, C and B are the restrictions to $\mathfrak{M}$ and $\mathfrak{J}_{1/2}$ of an element of $G(\mathfrak{J}/\Phi e)$.

All involutory automorphisms of $\mathfrak{J}$ are determined. An exceptional simple Jordan algebra which possesses involutory automorphisms is shown to be reduced.

J. L. Tits (Bonn)

Jacobson, N. 3066

Some groups of transformations defined by Jordan algebras. III.

J. Reine Angew. Math. **207** (1961), 61–85.

We keep the notations of the review above [#3065]. The author now investigates the group $L(\mathfrak{J})$ of all norm-preserving linear transformations of $\mathfrak{J}$ ($L(\mathfrak{J})$ is the group of rational points of an algebraic group of type E_6; other forms of E_6 have been considered by R. Steinberg [Pacific J. Math. **9** (1959), 875–891; MR **22** #79] and the reviewer [see the review above; Algebraical and Topological Foundations of Geometry (Proc. Colloq., Utrecht, 1959), pp. 175–192, Pergamon, Oxford, 1962; MR **25** #4039]). The main results are that $L(\mathfrak{J})$ is irreducible on $\mathfrak{J}$ and that the quotient of $L(\mathfrak{J})$ by its center C, which is of order 3 (it consists of the scalar multiplications by cubic roots of unity), is a simple group. The cases where $\mathfrak{J}$ is split and non-split are treated separately. In the split case, the proof relies on the fact that the automorphism group of $\mathfrak{J}$ is simple [see #3065 above]. Interesting auxiliary results in that case are that every element in $\mathfrak{J}$ is contained in a subalgebra isomorphic with the Jordan algebra of 3×3 matrices over Φ, and that $L(\mathfrak{J})$ is transitive on the set of all elements of norm 1 in $\mathfrak{J}$. In the non-split case, the proof is of a geometric nature. A Freudenthal-type construction of the projective plane over $\mathfrak{C}$ is given (about this construction, cf. also T. A. Springer [Nederl. Akad. Wetensch. Proc. Ser. A **63** (1960), 74–101; MR **23** #A3492]); the group $L(\mathfrak{J})/C$, which operates effectively on that plane, is shown to coïncide with the little projective group (group generated by the elations), which, in turn, is proved to be simple. The group $L(\mathfrak{J})$ and certain invariant subgroups are also studied for some special simple Jordan algebras $\mathfrak{J}$.

J. L. Tits (Bonn)

Gaĭnov, A. T. 3067

Binary Lie algebras of lower ranks. (Russian)

Algebra i Logika Sem. **2** (1963), *no.* 4, 21–40.

The paper is concerned with binary Lie algebras of ranks 3 and 4. It is first shown that a binary Lie algebra of rank 3 over any field is in fact a Lie algebra. This comes as a corollary of a result which states that there are only two non-isomorphic anticommutative algebras of given rank n with the property that every pair of linearly independent elements generates a subalgebra of rank 2. One is null and the other is given by the multiplication table $e_ie_j = 0$ for $1 \leqq i < j \leqq n-1$ and $e_ie_n = 0$ for $1 \leqq i \leqq n-1$. Both are, of course, Lie algebras. The rank 4 case is more complicated; multiplication tables are given for all binary Lie (but not Lie) algebras of rank 4 over an arbitrary field F of characteristic different from 2. There is a one-to-one correspondence $A_\alpha \leftrightarrow \alpha$ between these algebras and the elements α of F; α comes into the multiplication table for A_α in a simple and pleasant way. A corollary is that there exists just one Mal'cev algebra of rank 4 which is not Lie.

J. Wiegold (Cardiff)

Gaĭnov, A. T. 3068

Alternative algebras of rank 3 and 4. (Russian)

Algebra i Logika Sem. **2** (1963), *no.* 4, 41–46.

Here we have a discussion of the alternative algebras of ranks 3 and 4 over a field of characteristic prime to 6. For rank 3, all such algebras are associative; this is a corollary of a result of Albert [Trans. Amer. Math. Soc. **64** (1948), 552–593; MR **10**, 349] which says that an alternative ring of characteristic prime to 6 has a Lie commutator ring if and only if it is associative, and one of the author [see the preceding review #3067] that a binary Lie algebra of rank 3 is Lie. It is shown that there are exactly two non-isomorphic alternative non-associative algebras of rank 4 over a field of characteristic prime to 6, and that these are anti-isomorphic. The (very simple) multiplication tables are given.

J. Wiegold (Cardiff)

Levin, J. J.; Shatz, S. S. 3069

Riccati algebras.

Duke Math. J. **30** (1963), 579–594.

Let Γ be a fixed $m \times n$ matrix over the commutative associative ring with unity K, in which 2 is a unit. To each such matrix the authors associate a pair of algebras $\mathscr{A}$ and $\mathscr{R}$. The additive group of each of these algebras is the module $\mathscr{M}_m{}^n(K)$ of $n \times m$ matrices over K, and their

multiplications are defined by $A * B = A\Gamma B$ and $A \circ B = \frac{1}{2}(A\Gamma B + B\Gamma A)$, respectively, for $A, B \in \mathscr{A}$. Juxtaposition is ordinary matrix multiplication. In keeping with the Ricatti equation $dX/dt = X\Gamma X$, the algebra $\mathscr{R}$ is called a Ricatti algebra over K. $\mathscr{A}$ is an associative algebra and $\mathscr{R}$ is the attached Jordan algebra.

Two matrices Γ and $\hat{\Gamma} \in \mathscr{M}_n^m(K)$ are equivalent if there exist invertible matrices $W \in \mathscr{M}_n^m(K)$ and $Z \in \mathscr{M}_m^n(K)$ such that $\hat{\Gamma} = W\Gamma Z$. Then Γ is equivalent to a matrix $\hat{\Gamma} \in \mathscr{M}_n^m(K)$ with blocks E_r ($r \times r$ identity matrix) and H along the diagonal and zeros elsewhere. H can be chosen so that all of its entries are non-units of K. Three hypotheses are introduced: (α) Γ is equivalent to a matrix with $H = 0$, (β) there exists a $Q \in \mathscr{M}_m^n(K)$ such that $\Gamma Q \Gamma = \Gamma$, and (γ) Γ is equivalent to a matrix $\hat{\Gamma}$ with $r \geqq 1$. Various relationships are examined between these restrictions and the existence of idempotents in $\mathscr{R}$.

Let $\mathscr{I} = \{A : \Gamma A \Gamma = 0, A \in \mathscr{M}_m^n(K)\}$. The following results are obtained relating the ideal structures of $\mathscr{A}$ and K. Theorem: Let (γ) be satisfied. Then (i) $\mathscr{A}$ is noetherian if and only if K is noetherian, (ii) there is 1-1 correspondence between the maximal ideals of $\mathscr{A}$ and of K, (iii) the intersection of the maximal ideals of $\mathscr{A}$ contains $\mathscr{I}$, (iv) $\mathscr{A}$ is local if and only if K is local, and (v) K is a field if and only if every proper ideal of $\mathscr{A}$ is contained in $\mathscr{I}$. Theorem: If (β) is satisfied, $\Gamma \neq 0$, and K is noetherian, then the following are equivalent: (i) K is a field, (ii) there exists an idempotent P such that $\mathscr{A}$ is generated, as an ideal, by any nonzero element in $P * \mathscr{A} * P$, (iii) if $\mathscr{J}$ is a proper ideal of $\mathscr{A}$, then $\mathscr{J} \subseteq \mathscr{I}$.

Examples are given for which various subsets of restrictions do or do not hold.

R. H. Oehmke (Princeton, N.J.)

Albert, A. A. 3070

On the collineation groups associated with twisted fields.
Calcutta Math. Soc. Golden Jubilee Commemoration Vol. (1958/59), *Part II, pp.* 485–497. *Calcutta Math. Soc., Calcutta,* 1963.

Let $\mathscr{K}$ be a field of degree n over the field $\mathscr{F}$ of $q = p^m$ elements, where p is the characteristic. Then $\mathscr{K}$ is cyclic over $\mathscr{F}$ and the transformation S defined by $x \to xS = x^q$ for $x \in \mathscr{K}$ generates the automorphism group of $\mathscr{K}$ over $\mathscr{F}$. Let c be any element of $\mathscr{K}$ which is not the $(q-1)$st power of any element in $\mathscr{K}$, i.e., $c \neq kS/k$. Define transformations $A_x^{(c)}$ and $B_y^{(c)}$ by $A_x^{(c)} = R(xS) - SR(cx)$ and $B_y^{(c)} = SR(y) - R(c(yS))$, where $R_x = R(x)$ and $L_x = L(x)$ are the right and left multiplication by x in $\mathscr{K}$, respectively. A division algebra $\mathscr{D}_c$ over $\mathscr{F}$ is defined which is the same vector space as $\mathscr{K}$ but which has a product defined by the right and left multiplication $R_x^{(c)}$ and $L_x^{(c)}$ in $\mathscr{D}_c$ as $R_{xB}^{(c)} = A^{-1}A_x^{(c)}$, $L_{yA}^{(c)} = B^{-1}B_y^{(c)}$, where $A = A_e^{(c)}$ and $B = B_e^{(c)}$ and e is the identity element of $\mathscr{K}$. $\mathscr{D}_c$ is known as a twisted field.

To each such $\mathscr{D}_c$ one can attach a projective plane, $\mathscr{M}(\mathscr{D}_c)$, whose elements are the triples $E_2 = (0, f, 0)$, $E_1 + mE_2 = (f, m, 0)$ and $aE_1 + bE_2 + E_3 = (a, b, f)$, where m, a, and b are arbitrary elements of $\mathscr{D}_c$ and f is the identity element of $\mathscr{D}_c$. The main result of the paper is to show that the group of collineations of $\mathscr{M}(\mathscr{D}_c)$, $\mathscr{G}(\mathscr{D}_c)$ is solvable. If $\mathscr{G}_\tau$ and $\mathscr{G}_\sigma$ are the subgroups of $\mathscr{G}(\mathscr{D}_c)$ consisting of translations and shears, respectively, then $\mathscr{G}_\tau$ and $\mathscr{G}_\sigma$ are abelian and the elementary group $\mathscr{G}_\varepsilon = \mathscr{G}_\tau\mathscr{G}_\sigma$ is a normal subgroup of $\mathscr{G}(\mathscr{D}_c) = \mathscr{G}$. In addition, $\mathscr{G}/\mathscr{G}_\tau\mathscr{G}_\sigma$ is isomorphic to the subgroup $\mathscr{G}_\gamma$ of all collineations γ such that $E_i^\gamma = E_i$. Hence $\mathscr{G}$ is solvable if and only if $\mathscr{G}_\gamma$ is solvable.

For each collineation γ there exist nonsingular linear transformations P, Q, U such that $(f, m, 0)^\gamma = (f, mP, 0)$ and $(a, b, f)^\gamma = (aQ, bU, f)$. It is shown that there exists a $g \neq 0$ in $\mathscr{D}_c$ such that $P = L_g U$ and $Q = UR^{-1}(gU)$. In fact, it can be shown that if V generates the group of all automorphisms W of $\mathscr{K}$ over $\mathscr{J}_p$ (the prime field of p elements) such that $\alpha W = \alpha = c(cS)(cS^2)\cdots(cS^{n-1})$, then $g(cV) = c(gS)$ and there exists an $a \in \mathscr{K}$ and λ such that $P = B^{-1}(V^\lambda R_a)B$. Thus $\mathscr{G}_\gamma$ is isomorphic to the group of all $V^\lambda R_a$ as a ranges over all nonzero elements of $\mathscr{K}$ and λ ranges over the values $0, 1, \cdots, n\rho - 1$, where ρ is determined by V. It is shown that the order of $\mathscr{G}_\gamma$ is $(n\rho - 1)(q^n - 1)$ and is solvable. Hence $\mathscr{G}(\mathscr{D}_c)$ is solvable. As a corollary to these results it is also shown that two twisted fields $\mathscr{D}_c$ and $\mathscr{D}_d$ are isotopic (and hence define isomorphic planes $\mathscr{M}(\mathscr{D}_c)$ and $\mathscr{M}(\mathscr{D}_d)$) if and only if there is an automorphism V of $\mathscr{F}$ over $\mathscr{F}_p$ such that $d(dS)\cdots(dS^{n-1}) = c(cS)\cdots(cS^{n-1})V$.

R. H. Oehmke (Princeton, N.J.)

HOMOLOGICAL ALGEBRA

See also 2985, 2986, 3424.

Calenko, M. S. 3071

Correspondences over a quasi-exact category. (Russian)
Dokl. Akad. Nauk SSSR **155** (1964), 292–294.

To any quasi-exact category K the author associates an I-category $\mathfrak{K}(K)$ of correspondences over K which satisfies the axioms (K1)–(K3) [D. Puppe, Math. Ann. **148** (1962), 1–30; MR **25** #5095]. The category K can be isomorphically embedded in $\mathfrak{K}(K)$ as the subcategory of all proper morphisms. *E. Lluis* (Mexico City)

Roĭter, A. V. 3072

On a category of representations. (Russian)
Ukrain. Mat. Ž. **15** (1963), 448–452.

Let Λ be a noetherian ring. The author says that Λ has a category of representations if there exist categories K_Λ and ${}_\Lambda K$ of right and left noetherian Λ-modules such that: (1) $R_\Lambda \in K_\Lambda$ and ${}_\Lambda R \in {}_\Lambda K$, where R_Λ and ${}_\Lambda R$ are the regular representations of Λ (i.e., Λ itself is considered as a right and left module); (2) If $A \subset B$, $B \in K_\Lambda$ [respectively, ${}_\Lambda K$], then $A \in K_\Lambda$ [${}_\Lambda K$]; (3) There is an anti-isomorphism between the categories K_Λ and ${}_\Lambda K$ which is exact on strong sequences. Example: A ring Λ with 1 such that the center of Λ contains a Dedekind ring M and Λ is a finitely generated torsion-free M-module. The objects in K_Λ [${}_\Lambda K$] are finitely generated torsion-free M-modules. The author proves several properties of such categories.

E. Lluis (Mexico City)

Stärk, Roland 3073

★Nullsysteme in allgemeinen Kategorien.
Von der Eidgenössischen Technischen Hochschule in Zürich zur Erlangung der Würde eines Doktors der Mathematik genehmigte Promotionsarbeit.
Dissertationsdruckerei Leemann AG, Zürich, 1963. 36 *pp.*

In developing a theory of group-like structures in general categories, Eckmann and the reviewer assumed that the underlying category possessed zero maps (or, as one says, that the category was pointed). That is to say, it is assumed that, given any two objects A, $B \in \mathscr{C}$, there is a morphism $o = o_{AB}: A \to B$, such that $fo = o$, $og = o$, whenever the compositions are defined. The system $\{o_{AB}\}$ is then unique. Now the category of sets is not pointed (since, for example, there are no functions $A \to \phi$). The author defines a left null-system for A on $\mathscr{C}$ as a system of morphisms $\lambda_A{}^X: X \to A$, A fixed, X variable, such that $\lambda_A{}^Y f = \lambda_A{}^X$ for all $f: X \to Y$. A right null system is defined similarly. Plainly, two left null systems $\{\lambda_A{}^X\}$, $\{\tau_A{}^X\}$ for A coincide if and only if $\lambda_A{}^A = \tau_A{}^A$. If A possesses a right null system $\{\rho_X{}^A\}$ and B possesses a left null system $\{\lambda_B{}^X\}$, then $\rho_B{}^A = \lambda_B{}^A$ and this morphism is called the zero map from A to B. In the category of sets the left null systems of A stand in one-one correspondence with the elements of A, while the empty set is the only object possessing a right null system.

In stating the group axioms in a category, the author observes that it is only necessary to suppose that the object A possesses a left null system. An H-object is then a triple (A, λ, m), where λ is a left null system, $m: A \times A \to A$, and $m\{1_A, \lambda_A{}^A\} = m\{\lambda_A{}^A, 1_A\} = 1_A$. The author proceeds to develop this generalization of the Eckmann-Hilton theory systematically. In particular, he gives a careful discussion of the dual factorizations of the canonical map from free (inverse) product to direct product, and shows that, here as elsewhere, the crucial element of the structure is an endomorphism $\tau: A \to A$ with $\tau^2 = \tau$ (provided, of course, that we confine attention, as we do in discussing length, to products of copies of A). The paper closes with a list of several examples.

P. J. Hilton (Ithaca, N.Y.)

Inasaridze, H. N. 3074

On the theory of extensions in categories. (Russian)

Soobšč. Akad. Nauk Gruzin. SSR **30** (1963), 537–544.

The author gives a theory of extensions for general categories with certain conditions. This will enable him to obtain suitable theories of extensions for categories of semi-groups with 1, left semi-modules, etc. This will be done in forthcoming papers. Let **A** be a category in which $\mathrm{Hom}(A, B)$ has a binary operation (not necessarily everywhere defined). For this operation several axioms are assumed. G is called a distinguished object if, for every D, $\mathrm{Hom}(D, G)$ is an abelian group. $G \xrightarrow{\alpha} D \xrightarrow{\beta} S$ is said to be an extension of S by the distinguished element G if the following conditions are satisfied: (1) $\beta = \mathrm{coker}\ \alpha$. (2) For any $f: F \to D$, $f': F \to G$ the operation is defined for the couple $f, f'\alpha$. (3) For any $\gamma, \gamma': F \to D$ such that $\gamma\beta = \gamma'\beta$ there exists a unique $h: F \to G$ such that $\gamma' = \gamma + h\alpha$. It follows that $\alpha = \ker \beta$, α is a monomorphism and β an epimorphism. The author gives a map from the set $E^1(S, G)$ of equivalence classes of extensions of S by G into $S^1\ \mathrm{Hom}(S, G)$. Conditions are given in order that this map becomes an epimorphism and an isomorphism. If $\mathrm{Hom}(A, B)$ are semi-groups, then $\{S^n T\}$ is a universal connected sequence of functors. *E. Lluis* (Mexico City)

Buchsbaum, David A. 3075

A generalized Koszul complex. I.

Trans. Amer. Math. Soc. **111** (1964), 183–196.

Soit R un anneau commutatif. À un homomorphisme $R^n \to R$ est classiquement associé un "complexe de Koszul". Étant donné un R-module B, on définit ici, au moyen des algèbres extérieures de B et de B^*, un complexe $T(B)$ associé à B. Étant donnés un homomorphisme $f: A \to B$ de R-modules et deux entiers p, q, un procédé de "mapping cylinder" permet de définir de même des complexes $K(f; p, q)$ généralisant celui de Koszul. Si E est un R-module et $f: A \to R^n$ un homomorphisme, on étudie en détail les complexes $K(f; 1, n)$, $E(F) = K(f; 1, n) \otimes E$ et $E(\Lambda f) = K(f; n, 1) \otimes E$. Pour $f: R^m \to R^n$, la relation $H_i(E(f)) = 0$ pour tout $i \geqq p$ $(p > 0)$ implique $H_i(E(\Lambda f)) = 0$ pour tout $i \geqq p$. La notion de suite E-régulière (ou E-suite) est ainsi généralisée: on dit qu'un homomorphisme $f: R^m \to R^n$ est E-régulier si $H_1(E(f(i))) = 0$ pour $i = 1, \cdots, m-n+1$, $f(i)$ désignant la restriction de f à R^{n+i-1}: on a alors $H_q(E(f)) = 0$ pour tout $q \geqq 1$. Sur un corps R, les homomorphismes R-réguliers sont ceux de rang maximum. Si R est un anneau local, la notion de E-régularité est indépendante des bases choisies dans R^m et R^n. *P. Samuel* (Paris)

Buchsbaum, David A.; Rim, Dock S. 3076

A generalized Koszul complex. II. Depth and multiplicity.

Trans. Amer. Math. Soc. **111** (1964), 197–224.

Soient R un anneau commutatif, E un R-module, et $f: A \to B$ un homomorphisme de R-modules. Pour tout entier $p \geqq 0$, on définit un complexe $K(\Lambda^p f)$ analogue à ceux de l'article précédent [#3075], et on l'étudie en détail lorsque A et B sont libres. On note $H_*(\Lambda^p f, E)$ $[H^*(\Lambda^p f, E)]$ l'homologie du complexe $K(\Lambda^p f) \otimes E$ $[\mathrm{Hom}(K(\Lambda^p f), E)]$. Les supports de ces modules d'homologie sont contenus dans $\mathrm{Supp}(\mathrm{coker} f) \cap \mathrm{Supp}(E)$.

Supposons désormais que f applique R^m dans R^n $(m \geqq n)$. Posons $I(f) = \mathrm{Ann}(\mathrm{coker}(\Lambda^p f))$, et notons $d(I(f), E)$ la longueur des suites E-régulières maximales contenues dans $I(f)$. Alors $d(I(f), E)$ est le plus petit entier q tel que $H^q(\Lambda^p f, E) \neq 0$ $(1 \leqq p \leqq n)$; caractérisation parallèle au moyen de l'homologie $H_*(\Lambda^p f, E)$. Si $d(I(f), R) = m-n+1$ et si R est un anneau de Macaulay, le module $\mathrm{coker}(\Lambda^p f)$ est équidimensionnel pour $p = 1, \cdots, n$; ceci généralise le théorème sur l'équidimensionalité des idéaux engendrés par les mineurs d'une matrice.

On suppose maintenant que $\mathrm{coker}(f) \otimes E$ est de longueur finie; notons $S^j(f)$ l'extension de f aux puissances symétriques j-ièmes. Alors la longueur $P_f(j, E)$ du module $\mathrm{coker}(S^j(f)) \otimes E$ est finie et est un polynôme en j pour j assez grand. Étude de ses différences finies. Soit $ej^d/d!$ le terme dominant de $P_f(j, E)$; alors les entiers e et $d-n+1$ ne dépendent que des modules E et $\mathrm{coker}(f)$; posons $e = e_E(M)$ où $M = \mathrm{coker}(f)$. Étude de la multiplicité $e_E(M)$: additivité par rapport à E, interprétation comme caractéristique d'Euler-Poincaré d'un complexe. La hauteur des idéaux premiers minimaux associés à $\mathrm{coker}(f)$ est majorée par $m-n+1$, ce qui généralise l'Hauptidealsatz de Krull.

Enfin on construit un complexe double pour montrer que, si $\mathrm{coker}(f) \otimes E$ est de longueur finie, la caractéristique d'Euler-Poincaré du complexe $H_*(\Lambda^p f, E)$ est égale à $\binom{n-1}{n-p} \Delta^m P_f(j, E)$ (où Δ^m désigne la m-ième différence). Application à des caractérisations des anneaux de Macaulay. *P. Samuel* (Paris)

Parr, J. T. 3077

Cohomology of cyclic groups of prime square order.

Bull. Amer. Math. Soc. **70** (1964), 427–428.

The author uses the classification of Z_p-representations of Z/p^2Z by the reviewer and I. Reiner [Ann. of Math. (2) **76** (1962), 73–92; MR **25** #3993] to enumerate the groups which occur as cohomology groups of Z/p^2Z with finitely generated coefficient modules. *A. Heller* (Urbana, Ill.)

GROUP THEORY AND GENERALIZATIONS

See also 3070, 3405.

Papy, Georges 3078

★Groups.

Translated by Mary Warner.

Macmillan & Co., Ltd., London; St. Martin's Press, New York, 1964. xvii + 220 *pp.* (32 *plates*) 42*s*.

This is a translation of the French edition [Dunod, Paris, 1961], which was reviewed earlier [MR **26** #1347].

Grindlinger, E. I. 3079

The word problem for a class of semigroups with a finite number of defining relations. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 77–85.

The word problem is solved for semigroups whose defining relations $A_i = B_i$, $1 \leqq i \leqq n$, satisfy two conditions. If an A_i or B_i is called a 'special word', these conditions read: (a) a subword of different special words has length less than half of the length of either of them; (b) different special words have different initial letters. It is shown incidentally that each word is equivalent to a finite number of words. The proofs are self-contained.

S. Stein (Davis, Calif.)

Grindlinger, Martin [Greendlinger, Martin] 3080

Solutions of the word problem for a class of groups by Dehn's algorithm and of the conjugacy problem by a generalization of Dehn's algorithm. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 507–509.

Let G be a group given by generators $a_1, \cdots, a_n$ and defining relations $R_1 = 1, \cdots, R_k = 1$, such that (1) each R_i is a reduced word, (2) the set of words R_i is closed under inverses and cyclic conjugates, (3) if R_i and R_j are not inverses of each other then less than $\frac{1}{6}$ of the letters of R_i are absorbed by cancellation in the product R_iR_j, (4) If R_i, R_j, R_h are placed along the sides of a triangle, cancellation does not take place at all three vertices. The author shows that the following algorithm may be used to solve the word problem for G. Given a word W, if one alternately (i) applies cancellation and (ii) replaces a syllable S by the reduced form of SR_i if this has shorter length than S, then this process terminates with the empty word if and only if $W = 1$ in G [cf. M. Dehn, Math. Ann. **72** (1912), 413–421]. The proof proceeds by showing that any relation in G can be expressed as a product of conjugates of R_i with 'not too much cancellation' taking place. A modification of the method leads to a solution of the conjugacy problem for G. *P. M. Cohn* (London)

Newman, Morris 3081

Free subgroups and normal subgroups of the modular group.

Illinois J. Math. **8** (1964), 262–265.

The main results of this paper are the following. Theorem 1: Let H be a nontrivial normal subgroup of the modular group Γ and let H be different from Γ, Γ^2, Γ^3. Then H is a free group. Lemma 1: A nontrivial subgroup of Γ is free if and only if it contains no elements of finite period.

These theorems are "easy consequences of the Kuroš subgroup theorem", which states that a nontrivial subgroup H of a free product G is itself a free product; each free factor is either a free group or a conjugate of a subgroup of one of the free factors of G. The modular group is the free product of a cyclic group of order two by one of order three. It is astonishing that these consequences have apparently not been noticed until now. The date of the Kuroš theorem is 1934.

In the remainder of the paper the author develops criteria for the freedom of $\Gamma_0(n)$ defined by $c \equiv 0 \pmod{n}$ in the matrices $(ab|cd)$ of Γ and of $\Gamma(m, n)$ defined by $b \equiv 0 \pmod{m}$, $c \equiv 0 \pmod{n}$. In particular, $\Gamma_0(n)$ is free when $n = 12t - 1$.

{Reviewer's remark: The results of this paper extend in an obvious way to a subclass of the class of F-groups; these are the finitely presented groups that admit faithful representations as discontinuous (finitely generated) Fuchsian groups. Because of the importance of these results, the desirability of obtaining a simplified proof of the Kuroš theorem is underlined in the case of free products with a finite number of factors.}

J. Lehner (College Park, Md.)

Pic, Gheorghe 3082

Kriterien für das Zerfallen von Gruppen über Subnormalteilern.

Math. Z. **83** (1964), 27–28.

The author proves the following results: (1) Let K $(\neq 1)$ be a subnormal subgroup of a group G. Let M_i $(1 \leqq i \leqq s)$ be maximal normal subgroups of G. Let $K \cap D = 1$ with $D = M_1 \cap \cdots \cap M_s$. Then there exists an intersection H of some of the M_i's such that $G = HK$ and $H \cap K = 1$. Put $M = M_1 \cap \cdots \cap M_r$ $(r \leqq s)$. Then H can be so chosen that $M = (H \cap M)(K \cap M)$ holds. (2) Let K be a subnormal subgroup of a group G. Let S_i $(1 \leqq i \leqq r)$ be minimal normal subgroups of G, each of which satisfies the minimal condition for subnormal subgroups. Let $G = KS$ with $S = S_1 \cup \cdots \cup S_r$. Then K is a direct factor of G.

N. Ito (Nagoya)

Zemmer, J. L. 3083

On a class of doubly transitive groups.

Proc. Amer. Math. Soc. **12** (1961), 644–650.

The following theorem is proved: If a doubly transitive group G is such that the subgroup fixing one point is abelian, G is the affine group $\{x \to ax + b\}$ of a field. It is first remarked that a strictly doubly transitive group is an independent ABA group with $B = Z_2$, in the sense of D. Gorenstein [Canad. J. Math. **11** (1959), 39–47; MR **21** #688]. The proof of the theorem uses this fact and consists in showing that the set of elements in G without fixed points is simply transitive, which allows one to use a characterisation of the affine groups of near-fields, given

by M. Hall, Jr. [Trans. Amer. Math. Soc. **54** (1943), 229–277; MR **5**, 72; ibid. **65** (1949), 473–474; MR **10**, 618]. {A slightly different and, it seems, somewhat shorter proof of the same theorem has been given by the reviewer [Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° **27** (1952), no. 2; MR **15**, 198]; since it is hidden in a remark (2, p. 47), it is not surprising that the author has overlooked it. The author points out to the reviewer that his method also establishes the following: Let G be a strictly doubly transitive group whose involutory elements have fixed points, let 0, 1 be two distinct points, and let 2 be the transform of 0 by the involution which fixes 1. Assume that the transformation which maps 0, 1 onto 0, 2 belongs to the center of the group of all transformations fixing 0. Then G is the affine group of a near-field.}

J. L. Tits (Bonn)

Mills, W. H. 3084

An application of linear programming to permutation groups.

Pacific J. Math. **13** (1963), 197–213.

Let X be a finite set and S the symmetric group acting on X. If σ and τ are in S, define $D(\sigma, \tau)$ to be the number of elements $g \in X$ such that $g\sigma \neq g\tau$. Furthermore, set

$$D_\sigma(\tau) = \min_{\lambda \in C(\sigma)} D(\tau, \lambda)$$

and

$$D_\sigma = \max_{\tau \notin C(\sigma)} D_\sigma(\tau)/D(\sigma\tau, \tau\sigma).$$

Improving earlier results of D. Gorenstein, R. Sandler and himself [same J. **12** (1962), 913–923; MR **26** #6243] the author shows that if $\sigma \in S$ is the product of m disjoint cycles of the same length n, $n > 1$, and if n is odd, then $D_\sigma \leqq (n-1)^2/(4n-6)$. If furthermore $m \geqq n-2$, then $D_\sigma = (n-1)^2/(4n-6)$.

H. Lüneburg (Mainz)

Dénes, J.; Pásztor, C. 3085

Sur un problème de substitution de P. Vermes. (Russian summary)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **7** (1962), 317–322.

Let E be the set of all positive rational integers. If F and H are subsets of E such that $x < y$ for all $x \in F$ and all $y \in H$, then we write $F < H$. We denote by S the group of all the unrestricted permutations of E. We call an $\alpha \in S$ a string if there exists a decomposition of E in finite subsets E_i such that $E_i < E_j$ for $i < j$ and such that $E_i\alpha = E_i$ for all i. A string α is called special if the cycles of α form such a decomposition of E. The authors prove that every $\gamma \in S$ is the product of two strings and every string is the product of two special strings. Hence every $\gamma \in S$ is the product of four special strings.

H. Lüneburg (Mainz)

Šidov, L. I. 3086

On the number of non-special subgroups of finite groups. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **32** (1963), 527–534.

Let G be a non-soluble finite group, $\rho(G)$ the number of different primes dividing the order of G and $\sigma(G)$ the number of proper non-nilpotent subgroups of G of different orders. Then the author proves the obvious inequality $\rho(G) \leqq \sigma(G)$. The elementary proof of this result could be considerably shortened by using a classical theorem of P. Hall about soluble groups.

Z. Janko (Canberra)

Sah, Chih-Han 3087

On the normality of Hall subgroups in finite groups.

Math. Z. **82** (1963), 243–246.

The author proves a necessary and sufficient condition for a Hall subgroup of a finite group G to be normal. This is then applied to give a criterion for the Sylow 2-group of G to be normal.

W. Feit (Ithaca, N.Y.)

Brauer, Richard 3088

On quotient groups of finite groups.

Math. Z. **83** (1964), 72–84.

Let G be a finite group, H a subgroup and $H_0 \lhd H$. A normal subgroup G_0 of G is a normal complement of H over H_0 if $G = G_0H$ and $H_0 = G \cap H$, and hence $G/G_0 \approx H/H_0$. The purpose of this paper is to investigate the question of the existence in G of normal complements of H over H_0. The main result is the following. Theorem 1: Let G be a finite group, $H_0 \lhd H \subseteq G$, and let π be the set of primes dividing $|H:H_0|$. Assume the following four conditions: (AI) $(|G:H|, |H:H_0|) = 1$; (AII) If h_1, h_2 are in H and are conjugate in G, then h_1H_0 and h_2H_0 are conjugate in H/H_0; (AIII) If h is a π-element of $H - H_0$ and if p is a prime in π which does not divide the order of h, then p does not divide $|C_G(h):C_H(h)|$; (AIV) Let $H_0, H_1, \cdots, H_n$ be the inverse images in H of the conjugate classes of H/H_0. For $i \neq 0$ let G_i be the set of g in G whose π-factor g_π is conjugate in G to an element of H_i and let $G_0 = G - \bigcup_{i=1}^n G_i$. Assume that there do not exist indices i, j with $0 \leqq i$, $j \leqq n$ such that $|G_i| > |G:H| \cdot |H_i|$, $|G_j| < |G:H| \cdot |H_j|$. Then H has a normal complement in G over H_0.

Let ψ be a class function on H. Define Φ_ψ on G by setting $\Phi_\psi|G_i = \psi|H_i$. Condition (AII) of Theorem 1 implies that Φ_ψ is a class function on G. Then a well-known result of the author and J. Tate [Ann. of Math. (2) **62** (1955), 1–7; MR **16**, 1087] is used to show that each Φ_ψ is a character of G. The proof of Theorem 1 is completed by letting G_0 be the intersection of the kernels of all Φ_ψ. The rest of the paper consists of deriving some consequences of Theorem 1, in particular, a result of H. Wielandt [Math. Nachr. **18** (1958), 274–280; MR **21** #2009] and also a result of Hall and Grün [M. Hall, Jr., *The theory of groups*, Theorem 14.4.6, Macmillan, New York, 1959; MR **21** #1996]. Also some more recent results are consequences of Theorem 1 [M. Suzuki, J. Math. Soc. Japan **15** (1963), 387–391; MR **28** #2157; C.-H. Sah, #3087 above].

W. Feit (Ithaca, N.Y.)

Trofimov, P. I. 3089

On the solvability of finite groups. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 192–200.

The author extends results he had obtained in an earlier paper [Mat. Sb. (N.S.) **33** (**75**) (1953), 45–72; MR **15**, 286]. Let ρ denote the number of classes of non-invariant conjugate subgroups of the finite group G, and τ the number of prime divisors of the order of G. Since G is known [loc. cit.] to be solvable if $\rho \leqq 6$, the author assumes $\rho \geqq 7$. Then $\tau \leqq \rho - 3$, and G is solvable if $\tau = \rho - 3$.

H. Salzmann (Los Angeles, Calif.)

Thompson, John G. 3090

2-signalizers of finite groups.

Pacific J. Math. **14** (1964), 363–364.

Let π be a set of primes and G a finite group. The subgroup A of G is a π-signalizer of G if and only if $|A|$ and $|G:N_G(A)|$ are π' numbers. Let $s_\pi(G) = \max|A|$, A ranging over the π-signalizers of G. Theorem 1: For each pair of integers m, n there are only finitely many (isomorphism classes of) finite groups G such that $|G|_2 \leqq m$ and $s_2(G) \leqq n$. Theorem 2: Suppose the S_2-subgroup T of G is abelian and $S_2(G) = 1$. Then $\mho^1(T) \lhd G$. Both these results are proved by induction. In Theorem 1 the special nature of the prime 2 is brought into play by using a well-known result of Brauer and Fowler [Ann. of Math. (2) **62** (1955), 565–583; MR **17**, 580]. However, Theorem 2 ultimately rests on a very deep unpublished result of M. Suzuki in which he classifies all groups where S_2-subgroups are T.I. sets. The author concludes by stating the following very interesting conjecture. If G is a simple group, then every 2-signalizer of G is abelian.

If a complete classification of the finite simple groups were available, this conjecture would of course be settled immediately. Such a classification still seems far in the future. However, the proof of Theorem 2 illustrates the fact that some of the known classification theorems are sufficiently strong to be used in the proof of general results about finite groups. *W. Feit* (Ithaca, N.Y.)

Berkovič, Ja. G. 3091

The influence of π-properties of subgroups on the properties of a finite group. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 14–21.

Diese Arbeit bringt Beweise für den größten Teil der anderweitig [Dokl. Akad. Nauk BSSR **8** (1964), 10–12; MR **28** #2150] ohne Beweise mitgeteilten Sätze.

R. Kochendörffer (Rostock)

Weichsel, Paul M. 3092

On critical p-groups.

Proc. London Math. Soc. (3) **14** (1964), 83–100.

Let G be a finite p-group. Let $\{A_i\}$ be the set of (non-isomorphic) proper subgroups and proper factor groups of G. Let $\{A_i\}^*$ be the smallest set of finite p-groups containing $\{A_i\}$ which is closed under the operations of forming finite direct products and taking subgroups and factor groups. If G does not belong to $\{A_i\}^*$, G is called critical. Then, in a sense, the theory of finite p-groups can be reduced to that of critical p-groups. Now some of the principal results of the author are as follows: (1) Let A and B be respectively the set of (non-isomorphic) proper subgroups and that of proper factor groups of G. If G is not critical, then G belongs to either A^* or B^*. (2) Let G be a metabelian p-group, $p > 3$, of class c. If $n(G) = c \geqq 3$ ($n(G)$ is the smallest number of elements needed to generate G), then G is not critical. (3) Let G be a 3-generator class 3 p-group with $p \neq 3$. Then G is not critical. (4) Let G be a 2-generator p-group. Then G is critical if and only if its center is cyclic. (5) Let G be a critical p-group. G is called basic if, whenever G belongs to $\{A_1, \cdots, A_k\}^*$ with $\{A_i\}^* = \{G\}^*$ for $i = 1, \cdots, k$ ($k > 1$), we have $\{G\}^* = \{A_j\}^*$ for some $j \leqq k$. Now if G is basic, then $M_V(G) \subseteqq \Phi(G)$ for all sets of reduced words V such that $V(G) > 1$, where $M_V(G)$ is the marginal subgroup of G for V [see P. Hall, J. Reine Angew. Math. **182** (1940), 130–141; MR **2**, 211] and $\Phi(G)$ is the Frattini subgroup of G. (6) Let G be a p-group with cyclic center and assume that for some V, $M_V(G) \subseteqq \Phi(G)$. If $G/M_V(G)$ is critical, then G is critical.

A theorem of G. Birkhoff [Proc. Cambridge Philos. Soc. **31** (1935), 433–454] which implies that if G is critical, then there exists a reduced word f such that $f(G) \supset 1$ but $f(A) = 1$ for all groups A of $\{A_i\}^*$, is used in principle.

N. Ito (Nagoya)

Edge, W. L. 3093

An orthogonal group of order $2^{13}\cdot 3^5\cdot 5^2\cdot 7$.

Ann. Mat. Pura Appl. (4) **61** (1963), 1–95.

Of the $2^8 - 1$ points in a projective space [the author, Proc. London Math. Soc. (3) **10** (1960), 583–603; MR **22** #11046] over the field $F = GF(2)$, 135 points lie on a non-singular ruled quadric $\mathfrak{S}$. The group A of automorphisms of this quadric is of order $2^{13}3^5 5^2 7$. It has a simple subgroup A^+ of index 2 and order 174, 182, 400 called $FH(8, 2)$ by Dickson, which fixes each of two systems of 135 solid rulings on $\mathfrak{S}$. Referred to certain simplexes Σ the equation of the quadric becomes $\sum x_i x_j = 0$. The $8 + 70 + 56 + 1 = 135$ points m on $\mathfrak{S}$ have, respectively, 1, 4, 5, 8 coordinates not zero. Those 9 with 1 or 8 coordinates not zero form one of 960 enneads that are transitively permuted by A. The subgroup fixing an ennead is the symmetric group of order 9!. All the types of subspaces are described and counted. For example, a j-plane is spanned by $3m$ points, no two conjugate. Given an m called m_1 there are 70 others in the tangent prime, 2 on each of 35 generators, that are conjugate to m_1, and 64 not conjugate to m_1. Of these 64, there are 28 not conjugate to a chosen one, m_2. Hence there are $8! = 135\cdot 64\cdot 28/6$ j-planes. The numbers of incidences between pairs of subspaces of each two types are tabulated. Then an exhaustive study is made of the classes of elements according to the types of subspaces that are latent for each. Permutation characters in the classes of the subgroup S_9 are given for the representation of degrees 120 and 135 induced by subgroups A_p and A_m, the former the exceptional Lie group E_7 and the latter of order $2^6 8!$ having representatives in 55 of the 67 classes which are tabulated. The fact that the group A_m is isomorphic to the central quotient group of a group of $2^7 8!$ monomial matrices was not noted. Finally the matrices of a Sylow subgroup of order 2^{13} are given explicitly, together with their distribution among the classes of A. *J. S. Frame* (E. Lansing, Mich.)

Wang, John S. P. 3094

On completely decomposable groups.

Proc. Amer. Math. Soc. **15** (1964), 184–186.

Let G be a torsion-free abelian group. A subgroup H of G is called regular if each element of H has the same types $T(x)$ in H as it has in G. Using a result of Baer [Fuchs, *Abelian groups*, p. 163, Publ. House Hungar. Acad. Sci., Budapest, 1958; MR **21** #5672], a short proof is given of the following theorem. Let G be a direct sum $\sum_{\lambda<\alpha} G_\lambda$ (α an ordinal), where G_λ has rank one for $\lambda > 0$ and $T(x) \geqq T(y)$ whenever $0 \neq x \in G_\lambda$, $0 \neq y \in G_\mu$ with $\lambda < \mu$. Suppose that H is a regular subgroup of G. Then $H = (H \cap G_0) \oplus \sum_{\lambda>0} H_\lambda$, where $H_\lambda \cong G_\lambda$ or $H_\lambda = 0$. As an application of this result it is shown that if G is a finite rank group

whose elements have types which form a chain, then G is completely decomposable if and only if G/H is finite for every regular subgroup H of maximal rank.

R. S. Pierce (Seattle, Wash.)

Pierce, R. S. 3095

Centers of purity in abelian groups.

Pacific J. Math. **13** (1963), 215–219.

This note adds to that by J. D. Reid [same J. **13** (1963), 657–664; MR **27** #5829]. Reid defined a centre of purity of an abelian group G as a subgroup H such that every subgroup of G maximal with respect to being disjoint from H is pure in G. Lemma 2.1 of Reid gives a sufficient condition for centres of purity; here it is shown to be also necessary in a primary abelian group. Hence the following characterization of centres of purity in an abelian p-group G: let $G[p]$ be the set of elements of order $\leq p$, $P_k = G[p] \cap p^kG$, $P_\infty = \bigcap P_k$, and $P_{\infty+1} = P_{\infty+2} = \{0\}$; then H is a centre of purity in G if and only if for some k, $0 \leq k \leq \infty$, $P_k \supseteq H[p] \supseteq P_{k+2}$. From this it follows that H is a centre of purity of an arbitrary abelian group G if and only if the torsion group of H is a centre of purity of the torsion group of G and either G/H is a torsion group or $H[p] \subseteq \bigcap p^nG$ for all primes p.

Hanna Neumann (Canberra)

Procházka, Ladislav 3096

Conditions for decomposition into a direct sum for torsion-free Abelian groups of rank two. (Russian. German summary)

Mat.-Fyz. Časopis Sloven. Akad. Vied **12** (1962), 166–202.

Let G be a torsion-free Abelian group of rank two. Then G is called decomposable if it is isomorphic to a direct sum of two rank one groups. The theorems of this paper give criteria for such a group to be decomposable. The conditions are expressed in terms of the Mal'cev invariants. Let Z_p be the ring of p-adic integers. Then $Z_p \otimes G$ (tensor product over the ring of integers) is a torsion-free Z_p-module of rank two, and consequently $Z_p \otimes G$ is isomorphic to a direct sum of two rank one Z_p-modules, which are either Z_p or its quotient field R_p. Thus, each G determines a partition of the set of rational primes into three classes: $\pi_0(G)$, consisting of all p such that $Z_p \otimes G \cong Z_p \oplus Z_p$; $\pi_1(G)$, consisting of all p such that $Z_p \otimes G \cong Z_p \oplus R_p$; and $\pi_2(G)$, consisting of all p such that $Z_p \otimes G \cong R_p \oplus R_p$. The groups G discussed in this paper are those for which $\pi_1(G) \neq \emptyset$. Let $\{x_1, x_2\}$ be a pair of independent elements of G. Suppose that $p \in \pi_1(G)$. Then $Z_p \otimes G = R_p(x_1 + \alpha(p)x_2) \oplus Z_p(p^{-u(p)}x_2)$, where $\alpha(p) \in R_p$ and $u(p)$ is a non-negative integer, or else $Z_p \otimes G$ has a similar form with x_1 and x_2 interchanged. The quantities $\alpha(p)$ and $u(p)$ are invariants of G and the basis $\{x_1, x_2\}$. Similar invariants are obtained for the primes $p \in \pi_0(G)$. It is shown that if either $\alpha(p)$ is irrational for some p in $\pi_1(G)$, or if there are three distinct primes p_1, p_2, p_3 in $\pi_1(G)$ for which the quantities $\alpha(p_1)$, $\alpha(p_2)$, and $\alpha(p_3)$ are different, then G is indecomposable. {See R. A. Beaumont and the reviewer [Mem. Amer. Math. Soc. No. 38 (1961), esp. p. 23; MR **24** #A162] for a slightly more general result.} The principal theorems of the paper deal with the case in which the set $\{\alpha(p) | p \in \pi_1(G)\}$ consists of one or two rational numbers. In these cases, criteria for decomposability are given in terms of the invariants associated with the primes in $\pi_0(G)$.

R. S. Pierce (Seattle, Wash.)

Gruenberg, Karl 3097

The residual nilpotence of certain presentations of finite groups.

Arch. Math. **13** (1962), 408–417.

Let F be a non-cyclic free group, R a normal subgroup of F, and R' the commutator subgroup of R. Supplementing a theorem of M. Auslander and R. C. Lyndon [Amer. J. Math. **77** (1955), 929–931; MR **17**, 709] the author shows that the hypercenter of F/R' is just the center. His main theorem is that if R is of finite index, then F/R' is residually nilpotent if and only if F/R is of prime power order. The proof of this theorem suggested the following one and it is also proved in this paper. If G is a finite group and $I(G)$ is the augmentation ideal in ZG, then G is of prime power order if and only if $\bigcap_{k=1}^{\infty} I(G)^k = 0$.

J. McLaughlin (Ann Arbor, Mich.)

Kemhadze, Š. S. 3098

Quasi-nilpotent groups. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 1003–1005.

The author defines a group G to be quasi-nilpotent if each finitely generated subgroup H is subinvariant in G, i.e., if there exists a possibly transfinite increasing normal series $H \lhd H_1 \lhd \cdots \lhd H_\nu = G$, and states without proof that each group G contains a unique maximal quasi-nilpotent normal subgroup, the quasi-nilpotent radical $k(G)$, which lies in general properly between the nil-radical and the locally nilpotent radical. G is quasi-nilpotent if and only if each cyclic subgroup is subinvariant, or if and only if G is locally nilpotent and has an increasing normal series with commutative factors. Groups with the latter property are called RN^*-groups in the Russian terminology. G is an RN^*-group if and only if there is an increasing characteristic series in G with quasi-nilpotent factors.

H. Salzmann (Los Angeles, Calif.)

Calenko, M. S. 3099

Isomorphic nilpotent products of nilpotent p-groups. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 225–236.

The paper is concerned with the connection between different decompositions of a nilpotent p-group into regular products, the emphasis being on the case of nilpotent products (in the sense of O. N. Golovin [Mat. Sb. (N.S.) **27** (**69**) (1950), 427–454; MR **12**, 672]). Sample theorems: If the commutator factor group of a nilpotent p-group is a direct product of cyclic groups, and if the group itself is an nth nilpotent product of non-trivial subgroups, then the group has no non-trivial mth nilpotent decompositions for any m other than n. If a finite p-group is a verbal product of semidirectly indecomposable groups and belongs to the variety defined by the set of words in question, then every regular decomposition of the group can be refined to a verbal decomposition isomorphic to the one originally considered. If a nilpotent p-group is an nth nilpotent product of finitely many groups of class less than n and certain further conditions are satisfied, then every nilpotent decomposition of the group has a refinement isomorphic to the original decomposition. Analogous results had been obtained for nilpotent decompositions of torsion-free nilpotent groups by A. L. Smel'kin [Sibirsk. Mat. Ž. **4** (1963), 1412–1425; MR **28** #1236].

L. G. Kovács (Canberra)

Neumann, B. H.; Taylor, Tekla 3100
Subsemigroups of nilpotent groups.
Proc. Roy. Soc. Ser. A **274** (1963), 1–4.

The general question of embedding a semigroup in a group has been considered by numerous investigators. It being well known that the semigroups which can be embedded in abelian groups are the cancellative abelian semigroups, the authors consider the rather natural and more general question asking for the semigroups which can be embedded in nilpotent groups.

The authors first formulate a certain law as follows: Define a sequence of words $q_1, q_2, \cdots$, in variables x, y, z, where z stands for the sequence of variables $z_1, z_2, \cdots$ by $q_1(x, y, z) = xy$, $q_{i+1}(x, y, z) = q_i(x, y, z)z_i q_i(y, x, z)$. Note q_i involves only $x, y, z_1, \cdots, z_{i-1}$. The law L_c is: $q_c(x, y, z) = q_c(y, x, z)$. Hence L_1 is the commutative law and L_2 is $xyz_1yx = yxz_1xy$. The main result is Theorem 1: The semigroup S can be embedded in a nilpotent group of class c if and only if it is cancellative and satisfies L_c. Obviously one then has the corollary: The group G is nilpotent of class c if and only if it satisfies L_c.

The authors remark that since L_c involves $c+1$ variables it is natural to ask whether laws in fewer variables are also sufficient. While Macdonald [Math. Z. **78** (1962), 175–188; MR **26** #213] showed there is a single law in c variables characterising the nilpotent groups of class c, when $c \geq 3$, the authors state an unpublished result of I. D. Macdonald, B. H. Neumann, M. F. Newman and J. Wiegold, that no set of laws in fewer than c variables suffices. The corresponding problem for semigroups is not considered.

Among the other results obtained is the following corollary: Let X be a set of generators of the group G, and let X^1 be X with the identity element adjoined. If L_c is satisfied for all substitutions of elements of X^1 for the variables, then G is nilpotent of class c.

The authors show the necessity for the inclusion of the identity element in the range of variables.

R. P. Hunter (University Park, Pa.)

Robinson, Derek J. S. 3101
Groups in which normality is a transitive relation.
Proc. Cambridge Philos. Soc. **60** (1964), 21–38.

A group G is called a T-group if $K \triangleleft H \triangleleft G$ implies $K \triangleleft G$. Finite T-groups have been studied by Gaschütz [J. Reine Angew. Math. **198** (1957), 87–92; MR **19**, 940] and others. The author studies in detail the structure of infinite solvable T-groups. Some of the results proved are: (1) A solvable T-group is metabelian; (2) A finitely generated solvable T-group is finite or abelian; (3) Let G be a periodic solvable T-group and let C be the centralizer of G'. If $[G:C]$ is countable then G splits over $[G', G]$.

B. Chang (Vancouver, B.C.)

Golovin, O. N. 3102
The lattice of polyverbal operations. (Russian)
Dokl. Akad. Nauk SSSR **153** (1963), 1238–1241.

The concept of polyverbal subgroups and operations was introduced by the author [same Dokl. **145** (1962), 967–970; MR **26** #180]. Here the set of polyverbal subgroups and the set of polyverbal operations are made into lattices and theorems about these lattices are stated. At the end of the paper some work of Baranovich is mentioned in which these concepts and many of these results are transferred to universal algebras.

R. R. Struik (Pueblo, Colo.)

Golovin, O. N. 3103
Multi-identity relations in groups and operations defined by them on the class of all groups. (Russian)
Trudy Moskov. Mat. Obšč. **12** (1963), 413–435.

Proofs are given here of results announced in the work of the author [Dokl. Akad. Nauk SSSR **145** (1962), 967–970; MR **26** #180]. In addition, examples of VV-operations are discussed; conditions under which VV-operations are associative are found; examples of both associative and non-associative VV-operations are given.

R. R. Struik (Pueblo, Colo.)

Bovdi, A. A. 3104
On the embedding of crossed products into fields. (Russian)
Dokl. Akad. Nauk SSSR **151** (1963), 1253–1255.

If for each element g in the group G there is defined an automorphism $(\alpha \to \alpha^{g\sigma})$ of the field D, and if $\rho: G \times G \to D$ is a factor system such that the multiplication $\pi: (g\alpha, h\beta) \to gh\rho_{g,h}\alpha^{h\sigma}\beta$ on $G \times D$ is associative, then the vector space D^G with multiplication induced by π is called the crossed product (G, D, ρ, σ). Theorem: If the normal subgroup H of G has a (transfinite) normal series whose factors are locally finite over their center, if G/H is an ordered group, and if (H, D, ρ, σ) has no divisors of zero, then (G, D, ρ, σ) can be embedded into a field, and (H, D, ρ, σ) has a field of right quotients.

H. Salzmann (Los Angeles, Calif.)

Roseblade, James E. 3105
The automorphism group of McLain's characteristically simple group.
Math. Z. **82** (1963), 267–282.

Let Λ be a partially ordered set and let K be a division ring. Let V be a vector space over K with basis elements v_λ, $\lambda \in \Lambda$. For $\lambda < \mu$ define the endomorphism $e_{\lambda\mu}$ of V by $v_\lambda e_{\lambda\mu} = v_\mu$, $v_\rho e_{\lambda\mu} = 0$ for $\rho \neq \lambda$. Let F denote the ring generated by all $e_{\lambda\mu}$. The set $M(\Lambda, K)$ consisting of all $1+f$, f in F, forms a group. The author is interested in studying the group $M = M(R, K)$, where R is the set of rational numbers and K is the field of p elements for some prime p. Let A be the automorphism group of M. If $\theta \in A$ and G_θ denotes the split extension of G by θ, then θ is called a Baer automorphism of G if $\{\theta\}$ is subnormal in G. If θ acts like an inner automorphism of G on every finitely generated subgroup of G, then θ is called a locally inner automorphism of G. Let S be the set of Baer automorphisms of M and let L be the set of locally inner automorphisms of M. The main result of the paper is Theorem 1. (i) S is a locally finite p-group. (ii) S is generated by its abelian normal subgroups. (iii) S is the Baer radical of A (i.e., the subgroup of A generated by all the subnormal nilpotent subgroups. (iv) M is a characteristic subgroup of S. (v) $S < L$. (vi) If p is odd, then L/S has no non-trivial locally nilpotent normal subgroups. The proof is reduced to questions in ring theory by observing that locally inner automorphisms are closely related to quasi-regular elements of a certain ring of zero triangular matrices.

W. Feit (Ithaca, N.Y.)

Nakano, Takeo 3106

A theorem on lattice ordered groups and its applications to the valuation theory.

Math. Z. **83** (1964), 140–146.

Suppose that N_i, $1 \leqq i \leqq k$, are normal subgroups of the group G; the system $(G; N_i, 1 \leqq i \leqq k)$ is said to satisfy the compatibility axiom if for each system of elements $a_i \in G$ satisfying $a_i \equiv a_j \pmod{N_i N_j}$, $i \neq j$, there exists an element $a \in G$ for which $a \equiv a_i \pmod{N_i}$. The author proves the following theorem for l-groups, which in turn implies among other facts the general Chinese remainder theorem for finite sets of valuations [P. Jaffard, Bull. Sci. Math. (2) **85** (1961), 1ère partie, 127–135; MR **27** #5828; P. Ribenboim, Math. Z. **68** (1957), 1–18; MR **19**, 1035; I. Yakabe, Mem. Fac. Sci. Kyushu Univ. Ser. A **17** (1963), 10–28; MR **28** #87]. "Let $N_1, N_2, \cdots, N_k$ be a sequence of normal subgroups of a group G such that the factor groups $\Gamma_i = G/N_i$ $(i = 1, 2, \cdots, k)$ are l-groups. Then the following two statements are equivalent: (I) The factor group $\Gamma = G/N$, where $N = N_1 \cap N_2 \cap \cdots \cap N_k$, is an l-group such that the canonical mappings: $xN \to xN_i$ are order-homomorphisms of Γ onto Γ_i. (II) For every pair i, j, the factor group $\Gamma_{ij} = G/N_i N_j$ is an l-group such that the canonical mapping: $xN_i \to xN_i N_j$ is an order-homomorphism of Γ_i onto Γ_{ij}, and the compatibility axiom holds in the system $\{G; N_1, N_2, \cdots, N_k\}$."

O. F. G. Schilling (Lafayette, Ind.)

Howie, J. M. 3107

The embedding of semigroup amalgams.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 55–68.

A sufficient condition for an amalgam of semigroups S_i with a single amalgamated subsemigroup U to be embeddable in the direct product of the S_i with amalgamated U is that U is "almost unitary" and "central" in each S_i. We say that U is central in S if $us = su$ for all $u \in U$ and $s \in S$; U is almost unitary in S if an idempotent element e can be adjoined to S such that $es \in S$ and $se \in S$ for all $s \in S$, and $eu = ue = u$ for all $u \in U$, and $esu \in U$ or $use \in U$ implies $ese \in U$ [this is not the author's definition, but a paraphrase]. This can be derived from the author's earlier theorem [Proc. London Math. Soc. (3) **12** (1962), 511–534; MR **25** #2139] that the amalgam can be embedded in a semigroup, and hence in its free product, if U is almost unitary in each S_i, together with a reduction theorem in the present paper, namely, that if U is central in each S_i and if the amalgam is embeddable in a semigroup, then also in its direct product. However, a direct proof is also given. A corollary of the result is that if U is almost unitary and central in each S_i, and if the amalgam is finite, then it is embeddable in a finite semigroup; it is not known whether this remains true if U is not central in each S_i. However, if U is "consistent" in each S_i, that is, if a product is in U only if its factors are, then one can adjoin a zero in the obvious way to the amalgam, and the result is a semigroup, and it follows that if, moreover, the amalgam is finite, then it can be embedded in a finite semigroup. The results are compared with corresponding facts about amalgams of groups. It should, however, be noted that the author's definition of a direct product of semigroups does not specialize to that for groups if the semigroups are groups; nor do his direct products coincide with those defined, for example, by A. H. Clifford and G. B. Preston [*The algebraic theory of semigroups*, Vol. I, Amer. Math. Soc., Providence, R.I., 1961; MR **24** #A2627]. Briefly, the author's direct products are free products modulo the congruence that makes the constituents commute elementwise; they can also be constructed by adjoining a unit element to each constituent, whether it possesses one already or not, then forming the cartesian product of the resulting semigroups, restricting it to the semigroup of elements of finite support, and finally removing the unit element again.

B. H. Neumann (Canberra)

Tamura, T. 3108

Operations on binary relations and their applications.

Bull. Amer. Math. Soc. **70** (1964), 113–120.

Let $\mathscr{B}$ be a complete lattice of binary relations on a set E. If P, Q are mappings of $\mathscr{B}$ into $\mathscr{B}$, $P \leqq Q$ means that $\rho P \subseteq \rho Q$ for all $\rho \in \mathscr{B}$. $P: \mathscr{B} \to \mathscr{B}$ is a semi-closure operation if P is isotone and $\rho \subseteq \rho P$ for all $\rho \in \mathscr{B}$. The set $\mathfrak{P}$ of all semi-closure operations on $\mathscr{B}$ forms a complete lattice relative to $\leqq$ and relative to iteration forms a semigroup compatibly ordered by $\leqq$. The author discusses the relation between $\mathscr{B}$ and $\mathfrak{P}$. For example, it is shown that if $\rho \in \mathscr{B}$, then there exists, for any subset $\mathfrak{A}$ of $\mathfrak{P}$, a minimal relation $\sigma \in \mathscr{B}$ such that $\sigma P = \sigma$ for all $P \in \mathfrak{A}$. The results are applied to the construction of equivalences and congruences of various kinds on groupoids and semigroups. A fuller discussion is promised later.

G. B. Preston (Clayton)

Armbrust, Manfred; Schmidt, Jürgen 3109

Zum Cayleyschen Darstellungssatz.

Math. Ann. **154** (1964), 70–72.

The following two theorems are proved. (1) Let A be a semigroup with identity. Let E be the semigroup of all mappings of A into A with the natural multiplication. Let F be the subset of E consisting of all the mappings of the form $f_b(x) = xb$, where $x, b \in A$. Then A is isomorphic to the centralizer of F in E. (2) Let A be a set containing at least two elements a and b. (If A consists of only one element, then the theorem becomes trivial.) Let E be as in (1). Let S be the complete symmetric group over A, and let G be a subgroup of S. Let f be the mapping of E into A defined as follows: $f(s) = s(a)$ if $s \in G$, $= s(b)$ if $s \in E - G$. Then G is the set of all elements of S satisfying the condition $t(f(s)) = f(ts)$.

N. Ito (Nagoya)

Ivan, Ján 3110

Simplicity and minimal ideals of a direct product of semigroups. (Russian. English summary)

Mat.-Fyz. Časopis Sloven. Akad. Vied **13** (1963), 114–124.

Some simple considerations in the direction indicated by the title.

J. Wiegold (Cardiff)

Baayen, P. C.; Hedrlín, Z. 3111

On the product and sum of a system of transformation semigroups.

Comment. Math. Univ. Carolinae **4** (1963), 29–42.

The authors define the notions of (direct) product, and of restricted direct product (which they call the sum) of an

arbitrary family $\{F_\alpha\}$ of semigroups of mappings of sets $\{X_\alpha\}$ into themselves, where the sets X_α are not necessarily pairwise disjoint. Using the notion of product, they show that every commutative semigroup F of mappings of a set X into itself which contains the identity map is contained in a unique maximal semigroup of mappings with the same invariant sets as F. If this maximal semigroup G has an element g such that for every h in G there is a k in G such that $g=hk$, and if every element of G has at least one fixed point, then all mappings in G have a common fixed point.

They show also that every commutative semigroup of mappings of a set X into itself is a subdirect product of algebraically generated semigroups. By an algebraically generated semigroup is meant the set of all left multiplications of an abstract semigroup with unit.

O. Frink (University Park, Pa.)

Hedrlín, Zdeněk 3112

Two theorems concerning common fixed point of commutative mappings.

Comment. Math. Univ. Carolinae **3** (1962), *no.* 2, 32–36.

The author considers maximal commutative semigroups F of mappings of a set X into itself, each mapping having at least one fixed point. He shows that if X is a finite set, then the members of F have precisely one common fixed point. If X is infinite, and F satisfies one additional condition, then also the members of F have precisely one common fixed point. The additional condition is that there is in F at least one mapping f such that for every g in F there is an h in F such that $f=gh$.

He shows also that if f and g are commutative mappings of a set X into itself, and f has precisely n fixed points, where n is finite and greater than zero, then f and g^k have at least one common fixed point for some natural number k with $k\leq n$.

O. Frink (University Park, Pa.)

Šaĭn, B. M. 3113

One-sided nilpotent semigroups. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 1 (115), 187–189.

A semigroup G is a nil-semigroup [left nil-semigroup] if some power of each element of G is a zero [left zero] of G; G is a nilpotent semigroup [left nilpotent semigroup] of rank n if the product of any n elements of G is a zero [left zero] of G. The results are that: Every left nil-semigroup [left nilpotent semigroup] is a union of pairwise disjoint nil-semigroups [nilpotent semigroups]. The proofs are elementary.

J. Wiegold (Cardiff)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 3223, 3295, 3324, 3335, 3463, 3464, 3465, 3675.

Fell, J. M. G. 3114

Weak containment and induced representations of groups. II.

Trans. Amer. Math. Soc. **110** (1964), 424–447.

From the author's introduction: "In Acta Math. **106** (1961), 233–280, we have outlined a program for determining the structure of an arbitrary C^*-algebra A in terms of its primitive images. The first step in this program is always to determine the topological space $\hat{A}$ of all equivalence classes of irreducible representations of A, with the hull-kernel topology. If A is the group C^*-algebra of a locally compact group, $\hat{A}$ becomes the topological space $\hat{G}$ of equivalence classes of irreducible unitary representations of G. How is $\hat{G}$ to be analyzed? By Mackey's general theory [ibid. **99** (1958), 265–311; MR **20** #4789], the elements of $\hat{G}$ are obtained (under certain broad conditions) by inducing from representations of subgroups. It would be desirable to supplement Mackey's theory so as to obtain not merely the elements of $\hat{G}$ but also their topology, in terms of the corresponding entities on subgroups. As a first step [Part I] in this program we proved [Canad. J. Math. **14** (1962), 237–268; MR **27** #242] the continuity of the process of inducing a representation of a subgroup K up to a representation of G. However, this kind of theorem, in which the subgroup K is fixed, is inadequate; we need to consider the case in which not only the representation of K but even K itself can vary continuously. The main purpose of the present paper is to formulate and prove such a generalization. . . . We shall introduce a topology into the set $\mathscr{S}(G)$ of all pairs $\langle K, T\rangle$, where K is a (variable) closed subgroup of G, and T is a unitary representation of K, and then prove that the result ${}_LU^T$, of inducing the representation T of K up to a representation of a subgroup L containing K, is continuous in all three variables T, K, and L. A similar theorem holds for the operation of restriction: the result $T|L$ of restricting T to a subgroup L contained in K is continuous in T, K, and L [the author, Trans. Amer. Math. Soc. **94** (1960), 365–403; MR **26** #4201]."

The author also defines a "topological" generalization of the classical Reciprocity Theorem of Frobenius, and shows that it holds for certain special cases of non-compact groups and fails for others. Related results on the structure of $\hat{G}$ may be found in the work of Glimm [Pacific J. Math. **12** (1962), 885–911; MR **26** #3819].

R. T. Prosser (Lexington, Mass.)

Gel'fand, I. M.; Pjateckiĭ-Šapiro, I. I. 3115

Automorphic functions and the theory of representations. (Russian)

Trudy Moskov. Mat. Obšč. **12** (1963), 389–412.

Let G be a real semi-simple Lie group, Γ a discrete subgroup and $X=\Gamma\backslash G$. The chief purpose of this paper is to prove the main result of a previous note [Dokl. Akad. Nauk SSSR **147** (1962), 17–20; MR **26** #260], which asserts that, under suitable assumptions on Γ, a certain subspace of $L_2{}^0(X)$ of $L_2(X)$ is a discrete sum of irreducible representations of G, each occurring with finite multiplicity. Let $Z\cdot A$ be a solvable group occurring in the definition of Siegel domain, and U its intersection with a Siegel domain. A series of reductions brings this down to showing that an integral operator on U with kernel $\sum\varphi(g_1{}^{-1}\cdot\delta\cdot g_2)$ $(\delta\in\Gamma\cap Z)$, where $\varphi=\varphi_1*\tilde{\varphi}_1$, and φ_1 is smooth with small support, is of the trace class. This follows from an estimate of the number of non-zero terms and of certain derivatives when $g_1=z\cdot a$, $g_2=z_0\cdot a$ $(a\in A,\ z,z_0\in Z)$. In an introduction, the authors also give a rough outline of their study of $L_2(X)$ [see also Gel'fand, Proc. Internat. Congr. Mathematicians (Stockholm, 1962), pp. 74–85, Inst. Mittag-Leffler, Djursholm, 1963].

A. Borel (Paris)

Lomont, J. S.; Moses, H. E. 3116
Exponential representation of complex Lorentz matrices. (Italian summary)
Nuovo Cimento (10) **29** (1963), 1059–1067.
Authors' summary: "It is shown that every proper, complex Lorentz matrix L can be expressed in the form $L=\sigma\exp[AG]$, where A is an antisymmetric matrix, G is the metric matrix, and $\sigma=\pm1$. It is then shown that not every proper, complex Lorentz matrix L can be expressed in the form $L=\exp[AG]$, with A and G as above."
K. Maurin (Warsaw)

Wang, Hsien-Chung 3117
Some geometrical aspects of coset spaces of Lie groups.
Proc. Internat. Congr. Math. 1958, *pp.* 500–509. *Cambridge Univ. Press, New York*, 1960.
This is a brief and very clear account of results on homogeneous spaces with various types of geometrico-differential structures. The author first shows, on two examples (tensor fields and connections), how invariant structures on coset spaces G/H of Lie groups can be expressed in terms of the Lie algebras of G and H. Thereafter, the attention is mainly focused on properties of the "C-spaces": these are the simply connected compact homogeneous complex analytic manifolds, which have been completely classified and studied by the author [Amer. J. Math. **76** (1954), 1–32; MR **16**, 518]. Although the paper is essentially expository, some interesting new aspects of known results are brought into light. For instance, in view of the fact that any n-complex-dimensional complex torus is a quotient of $\mathbf{C}^{*2n}$ by a connected analytic subgroup, it is remarked that, given a C-space A, which is known to be a principal bundle over a rational algebraic variety with a torus as the group, there exists a uniquely defined C-space having the same underlying space, the same real analytic bundle structure and the same complex analytic base as A, and whose group is an arbitrarily preassigned complex torus (of the right dimension).
J. L. Tits (Bonn)

Yamaguchi, Satoru 3118
Semi-simplicity in infinite-dimensional Lie groups.
Mem. Fac. Sci. Kyushu Univ. Ser. A **17** (1963), 164–167.
The author states and proves the following theorem. The theorem is well known in the finite-dimensional case and the proof is identical in the infinite-dimensional case. Theorem: Let G be a simply connected Lie group, g its Lie algebra and suppose g is the direct sum of closed ideals g_1 and g_2. Let G_i be the Lie subgroup of G corresponding to g_i. Then G_i is simply connected and $G=G_1\times G_2$.
R. S. Palais (Waltham, Mass.)

Gu, Čao-hao [Ku, Chao-Hao] 3119
On reducibility of infinite groups of Cartan mappings. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1958**, *no.* 4 (5), 60–66.
The author considers the problem of reducibility of Cartan's infinite transitive transformation group G. Such a group, defined by a first-order differential equation in the n-dimensional euclidean space E_n, is said to be reducible if E_n admits a coordinate system (x^i) in terms of which the finite equations of G have the form

$$x'^{i_1}=f^{i_1}(x^1,\cdots,x^q),$$
$$x'^{i_2}=f^{i_2}(x^{q+1},\cdots,x^n)$$
$$(i_1=1,2,\cdots,q,\quad i_2=q+1,q+2,\cdots,n),$$

with mutually independent functions f^{i_1} and f^{i_2}. The author proves the following theorems. (1) A group G is reducible in the above sense when the specially defined adjoint linear group L does not admit a stable contravariant or covariant vector and is given by a product of two linear groups L' and L'', which are in the subspaces E_q and E_{n-q} and do not possess a common direction. (2) If the adjoint linear group L has mobile vectors, the given group G is semireducible. A special class of semi-reducible infinite groups is defined.
In an earlier paper with G. I. Kručkovič [Dokl. Akad. Nauk SSSR **120** (1958), 1183–1186; MR **21** #5221] the author determined a criterion of reducibility of homogeneous Riemannian spaces. *T. P. Anđelić* (Belgrade)

FUNCTIONS OF REAL VARIABLES

See also 3044, 3148, 3307, 3317, 3322.

Diaz, J. B.; Výborný, R. 3120
On mean value theorems for strongly continuous vector valued functions.
Contributions to Differential Equations **3** (1964), 107–118.
If B is a normed linear space, then a function $x:[a,b]\to B$ is strongly continuous at t_0 if $\lim_{t\to t_0}\|x(t)-x(t_0)\|=0$. The main purpose of this paper is to prove the following theorem: If $x:[a,b]\to B$ is strongly continuous, then there exists $c\in(a,b)$ such that one either has

$$\left\|\frac{x(b)-x(a)}{b-a}\right\|\leqq\left\|\frac{x(c+h)-x(c)}{h}\right\|$$

whenever both $h>0$ and $a\leqq c+h\leqq b$ or

$$\left\|\frac{x(b)-x(a)}{b-a}\right\|\leqq\left\|\frac{x(c)-x(c-h)}{h}\right\|$$

whenever both $h>0$ and $a\leqq c-h\leqq b$.
W. P. Ziemer (Bloomington, Ind.)

Šarkovskiĭ, A. N. 3121
Co-existence of cycles of a continuous mapping of the line into itself. (Russian. English summary)
Ukrain. Mat. Ž. **16** (1964), 61–71.
Let C be the class of all continuous, real-valued functions $f(x)$ on $(-\infty,+\infty)$. A point a is called a fixed point of order k of a function $f(x)$ if $f^k(a)=a$ and $f^j(a)\neq a$ for $j=1,\cdots,k-1$, where $f^1(x)=f(x)$, $f^{i+1}(x)=f(f^i(x))$. The relation $\prec$ is defined for positive integers as follows: $m\prec n$ if for any $f\in C$ the existence of a fixed point of order m of f implies the existence of a fixed point of order n of f, but not conversely. It is proved that the set of positive integers is ordered by the relation $\prec$ and $3\prec5\prec7\prec\cdots\prec3\cdot2\prec5\cdot2\prec\cdots\prec3\cdot2^2\prec5\cdot2^2\prec\cdots\prec2^3\prec2^2\prec2\prec1$. This greatly improves some of the author's earlier results [same

Ž. **12** (1960), 484–489; MR **25** #353; Dokl. Akad. Nauk SSSR **139** (1961), 1067–1070; MR **25** #352]. The proof is very long, but entirely elementary.

M. Kuczma (Katowice)

Kozlovcev, S. G. 3122

On the structure of measurable functions without an asymptotic derivative. (Russian)

Mat. Sb. (N.S.) **63 (105)** (1964), 284–308.

If f is a finite real function on $(0, 1)$, $x_0 \in (0, 1)$, and if A, B are arbitrary real numbers, we denote by $E(f, x_0, A, B)$ the set of all $x \in (x_0, 1)$ such that $A < (f(x) - f(x_0))/(x - x_0) < B$. The author constructs a finite measurable function f on $(0, 1)$ such that, for almost all $x_0 \in (0, 1)$, the following assertion is true: If $0 < A < B < \infty$, then x_0 is a point of dispersion for $E(f, x_0, A, B)$ and the right upper density of $E(f, x_0, -B, -A)$ at x_0 is $\geq (B - A)/B$.

J. Mařík (Prague)

Neugebauer, C. J. 3123

On a paper by M. Iosifescu and S. Marcus.

Canad. Math. Bull. **6** (1963), 367–371.

Let $I_0 = [0, 1]$, and let R be the set of real numbers. The author proves the following theorems. Theorem 1: There exist $f, g : I_0 \to R$, bounded on I_0, such that f is approximately continuous on I_0, g is a derivative on I_0, and $f^2 + g^2$ does not possess the Darboux property on I_0. Theorem 2: There exist $f, g : I_0 \to R$ such that f is approximately continuous on I_0, g is a derivative on I_0, and $f + g$ does not possess the Darboux property. Theorem 3: Let $f : I_0 \to R$ be approximately continuous on I_0 and let $g : I_0 \to R$ be a derivative on I_0. If either f or g is bounded on I_0, then $h = f + g$ possesses the Darboux property on I_0. {The paper by Iosifescu and the reviewer cited in the title is in same Bull. **5** (1962), 129–132 [MR **25** #3129].}

S. Marcus (Bucharest)

Radziszewski, K. 3124

Sur un théorème de l'Hospital.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 731–735.

S'il existe une suite $t_n \to t$, telle que

$$g'(t) = \lim \frac{g(t_n) - g(t)}{(t_n - t)},$$

$g'(t)$ est un nombre dérivé de g en t. Par $Df(t)/Dg(t)$ nous désignerons l'ensemble des nombres $w(t, \{t_n\})$, $w(t, \{t_n\}) \in [-\infty, +\infty]$, définis de la manière suivante:

$$w(t, \{t_n\}) = \lim_{t_n \to 0} \frac{f(t + t_n) - f(t)}{g(t + t_n) - g(t)}$$

si

$$\lim_{t_n \to 0} \frac{f(t + t_n) - f(t)}{t_n} = \infty, \quad \lim_{t_n \to 0} \frac{g(t + t_n) - g(t)}{t_n} = \infty,$$

$$w(t, \{t_n\}) = \frac{\lim\,[f(t + t_n) - f(t)]/t_n}{\lim\,[g(t + t_n) - g(t)]/t_n}, \qquad t_n \to 0,$$

dans les autres cas, pour une suite fixée $\{t_n\}$ de nombres (ici t_n parcourt les mêmes valeurs au numérateur et au dénominateur d'une telle manière que toutes ces limites existent). L'ensemble $Df(t)/Dg(t)$ est une fonction de t. Nous disons que l'ensemble $Df(t)/Dg(t)$ tend vers un nombre A, lorsque $t \to a$, et par conséquent nous écrivons $\lim_{t \to a} Df(t)/Dg(t) = A$ si pour chaque $w(t, \{t_n\}) \in Df(t)/Dg(t)$ on a $\lim_{t \to a} w(t, \{t_n\}) = A$.

Théorème 1: (a) Si les fonctions f et g sont définies et continues dans un intervalle ouvert (a, b), $a < b$, n'étant pas identiques à un nombre constant dans aucun entourage du point a, (b) si pour tous les nombres dérivés $g'(t)$, $t \in (a, b)$, on a $g'(t) \geq 0$, (c) s'il existe une limite (finie ou infinie) $\lim_{t \to a} Df(t)/Dg(t) = K$, alors il existe une limite $\lim_{t \to a} f(t)/g(t) = A$ (finie ou infinie). Théorème 2: Si dans le Théorème 1 les fonctions f et g sont continues dans un intervalle fermé $[a, b]$ et $f(a) = g(a) = 0$, alors $A = K$. Théorème 3: Si dans le Théorème 1 $\lim f(t) = \lim g(t) = -\infty$, alors $A = K$.

S. Marcus (Bucharest)

Davies, G. S.; Petersen, G. M. 3125

On an inequality of Hardy's. II.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 35–40.

The main theorem of Petersen's previous paper [same J. (2) **13** (1962), 237–240; MR **25** #4048] is generalized from the case of integral p to real $p > 1$, and an analogue of this theorem is proved for $0 < p < 1$. This main theorem is also generalized in another direction, and an integral analogue of the resulting theorem is proved.

A. E. Danese (Buffalo, N.Y.)

Basu, S. K. 3126

On a class of completely monotone functions.

Calcutta Math. Soc. Golden Jubilee Commemoration Vol. (1958/59), *Part I*, *pp.* 105–110. *Calcutta Math. Soc., Calcutta*, 1963.

Let $f(t)$ be of class C^∞ for $t \geq 0$ and $f(t) = \int_0^\infty e^{-tx} d\rho(x)$, where $d\rho(x) \geq 0$. It is known that

$$f_k(t) = \left\{ f(t) - \sum_{j=0}^{k-1} f^j(0) t^j / j! \right\} (-t)^{-k}$$

is completely monotone for $t \geq 0$ [J. B. Rosser, Duke Math. J. **15** (1948), 313–331; MR **9**, 572]. The author reproves this fact and obtains the representation $f_k(t) = \int_0^\infty e^{-tx} d\chi_k(x)$, where $\chi_k(x) = -\int_x^\infty (t - x)^k d\rho(t)/k!$.

P. Hartman (Baltimore, Md.)

Albuquerque, J. 3127

Sur les fonctions implicites définies par système d'équations.

Univ. Lisboa Revista Fac. Ci. A (2) **6** (1957/58), 269–280.

The present paper is concerned with implicit-function theorems. The main theorem as it is given is in error, since it does not hold for a system of the form

$$f_i = a_{ij} x_j + b_{ij} y_j = 0 \qquad (i, j = 1, \cdots, n),$$

where $A = (a_{ij})$ is nonsingular and $B = (b_{ij})$ is singular, and $(x_0, y_0) = (0, 0)$. The theorem is correct if one makes the additional assumption that the matrix $(\partial f_i / \partial y_j)$ is nonsingular at (x_0, y_0). The remaining results in the paper are standard results in implicit-function theory.

M. R. Hestenes (Los Angeles, Calif.)

Il'in, V. P. 3128

Some inequalities between the norms of partial derivatives of functions of several variables. (Russian)

Dokl. Akad. Nauk SSSR **152** (1963), 262–265.

Il s'agit du problème suivant : On désigne par f une fonction indéfiniment différentiable dans un ouvert Ω de R^n ; on se donne pour $i=0, 1, \cdots, n$, $r_i=(l_1^i, \cdots, l_n^i)$ où les l_j^i sont entiers $\geqq 0$. On cherche les réels $q \geqq p \geqq 1$ et les $\rho=(v_1, \cdots, v_n)$, v_i entiers $\geqq 0$ tels qu'il existe une constante finie C indépendante de f telle que

$$(1) \qquad \|D^\rho f\|_{L_q(\Omega)} \leqq C \sum_{i=0}^{n} \|D^{r_i} f\|_{L_p(\Omega)},$$

où $D^{r_i}=(\partial/\partial x_1)^{l_1^i} \cdots (\partial/\partial x_n)^{l_n^i}$.

Les résultats obtenus font suite à ceux de Sobolev, Nikol'skiĭ et l'auteur et tiennent en trois théorèmes énonçant des conditions suffisantes pour que l'on ait (1).

Voici une forme simplifiée du théorème 1 : Soit s un entier $s \leqq n$ et supposons ρ et les r_i vérifier : (1^0) Si $i=0, 1, \cdots, s-2$, $l_j^i \geqq v_j$ pour $j=1, 2, \cdots, i$; $l_{i+1}^i \leqq v_{i+1}$, $l_j^i=v_j$ pour $j=i+2, \cdots, n$; $l_j^{s-1} \geqq v_j$ $(j=1, \cdots, s-1)$, $l_j^{s-1} \leqq v_j$ $(j=s, \cdots, n)$ pour $i=s, \cdots, n$: pour chaque $j=1, \cdots, s-1$ soit $l_j^i > v_j$ $(i=s, \cdots, n)$ soit $l_j^i=v_j$ $(i=s, \cdots, n)$ et soit (a) $l_i^i > v_i$ $(i=s, \cdots, n)$, $l_j^i \leqq v_j$ $(j=s, \cdots, n ; j \neq i ; i=s, \cdots, n)$, soit (b) $l_j^i=v_j$ $(i=s, \cdots, n ; j=s, \cdots, n)$. ($2^0$) Si les r_i $(i=s, \cdots, n)$ satisfont (a), on suppose l'existence de nombres $\chi_i > 0$ $(i=s, \cdots, n)$ tels que $\sum_s^n v_j\chi_j < \sum_s^n l_j^i\chi_j$ pour $i=s, \cdots, n$: alors, on a (1) pour $q=p \geqq 1$, pour des ouverts Ω d'un type précisé. *H. Morel* (Marseille)

Zahorska, Helene 3129

Über die singulären Punkte einer Funktion der Klasse C_∞.

Acta Math. Acad. Sci. Hungar. **15** (1964), 77–94.

Let $f(x, y)$ be of class C^∞ in the (x, y)-plane E^2; $P=P(x, y ; h, k)=\sum\sum a_{nm}(x, y)h^n k^m$ the formal Taylor expansion of f at (x, y); $r_a(x, y)=\sup\{r : P$ is absolutely convergent for $|h| \leqq r$, $|k| \leqq r\}$; if $r_a(x, y) > 0$, let $r_c(x, y)=\sup\{r : 0 \leqq r < r_a,\ P=f(x+h, y+k)$ for $|h| \leqq r$, $|k| \leqq r\}$; $A(f)=\{(x, y) : r_a(x, y) > 0,\ r_c(x, y) > 0\}$, $B(f)=\{(x, y) : r_a(x, y)=0\}$, and $C(f)=\{(x, y) : r_a(x, y) > 0, r_c(x, y)=0\}$. It is shown that $A(f)$ is an open set, $B(f)$ is a G_δ-set, and $C(f)$ is an F_σ-set of the first category. Also, if C is any closed, nowhere dense set in E^2, then there exists an $f(x, y) \in C^\infty$ such that $C(f)=C$ and $A(f)=E^2-C$.

P. Hartman (Baltimore, Md.)

Neuberger, J. W. 3130

A quasi-analyticity condition in terms of finite differences.

Proc. London Math. Soc. (3) **14** (1964), 245–259.

Let G_t denote the class of all continuous real functions $f(x)$ on $(0, 1)$ such that if $0 < u < v < 1$, then for some positive number L,

$$\left|\sum_{i=0}^{n} C(n, i)(-1)^{n-i} f(u+i(v-u)/n)\right| < Lt^n, \quad n = 1, 2, \cdots.$$

The author proves that for $1 \leqq t < 2$, G_t is a quasi-analytic collection in the sense that if $f \in G_t$, $g \in G_t$ and $f(x)=g(x)$ for all x in a subinterval of $(0, 1)$, then $f=g$. It is pointed out that if $f(x)$ has a power series expansion about 0 which converges on $[0, 1)$, then $f \in G_1$.

T. Kövari (College Park, Md.)

MEASURE AND INTEGRATION

See also 3271, 3296, 3297, 3326, 3336, 3412, 3413, 3470, 3696.

Riečan, Beloslav 3131

A note on the construction of measure. (Slovak. Russian and English summaries)

Mat.-Fyz. Časopis Sloven. Akad. Vied **12** (1962), 47–59.

Let $\mathbf{K}$ be a family of subsets of a metric space X such that the empty set 0 belongs to $\mathbf{K}$. Let μ be a non-negative function on $\mathbf{K}$ with the following properties: (a) $\mu(0)=0$; (b) if $K_0, K_1, \cdots \in \mathbf{K}$, $K_0 \subset \bigcup K_n$ $(n > 0)$, then $\mu(K_0) \leqq \sum \mu(K_n)$ $(n > 0)$; (c) if $K \in \mathbf{K}$, $\mu(K) < \infty$ and if ε is a positive number, then there exist $K_n \in \mathbf{K}$ such that $\bigcup K_n \supset K$, diam $K_n < \varepsilon$ and $\mu(K)+\varepsilon > \sum \mu(K_n)$ $(n=1, 2, \cdots)$. For every $A \subset X$ put $\mu^*(A)=\inf \sum \mu(K_n)$, where $K_n \in \mathbf{K}$, $A \subset \bigcup K_n$ $(n=1, 2, \cdots)$. Then μ^* is an outer measure; every Borel set is μ^*-measurable and $\mu^*(K)=\mu(K)$ for every $K \in \mathbf{K}$. Further, the author constructs functions $\bar\mu$ and $H[\mu, \mathbf{K}]$ in the following manner: If U is open, put $\lambda(U)=\sup \sum \mu(K_n)$, where $K_m \cap K_n=0$ for $m \neq n$, $K_n \in \mathbf{K}$, $K_n \subset U$ $(n=1, 2, \cdots)$; for an arbitrary $A \subset X$ put $\bar\mu(A)=\inf \lambda(U)$ (U open, $U \supset A$). For every $A \subset X$ and every $\varepsilon > 0$ put $H_\varepsilon[\mu, \mathbf{K}](A)=\inf \sum \mu(K_n)$, where $\bigcup K_n \supset A$, diam $K_n < \varepsilon$, $K_n \in \mathbf{K}$ $(n=1, 2, \cdots)$ and for every $A \subset X$ put $H[\mu, \mathbf{K}](A)=\sup H_\varepsilon[\mu, \mathbf{K}](A)$ $(\varepsilon > 0)$. These functions are used for the construction of measures. Finally, some sufficient conditions are given for the equality of the functions μ^*, $\bar\mu$ and $H[\mu, \mathbf{K}]$. *J. Mařík* (Prague)

Riečanová, Zdena 3132

On Carathéodory outer measure. (Russian. English summary)

Mat.-Fyz. Časopis Sloven. Akad. Vied **12** (1962), 246–252.

Let X be a locally compact Hausdorff topological space and let μ be an outer measure on the family of all subsets of X. Suppose that the following condition is fulfilled: If G_1, G_2 are disjoint open subsets of X and if the closure of the set A_j is contained in G_j $(j=1, 2)$, then $\mu(A_1 \cup A_2)=\mu(A_1)+\mu(A_2)$. Let the set S belong to the smallest σ-ring including all compact G_δ-sets in X. Then S is μ-measurable.

J. Mařík (Prague)

Cartan, H.; Dixmier, J.; Revuz, A.; Dubreil, P.; Lichnerowicz, A. 3133

★Problèmes de mesure.

Conférences organisées par la Société Mathématique de France et par l'Association des Professeurs de Mathématiques de l'Enseignement Public, à l'intention spéciale des professeurs et prononcées à l'Institut Henri-Poincaré du 15 novembre 1957 au 22 mai 1958. Monographies de "L'Enseignement Mathématique", No. 10.

Institut de Mathématiques, Université, Geneva; Association des Professeurs de Mathématiques de l'Enseignement Public, Paris; 1962. 87 *pp.* 11 *fr. s.*

This is a set of five expository articles, two of which have nothing to do with measure. Cartan discusses volumes of polyhedra in the context of Jordan measure. He gives a concise self-contained statement of the main definitions, theorems, and proofs; the presentation includes general

theorems (every bounded convex open set in R^n is "cubable") and special computations (the volume of the unit ball in R^{2p} is $\pi^p/p!$). Dixmier approaches the concept of angle (in the plane) via linear algebra. He defines angle for an ordered pair of unit vectors as the unique proper rotation that carries the first onto the second. There is a natural isomorphism between proper rotations and complex numbers of modulus 1; the trigonometric functions of an angle are defined via this isomorphism. Revuz's article on integration is the longest (29 pages). It overlaps Cartan's somewhat in its discussion of Jordan measure and Riemann integration; in addition, it treats outer measure, the Carathéodory definition of measurability, and the classical approach to Lebesgue integration, as well as the linear functional point of view and its relation to the classical approach. Dubreil gives a historical account of some aspects of algebra from Lagrange to Steinitz, via Abel and Galois. Lichnerowicz treats tensor calculus beginning with the definition of a vector space, going through the definition of tensor product, developing some tensor algebra, and ending, breathlessly, with a mention of the Lorentz group and the Maxwell-Lorentz equations.

P. R. Halmos (Ann Arbor, Mich.)

Rudin, Walter 3134

An arithmetic property of Riemann sums.

Proc. Amer. Math. Soc. **15** (1964), 321–324.

Let f be a real-valued periodic function with period 1 on the real line and let $(M_nf)(x)=(1/n)\sum_{i=1}^{n}f(x+i/n)$, $n=1, 2, \cdots$. If f is Riemann integrable on $[0, 1]$ and $\int f$ denotes its integral, then (1) $\lim_{n\to\infty}(M_nf)(x)=\int f$ for every x. The situation is different concerning Lebesgue integrable functions. The author proves that there are bounded measurable functions f for which (1) is false for every x. Some number-theoretic aspects of this situation are also discussed.

A. Mallios (Athens)

Williamson, J. H. 3135

★Lebesgue integration.

Holt, Rinehart and Winston, New York, 1962. viii+117 *pp*. $3.75.

The book gives a concise, straightforward presentation of the basic facts concerning Lebesgue integration, largely in R^n, based on Euclidean volume. Its coverage can be gleaned from the following table of contents. (1) Sets and functions: generalities; countable and uncountable sets; sets in R^n; compactness; functions. (2) Lebesgue measure (in R^n): preliminaries; the class $\mathcal{J}$ of countable unions of intevals; measurable sets; sets of measure zero; Borel sets and nonmeasurable sets. (3) The integral, I: Definition, based on dissection (subdivision) into countable sets of measurable sets and upper and lower integrals; elementary properties; measurable functions; complex and vector functions; other definitions of integral (measure of ordinal set). (4) The integral, II: Convergence theorems (Lebesgue dominated, and Fatou); Fubini's theorem; approximation to integrable functions; the L_p spaces; convergence in the mean; Fourier theory. (5) Calculus: change of variable; differentiation of integrals of functions of one variable; integration of derivatives; integration by parts. (6) More general measures: Borel measures; signed measures and complex measure; absolute continuity, Lebesgue decomposition theorem; measures, functions and functionals; norms, Fourier transforms and convolution products. There are exercises at the close of each chapter containing ideas not developed in the text. There are no references.

T. H. Hildebrandt (Ann Arbor, Mich.)

Tevzadze, N. R. 3136

Lebesgue points of a function of two variables. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **32** (1963), 17–22.

Let $R=[a, b]\times[c, d]$ and let f be a real function defined on R. A point $(x, y)\in R$ is said to be a Lebesgue point of f if

$$\lim_{t\to 0} t^{-1}\int_x^{x+t} du \int_c^d |f(u, y)-f(x, y)|\, dy = 0,$$

$$\lim_{\tau\to 0} \tau^{-1}\int_y^{y+\tau} dv \int_a^b |f(x, v)-f(x, y)|\, dx = 0.$$

It is proved that if f is integrable, then almost every point of R is a Lebesgue point. Then some consequences concerning double Fourier series are derived.

N. Dinculeanu (Bucharest)

Onicescu, Octav 3137

Sur les fonctions somme au sens large et sur les corrélations somme.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 501–503.

Soient $\{E, \mathscr{K}\}$ un champ, P une mesure finie sur $\mathscr{K}$, f une fonction mesurable et $\sigma_\alpha=\{\xi; |f(\xi)|\leqq\alpha\}$ pour chaque $\alpha\geqq 0$. Pour $\omega\in\mathscr{K}$ et $\alpha\geqq 0$ on pose

$$F(\sigma_\alpha\cap\omega) = \int_{\sigma_\alpha\cap\omega} f(\xi)\, dP(\xi).$$

On définit la fonction somme supérieure $\bar{F}(\omega)=\limsup_{\alpha\to\infty} F(\sigma_\alpha\cap\omega)$ et la fonction somme inférieure $\underline{F}(\omega)=\liminf_{\alpha\to\infty} F(\sigma_\alpha\cap\omega)$, qui sont égales pour les ensembles $\omega\in\mathscr{K}$ telles que $f\varphi_\omega$ est P-intégrable.

On définit de la même manière la connexion et la corrélation supérieure ou inférieure de deux fonctions.

N. Dinculeanu (Bucharest)

Jacobs, K. 3138

★Lecture notes on ergodic theory, 1962/63. Parts I, II.

Matematisk Institut, Aarhus Universitet, Aarhus, 1963. *Part I*: vii+*pp*. 1-207; *Part II*: *pp*. 208-505. $6.00.

As is well known, the study of ergodic theory began with certain problems in statistical mechanics. The origins of the subject explain in part its wide appeal and interest, but it is its later development in measure theory which accounts for the major portion of current attention paid to it. Its history does serve, however, to give it a certain flavor of vigorous reality. Ergodic theory centers around the study of problems related to measure-theoretic properties which are invariant under measurable transformations. It is, consequently, the purest form of measure theory. There is a starkness which is reminiscent of number theory: the important results can be understood without elaborate preparation, and the proofs are combinatorial and difficult. This facet of the subject is particularly apparent in the non-topological considerations emphasized in the book under review.

The author has succeeded in performing an immense task in writing this book. It is almost completely different from his earlier *Neuere Methoden und Ergebnisse der Ergodentheorie* [Springer, Berlin, 1960; MR **23** #A1000]. It is based on lectures given at the universities of Aarhus and Göttingen in 1962–63, and is written in the style of lecture notes.

The first five chapters are an introduction to the main part of the text and are written in an especially informal and refreshing style. Chapter 6, which includes the work of F. B. Wright and M. Kac, is on recurrence and is the basis for the results on induced automorphisms. Chapter 7 is on the invariant measure problem and, of course, includes the work of Hajian-Kakutani, Hopf, and Ornstein. Chapter 8 is on the mean ergodic theorem, and discusses the work of Dunford-Schwarz, Eberlein, and Kakutani-Yosida, as well as that of Jacobs. The chapter ends with the de Leeuw-Glicksberg theory, almost periodic functions, and harmonic analysis.

The concept of mixing, together with various unsolved problems relating to it, is discussed thoroughly and clearly in Chapter 9. Chapter 10 contains the work of Halmos, Kolmogorov-Sinai, Pinsker, Rohlin, and Sinai on invariants, entropy and related questions. There is also a discussion of the Blum-Hanson generalization of Meshakin's result on Bernoulli schemes. Chapter 11 deals with pointwise ergodic theory and is extremely well-written.

The author has apologetically left out the work of G.-C. Rota and A. and C. Ionescu-Tulcea on ergodic and martingale theorems, and all work on flows, such as that of Révèsz and Auslander-Green-Hahn. He has necessarily also omitted work of Ionescu-Tulcea, Maharam, Sucheston, and others, which is related to work discussed but which appeared too late for inclusion.

The book is highly entertaining and informative and is the best reference and guide to the literature. It can be highly recommended to anyone interested in an account of ergodic theory since Hopf's famous monograph [*Ergodentheorie*, J. Springer, Berlin, 1937]. This work is an accomplishment of which the author can feel justly proud, and which will do a great deal to explain what ergodic theory is all about. *R. V. Chacon* (Providence, R.I.)

Varadarajan, V. S. 3139

Groups of automorphisms of Borel spaces.

Trans. Amer. Math. Soc. **109** (1963), 191–220.

The main result of this paper is a generalization of the Krylov-Bogoljubov sharpening of von Neumann's theorem on the decomposition of invariant measures into ergodic parts. Krylov and Bogoljubov deal with the action of the real line on a metric space X and show that X can be decomposed into invariant sets each of which admits a unique invariant probability measure. Any invariant probability measure may be built up from these by integration. Though it has been known for some time that the original von Neumann theorem holds for much more general transformation groups, this is not the case for the refinement of Krylov and Bogoljubov. It has not been clear how to generalize the argument in which Krylov and Bogoljubov invoke the ergodic theorem, and an example due to Kolmogorov suggests that non-commutativity is a really serious obstacle.

The author finds an appropriate substitute for the ergodic theorem and proves a Krylov-Bogoljubov type decomposition theorem for any separable locally compact group acting measurably on a standard Borel space in such a manner that at least one invariant probability measure exists. These results are obtained in Section 4. Sections 1 and 2 include the introduction and various preliminaries. Section 3 is devoted to a detailed analysis of the connection between topological transformation groups and Borel transformation groups. In Section 5 the author drops the hypothesis that X be a standard Borel space and investigates the extent to which various consequences of his main theorem continue to hold. The final Section 6 contains a number of examples illustrating the need for various hypotheses and otherwise illuminating the theory. The example of Kolmogorov alluded to above is one of those presented. *G. W. Mackey* (Cambridge, Mass.)

Yosida, Kôsaku 3140

Abelian ergodic theorems in locally convex spaces.

Ergodic Theory (Proc. Internat. Sympos., Tulane Univ., New Orleans, La., 1961), pp. 293–299. *Academic Press, New York,* 1963.

In this paper the author states three abelian ergodic theorems and gives some applications. No proofs are given. Theorems 1 and 2 were proved by the author in an earlier paper [Proc. Japan Acad. **37** (1961), 422–425; MR **25** #3375]. The author points out that Theorem 3 is essentially due to H. L. Trotter but that it is also an easy consequence of Theorem 2. *F. Hahn* (New Haven, Conn.)

Almgren, F. J., Jr. 3141

An isoperimetric inequality.

Proc. Amer. Math. Soc. **15** (1964), 284–285.

Let f be Lipschitzian from a k-cube I^k into euclidean R^n such that $|f(x)-f(y)|\geqq b$ whenever x and y lie on opposite $(k-1)$-dimensional faces of I^k. Then the k-volume of f is at least b^k. For $k=2$ this was proved by S. Bergman and D. C. Spencer [Duke Math. J. **9** (1942), 757–762; MR **4**, 243]. The author obtains the theorem as a consequence of a similar result about chain maps from the cell complex of I^k into the chain complex of integral currents in R^n.

W. H. Fleming (Providence, R.I.)

Naas, Josef 3142

Kurveninvarianten des euklidischen Raumes E_n, die durch Fundamentallängen dargestellt werden.

Math. Nachr. **24** (1962), 349–370.

L'auteur généralise la notion de courbe à l'aide des distributions. Une courbe généralisée est localement une dérivée d'ordre p au sens des distributions d'une courbe continue $x:t\to x(t)\in R^n$, ce qui amène naturellement à passer au quotient dans l'ensemble des (x,p), p désignant l'ordre de dérivation; la notion de convergence au sens des distributions permet de définir une notion de convergence dans l'espace des courbes généralisées.

Une courbe généralisée est ainsi, brièvement dit, localement une dérivée d'ordre p d'une courbe une fois continûment différentiable $x:t\to x(t)$ possédant une longueur euclidienne dont la dérivée d'ordre p au sens des distributions sera une longueur fondamentale de la courbe généralisée; là aussi il convient de passer naturellement au quotient.

L'auteur définit alors une notion de "ebene Länge"

dont il démontre qu'elle caractérise les classes de courbes prises modulo les déplacements et les changements de variable, s'aidant en celà de la théorie de Frenet.

L'introduction ainsi faite de la théorie des distributions dans un chapitre de géométrie différentielle classique permet de prendre en considération des courbes non-régulières, par exemple continues non-différentiables.

H. Morel (Marseille)

FUNCTIONS OF A COMPLEX VARIABLE

See also 3006, 3011, 3235, 3238, 3282, 3338.

Ridder, J. 3143

Hinreichende Bedingungen für die Analytizität komplexwertiger Funktionen.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 1–9.

Let $f(z)$ be continuous in the bounded domain B. It is proved that $f(z)$ is analytic in B if the following two conditions are satisfied. (1) There exists a set E that is the union of countably many closed sets of finite linear measure such that at each point $z_0 \in B-E$ there meet three Jordan arcs J_j $(j=1,2,3)$ that lie in sectors that are disjoint, except for their center z_0, such that $|(f(z)-f(z_0))/(z-z_0)|$ has a finite upper limit as $z\to z_0$, $z\in J_j$. (2) For almost all points z_0 of B there are three sequences (z_{jn}) $(j=1,2,3)$ with $z_{jn}\to z_0$ and $\arg(z_{jn}-z_0)\to\alpha_j$ $(0\leq\alpha_1<\alpha_2<\alpha_3<2\pi)$ as $n\to\infty$ such that $\lim_{n\to\infty}\arg[(f(z_{jn})-f(z_0))/(z_{jn}-z_0)]$ exists and is independent of j.

Condition (2) can be replaced by a condition involving the limit of the modulus on three sequences, and also by a condition involving the limit of the difference quotient itself on two sequences. *Ch. Pommerenke* (Göttingen)

R.-Salinas, Baltasar 3144

Solution of the problem of equivalence of classes of asymptotic series. (Spanish)

Rev. Acad. Ci. Zaragoza (2) **16** (1961), *no.* 1, 47–51.

Let E be a subset of the entire complex plane and z_0 a point in the closure of E. Let $\{m_n\}$ be a sequence of positive numbers. $C_{z_0}\{m_n\}$ is the class of complex functions $f(z)$ having asymptotic expansion $f(z)=\sum_{k=0}^{\infty}a_k(z-z_0)^k$ for all z in E, with $|f(z)-\sum_{k=0}^{n-1}a_k(z-z_0)^k|\leq Mq^nm_n|z-z_0|^n$, where M, q are constants depending on $f(z)$. The author states without proof some results on the problem of equivalence of the classes $C_0\{m_n\}$ and $C_0\{m_n'\}$. A sufficient but not necessary condition is that $0<\liminf(m_n'/m_n)^{1/2}\leq\lim(m_n'/m_n)^{1/2}<+\infty$.

If E is connected and bounded, with B that component of its complement which contains the point at infinity, then $w(z)$ maps B analytically on the exterior of the unit circle. If 0 is a frontier point of E, the sequence $\{m_n\}$ is regularized as follows: Set $\sigma(r)=\mathrm{Sup}\{\log|w(z)|\,\big|\,|z|=r$, z in $B\}$, $\phi(t)=\mathrm{Inf}_{r>0}\,e^{t\sigma(r)}/r$, $S(t)=\mathrm{Sup}_{n\leq t}\,\phi(t/n)^n/m_n$, $\overline{m}_n=\mathrm{Sup}_{t\geq n}\,\phi(t/n)^n/S(t)$. The author then states the following theorem: $C_0\{m_n\}\subset C_0\{m_n'\}$ if and only if

$$\limsup_{n\to\infty}(\overline{m}_n/m_n')^{1/2}<+\infty.$$

A similar condition is given when B is connected but not bounded. *A. A. Armendáriz* (New Orleans, La.)

Sunyer Balaguer, F. 3145

Generalization of the method of Wiman and Valiron to a class of Dirichlet series. (Spanish)

Actas 2.ª Reunión Mat. Españoles (1961), *pp.* 43–47. *Seminario Matemático de Zaragoza, Zaragoza*, 1962.

If $s=\sigma+it$, let $f(s)=\sum c_ne^{-\lambda_ns}$ be an entire function with $0\leq\lambda_n\leq\lambda_{n+1}$, and $\liminf_{n\to\infty}(\lambda_{n+1}-\lambda_n)>0$. If $\mu(\sigma)=\max(|c_n|e^{-\lambda_n\sigma})$ and $N=N(\sigma)$ the largest integer such that $\mu(\sigma)=|c_N|e^{-\lambda_N\sigma}$, then, letting $\lambda(\sigma)=\lambda_{N(\sigma)}$, define the function $F(s)=\sum e^{H(\lambda_n)}e^{-\lambda_ns}$, where $H(\lambda)$ goes to infinity and $H(\lambda)/\lambda$ goes to zero as λ increases to infinity.

The author announces without proof one main theorem and several corollaries. The main theorem follows. If $H(\lambda)$ has derivative decreasing to zero as λ increases to infinity, then given any number σ it is generally possible to find two numbers k, q such that $\sigma=q+H'(\lambda(\sigma))$ and such that the largest terms of $f(\sigma)$ and of $kF(\sigma-q)$ are equal and correspond to the same λ_N. Moreover, the other terms of $f(\sigma)$ are dominated by the corresponding terms of $kF(\sigma-q)$. If $E(v)$ is the set of $\sigma>v$ for which the above is false, then $E(v)$ consists of at most $N(v)$ segments whose total length is bounded by a quantity independent of v. A value σ for which the result holds is an ordinary value. For ordinary values the author states some corollaries: If $A(\sigma,f)$ is the maximum of the real part of f, then $\lim_{\sigma\to-\infty}A(\sigma,f)/M(\sigma,f)=1$. If σ is an ordinary value, then $\lim_{\sigma\to-\infty}M(\sigma,f')/\lambda(\sigma)M(\sigma,f)=1$.

A. A. Armendáriz (New Orleans, La.)

Král, Josef 3146

On cyclic and radial variations of a plane path.

Comment. Math. Univ. Carolinae **4** (1963), 3–9.

In this paper a set of theorems is formulated (without proof) concerning the concepts of the cyclic and radial variation of a path introduced by the author in an earlier paper [same Comment. **3** (1962), no. 1, 3–10; MR **28** #2227]. *D. Mitrović* (Zagreb)

Kral, Iosef [Král, Josef] 3147

Angular limit values of integrals of Cauchy type. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 32–34.

Let C be a smooth oriented arc in the euclidean plane E_2 and λM be the length (linear measure) of the set $M\subset C$. Let φ be a bounded function (λ)-integrable on C and p be a bounded, non-negative lower-semicontinuous function defined on C. The author investigates the behavior of the integral

$$\phi_\eta(z)=\int_C(\varphi(\zeta)-\varphi(\eta))/(\zeta-z)\,d\zeta,\quad \eta\in C,\ z\in E_2\setminus C,$$

as z tends to η along non-tangential paths, under one of two following conditions:

$$\varphi(\zeta)-\varphi(\eta)=O(p(\zeta)),\quad \zeta\to\eta,$$
$$\varphi(\zeta)-\varphi(\eta)=o(p(\zeta)),\quad \zeta\to\eta.$$

The results are given without proofs.

D. Mitrović (Zagreb)

Schoenfeld, Lowell 3148

The evaluation of certain definite integrals.

SIAM Rev. **5** (1963), 358–369.

The author defines a modified principal-value integral. Given a complex function $f(x)$, bounded in a neighborhood of each point of $0<x<\infty$ except for $a_1, a_2, \cdots, a_n$ $(0<a_1<\cdots<a_n<\infty)$, one forms $\int_K f(z)dz$, where K is the path leading from δ_0 $(0<\delta_0<a_1)$ to ρ $(>a_n)$ along the x-axis, except that around each a_j the x-axis is replaced by a small semi-circle in the upper half-plane with center a_j and radius δ_j. Then P.V.* $\int_0^\infty f(x)dx$ is defined as the limit of the contour integral $\int_K f(z)dz$ (if this limit exists) when $\delta_0, \delta_1, \cdots, \delta_n$ tend to 0^+ and $\rho\to\infty$. The main theorem is the following. Let $Q(z)$ be an entire function and let $F(z)$ be analytic in the plane with at most a finite number of exceptions. Let $zF(-z)Q(\log z)$ tend to d as z tends to 0, and let it tend to D as $|z|$ tends to ∞. Then

$$\text{P.V.*}\int_0^\infty F(x)\{Q(\log x+\pi i)-Q(\log x-\pi i)\}\,dx = 2\pi i(R-r-D+d),$$

where R is the sum of the residues of $F(-z)Q(\log z)$ at its singularities not on the non-positive real axis and r is the sum of the residues of $F(z)Q(\log z+\pi i)$ at its singularities on the positive real axis. The author proves then a few corollaries, and finally shows an application, namely, that $Q=\int_0^\infty xe^{-2x}\psi^2(x)dx=\frac{1}{3}$, where

$$\psi(x)\equiv\int_0^{\pi/2}\{1-\exp[x(1-\operatorname{cosec}\vartheta)]\}\sec^2\vartheta\,d\vartheta.$$

A numerical integration for Q gave a value of 0.3333, and consequently W. L. Bade conjectured [Problem 61-9, same Rev. **3** (1961), 329] that $Q=\frac{1}{3}$. For other derivations (by a number of solvers) of this equality see same Rev. **5** (1963), 157–159. *O. Shisha* (Dayton, Ohio)

Fuchs, W. H. J. 3149

Proof of a conjecture of G. Pólya concerning gap series.
Illinois J. Math. **7** (1963), 661–667.

Let $\{\lambda_n\}$ be an increasing sequence of non-negative integers satisfying the Fabry gap condition: $\lambda_n/n\to\infty$. Let $f(z)=\sum_0^\infty c_n z^{\lambda_n}$ be an entire function of finite order, and write $M(r)=\max_{|z|=r}|f(z)|$, $L(r)=\min_{|z|=r}|f(z)|$. The author proves that under these conditions, for every $\varepsilon>0$,

$$\log L(r) > (1-\varepsilon)\log M(r)$$

holds outside a set of logarithmic density zero. In particular:

$$\limsup \frac{\log L(r)}{\log M(r)} = 1,$$

which proves a well-known conjecture of G. Pólya, unsolved since 1929. *T. Kövari* (College Park, Md.)

Èzrohi, T. G. 3150

On a class of functions univalent in the domain $1<|z|<\infty$. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 166–172.

Let L^* denote the class of functions $f(z)=z+\sum_{n=1}^\infty a_n z^{-n}$ in $1<|z|<\infty$ for which $\sum_{n=1}^\infty n|a_n|\leq 1$. For the class L^* the author proves that in this domain (a) the functions are univalent, (b) the image of the circle $|z|=r$ is starlike, (c) the image of the circle $|z|=r$ is convex for $r\geq r_0$, where r_0 is a root of $1+r_0^2=2r_0^2\ln r_0$. Further, he obtains sharp bounds for $|f|$, $|f'|$, $\Re(zf'/f)$, $\Re(z^2f'/f^2)$, arg f', and arg (f/z). *A. W. Goodman* (Lexington, Ky.)

Constantinescu, Corneliu; Cornea, Aurel 3151

★Ideale Ränder Riemannscher Flächen.
Ergebnisse der Mathematik und ihrer Grenzgebiete, N. F., Bd. 32.
Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963. viii + 244 *pp.* *DM* 68.00.

It is a commonplace that the mathematical output of the present time is very great. For that special area of Riemann surface theory to which the monograph under review addresses itself some idea of the amount of work that has been carried out in the last few years may be had by comparing the comprehensive (and at the time of publication, up-to-date) bibliography of Ahlfors and Sario [*Riemann surfaces*, Princeton Univ. Press, Princeton, N.J., 1960; MR **22** #5729] with the bibliography of the present book (1963). The increment is far from negligible. The roots of *Ideale Ränder Riemannscher Flächen* reach back quite far. Just what notion of an ideal boundary is apposite is dictated by the nature of the problem to be treated. In the earlier days of abstract Riemann surface theory the ideal boundary was little more than a device to treat the limiting behavior of maps and functions and by a (punctured) neighborhood of the ideal boundary of a non-compact surface was meant a set whose complement was relatively compact. This notion was formalized by the Alexandroff compactification. Among other early notions of compactification introduced in Riemann surface theory was that of Kerékjártó-Stoïlow. More refined notions were brought in via potential theory and the theory of harmonic functions. A paper of basic importance was that of R. S. Martin [Trans. Amer. Math. Soc. **49** (1941), 137–172; MR **2**, 292]. In it the Martin compactification was introduced. This paper has turned out to be the impetus of much fundamental research in the study of boundary problems for Riemann surfaces and cognate manifolds. Here mention should be made of the work of Brelot, Choquet, Doob, Kuramochi, Naim, Ohtsuka, Parreau, and the authors of the present monograph.

A result of G. C. Evans [Monatsh. Math. Phys. **43** (1936), 419–424] states the existence of harmonic functions in euclidean space which become positively infinite on a closed set of zero capacity. Z. Kuramochi considered in a series of important papers the Evans problem and allied questions for parabolic Riemann surfaces and established the existence of a harmonic function on a parabolic surface less a point which becomes positively infinite on the Alexandroff compactification. These investigations of Kuramochi introduced the compactification bearing his name. Other compactifications that have played a fundamental role are those of Royden and Wiener.

The authors center their interest on the following classes of ideal boundaries: (1) ideal boundaries that serve in extending the classical results of the plane, e.g., those of Martin and Kuramochi; (2) ideal boundaries serving to simplify proofs, e.g., of Royden and Wiener; (3) ideal boundaries of type B (for problems involving bounded harmonic functions), e.g., of Martin and Wiener; (4) ideal boundaries of type D (for problems involving harmonic functions with bounded Dirichlet integral), e.g., of Kuramochi and Royden.

The monograph is essentially self-contained and while a familiarity with one of the modern treatises on Riemann surfaces is expected, only the most elementary facts are presupposed. The value of *Idealer Ränder* is greatly enhanced by its "ab ovo" character since the advances of the recent years have proceeded rapidly and have placed considerable technical demands on the reader. The access that is here afforded to the field makes the book very welcome to mathematicians having allied interests.

The contents follow: (0) Auxiliary concepts and notations; (1) Superharmonic functions; (2) The class HP (which is the class of functions on a Riemann surface representable as the difference of non-negative harmonic functions); (3) The Dirichlet problem; (4) Potential theory; (5) Energy and capacity; (6) Wiener functions; (7) Dirichlet functions; (8) Ideal boundaries; (9) Q-ideal boundaries; (10) Q-Fatou maps; (11) Classes of Riemann surfaces; (12) Extension of a potential theory (exposition of the common parts of the theories of the Martin compactification and of the Kuramochi compactification); (13) The Martin ideal boundary; (14) Behavior of analytic maps on the Martin boundary; (15) Completely superharmonic (vollsuperharmonisch) functions; (16) The Kuramochi boundary; (17) Potential theory on the Kuramochi compactification; (18) Behavior of Dirichlet maps on the Kuramochi ideal boundary; (19) The boundary behavior of analytic maps of the unit disk.

M. H. Heins (Berkeley, Calif.)

Potjagaĭlo, D. B. 3152

The sewing problem for two semi-circles. (Russian)

Dokl. Akad. Nauk SSSR **141** (1961), 32–35.

Let $\varphi(x)$, $-\rho \leq x \leq \rho$, be a continuous, monotonically increasing function, $\varphi(\pm\rho) = \pm\rho$, $\rho \leq \infty$ (the "sewing" function). For every continuously differentiable sewing function $\varphi(x)$, with $\varphi'(x) > 0$, the author constructs a Riemann surface F with the following property. After cutting a sheet of F along the real axis, F is divided into two simply-connected parts F_r, $r = 1, 2$, on which quasiconformal mappings g_r exist, with $g_r(F_r) = \{z|(-1)^r \operatorname{Im}(z) \geq 0, |\operatorname{Re}(z)| < \rho\}$, $g_2[g_1^{-1}(x)] = \varphi(x)$. F is obtained by pasting planes together at an explicitly given set of horizontal slits. If $\rho < \infty$, then F is of hyperbolic type.

E. Reich (Minneapolis, Minn.)

Ikoma, Kazuo 3153

A criterion for a set and its image under quasiconformal mapping to be of α $(0 < \alpha \leq 2)$-dimensional measure zero.

Nagoya Math. J. **22** (1963), 203–209.

Let E be a compact set in the complex plane, and let the complement of E be a domain G. Let D be a domain containing E, and let $w(z)$ be a K-quasiconformal mapping of D. The author defines what he means by a family of rings $\{R_n{}^j\}$ inducing an exhaustion of G, and uses this to develop a criterion for both E and E', the image of E, to have α $(0 < \alpha \leq 2)$-dimensional measure zero. The criterion is extremely complicated and cannot be stated in this review. He extends his argument to the case where E lies in a segment to get a criterion for both E and E' to have α $(0 < \alpha \leq 1)$-dimensional measure zero. He then gives two examples of sets which satisfy his criteria.

S. B. Agard (Stanford, Calif.)

Caraman, Petru 3154

Contributions à la théorie des représentations quasi-conformes n-dimensionnelles.

Rev. Math. Pures Appl. **6** (1961), 311–356.

Let $T = f(t)$ be a homeomorphism of a domain D in n-space onto a domain Δ in n-space. The paper begins with an extensive discussion of the behavior of f in the neighborhood of a point of differentiability. The author then introduces the characteristics of an ellipsoid $E_h[(C), t_0]$ at a point t_0, and discusses what it means for the mapping f to have one set of characteristics (C), or to have two sets of characteristics (C) and (C') in D. In particular, he considers what kinds of analogues of the well-known Beltrami system the vector function f can satisfy. Finally, the author defines the notion of quasiconformality for a mapping with two sets of characteristics (C) and (C'), and then proves that a quasiconformal mapping is almost everywhere differentiable. The proof appeals to the n-dimensional form of the Rademacher-Stepanoff theorem.

F. W. Gehring (Stanford, Calif.)

Caraman, Petru 3155

Le jacobien et les dilatations des représentations quasi conformes à n-dimensions. (Romanian. Russian and French summaries)

Acad. R. P. Romîne Fil. Iaşi Stud. Cerc. Şti. Mat. **13** (1962), 61–86.

Let $T = f(t)$ be a quasiconformal mapping of a domain D in n-space with Jacobian $|J(t)|$. The author proves that

$$(1) \qquad \int \cdots \int_E |J| \, dx^1 \cdots dx^n = mf(E)$$

for each measurable set $E \subset D$. Hence f maps sets of measure zero into sets of measure zero and $|J(t)| > 0$ almost everywhere in D. The author goes on to investigate the sets where $\Lambda_f(t) = \infty$, $\lambda_f(t) = 0$, $\Lambda_{f^{-1}}(t) = \infty$, $\lambda_{f^{-1}}(t) = 0$; here

$$\Lambda_f(t) = \limsup_{t \to 0} \frac{|f(t+h) - f(t)|}{|h|},$$

$$\lambda_f(t) = \liminf_{t \to 0} \frac{|f(t+h) - f(t)|}{|h|}.$$

He also studies the relations between $|J|$, Λ_f, and λ_f at points where f is differentiable, and at points where f is differentiable with $|J| > 0$, and he shows that $|J|^{2/n}$ is locally integrable in D. The author extends a theorem of Fedorov [Mat. Sb. (N.S.) **34** (**76**) (1954), 417–428; MR **15**, 949] from 3 to n dimensions, and the paper concludes with remarks on the behavior of infinitesimal ellipsoids under a quasiconformal mapping and its inverse. {The reviewer could not follow the proof of (1).}

F. W. Gehring (Stanford, Calif.)

Fritzsche, Reiner 3156

Beitrag zur Theorie der Modulfunktionen 2. Grades.

Math. Ann. **154** (1964), 135–146.

This is an abridged version of the author's 1962 dissertation (Martin-Luther-Univ. Halle-Wittenberg, Halle), which was reviewed earlier [MR **28** #251].

Mizumoto, Hisao 3157

Notes on fundamental regions of covering transformation groups.

Math. J. Okayama Univ. **12** (1963/64), 49–62.

Let G be the group of transformations of the complex plane Z generated by T_1 and T_2, where $T_1z=z+1$ and $T_2z=z+i$. Let $F_0=\{z|0\leqq x\leqq 1,\ 0\leqq y\leqq 1\}$. Let $\alpha_1=\{z|y=0,\ 0\leqq x\leqq 1\}$; $\alpha_2=\{z|x=0,\ 0\leqq y\leqq 1\}$. Let K be a bounded set in Z consisting of a finite number of continua or isolated points such that $Z-K$ is a domain, no two points of K are equivalent under G, and K contains no Gaussian integer. The main theorem asserts the existence of a fundamental region F of G and a homeomorphism f of F_0 onto F such that (a) 0, 1, i, and $1+i$ are fixed points of f, (b) $f\circ T_1(z)=T_1\circ f(z)$ for $z\in\alpha_2$, $f\circ T_2(z)=T_2\circ f(z)$ for $z\in\alpha_1$ and (c) $K\subset$ interior F.

R. Accola (Providence, R.I.)

Kjellberg, Bo 3158

A theorem on the minimum modulus of entire functions.

Math. Scand. **12** (1963), 5–11.

The following theorem is established: Let f be a non-constant entire function and let λ be a positive real number less than one. Then either the set of r satisfying $\log m(r)>\cos\pi\lambda\log M(r)$ is not bounded above or else $\lim_{r\to\infty}r^{-\lambda}\log M(r)$ exists and is positive (finite or infinite). Here $m(r)=\min_{|z|=r}|f(z)|$ and $M(r)=\max_{|z|=r}|f(z)|$. Earlier results of the author [Math. Scand. **8** (1960), 189–197; MR **23** #A3264] are thereby refined. The present result establishes a conjecture made by Hayman in his review [MR **23** #A3264] of the cited paper. The proof of the present result is based upon repeated use of the Poisson integral for a half-plane to obtain the inequality

$$(1)\qquad \log\frac{M(R)}{R^\lambda}<\int_0^\infty\frac{\log M(r)}{r^\lambda}K(r,R)\,dr,\qquad R>0,$$

where

$$K(r,R)=\frac{2}{\pi^2}(1+\cos\pi\lambda)\left(\frac{r}{R}\right)^\lambda\frac{R\log(R/r)}{R^2-r^2},$$

in the case where $0<\lim\inf_{r\to\infty}r^{-\lambda}\log M(r)\leqq\lim\sup_{r\to\infty}r^{-\lambda}\log M(r)<+\infty$.

From (1) the existence of $\lim_{r\to\infty}r^{-\lambda}\log M(r)$ is established.

M. H. Heins (Berkeley, Calif.)

Essén, Matts 3159

Note on "A theorem on the minimum modulus of entire functions" by Kjellberg.

Math. Scand. **12** (1963), 12–14.

In this note it is shown that the conclusion of the paper reviewed above [#3158] may be deduced from the cited integral inequality by means of the following general theorem, a proof of which is given: Suppose that $\varphi\in L^\infty$ and that φ is slowly decreasing as $x\to+\infty$. Suppose also that the non-negative function $K\in L^1(-\infty,+\infty)$ and is such that $\int_{-\infty}^{+\infty}K(x)\,dx=1$, $\int_{-\infty}^{+\infty}|x|K(x)\,dx<+\infty$, and $\int_{-\infty}^{+\infty}xK(x)\,dx=m\neq 0$. If $\varphi-\varphi*K\leqq 0$, then $\lim_{x\to+\infty}\varphi$ exists.

M. H. Heins (Berkeley, Calif.)

Marcun, N. F. 3160

Certain types of interference for entire functions of finite degree. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 270–272.

If $f(z)$ is an entire function of exponential type σ with $|f(k\pi/\sigma)|\leqq M$ and $f(x)=o(|x|)$, it does not necessarily follow that $f(x)$ is bounded. However, it does follow that certain linear combinations $\sum a_kf(x+\tau_k)$ are bounded (this is the phenomenon of interference). For example, if

$$2L[f]=f\left(x-\frac{m\pi}{2\sigma}\right)+f\left(x+\frac{m\pi}{2\sigma}\right),$$

then

$$(*)\qquad \sup_f|L[f]|=2\pi^{-1}M\log m+O(1)$$

[Timan, *Theory of approximation of functions of a real variable* (Russian), p. 195, Fizmatgiz, Moscow, 1960; MR **22** #8257]. The author tries

$$L[f]=2^{-n}\sum_{\nu=0}^{n}\binom{n}{\nu}f\left(x+\frac{\nu m\pi}{\sigma}\right)$$

and finds (*) again, so that taking more terms does not increase the amount of interference. However,

$$\frac{1}{3}\sup\left|f\left(x-\frac{2m\pi}{3\sigma}\right)+f(x)+f\left(x+\frac{2m\pi}{3\sigma}\right)\right|=\frac{4M}{3\pi}\log m+O(1),$$

which shows more interference than (*). Still more improvement occurs for sums with n terms. In addition, (*) holds when

$$2L[f]=f(x-m\pi/\sigma)-f(x+m\pi/\sigma);$$

for $m=1$ the upper bound is at most $32M/(3\pi)$, which is smaller than the bound given by Bernšteĭn [*Collected Works* (Russian), Vol. II, p. 446, Izdat. Akad. Nauk SSSR, Moscow, 1954; MR **16**, 433; Ivz. Akad. Nauk SSSR Ser. Mat. **12** (1948), 421–444; MR **10**, 363].

R. P. Boas, Jr. (Evanston, Ill.)

Sato, Daihachiro; Straus, Ernst G. 3161

Rate of growth of Hurwitz entire functions and integer valued entire functions.

Bull. Amer. Math. Soc. **70** (1964), 303–307.

Let $g(z)$ and $f(z)$ be entire functions. The function $g(z)$ is integer-valued if $g(n)$ is an integer for $n=0,1,2,3,\cdots$. Similarly, $f(z)$ is a Hurwitz function if $f^{(n)}(0)$ is an integer for $n=0,1,2,3,\cdots$ ($f^{(0)}(z)\equiv f(z)$). The authors also consider two-point Hurwitz functions, which are Hurwitz entire functions at two consecutive integers, say $z=0$ and $z=1$. The authors state seven theorems concerning the above functions. Their purpose is to find, as accurately as possible, a dividing line for the growth of such functions such that, below this line, one finds only polynomials. Two of the simplest results of the authors may be stated as follows. Let $\phi(r)=\max_n\{r^n/\Gamma(n+1)\}$ $(r\geqq 0)$. (I) A Hurwitz entire function is a polynomial if $M(r)=\max_{|z|=r}|f(z)|<\phi(r)+r^N$ for some N and all $r>r_0$. (II) There exists a denumerable infinite set of transcendental, integer-valued entire functions which satisfy $M(r)<2^r-r^N$ for any fixed N and all $r>r_0$.

The authors give brief sketches of their proofs.

A. Edrei (London)

Džafarov, A. S. 3162

Some theorems on best approximation of functions by entire functions of finite degree. (Russian. Azerbaijani summary)

Akad. Nauk Azerbaĭdžan. SSR Dokl. **19** (1963), no. 10, 3–7.

A function F of one variable belongs to $L_{p,q}$ if $F(y+t)$ belongs to $L_q(0, 2\pi)$ in t and its L_q norm belongs to $L^p(-\infty, \infty)$. The author considers the class of functions F of n variables such that F/φ belongs to L_{p_1,q_1} in the first variable, its norm belongs to L_{p_2,q_2} in the second variable, and so on, for a suitable class of weights φ. He discusses the approximation to such functions by entire functions of exponential type, stating four rather complicated theorems connecting the degree of approximation with the degree of differentiability of the function.

R. P. Boas, Jr. (Evanston, Ill.)

San-Juan, R. 3163

Equivalence of Davis theory with that of Carleman for certain domains.

J. Math. Pures Appl. (9) **42** (1963), 167–193.

Let G be a domain whose boundary C is a rectifiable Jordan curve which has finite total length, passes through 0 and contains 1 as an interior point. Let a sequence of positive numbers $\{m_n\}$ be given, and designate by $C_0(m_n, G)$ and $D_0(m_n, G)$ the sets of functions $f(z)$ that are regular in G and satisfy respectively $\sup_{z\in G}|f(z)z^{-n}| < ck^n m_n$ or

$$\int_C |f(z)z^{-n}|^2|dz| < c_1 k_1^{2n} m_n^2 \qquad (n = 0, 1, \cdots).$$

The constants c, c_1, k, k_1 are independent of n, and the integral is interpreted as $\lim_{r\to 1^-} \int_{C_r}$, where C_r are level curves interior to G. The sequence $\{m_n\}$ is said to verify the condition $C(G)$ or $D(G)$ if the sets $C_0(m_n, G)$ or $D_0(m_n, G)$ are, respectively, empty. These conditions are relevant to the study of uniqueness classes for asymptotic expansions (in the sense of Poincaré) of analytic functions in general domains. The first condition was employed by T. Carleman [*Les fonctions quasi analytiques*, Gauthier-Villars, Paris, 1926] in the case of the circle $|z-1| < 1$ and by A. Ostrowski [Acta Math. **53** (1929), 181–266] in the case of more general domains. The second condition was introduced by the reviewer [Pacific J. Math. **7** (1957), 849–859; MR **19**, 268].

Employing bounds which he had previously established, together with well-known properties of conformal maps, the author establishes the equivalence of conditions $C(G)$ and $D(G)$ for some extensive classes of domains G.

P. J. Davis (Providence, R.I.)

Sabitov, I. H. 3164

A general boundary-value problem for the linear conjugate on the circle. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 124–129.

Let Γ denote the unit circle $|z|=1$, D^+ its interior and D^- its exterior. Consider the problem of determining functions $\varphi^+(z)$ and $\varphi^-(z)$, analytic, respectively, in D^+ and D^-, with boundary values $\varphi^+(t)$, $\varphi^-(t)$ satisfying on Γ a relation of the form

$$(1) \quad \varphi^+(t) = a(t)\varphi^-(t)+b(t)\overline{\varphi^-(t)}+c(t), \qquad \varphi^-(\infty) = 0.$$

The functions a, b, c are given and are assumed Hölder-continuous on Γ. The solvability of this problem is completely determined, when (*) $|a(t)|>|b(t)|$, by the value of $k=\operatorname{ind}_\Gamma a(t)$ [see L. G. Mihaĭlov, Dokl. Akad. Nauk SSSR **139** (1961), 294–297; MR **25** #3178]. When (**) $|a(t)|\equiv|b(t)|$, solvability depends also on a knowledge of $\operatorname{ind}_\Gamma b(t)$. The author's chief concern in the present paper is the statement and proof of a theorem concerning the solvability of the problem when neither (*) nor (**) is valid on Γ.

J. F. Heyda (King of Prussia, Pa.)

Govorov, N. V. 3165

The Riemann boundary-value problem with infinite index. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 1247–1249.

Both the homogeneous $[\phi^+(t)=G(t)\phi^-(t)]$ and the non-homogeneous $[\phi^+(t)=G(t)\phi^-(t)+g(t)]$ Riemann boundary problems are considered for a domain D with boundary $[1, \infty]$. The new item of interest here is that $G(t)$ in each case has infinite index, that is to say, $G(t)$ is continuous and non-vanishing on $[1, \infty)$ but with $\lim \arg_{t\to\infty} G(t) = +\infty$. Explicit solutions are given; proofs are not included.

J. F. Heyda (King of Prussia, Pa.)

Żakowski, W. 3166

Problème aux limites d'Hilbert-Haseman généralisé.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 511–515.

In the generalized Hilbert-Haseman problem one seeks, relative to domains S^+, S^- into which a closed Jordan curve L divides the plane, piecewise holomorphic functions $\Phi_k(z)\equiv[\Phi_k^+(z)$ for $z\in S^+$, $\Phi_k^-(z)$ for $z\in S^-]$, $k=1, 2, \cdots, n$, whose boundary values for $t\in L$ satisfy the equations

$$\Phi_k^+[\alpha_k(t)] = G_k(t)\Phi_k^-(t)+g_k(t) + F_k\{t, \Phi_1^+[\alpha_1(t)], \Phi_1^+(t), \Phi_1^-[\alpha_1(t)], \Phi_1^-(t), \cdots, \Phi_n^+[\alpha_n(t)], \Phi_n^+(t), \Phi_n^-[\alpha_n(t)], \Phi_n^-(t)\},$$

$k=1, 2, \cdots, n$, where $\alpha_k(t)$, $G_k(t)$, $g_k(t)$ and F_k are given functions defined on L. With an additional restriction on L and suitable restrictions on the given functions the author shows that a solution of the problem exists and may be obtained by a method of successive approximations for which a specific formulation is supplied.

J. F. Heyda (King of Prussia, Pa.)

Rudin, Walter 3167

Essential boundary points.

Bull. Amer. Math. Soc. **70** (1964), 321–324.

Beginning with any compact set E of one-dimensional Lebesgue measure zero on the real axis the author constructs a region D on the Riemann sphere so that every point z_0 in E is an essential boundary point of D (i.e., some bounded analytic function on D fails to be analytically extendable to an open set containing z_0 and D) but so that the points of E behave in some ways like interior points of D. In Theorem I he employs a Cauchy integral formula to extend each bounded analytic function f on D (together with its derivatives) to $D\cup E$ and proves that the restriction of f to the real axis lies in a quasi-analytic class in the sense of Carleman [*Les fonctions quasi analytiques*, Gauthier-Villars, Paris, 1926]. Using this and the inequalities leading to it, he shows that if f is non-constant, then it cannot achieve its maximum modulus at a point of E (so in particular, as is well known, E is a Painlevé null set); and if f vanishes on an infinite set in E, it is identically zero. As corollaries he shows how to obtain examples of sup-norm algebras in which each

function is determined on a "very small" subset of the Šilov boundary, and proves (Theorem II) that each evaluation at a point of E is a β-continuous [see Buck, Michigan Math. J. **5** (1958), 95–104; MR **21** #4350] homomorphism of the algebra of bounded analytic functions on D. *Max Weiss* (Princeton, N.J.)

POTENTIAL THEORY

See also 3243, 3255, 3256, 3279, 3873.

Ivanov, V. K.; Kazakova, L. È. 3168

An application of analytic functions to the inverse problem of potential theory. (Russian)

Sibirsk. Mat. Ž. **4** (1963), 1311–1317.

Let Ω be the unit circle with the center at the origin. Suppose that there is a mass distributed inside Ω. The paper deals with the problem of finding the mass and its center of gravity if the partial derivatives of the logarithmic potential of the mass are known on the segment $-a \leqq x \leqq a$, $y = h$, $h > 1$.

P. Saworotnow (Washington, D.C.)

Solomencev, E. D. 3169

A Phragmén-Lindelöf type theorem for harmonic functions in space. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 765–766.

The following theorem of Phragmén-Lindelöf type is established: Let D denote a region contained in the set

$$\{|x| \leqq a\} \times \{|y| \leqq b\} \times \{|t| < +\infty\}$$

of euclidean three-space. It is supposed that the frontier Γ of D is piecewise smooth. It is supposed that u is a non-negative harmonic function on the δ-neighborhood of D, δ being a fixed positive number. If $\sup_\Gamma u \leqq M$ and

$$\max_{(x,y)} u(x, y, t) < c \exp \frac{[\exp \pi(1-\varepsilon)|t|]}{2q},$$

$\varepsilon > 0$, $q^{-2} = a^{-2} + b^{-2}$, then $\sup_D u \leqq M$. Generalizations to spaces of higher dimensions are indicated.

M. H. Heins (Berkeley, Calif.)

Neckermann, Ludwig 3170

★Über in Kegelbereichen harmonische Funktionen.

Inauguraldissertation zur Erlangung der Doktorwürde der Hohen Naturwissenschaftlichen Fakultät der Julius-Maximilians-Universität Würzburg, Würzburg, 1962. 65 *pp.*

The author investigates the eigenfunctions of the Legendre equation for the boundary conditions $y(x) = O(1)$ at $x = 1$, $y(c) = 0$, where $-1 < c < 1$. These occur in connection with potential theory in a circular cone; in the second part of the thesis the author applies his results to the theory of the growth of harmonic functions in a cone. If P_{λ_n} is the nth eigenfunction, the author shows in particular that $P_{\lambda_n}(x)$ has n zeros in (c, 1) and that the P_{λ_n} are orthogonal; he obtains an asymptotic formula for the normalizing factor. He proves completeness of the P_{λ_n} and convergence theorems for the expansions of functions with derivatives of bounded variation and of step functions. In the second part he considers harmonic functions $u(P)$ in a cone of half-angle $\gamma = \cos^{-1} c$. His main theorem is the following result of Phragmén-Lindelöf type. Suppose that $\limsup u(P) \leqq 0$ on the boundary; let $M(R) = \max u(P)$ on the intersection of the cone with a spherical surface of radius R and center at the vertex. Then $M(R)R^{-\lambda_0} \rightarrow L \geqq 0$ and $u(P) \leqq LP_{\lambda_0}(\cos\theta)r^{\lambda_0}$ if L is finite.

R. P. Boas, Jr. (Evanston, Ill.)

Dümmel, Siegfried 3171

Ein Satz über die ersten und zweiten partiellen Ableitungen von verallgemeinerten Potentialen.

Math. Nachr. **25** (1963), 311–318.

Let U be the generalized potential $\int_A \Phi(r(x, \xi))\, d\phi_\xi$. Here r is the Euclidean distance in R^n (space of real n-tuples), $A \subset R^n$ is a bounded set, while ϕ is a signed Borel measure on A. Finally, Φ is continuous for $r > 0$, and, together with its derivatives, satisfies monotonicity and convexity conditions in a right neighborhood of $r = 0$. The theorem of the title then asserts the existence of first and second partial derivatives of U at a point $x_0 \in R^n$ under suitable further restrictions. For example, if

$$\lim_{\rho\to 0} \int_{A\cap S_\rho(x_0)} |\Phi(r(x_0, \xi))|\, d|\phi_\xi| = 0,$$

and if

$$\lim_{\rho\to 0} \int_{A\cap S_\rho(x)} |\Phi'(r(x, \xi))|\, d|\phi_\xi| = 0$$

uniformly for x in a neighborhood of x_0, then the derivatives $\partial U/\partial x_i$ $(i = 1, 2, \cdots, n)$ exist at x_0. It is also permissible to differentiate under the integral sign. The corresponding requirements for the existence of the second partials are similar.

E. C. Schlesinger (New London, Conn.)

SEVERAL COMPLEX VARIABLES

See also 3006, 3043, 3115, 3338, 3339.

Ševčenko, V. I. 3172a

An integral representation of a vector which is holomorphic in a sphere. (Russian)

Dokl. Akad. Nauk SSSR **153** (1963), 1276–1279.

Ševčenko, V. I. 3172b

A boundary-value problem for a vector which is holomorphic in a half-space. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 276–278.

Both these papers have to do with the following situation. Let x_1, x_2, z be coordinates for E^3, and $V = (p, u, v, w)$ a four-dimensional vector-valued function on a domain D in E^3. Let $\zeta = x_1 + ix_2$ be considered as a complex variable, and let $U = u + iv$, $W = w + ip$. V is called a holomorphic function if the following equations are satisfied: $U_\zeta + W_z = 0$, $U_z - W_{\bar\zeta} = 0$. In the first paper, the domain of definition is taken to be the ball, and it is assumed that V has Hölder-continuous boundary values on the sphere. In this case an integral formula over the sphere is given. In the second paper, D is taken to be the half-space $x_3 > 0$, and a boundary-value problem is discussed.

H. Rossi (Princeton, N.J.)

Vladimirov, V. S.; Širinbekov, M. 3173
On the construction of envelopes of holomorphy for Hartogs domains. (Russian)
Ukrain. Mat. Ž. **15** (1963), 189–192.
Represent the coordinates of C^{n+1} in the form (w, z), $w \in C$, $z = (z_1, \cdots, z_n) \in C^n$. Let D be a domain in C^n, and $R(z)$ a semicontinuous function on D. Then $G_D = \{(w, z) \in C^{n+1};\ |w| < R(z),\ z \in D\}$ is the Hartogs domain over D defined by R. Let $H(D)$ be the envelope of holomorphy of D, and let $V^*(z)$ be the greatest plurisubharmonic minorant in $H(D)$ of the function $v(z)$, defined in D as $-\ln R(z)$ and in $H(D) - D$ as $+\infty$. The authors prove that $H(G_D) = \{(w, z);\ |w| < e^{-V^*(z)},\ z \in H(D)\}$.
H. Rossi (Princeton, N.J.)

Villani, Vinicio 3174
Frontiere di Silov negli spazi complessi e applicazioni olomorfe proprie.
Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 333–348.
Let X be a complex space, and K a compact subset of X, which is the closure of an open set D. Let $A(K)$ be the algebra of continuous functions on K which are holomorphic on D. Let $S(K)$ represent the Šilov boundary of $A(K)$, i.e., the minimum closed set on which the modulus of every function in $A(K)$ attains its maximum. Now suppose X and Y are complex spaces on which the holomorphic functions separate points, and $\pi: X \to Y$ a proper holomorphic mapping, and let B be a relatively compact domain on Y, let $K = \bar{B}$, and let K' be the closure of $\pi^{-1}(B)$. The following theorem is proven: If Y is irreducible at every point of K, then $S(K) = \pi(S(K'))$ and $\pi^{-1}(S(K)) = S(K')$. The theorem is proven in two parts; first the case where X is the normalization of Y, and second where Y is assumed to be normal. The proof, in each case, is based on the fact that $A(K')$ is an algebraic extension of $A(K)$. It is clear that the assumption that Y is everywhere irreducible is necessary. In the process the following interesting lemma is proven. A Galois analytic cover is an analytic cover $\pi: X \to Y$ with the property that for every pair x', $x \in X$, such that $\pi(x) = \pi(x')$, there is an automorphism $\sigma: X \to X$ such that $\pi\sigma = \pi$ and $\sigma(x) = x'$. Lemma: If $\pi: X \to Y$ is an analytic cover of normal spaces, then there exists a complex space Z, analytic covers $\rho: Z \to X$, $\tau: Z \to Y$ such that $\pi\rho = \tau$ and $\tau: Z \to Y$ is Galois. *H. Rossi* (Princeton, N.J.)

Andreotti, Aldo; Norguet, François 3175
Problème de Levi pour les classes de cohomologie.
C. R. Acad. Sci. Paris **258** (1964), 778–781.
The following fact is a well-known observation of E. E. Levi. Let D be a domain in C^n, and x a boundary point of D. If there is a holomorphic function defined in $D \cap U$, where U is a neighborhood (in C^n) of x, which has no holomorphic extension to a neighborhood of x, then D is weakly pseudoconvex at x. Conversely, if D is strongly pseudoconvex at x, then there exists such a function defined in a neighborhood U of x; in fact, U can be chosen so that $D \cap U$ is holomorphically convex. Levi's problem was to show that if D is strongly pseudoconvex at every boundary point, then D is holomorphically convex (or, what is the same, to find a function holomorphic in all of D which is not extendable through x). Levi's problem has many solutions, notably that of Grauert [Ann. of Math. (2) **68** (1958), 460–472; MR **20** #5299]. Andreotti and Grauert have extended these results to the q-pseud[oconvex] convex spaces [Bull. Soc. Math. France **90** (1962), 19[3]–259; MR **27** #343]. In the present paper the authors u[se] these results to show that, on a strongly q-pseudoconv[ex] manifold Y (with boundary) the assertions of the Le[vi] problem can be solved, with the $(q-1)$-cohomology class[es] $(H^{q-1}(Y, \mathcal{O}))$ taking the place of the holomorphic functio[ns] $(H^0(Y, \mathcal{O}))$.

More specifically, let Y be a relatively compact doma[in] in a complex space X, let $x \in bY$, and let F be a sheaf [on] X. We denote by $H_+{}^r(Y \cup \{x\}, F)$ the inductive limit [of] the groups $H^r(Y \cup U, F)$ as U runs through the neighbo[r]hoods of x. An element of $H^r(Y, F)$ is extendable at [x] if it is in the image of the natural homomorphis[m] $H^r(Y \cup \{x\}, F) \to H^r(Y, F)$. The following theorems a[re] stated. Theorem: Let Y be an open subset of a compl[ex] space X, and x a regular point of bY. If F is a locally fr[ee] sheaf on X, and $H^r(Y \cap U, F)$ has a non-extendab[le] element (U some neighborhood of x), then Y is weak[ly] q-pseudoconvex at x, with $q \leqq r+1$. Conversely, if Y [is] strongly q-pseudoconvex and, in addition, the Levi for[m] has $q-1$ negative eigenvalues at x, then $H^{q-1}(Y, F)$ h[as] a non-extendable element. Theorem: If X is a manifol[d] and the Levi form of a defining function for Y is no[n]degenerate and has $q-1$ negative eigenvalues everywhe[re] (on bY), then Y is strongly q-pseudoconvex. If F is [a] locally free sheaf on X, there is an element in $H^{q-1}(Y, \ldots$ which is extendable through no point of bX.
H. Rossi (Princeton, N.J.)

Andreotti, Aldo; Narasimhan, Raghavan 31[76]
Oka's Heftungslemma and the Levi problem for compl[ex] spaces.
Trans. Amer. Math. Soc. **111** (1964), 345–366.
The following theorem is proven. Let X be a comple[x] space and p a strongly pseudoconvex function on X su[ch] that for any $\alpha > 0$ the set $\{x \in X;\ p(x) < \alpha\}$ is relative[ly] compact in X. Then X is a Stein space. There are proofs [of] this theorem in the literature, but the only elementa[ry] proof for domains over C^n is that of Oka [Japan. J. Mat[h.] **23** (1953), 97–155; MR **17**, 82]. The present proof is f[or] the general case and is elementary. The main techniq[ue] is a refinement of Oka's "Heftungslemma", used in h[is] argument. The following theorems are also prove[n]. Theorem: On a K-complete space, any relatively compa[ct] open set which is pseudoconvex with a globally define[d] boundary is a Stein space. Theorem: On a Stein space wi[th] isolated singularities, any open set which is locally [a] Stein space is also globally a Stein space.
H. Rossi (Princeton, N.J.)

Hervé, Michel 31[77]
Quelques propriétés des applications analytiques d'u[ne] boule à m dimensions dans elle-même.
J. Math. Pures Appl. (9) **42** (1963), 117–147.
Dans l'espace numérique complexe $\mathbf{C}^n$, soit B_n la boul[e] unité ouverte $|x| < 1$. Il est connu que, pour toute applica[tion] holomorphe $f: B_n \to B_n$ telle que $f(0) = 0$, on [a] $|f(x)| \leqq |x|$ (c'est un cas particulier d'un résultat valab[le] pour tout ouvert borné convexe et cerclé). L'aute[ur] montre que les $x \in B_n$ tels que $|f(x)| = |x|$ $[f(x) = x]$ so[nt] ceux de $B_n \cap V$ $[B_n \cap W]$, où V $[W]$ désigne un sou[s-]espace vectoriel (complexe) de $\mathbf{C}^n$. Corollaire: pour tou[t]

application holomorphe $f: B_n \to B_n$, l'ensemble des points invariants par f est de la forme $B_n \cap W$, où W est une variété linéaire affine, éventuellement vide.

Théorème (cf. théorème 6): soit f une application holomorphe $B_n \to B_n$. Si f n'a pas de point fixe, la suite des itérées f^n converge (uniformément sur tout compact) vers une constante, dont la valeur est un point de la sphère ∂B_n. Si f a au moins un point fixe, il existe un plus grand sous-espace analytique $Y \subset B_n$, contenant les points fixes de f, tel que $f|Y$ soit un automorphisme de Y; Y est l'intersection de B_n avec une variété linéaire affine; toute limite (uniforme sur tout compact) d'itérées f^{n_k} ($n_k < n_{k+1}$) applique B_n sur Y et induit un automorphisme de Y; la suite (f^n) elle-même n'est convergente que si $f|Y$ est l'identité.

L'auteur étend aussi à la boule B_n le lemme de Pick et les théorèmes de Julia-Carathéodory sur la dérivée angulaire au bord du disque; il étudie en détail les cas-limites où les inégalités deviennent des égalités.

H. Cartan (Paris)

SPECIAL FUNCTIONS

See also 2999, 3048, 3170, 3296, 3529a-b, 3576, 3773.

Laričeva, L. S. 3178

A limiting precision estimate for asymptotic expansions of a certain class of functions. I. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1963**, *no.* 6 (37), 109–115.

Supposons que $f(t)$ admet la transformée de Laplace de la forme $f^* = (p-p_0)^{\gamma-1}\varphi(p)$, $0<\gamma<1$, p, p_0: les nombres complexes (p_0 fixe), $\varphi(p)$ ayant un nombre fixe des poles ξ_k et satisfaisant aux certaines conditions portant sur l'allure asymptotique pour $|p| \to \infty$ et aux positions des poles, et soit $\varphi(p)$ telle que $f^*(p) \to 0$ pour $|p| \to \infty$, $\operatorname{Re} p \leq \operatorname{Max}\{\operatorname{Re} p_0, \operatorname{Re} \xi_k\}$. Alors la fonction $F(t) = e^{-p_0 t}\{f(t) - \sum_k \operatorname{res} f^*(\xi_k)e^{\xi_k t}\}$ admet le développement asymptotique de la forme

$$\sum_0^\infty \varphi^{(n)}(p_0)[n!\Gamma(-\gamma-n+1)]^{-1}t^{-\gamma-n}.$$

En posant

$$r_N(t) = \left| F(t) - \sum_{n=0}^N \frac{\varphi^{(n)}(p_0)}{n!\Gamma(-\gamma-n+1)} t^{-\gamma-n} \right|$$

on détermine $N = N_t$ pour lequel r_{N_t} prend la valeur minimale, et on donne alors le comportement asymptotique de r_{N_t}.

M. Tomić (Belgrade)

Olsson, Per O. M. 3179

Integration of the partial differential equations for the hypergeometric functions F_1 and F_D of two and more variables.

J. Mathematical Phys. **5** (1964), 420–430.

For the system of partial differential equations satisfied by Appell's series

$$F_1(a, b, b', c, x, y) = \sum_{m,n=0}^\infty \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!} x^m y^n,$$

$$(a)_k = \Gamma(a+k)/\Gamma(a), \qquad |x| < 1, |y| < 1,$$

sixty series in terms of F_1, representing ten distinct solutions, are listed in the monograph by P. Appell and J. Kampé de Fériet [*Fonctions hypergéométriques et hypersphériques*, Gauthier-Villars, Paris, 1926]. Another sixty series in terms of Horn's series G_2, representing fifteen new distinct solutions, have been obtained by the reviewer [Acta Math. **83** (1950), 131–164; MR **12**, 257] from Picard's contour integral solutions.

The author now obtains the G_2-solutions by re-writing F_1 as an infinite sum of Gauss's hypergeometric series ${}_2F_1$ and then applying the known transformation and analytic continuation formulae for ${}_2F_1$. In this manner he obtains not only the complete set of solutions but also the connection formulae between them. He also indicates the application of the same process to Lauricella's triple series F_D.

A. Erdélyi (Pasadena, Calif.)

Bhonsle, B. R. 3180

On Rice's polynomials, $H_n(\xi, p, v)$.

Proc. Nat. Acad. Sci. India Sect. A **32** (1962), 175–178.

In the first part of this paper, Rice's polynomial is defined in terms of generalised hypergeometric functions, and two expansions in series are given. The first is an expansion of the Gauss function in terms of the H-functions, and the second is an expansion of the confluent hypergeometric function as a series of products of H-functions and Bessel functions.

In the second part of the paper, two integrals for the H-function are given; the first is an integral of a generalised hypergeometric function, and the second is an integral of a Bateman polynomial.

L. J. Slater (Cambridge, England)

Verma, Arun 3181

General expansions involving E-functions.

Math. Z. **83** (1964), 29–36.

The author obtains in this paper some expressions, too complicated to reproduce here, involving E-functions. As special cases, several results of MacRobert and Ragab are obtained.

W. A. Al-Salam (Durham, N.C.)

Carlitz, L. 3182

The coefficients of the reciprocal of a Bessel function.

Proc. Amer. Math. Soc. **15** (1964), 318–320.

Let $\omega_n(\nu)$ be defined by

$$\left[\sum_{n=0}^\infty \frac{(-1)^n x^n}{n!\Gamma(\nu+n+1)}\right]^{-1} = \sum_{n=0}^\infty \frac{\omega_n(\nu)x^n}{n!\Gamma(\nu+n+1)},$$

in which ν is an arbitrary real or complex number, other than a negative integer. Then the author proves that the $\omega_n(\nu)$ cannot satisfy a linear recurrence relation of fixed order with polynomial coefficients in n.

F. W. J. Olver (Washington, D.C.)

Rangarajan, S. K. 3183

On "A new formula for $P_{m+n}^m(\cos\alpha)$".

Quart. J. Math. Oxford Ser. (2) **15** (1964), 32–34.

The author points out that a formula obtained by

G. M. Yadao [same J. (2) **13** (1962), 29–30; MR **25** #4148] is incorrect. The corrected formula reads

$$\left(\frac{\sin\beta}{\sin\alpha}\right)^{m+n} P^m_{m+n}(\cos\alpha) = \sum_{r=0}^{n} \binom{2m+n}{r} \left(\frac{\sin(\beta-\alpha)}{\sin\alpha}\right)^r P^m_{m+n-r}(\cos\beta).$$

The mistake is due to the use of an incorrect formula given by C. A. T. Truesdell [*An essay toward a unified theory of special functions based upon the functional equation* $\partial/\partial z F(z, \alpha) = F(z, \alpha+1)$, p. 101, Princeton Univ. Press, Princeton, N.J., 1948; MR **9**, 431].

It is noted that the evaluation of a certain contour integral on p. 105 of this reference is also incorrect.

L. Carlitz (Durham, N.C.)

Chatterjea, S. K. 3184

On some classical orthogonal polynomials.

Rev. Mat. Hisp.-Amer. (4) **23** (1963), 196–202.

The following inequalities for Legendre polynomials are established:

(1) $n(n+1)\Delta_n(x) + (1-x^2)P_n(x)P'_{n-1}(x) \geq 0,$

$$n \geq 1, |x| \leq 1,$$

$$\Delta_n(x) = P_n^2(x) - P_{n+1}(x)P_{n-1}(x),$$

(2) $nP_n'(x)P_{n-1}(x) - (n+1)P_n(x)P'_{n-1}(x) > 0, \quad n \geq 1,$

together with the analogue of (1) for Laguerre polynomials.

The inequality (2) is stronger than

$$p_n'(x)p_{n-1}(x) - p_n(x)p'_{n-1}(x) > 0$$

(valid for all orthogonal polynomials) for $p_n(x) = P_n(x)$.

A. E. Danese (Buffalo, N.Y.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 3250, 3269, 3300, 3342, 3345, 3346, 3349, 3434, 3477, 3487, 3559, 3565, 3566, 3888, 3889, 3891, 3892, 3893, 3894, 3895, 3896, 3897.

★**Proceedings of the International Symposium on Non-linear Vibrations. Vol. II: Qualitative methods in the theory of non-linear vibrations [Труды Международного Симпозиума по Нелинейным Колебаниям. Том II: Качественные методы теории нелинейных колебаний].** 3185a

International Union of Theoretical and Applied Mechanics, Kiev, 12–18 September 1961.

Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963. 542 *pp.* 3.30 *r.*

★**Proceedings of the International Symposium on Non-linear Vibrations. Vol. III: Applications of the methods of non-linear vibrations to the problems of physics and technology [Труды Международного Симпозиума по Нелинейным Колебаниям. Том III: Приложения методов теории нелинейных колебаний к задачам физики и техники].** 3185b

International Union of Theoretical and Applied Mechanics, Kiev, 12–18 September 1961.

Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963. 515 *pp.* 3.00 *r.*

These are the second and third volumes of the proceedings of the 1961 symposium mentioned in the title; as in the case of the first volume [1961; MR **27** #5972], the papers will be reviewed individually.

★**Approximate methods of solving differential equations [Приближенные методы решения дифференциальных уравнений].** 3186

Ju. A. Mitropol'skiĭ, Editor-in-Chief.

Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963. 155 *pp.* 0.64 *r.*

Most papers in this collection will be reviewed individually.

Chalkley, Roger 3187

A certain homogeneous differential equation transformation.

Arch. Math. **14** (1963), 186–192.

Consider the differential equation $Q=0$, where Q is the quadratic form $aY_1^2 + bY_1Y_2 + \cdots + fY_3^2$ in the expressions $Y_1 = y''y + g_2y'^2 + g_1y'y + g_0y^2$, $Y_2 = y'(h_1y' + h_0y)$, $Y_3 = (h_1y' + h_0y)y$, and $a, \cdots, g_0, \cdots, h_1$ are elements of a differential field of characteristic 0 with $a \neq 0$ and with h_0, h_1 not both 0. The author shows how to solve $Q=0$ by solving in succession two differential equations of order 1.

E. R. Kolchin (New York)

Wouk, Arthur 3188

On the Cauchy-Picard method.

Amer. Math. Monthly **70** (1963), 158–162.

The author applies the principle of contraction mappings to the equation $dy/dx = f(x, y, \mu)$ with parameter μ. Using existence and Lipschitz assumptions on f and its mixed partial derivatives, he establishes existence, uniqueness, and continuity of solutions $y(x, \mu)$ and their mixed partial derivatives.

W. J. Coles (Salt Lake City, Utah)

Grudo, Ė. I. 3189

Construction of solutions of order $\nu-0$ of first-order differential equations. (Russian)

Dokl. Akad. Nauk BSSR 8 (1964), 13–17.

Consider the equation $dy/dx = P(x, y)/Q(x, y)$, where P and Q are analytic and vanish at the origin. A solution $y(x)$ of order $\nu-0$ has the property that $y/x^m \to 0$ for $m<\nu$ and $\to\infty$ for $m \geq \nu$ when $x, y \to 0$. The author constructs expansions in convergent series for these solutions.

W. A. Coppel (Canberra)

Mangeron, D. 3190

The integral equation method in nonlinear mechanics.

Acta Math. Sinica **12** (1962), 179–180 (*Chinese*); *translated as Chinese Math.* **3** (1963), 194–195.

The author describes, very briefly and vaguely, a method for solving the initial-value problem for a class of second-order equations. The given problem is converted into an integral equation of Volterra type, and the latter is then solved by asymptotic methods; the author claims that in some cases his procedure is suitable for effective numerical computation.

Bernard Epstein (New York)

Hille, Einar 3191
Green's transforms and singular boundary value problems.
J. Math. Pures Appl. (9) **42** (1963), 331–349.
The author studies H. Weyl's singular boundary problem $u'' - [\lambda + q(x)u] = 0$ $(0 \leqq x < \infty)$, $\cos \alpha u(0, \lambda) + \sin \alpha u'(0, \lambda) = 0$, $u(x, \lambda) \in L_2(0, \infty)$. Here $q(x)$ is assumed to be real and continuous. The Green's transform of the boundary problem is

$$[\overline{u(x, \lambda)}u'(x, \lambda)]_a{}^b - \int_a^b |u'(s, \lambda)|^2\,ds - \int_a^b [\lambda + q(s)]|u(s, \lambda)|^2\,ds = 0.$$

Taking the imaginary part, one obtains an expression which is basic in Weyl's methods. The author proposes to exploit the companion expression which is the real part of the Green's transform. With it he obtains new proofs of known facts about the boundary problem, especially when $q(x)$ is bounded below, e.g., some properties of the spectrum, and the behavior at ∞ of the solution u. The following result seems to be new: Let $m(\lambda, \alpha)$ be Weyl's limit point function and let $\lambda = i\nu$ be pure imaginary. Then when $\alpha \not\equiv 0 \pmod{\pi}$, the expression $\nu^{1/2}[m(i\nu, \alpha) + \cot \alpha]$ is bounded away from zero and infinity as $\nu \to +\infty$. When $\alpha = 0$, the expression $\nu^{-1/2}|m(i\nu, 0)|$ is similarly bounded. The proofs are classical, in the spirit of Weyl.
Robert McKelvey (Madison, Wis.)

Nehari, Zeev 3192
On an inequality of P. R. Beesack.
Pacific J. Math. **14** (1964), 261–263.
This paper contains a much simpler derivation of an inequality originally obtained by Beesack [same J. **12** (1962), 801–812; MR **26** #2672].
P. Cooperman (Teaneck, N.J.)

Cole, Randal H. 3193
General boundary conditions for an ordinary linear differential system.
Trans. Amer. Math. Soc. **111** (1964), 521–550.
Matrix differential systems of the type

$$Y' = P(x, \lambda)Y + B(x),$$
$$\sum_{h=1}^{m} A_h(\lambda)Y(a_h, \lambda) + \int_a^b F(x, \lambda)Y(x, \lambda)\,dx = C(\lambda),$$

where the components of the coefficient matrices and vectors are polynomials in λ and indefinitely differentiable in x, are treated. The boundary conditions are rewritten in terms of Stieltjes integrals and are used in this form. The Green's matrix for the homogeneous system is developed, and adjoint systems are introduced. The adjoint systems are shown to involve interface conditions at the "internal" boundary points, a_h. The given and adjoint systems are shown to have the same characteristic values and the same order of compatibility for each of these. The structure of the Green's matrix is analysed, and residues for this at poles associated with the characteristic values are examined. Regularity conditions for the boundary-value problem are introduced, and a formal expansion for a vector function $f(x)$ is obtained. Under the assumption that $f(x)$ has bounded and integrable derivatives of sufficiently high order, the formal expansion is shown to converge uniformly to $f(x)$ on any sub-interval of $[a, b]$ on which the boundary-value problem is regular.
W. M. Whyburn (Chapel Hill, N.C.)

Levitan, B. M. 3194
Determination of a Sturm-Liouville differential equation in terms of two spectra. **(Russian)**
Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 63–78.
Proofs are given of results previously announced by the author [Dokl. Akad. Nauk SSSR **150** (1963), 474–476; MR **27** #379]. Let $\{\lambda_n\}$ be the eigenvalues and $\{\psi_n(x)\}$ the corresponding eigenfunctions (normed by the condition $\psi_n(0) = 1$) of the problem (P): $y'' + [\lambda - q(x)]y = 0$, $y'(0) - hy(0) = 0$, $y'(\pi) + Hy(\pi) = 0$, where q is real and continuous and H, h are real numbers. Let $\{\mu_n\}$ be the eigenvalues of the problem (P_1), which is the same as (P) except that H is replaced by $H_1 \neq H$. G. Borg [Acta Math. **78** (1946), 1–96; MR **7**, 382] has shown that the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ uniquely determine q. M. G. Krein [Dokl. Akad. Nauk SSSR **76** (1951), 21–24; MR **12**, 613; ibid. **76** (1951), 345–348; MR **13**, 43] studied the problem of construction of (P) and (P_1) given $\{\lambda_n\}$ and $\{\mu_n\}$. I. M. Gel'fand and the author [Izv. Akad. Nauk SSSR Ser. Mat. **15** (1951), 309–360; MR **13**, 558] also gave a method of construction of (P) in terms of the asymptotic expansions of λ_n and $\alpha_n = \int_0^\pi \psi_n{}^2(x)\,dx$. In the present article the author shows how one may obtain the asymptotic expansion of α_n from the expansions of λ_n and μ_n. This is done by studying the asymptotic behavior of $\Phi_2(\lambda_k)$ and $\Phi_1'(\lambda_k)$ for large k, where $\Phi_1(\lambda) = \prod_{n=0}^{\infty}(1 - \lambda/\lambda_n)$ and $\Phi_2(\lambda) = \prod_{n=0}^{\infty}(1 - \lambda/\mu_n)$, and by using the equation $\alpha_k = A\Phi_2(\lambda_k)\Phi_1'(\lambda_k)$. These results are used to prove a theorem concerning conditions under which two sequences of numbers $\{\lambda_n\}$ and $\{\mu_n\}$ are eigenvalues of problems (P) and (P_1). *R. C. Gilbert* (Fullerton, Calif.)

Levitan, B. M. 3195
Calculation of the regularized trace for the Sturm-Liouville operator. **(Russian)**
Uspehi Mat. Nauk **19** (1964), *no.* 1 (115), 161–165.
Let $\{\lambda_n\}$ be the sequence of eigenvalues of the Sturm-Liouville boundary-value problem $y'' + \{\lambda - q(x)\}y = 0$ $(0 \leqq x \leqq \pi)$, $y'(0) - hy(0) = 0$, $y'(\pi) + Hy(\pi) = 0$, where q is continuously differentiable on $[0, \pi]$ and h, H are finite real numbers. A new method is presented for calculation of the regularized sums $\sum_{n=0}^{\infty} \lambda_n{}^s$ $(s = 1, 2, \cdots)$. If $\varphi(x, \lambda)$ is the solution of the differential equation satisfying $\varphi(0, \lambda) = 1$, $\varphi'(0, \lambda) = h$, then the eigenvalues λ_n are roots of the entire analytic function $\varphi'(\pi, \lambda) + H\varphi(\pi, \lambda)$. Hence,

$$\varphi'(\pi, \lambda) + H\varphi(\pi, \lambda) = A \prod_{n=0}^{\infty} (1 - \lambda/\lambda_n).$$

The method consists of studying the asymptotic behavior of both sides of this equation for large negative $\lambda = -\mu$ and then identifying coefficients of like powers of μ.
R. C. Gilbert (Fullerton, Calif.)

Kac, I. S. 3196
Spectral multiplicity of a second-order differential operator and expansion in eigenfunction. **(Russian)**
Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 1081–1112.
Detailed proofs are given of results previously announced by the author [Dokl. Akad. Nauk SSSR **145** (1962), 510–513; MR **26** #2906]. Let $\tau_i(\lambda)$, $i = l, r$, be two non-decreasing

functions such that $\tau_i(0)=0$, $\tau_i(t)=\frac{1}{2}[\tau_i(t-0)+\tau_i(t+0)]$, $\int_{-\infty}^{\infty}(1+t^2)^{-1}\,d\tau_i(t)<\infty$. Let $\omega_i(\lambda)=\alpha+\beta\lambda+\int_{-\infty}^{\infty}[(t-\lambda)^{-1}-t(1+t^2)^{-1}]\,d\tau_i(t)$, where $\beta\geqq 0$ and α are real constants. Let $\Omega_{11}(\lambda)=\omega_l(\lambda)\omega_r(\lambda)[\omega_l(\lambda)+\omega_r(\lambda)]^{-1}$, $\Omega_{22}(\lambda)=-[\omega_l(\lambda)+\omega_r(\lambda)]^{-1}$, $\Omega_{12}(\lambda)=\Omega_{21}(\lambda)=\omega_l(\lambda)\times[\omega_l(\lambda)+\omega_r(\lambda)]^{-1}$, $\sigma_{ij}(t)=\lim_{\eta\to+0}(1/\pi)\int_0^t \operatorname{Im}\Omega_{ij}(s+i\eta)\,ds$ $(-\infty<t<\infty;\ i,j=1,2)$, $S(t)=\|\sigma_{ij}(t)\|_{i,j=1}^2$. If T_s is the operator of multiplication by the independent variable in the space $L_2(S)$ and if $Q[\tau_i]$ is the set of points in which τ_i has a finite symmetric derivative different from zero, it is shown, among other things, that T_S has a simple spectrum when the Lebesgue measure of $K=Q[\tau_l]\cap Q[\tau_r]$ is zero, and that the multiplicity of the spectrum of T_S is two when the Lebesgue measure of K is positive. These results are immediately applicable to the case of a Sturm-Liouville differential operator which is in the limit point case at both ends of the interval of definition if τ_l and τ_r are taken to be the spectral functions at the left and right end-points, respectively. The results of the paper are also used in connection with a Sturm-Liouville operator on an interval $[a,b)$ for the generalization of an expansion theorem of A. V. Štraus [Izv. Akad. Nauk SSSR Ser. Mat. **20** (1956), 783–792; MR **19**, 277].

R. C. Gilbert (Fullerton, Calif.)

Turaev, A. T. 3197

A qualitative investigation of the equation

$$\frac{dy}{dx}=\frac{-x^5+Px^2y+Qxy^2+My^3}{y+Ax^3+Bx^2y+Cxy+Dy^3}$$

in the large. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 6, 30–34.

The author investigates the complete phase portrait of the equation in the title for the two cases in which the origin is a centre, namely, $P+3A=B+Q=3M+C=0$ and $A=P=M=C=0$.

W. A. Coppel (Canberra)

Bylov, B. F. 3198

The near-reducibility of a system of linear differential equations having different characteristic indices. (Russian)

Sibirsk. Mat. Ž. **4** (1963), 1241–1262.

Consider the system (1) $\dot{x}=A(t)x$, where $A(t)$ is a continuous uniformly bounded $n\times n$ matrix for $-\infty<t<\infty$. Assume that there exist n distinct real numbers $\lambda_1<\lambda_2<\lambda_3<\cdots<\lambda_n$ and n solutions $x_i(t)$ of (1) such that for each i the limit

$$\lim_{(t-\tau)\to\infty}\frac{1}{t-\tau}\frac{\|x_i(t)\|}{\|x_i(\tau)\|}=\lambda_i$$

exists uniformly in τ. Under these assumptions the author proves that $A(t)$ is nearly reducible (approximately similar) to the constant matrix $\operatorname{diag}(\lambda_1,\lambda_2,\cdots,\lambda_n)$.

J. C. Lillo (W. Lafayette, Ind.)

Loud, W. S. 3199

Behavior of the period of solutions of certain plane autonomous systems near centers.

Contributions to Differential Equations **3** (1964), 21–36.

The author considers the center-focus problem for the real system $\dot{x}=P(x,y)$, $\dot{y}=Q(x,y)$, where P and Q are analytic and where the origin is an isolated singular point. He establishes necessary conditions in order that the origin be a center. His results are obtained by an implicit function approach, as contrasted with series methods of Poincaré and others. In case the origin is a center, the behavior of the period of those periodic solutions in a small neighborhood of the origin is determined. More specifically, a necessary condition is found in order that the center be isochronous. The paper concludes with a discussion of the system (1) $\dot{x}=-y+Ax^2+Bxy+Cy^2$, $\dot{y}=x+Dx^2+Exy+Fy^2$. By the adaptation of certain criteria due to Saharnikov and Urabe, the author is able to state precisely conditions under which any center associated with (1) is isochronous. The reviewer wishes to remark that the paper is easily read and exhibits several nice techniques.

J. C. Wilson (Carbondale, Ill.)

Hale, J. K.; Seifert, G. 3200

Bounded and almost periodic solutions of singularly perturbed equations. (Russian summary)

Qualitative methods in the theory of non-linear vibrations (Proc. Internat. Sympos. Non-linear Vibrations, Vol. II, 1961), pp. 427–432. *Izdat. Akad. Nauk Ukrain. SSR, Kiev.* 1963.

Consider the systems

(1) $\dot{x}=f(t,x,y,\varepsilon)$, $\varepsilon\dot{y}=g(t,x,y,\varepsilon)$,

(2) $\dot{x}=f(t,x,y,0)$, $0=g(t,x,y,0)$ $(\dot{}=d/dt)$,

where $\varepsilon>0$ is a real parameter, $t\in R$, $x,f\in R_n$, $y,g\in R_m$, and assume that (2) has a solution $x=\theta(t)$, $y=\chi(t)$ which is almost periodic and uniformly continuous for $t\in R_1$. Employing a technique of J. J. Levin [Duke Math. J. **23** (1956), 609–620; MR **19**, 305] used by him for a finite interval (but without any assumption of periodicity or almost periodicity) the authors give sufficient conditions for the existence of numbers ε_0, $\rho>0$ such that for $0<\varepsilon\leqq\varepsilon_0$ the system (1) has a unique solution $x=p(t,\varepsilon)$, $y=q(t,\varepsilon)$ in the region $|x-\theta(t)|+|y-\chi(t)|\leqq\rho$, $t\in R_1$, and $|p(t,\varepsilon)-\theta(t)|+|q(t,\varepsilon)-\chi(t)|\to 0$ as $\varepsilon\to 0$, uniformly in t, $t\in R_1$. Both here and in the analysis of Levin a lemma of Flatto and Levinson [J. Rational Mech. Anal. **4** (1955), 943–950; MR **17**, 849] on the behaviour of solutions of a singularly perturbed linear system having a bounded continuous coefficient matrix with all eigenvalues having real part negative and bounded away from zero plays a central role. The authors also prove that if a certain auxiliary matrix (already needed for the first result) is almost periodic, then a suitable application of a result of Amerio [Ann. Mat. Pura Appl. (4) **39** (1955), 97–119; MR **18**, 128] shows that the solution $p(t,\varepsilon)$, $q(t,\varepsilon)$ is almost periodic.

The present result does not give the one of Flatto and Levinson in case the solution $\theta(t)$, $\chi(t)$ of (2) is assumed periodic; the reason is that one hypothesis (H_4) needed in the present work is much stronger than the corresponding one required in the periodic case.

J. A. Nohel (Madison, Wis.)

Derwidué, L. 3201

Sur l'existence de solutions périodiques des équations différentielles quasi-linéaires à coefficients périodiques. I, II.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 346–351; *ibid.* (5) **49** (1963), 461–469.

The author considers the integral equation

$$(1) \qquad y(x) = \lambda \int_0^\omega G(x, \xi)P[\xi, y(\xi)]\, d\xi$$

where $G(x, \xi)$ denotes the Green's function of an nth order regular formal differential operator $L(y)$ with periodic coefficients (with common period ω) and where $P[x, y(x)]$ denotes a function of $x, y, y', \cdots, y^{(n-1)}$ which is Lipschitzian with respect to y and its derivatives, and continuous and periodic with period ω with respect to x. In the first paper the author shows that the integral equation (1) admits a unique continuous periodic solution of period ω provided only that $|\lambda|$ is sufficiently small and that $L(y)=0$ has no nontrivial solution of period $k\omega$, for some integer k. In the second paper the author applies the topological method of Leray and Schauder [Ann. Sci. École Norm. Sup. (3) **51** (1934), 45–78] to establish the existence of at least one periodic solution with period ω.

G. H. Meisters (Boulder, Colo.)

Rozenvasser, E. N. 3202

On the exact determination of oscillations in piecewise-linear continuous systems.

Avtomat. i Telemeh. **23** (1962), 1414–1420 (*Russian. English summary*); *translated as Automat. Remote Control* **23** (1963), 1328–1334.

In the present paper the problem of determining a periodic solution of a piecewise linear system is investigated. The system is of the form $dx/dt = A_i x + b_i$ for $t_{i-1} + pT \leqq t < t_i + pT$, $i = 1, \cdots, m$, $T = t_m - t_0$, $p = 0, \pm 1, \cdots$. A_i are constant matrices, b_i constant column vectors, t_i and T are constants to be determined.

The usual approach to this problem is to form the solutions $x_i(t)$ of the system on each interval $[t_{i-1}, t_i]$ and then require that (*) $x_i(t_{i-1}) = x_{i-1}(t_{i-1})$ $(i = 2, \cdots, m)$ and $x_1(t_0) = x_m(t_m)$. This leads to a linear nonhomogeneous system $(E-U)x_0 = M$ for the initial condition $x_0 = x_1(t_0)$ in which the matrix $(E-U)$ and vector M depend on t_i. From this system and some additional relations which determine the transition from one segment of the non-linear characteristic to another, the unknown initial condition can be eliminated, and in this manner equations for the period are obtained which contain only the unknown t_i. This procedure depends essentially on the rank of the matrix $(E-U)$. However, if one requires the right-hand sides of the system to be continuous along the periodic solution sought, which is the case considered in the present paper, then, as the author proves, the determinant of this matrix is identically zero, which is an additional difficulty.

Thus the author proposes another approach. The idea consists in requiring that $x_i'(t_{i-1}) = x_{i-1}'(t_{i-1})$ and $x_1'(t_0) = x_m'(t_m)$ $('=d/dt)$ rather than (*), which results in a considerable simplification of the final expressions.

C. Olech (Kraków)

Wilson, J. C. 3203

Algebraic periodic solutions of $\ddot{x} + f(x)\dot{x} + g(x) = 0$.

Contributions to Differential Equations **3** (1964), 1–20.

The author gives a necessary and sufficient condition for the system $x' = y$, $y' = -f(x)y - g(x)$, where f and g are rational functions, to have an integral $y^2 + A(x)y + B(x) = 0$, where A and B are rational functions and $A^2 - 4B$ is not a perfect square. He thus obtains a class of equations $x'' + f(x)x' + g(x) = 0$ with algebraic periodic solutions. This class has non-empty intersection with the class of equations satisfying the conditions of Levinson and Smith for the existence of a unique periodic solution. Hence the limit cycle of, for example, van der Pol's equation can be approximated by the algebraic periodic solution of a related equation. {Reviewer's remark: The second sentence of the proof of Theorem 3.1 is unjustified. However, it may be shown that the only polynomial solutions of (3.3) are $A = ax$, $B = x^2$ $(a \neq \pm 2)$; $A = 0$, $B = x^2 + b$, and $A = 2\sqrt{2}(x+c)$, $B = 2x(x+c)$.}

W. A. Coppel (Canberra)

Samoĭlenko, A. M. 3204

On periodic solutions of differential equations with non-differentiable right-hand sides. (Russian)

Ukrain. Mat. Ž. **15** (1963), 328–332.

The author considers the question of existence of periodic solutions of $dx/dt = \varepsilon X(t, x)$, where X is continuous and periodic in t of period 2π. His argument uses the averaged equation $d\xi/dt = \varepsilon \bar{X}(\xi)$, where $\bar{X}(\xi) = (2\pi)^{-1} \int_0^{2\pi} X(t, \xi)\, dt$, and it generalizes similar results of Ju. A. Mitropol'skiĭ [Dokl. Akad. Nauk SSSR **128** (1959), 1118–1121; MR **22** #798] and Ju. A. Mitropol'skiĭ and O. B. Lykova [Bul. Inst. Politehn. Iași (N.S.) **6 (10)** (1960), no. 3–4, 7–12; MR **24** #A3344].

G. R. Sell (Cambridge, Mass.)

Kharasakhal, V. Kh. [Harasahal, V. H.] 3205

On quasi-periodic solutions of systems of ordinary differential equations.

Prikl. Mat. Meh. **27** (1963), 672–682 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1019–1034.

The first part of this paper is an elaboration of the results previously announced by the author [Dokl. Akad. Nauk SSSR **146** (1962), 1290–1293; MR **25** #5233]. These results concern the linear equation $dx/dt = P(t)x$, where $x \in R^n$ and $P(t)$ is a quasi-periodic matrix. In the second part, the author considers the non-linear equation

$$(1) \qquad \frac{dx}{dt} = f(t, x) + \alpha F(t, x, \alpha),$$

where $t \in R$, $x \in G$ (G is a domain in R^n) and α is a small parameter. f and F are assumed to be quasi-periodic in t. By using techniques similar to those used for the linear equation, he proves theorems concerning the existence of periodic (quasi-periodic) solutions of (1) for α sufficiently small.

G. R. Sell (Cambridge, Mass.)

Mařík, Jan; Ráb, Miloš 3206

Nichtoszillatorische lineare Differentialgleichungen 2. Ordnung. (Russian summary)

Czechoslovak Math. J. **13 (88)** (1963), 209–225.

The authors investigate solutions of nonoscillatory differential equations (1) $y'' + p(x)y' + q(x)y = 0$ $(p, q \in C_0)$ in connection with the solutions of a Riccati equation (2) $\delta(x)z' + z^2 + P(x)z + Q(x) = 0$. An equation (1) is called nonoscillatory if every nontrivial solution of (1) has only a finite number of zeros in the interval under consideration $J = \langle a, \infty)$, where a is a real number. The theorems stated in this paper give relations between a solution of (1) and a solution of (2), necessary and sufficient conditions that equation (1) is nonoscillatory, and asymptotic properties

of solutions of (1), in particular, for solutions of $y''-A(x)y=0$. A part of the results are generalisations of theorems found by P. Hartman and A. Wintner [Amer. J. Math. **77** (1955), 45–86; errata, ibid. **77** (1955), 404; errata, ibid. **77** (1955), 932; MR **16**, 590].

R. F. Albrecht (Munich)

Ku, Y. H. 3207

On nonlinear oscillations in electromechanical systems. (Russian summary)

Applications of the methods of non-linear vibrations to the problems of physics and technology (Proc. Internat. Sympos. Non-linear Vibrations, Vol. III, 1961), pp. 180–199. Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963.

The paper recapitulates the well-known tensorial equations of rotating electric machinery during transient operation and steady hunting. Both holonomic and non-holonomic reference axes are considered.

G. Kron (Schenectady, N.Y.)

Belohorec, Štefan 3208

Non-linear oscillations of a certain non-linear second-order differential equation. (Slovak. English summary)

Mat.-Fyz. Časopis Sloven. Akad. Vied **12** (1962), 253–262.

The author studies the oscillatory and asymptotic properties of the equation (1) $y''+\sum_{i=1}^{n} f_i(x)y^{N_i}=0$, where $0<N_i<1$, $N_i=p_i/q_i$, p_i, q_i are odd numbers and $f_i(x)$ is continuous and non-negative in $(0,\infty)$. It is proved that a bounded non-oscillatory solution of (1) exists if and only if $\int^{\infty}\sum_{i=1}^{n} xf_i(x)\,dx<\infty$ and a solution $y(x)\sim cx$ $(c>0)$ exists if and only if $\int^{\infty}\sum_{i=1}^{n} x^{N_i}f_i(x)\,dx<\infty$.

Let $y(x)$ be an oscillatory solution of (1) and let $\{x_m\}$, $\{x_m'\}$ be increasing sequences of points, $y(x_m)=0$, $y'(x_m')=0$. Let $f_i'(x)$ be non-positive functions and let an index j exist in such a way that $f_j'(x)<0$ for $x>0$. Then the sequence $\{|y'(x_m)|\}$ is decreasing and the sequence $\{|y(x_m')|\}$ is increasing. Under the additional assumption

$$\int^{\infty}\sum_{i=1}^{n} xf_i(x)\,dx<\infty,$$

every solution of (1) is non-oscillatory. If the functions $f_i(x)$ have continuous derivatives $f_i'(x)\geqq 0$ for $x>0$ and if, for an index j, $f_j'(x)>0$ holds, then the sequence $\{|y'(x_m)|\}$ is increasing, the sequence $\{y(x_m')\}$ decreasing and every solution of (1) is bounded

M. Ráb (Brno)

Osinskiĭ, Z. [Osiński, Z.] 3209

Vibrations of a system with one degree of freedom with non-linear internal friction and relaxation. (Russian. English summary)

Applications of the methods of non-linear vibrations to the problems of physics and technology (Proc. Internat. Sympos. Non-linear Vibrations, Vol. III, 1961), pp. 314–325. Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963.

Author's summary: "In this paper a mechanical system is considered with nonlinear internal friction and relaxation. The motion of this system can be described by a third-order differential equation (for a system with one degree of freedom):

$$(1)\qquad \dddot{\varepsilon}+\kappa\ddot{\varepsilon}+\alpha\dot{\varepsilon}+\omega^2\kappa\varepsilon=F(t).$$

In the free vibrations case the solution is obtained by small parameter method. The author has investigated influence of nonlinear components (describing the ela resistance, internal friction resistance and relaxat resistance) on the motion of the system and in partic on the displacement of the null axis and on the logarith decrement. An approximate solution is obtained in case of forced resonance vibrations. The resonance cu and the problem of the number of stable and unsta periodic solutions have been investigated."

Leighton, Walter 3

On the construction of certain Liapunov functions.

Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 124–125.

A more complete version appears in Contributions Differential Equations **2** (1963), 367–383 [MR **27** #388

J. P. LaSalle (Baltimore, M

Klimušev, A. I. 3

Uniform asymptotic stability of systems of lin differential equations with a small parameter. (Russi

Sibirsk. Mat. Ž. **5** (1964), 94–101.

Given a differential system (1) $\dot{x}=A(t)x+B(t)y$, μ $C(t)x+D(t)y$ $(\cdot=d/dt)$, where μ is a small positive para eter and $A(t)$, $B(t)$, $C(t)$, $D(t)$ are matrices which are c tinuous and bounded in $[t_0,\infty)$, together with their fi order derivatives. Assume $\det D(t)\neq 0$. Then the degen ate system of (1) can be written in the form (2) $\dot{x}=R(t$ $y=S(t)x$. Let $(x(t,\mu),y(t,\mu))$ and $(x(t),y(t))$ be, respective the solutions of (1) and (2) such that $x(t_0,\mu)=$ $y(t_0,\mu)=y_0$ and $x(t_0)=x_0$, $y(t_0)=S(t_0)x_0$. In the pres paper, the following theorem is proved. "If $\dot{x}=R(t)x$ a $\dot{y}=D(t)y$ are both uniformly asymptotically stable, t (1) is uniformly asymptotically stable for sufficiently sm μ. Further, for any $Q>0$ and $\varepsilon>0$, there exists $\mu_0>0$ su that $\|x(t,\mu)-x(t)\|$, $\|y(t,\mu)-y(t)\|<\varepsilon$ for any $\mu<$ whenever $\|y(t_0,\mu)-y(t_0)\|<Q$ and $t>t_1(Q,\varepsilon)$." Let $v(t,$ and $w(t,y)$ be, respectively, the Liapunov functions of t quadratic forms for the equations $\dot{x}=R(t)x$ and $\dot{y}=D(t$ Then, for sufficiently small $\mu>0$, $u(t,\xi,\eta)=v(t,\xi)+w(t,$ becomes a Liapunov function for the linear different equation with respect to $\xi=x(t,\mu)-x(t)$ and $\eta=y(t,\mu$ $S(t)x(t,\mu)$. This fact is proved first, and then the theor is proved in the usual way.

M. Urabe (Fukuol

Popov, V. M. 3

A critical case of absolute stability.

Avtomat. i Telemeh. **23** (1962), 3–24 (*Russian. Engl summary*); *translated as Automat. Remote Control* (1962), 1–21.

Consider a system of ordinary differential equations of t form

$$(1)\qquad dz/dt=Az+a\varphi(\sigma),\quad \sigma=p^*z,$$

where z, a are real column n-vectors, A is a real n matrix, p^* is a row n-vector. The function φ is defin and continuous for all σ and satisfies the conditions

$$(2)\qquad \varphi(0)=0,\quad \varphi(\sigma)\sigma>0 \text{ for } \sigma\neq 0.$$

Problem: Find sufficient conditions on A, a and p whi guarantee that (1) is absolutely stable, that is, the triv

solution of (1) is asymptotically stable in the large for any function φ satisfying (2).

In a previous paper [Avtomat. i Telemeh. **22** (1961), 961–979; MR **24** #A3394] the author considered the case when A has all eigenvalues with negative real part. The present paper is concerned with a critical case, namely, it is assumed that (3) A has two zero eigenvalues with non-simple elementary divisors while the remaining eigenvalues have negative real part.

Let $G(s) = -p^*(sE-A)^{-1}a$, where E is the unit matrix. $G(s)$ is called the transfer function of the linear part of the system. In terms of this function the author gives three theorems each having as conclusion the asymptotic stability in the large of the trivial solution of (1). Thus below we state only the assumptions of each of them, respectively.

(I) We assume that (2), (3) and the inequalities (4) $\lim_{\omega\to\infty} \operatorname{Re} \omega^2 G(i\omega) < 0$ and $\operatorname{Re} i\omega G(i\omega) \geqq 0$ hold for all real positive ω ($i=\sqrt{-1}$) and $\lim \int_0^\sigma \varphi(\sigma)\,d\sigma = +\infty$ as $\sigma\to+\infty$ as well as $\sigma\to-\infty$. (II) We assume (2), (3), (4), the inequality $\lim_{\omega\to\infty} \operatorname{Re} i\omega G(i\omega) > 0$, and the following condition: $\lim\sup(|\varphi(\sigma)| + \int_0^\sigma \varphi(\sigma)\,d\sigma) = +\infty$ as $\sigma\to+\infty$ and $\sigma\to-\infty$. (III) Here it is assumed that the trivial solution of (1) is stable in the large, conditions (2) and (3) hold, the first inequality of (4) and the second with "$\geqq$" being replaced by "$>$" are satisfied.

In an appendix the author shows that the second inequality of (4) is necessary for the existence of a Lyapunov function for (1), which is the sum of a quadratic form plus an integral of the nonlinearity. *C. Olech* (Kraków)

Brunovský, Pavol 3213

On Emden-Fowler's equation in the case $n<1$. (Slovak. Russian and English summaries)

Mat.-Fyz. Časopis Sloven. Akad. Vied **12** (1962), 60–80.

The author studies the asymptotic properties of solutions of the Emden-Fowler equations (1) $d^2u/dt^2 - t^\sigma u^n = 0$, (2) $d^2v/dt^2 + t^\sigma v^n = 0$; here σ and n are constants, $0<n<1$, $n=p/q$, where p, q are odd numbers. The method of investigation and results of this paper are similar to those which have been published in the monograph of R. Bellman [*Stability theory of differential equations*, McGraw-Hill, New York, 1953; MR **15**, 794] in the case $n>1$.

Put

$$\omega = (\sigma+2)/(1-n),$$

$$\gamma = [(\sigma+2)(\sigma+n+1)/(1-n)^2]^{1/(n-1)}$$

and let c denote a positive constant. The following asymptotic formulas are valid for every positive solution [v] of (1) [(2)]: If $\sigma+2<0$, then $u\sim\gamma t^\omega$ or $u\sim ct$ or $u-c\sim c^n t^{\sigma+2}/(\sigma+1)(\sigma+2)$. If $\sigma+n+1>0$, then $u\sim\gamma t^\omega$. If $\sigma+n+1<0\leqq\sigma+2$, then $u\sim ct$. If $\sigma+n+1=0$, then $\log u\sim\log t$. If $\sigma+2<0$, then $v\sim ct$ or $v-c\sim -c^n t^{\sigma+2}\times(\sigma+1)^{-1}(\sigma+2)^{-1}$. If $\sigma+2=0$, then $v\sim ct$. If $\sigma+n+1<0<\sigma+2$, then $v\sim-\gamma t^\omega$ or $v\sim ct$. If $\sigma+n+1\geqq0$, then the equation (2) has no positive solution. *M. Ráb* (Brno)

[illegible]e, E. R. 3214

On the related-equation method of asymptotic approximation (W.K.B. or A-A method). I. A proposed new existence theorem.

Quart. J. Mech. Appl. Math. **17** (1964), 105–124.

This is a review of the theory of asymptotic approximation, for $\lambda\sim\infty$, to solutions of linear differential equations of the second order involving a parameter λ. The focus of the paper is the difficulty of obtaining a simple uniform approximation to a solution which is large in certain ranges of the variable and small in others; and a method, involving the introduction of adjustable constants into the approximation, is suggested to overcome this difficulty. The author states that the method can be justified for approximations in terms of exponentials or Airy functions over finite ranges, and suggests physical grounds for its more general validity [cf. C. A. Rogers, Proc. London Math. Soc. (3) **8** (1958), 609–620 (610!); MR **21** #847]. *T. M. Cherry* (Melbourne)

Minc, R. M. 3215

A study of certain basic types of complex equilibrium states in three-dimensional space. (Russian)

Mat. Sb. (N.S.) **63 (105)** (1964), 169–214.

This paper contains the proofs of results announced earlier concerning the nature of the trajectories of the system, $\dot x = Ax + f(x)$, x a 3-vector, near $x=0$ in the case that one root of the constant matrix A is 0, while the other two roots have non-zero real part (see the author [Dokl. Akad. Nauk SSSR **111** (1956), 535–537; MR **18**, 897; ibid. **124** (1959), 1215–1218; MR **21** #2101]).

C. S. Coleman (Baltimore, Md.)

Vu, Tuan 3216

On the operational theory of I. Z. Štokalo for solving linear differential equations. (Russian)

Ukrain. Mat. Ž. **15** (1963), 303–305.

Le travail contient quelques résultats (sans démonstration) concernant les systèmes différentiels de la forme

$$(*)\qquad dx/dt - [A+\varepsilon f(t)]x = g(t)\exp(pt),$$

où A est une matrice constante, $f(t)$ est une matrice continue et bornée sur l'axe réel, $g(t)$ un vecteur continu et borné et ε un petit paramètre. Si le nombre p est différent de toutes les parties réelles des racines caractéristiques de la matrice A, alors il existe une seule solution de (*), $x=\xi(t,p,\varepsilon)\exp(pt)$, telle que $\xi(t,p,\varepsilon)$ soit borné sur l'axe réel. D'autres résultats, semblables au précédent, sont donnés. *C. Corduneanu* (Iaşi)

Barbashin, E. A. [Barbašin, E. A.]; Tabueva, V. A. 3217

Theorem on the stability of the solution of a third order differential equation with a discontinuous characteristic.

Prikl. Mat. Meh. **27** (1963), 664–671 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1005–1018.

For the differential equation

$$(*)\qquad \dddot x + F(x,\dot x,\ddot x,t) + Kx\operatorname{sgn}[x(\ddot x - \varphi(x,\dot x))] = 0,$$

let K be a positive constant and, in the region

$$\{(t,x,\dot x,\ddot x)|\ |x|<\infty, |\dot x|<\infty, |\ddot x|<\infty, 0\leqq t<\infty\},$$

F is continuously differentiable in each of its arguments and is bounded as a function of t, whereas φ is continuous and has piecewise continuous first and second derivatives with respect to x, $\dot x$. Furthermore, let F, φ satisfy: (a) $|p^2F(x,y/p,z/p^2,tp)| < A(x,y,z)$, $|p\varphi(x,y/p)| < B(x,y)$; (b) $\varphi(0,0)=0$, $x\varphi(x,0)<0$ for $x\neq0$, $y[\varphi(x,y)-\varphi(x,0)]<0$

for $y \neq 0$, $\int_{\pm\infty}^{0} \varphi(x, 0)\,dx = \infty$, where p is a sufficiently small parameter and A, B are continuous. Theorem: Given $\varepsilon > 0$ and a bounded region, G, of the phase space of (*), there is a $K_0 > 0$ such that for every $K \geq K_0$, any solution of (*) with initial values in G satisfies $|x(t)| < \varepsilon$, $|\dot{x}(t)| < \varepsilon$, $|\ddot{x}(t)| < \varepsilon$ for all sufficiently large values of t.
T. F. Bridgland, Jr. (Columbia, S.C.)

Mozart-Boisvert, Louis [Boisvert, Louis-Mozart] 3218
Sur l'existence de solution périodique dans une équation différentielle non linéaire autonome d'ordre 3.
C. R. Acad. Sci. Paris **257** (1963), 2237–2239.

The equation under consideration is

$$a\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx + qx^3 = 0,$$

with a nonlinear term of Duffing type, which arises, e.g., in the study of self-adjusted systems.

For k positive and q negative a $\gamma > 0$ and the corresponding periodic solution are obtainable by classical methods, and several examples are described graphically. The author then exhibits graphically periodic solutions obtained by means of an analog computer in several examples where k is negative and q is positive. These solutions are of two types involving even and odd harmonics, respectively. A brief discussion is given of the nature of such periodic solutions, the conditions under which they may arise, and their physical significance.
D. V. V. Wend (Salt Lake City, Utah)

Romiti, Ario 3219
Un'estensione dei metodi d'indagine dei sistemi lineari per lo studio del comportamento asintotico dei servomeccanismi non lineari.
Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 156–160.

Consider the differential equation

$$(1) \qquad y^{(n)} + \sum_{i=1}^{n} f_i y^{(i-1)} = 0,$$

where $f_i = a_i - b_i\varphi(x)$, a_i and b_i are constants,

$$x = \sum_{i=1}^{n} b_i y^{(i-1)},$$

and $\varphi(x)$ is a non-linear function satisfying the condition (2) $0 < \varphi(x) < c$. For the absolute stability of system (1) (i.e., asymptotic stability for all $\varphi(x)$ satisfying (2)), it is obviously necessary that all linearized equations (i.e., equation (1) for all constant $\varphi(x)$ in the range (2)) be stable, but this condition is in general not sufficient. In the present paper special cases are considered in which the behaviour of the non-linear equation is the same as the linearized. Earlier results in this direction were given by the author in a previous paper [Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. **97** (1962/63), 431–456; MR **27** #423].
E. O. Roxin (Baltimore, Md.)

Proskuriakov, A. P. [Proskurjakov, A. P.] 3220
Stability of periodic solutions of quasilinear autonomous systems with one degree of freedom.
Prikl. Mat. Meh. **27** (1963), 559–564 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1963), 837–846.

The author is concerned with equations $x'' + p^2x = \mu f(x, x', \mu)$, where f is analytic in its arguments and μ is a small parameter. In a previous paper [Prikl. Mat. Meh. **25** (1961), 954–960; MR **24** #B1710] he obtained expansions for the periodic solutions in terms of fractional powers of μ when the equation $\int_0^{2\pi/p} f(x_0, x_0', 0) \sin pt\,dt = 0$ has double and triple roots. He now examines the stability of these periodic solutions by a detailed analysis of the coefficients in their expansions, and lists sufficient conditions for the case of double and triple roots as well as various subcases. *H. A. Antosiewicz* (Los Angeles, Calif.)

Skovronskiĭ, Ja. M. [Skowroński, J. M.]; Zemba, S. [Ziemba, S.] 3221
The domain of boundedness of motion of strongly non-linear non-autonomous mechanical systems with partially negative damping. (Russian. English summary)
Qualitative methods in the theory of non-linear vibrations (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. II*, 1961), *pp.* 356–364. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

Es werden 3 Sätze über die Beschränktheit der Lösungen eines Systems von nichtlinearen Differentialgleichungen $\ddot{q} + F(q, \dot{q}, t) = 0$ mit q und F als n-Vektoren bewiesen. Der Operator F soll dabei so beschaffen sein, daß auch die Möglichkeit negativer Dämpfungen zugelassen wird. In Ergänzung zu früheren Arbeiten derselben Verfasser wird jetzt gezeigt, daß unter gewissen Voraussetzungen die Lösungen des Gleichungssystems gleichmäßig beschränkt im Großen sind. Im Raume der Variablen q und $\dot{q}$ existiert ein endlicher geschlossener Bereich, in den nach endlicher Zeit alle Phasenbahnen einlaufen, wenn die Anfangswerte bestimmten Bedingungen genügen.
K. Magnus (Stuttgart)

Morgunov, B. I. 3222
Stationary resonance regimes of certain rotational motions. (Russian)
Dokl. Akad. Nauk SSSR **155** (1964), 277–280.

The author studies perturbations of certain solutions of equations of the form $y'' + Q(y) = 0$. The solutions are of rotating type, e.g., large energy motions of a simple pendulum. The system is perturbed by slowly varying parameters and is also forced. The system studied is

$$(1) \qquad \frac{d}{dt}[m(x)\dot{y}] + Q(x, y) = \varepsilon f(x, y, \dot{y}, \theta), \qquad \dot{x} = \varepsilon X(x, y, \dot{y}, \theta), \qquad \dot{\theta} = \nu(x) + \varepsilon\Theta(x, y, \dot{y}, \theta),$$

where x is a slowly varying vector parameter, y is a scalar, and θ is the phase of the external force. All functions in (1) have period 2π in y and in θ, and Q has mean value zero in y.

The system (1) is studied by a change of variables in which the new variables are the energy $E = \frac{1}{2}\dot{y}^2 + \int Q\,dy$ and the phase ψ of y. The principal term in $\dot{\psi}$ is $\omega(E, x)$, the frequency of the solution of the unperturbed system. The various resonant states studied correspond to rational ratios of ν and ω.

The technique used is given in a paper by Volosov and the author [same Dokl. **153** (1963), 559–561]. This is a variant of the method of averaging which approximates the resonant states and indicates their stability. In

particular, asymptotic behavior for large energy is studied [cf. N. N. Moiseev, Ž. Vyčisl. Mat. i Mat. Fiz. 3 (1963), 145–158; MR 27 #5978].

The results are illustrated with two examples, one the simple pendulum and the other a satellite orbit problem.

W. S. Loud (Minneapolis, Minn.)

Wei, J.; Norman, E. 3223

On global representations of the solutions of linear differential equations as a product of exponentials.

Proc. Amer. Math. Soc. 15 (1964), 327–334.

A linear differential equation $dU/dt = A(t)U$ is considered where $U(t)$ and $A(t)$ are linear operators not precisely defined. Let $\mathfrak{g}$ be a Lie algebra of linear operators with a basis $X_1, \cdots, X_n$, $n < \infty$. Assume that $A(t)$ is $\mathfrak{g}$-valued and that $U(0)$ is the identity operator. The authors continue Magnus's work [Comm. Pure Appl. Math. 7 (1954), 649–673; MR 16, 790]. It is noted that

$$(1) \qquad U(t) = (\exp g_1(t)X_1)\cdots(\exp g_n(t)X_n)$$

holds for $|t|$ sufficiently small. This paper shows (Theorem 1) how the g's may be obtained through differential equations. Theorem 2 states that, through a suitable choice of the basis, the g's may be found by quadrature and that (1) holds for all values of t. This result is shown to be valid also for the case of $\mathfrak{g}$ being the Lie algebra of the 2×2 real matrices (Theorem 3). Finally, a necessary condition for the global validity of (1) is discussed. For related work, see Wichmann [J. Mathematical Phys. 2 (1961), 876–880; MR 25 #292]. Theorem 2 overlaps with the reviewer's work under a somewhat different setup [Math. Ann. 146 (1962), 263–278, § 4; MR 25 #4038].

K.-T. Chen (New Brunswick, N.J.)

Chen, Kuo-Tsai 3224

Equivalence and decomposition of vector fields about an elementary critical point.

Amer. J. Math. 85 (1963), 693–722.

If $X = \sum a^i(x)\partial/\partial x^i$ is a vector field, $x=0$ is called a critical point if $a^i(0)=0$ for $i=1,\cdots,n$; it is called an elementary critical point if the eigenvalues of $(\partial a^i(0)/\partial x^j)$ have non-vanishing real parts. The first result is the following: (I) Let X be a C^∞ vector field with an elementary critical point $x=0$. Then, in a vicinity of $x=0$, X has a decomposition $X=S+N$ where S, N are C^∞ vector fields such that $[S,N]=0$; in a suitable y-coordinate system $S=\sum\sum c_j^i y^j\partial/\partial y^i$, where (c_j^i) is similar to a diagonal matrix; finally, the linear part of N can be represented by a nilpotent matrix. The second result is (II) Let $X=\sum a^i(x)\partial/\partial x^i$ and $Y=\sum b^i(y)\partial/\partial y^i$ be C^∞ vector fields with $x=0$, $y=0$ as elementary critical points. Let $\hat a^i(x)$, $\hat b^i(y)$ be the formal Taylor expansions of $a^i(x)$, $b^i(y)$ at $x=0$, $y=0$. Then there exists a C^∞ transformation $x\to y$ of a neighborhood of $x=0$ onto a neighborhood of $y=0$ which carries X into Y if and only if there exists a formal (power series) transformation which carries the formal vector field $\sum \hat a^i(x)\partial/\partial x^i$ into $\sum \hat b^i(y)\partial/\partial y^i$. This latter result generalizes a theorem of Sternberg [same J. 80 (1958), 623–631; MR 20 #3336] dealing with the case when $b^i(y)$ is linear in y; the proof is similar to, but simpler than, that of Sternberg.

P. Hartman (Baltimore, Md.)

Caubet, J. P. 3225

Sur une étude qualitative de l'équation de Duffing.

Deux. Congr. Assoc. Française Calcul et Traitement Information (Paris, 1961), pp. 159–169. Gauthier-Villars, Paris, 1962.

Myškis, A. D. 3226

A differential-functional inequality. (Russian)

Uspehi Mat. Nauk 15 (1960), *no.* 4 (94), 157–161.

The author derives estimates for the solutions of the differential inequality

$$y'(x) \leq a(x)y(x) + b(x)y(x-\theta(x)),$$

where $\theta \geqq 0$, $b \leqq 0$, $a+b \geqq 0$. He proves that solutions are either non-positive and non-increasing for sufficiently large x or are positive on each interval where $y(x)$ takes its least value at the right end-point. Solutions of the first type can be estimated from above, those of the second type from below. This decomposition is applied to the solutions of $y'(x) = -\int_0^\infty y(x-s)\,dr(x,s)$ and earlier results of the author are sharpened [*Lineare Differentialgleichungen mit nacheilendem Argument*, p. 154, Deutscher Verlag der Wissenschaften, Berlin, 1955; MR 17, 497].

W. T. Kyner (Los Angeles, Calif.)

Myškis, A. D. 3227

Asymptotic estimate of solutions of systems of linear homogeneous differential equations with lagging argument. (Russian)

Uspehi Mat. Nauk 15 (1960), *no.* 4 (94), 163–167.

Two methods for estimating the solutions of the system of equations,

$$y_i'(x) = \sum_{j=1}^n \int_0^\infty y_j(x-s)\,dr_{ij}(x,s),$$
$$1 \leqq i \leqq n,\ A \leqq x < \infty,$$

are given. The first, an improvement of one given in the author's book [p. 29 of the book cited in #3226 above], provides an upper bound on the norm $\rho(x) = [\sum y_i(x)^2]^{1/2}$. The second method, an application of the result reviewed above [#3226], decomposes the set of solutions into the "fundamental class" which can be estimated in norm from below and into the class of "rapidly damped solutions" which can be estimated in norm from above. The sharper results of the second method are only valid for systems with sufficiently small lag.

W. T. Kyner (Los Angeles, Calif.)

Velikiĭ, A. P. 3228

On a representation for solutions of differential-difference equations with constant coefficients. (Russian)

Sibirsk. Mat. Ž. 5 (1964), 34–38.

The author formulates a procedure for solving a linear differential-difference equation with constant coefficients and integer differences, provided the initial function is a constant. The method, which considers successively the intervals $[0,1], [1,2], \cdots$, avoids solving a transcendental equation. As an example, the author solves, in closed form, the equation $y'(x)+Ay(x)+By(x-1)=0$ on $[0,\infty)$ with $y(x)=1$ on $[-1,0]$.

R. D. Driver (Albuquerque, N.M.)

Hale, J. K. 3229
Asymptotic behaviour of the solutions of differential-difference equations. (Russian summary)
Qualitative methods in the theory of non-linear vibrations (Proc. Internat. Sympos. Non-linear Vibrations, Vol. II, 1961), pp. 409–426. Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963.

In this expository article the author discusses asymptotic behaviour (boundedness, stability, etc.) of perturbed linear and nonlinear differential-difference equations, as well as such equations with "slowly varying parameters", with the aid of Liapunov functionals; specifically, a converse theorem of Yoshizawa (adapted for such equations) plays a central role. The author has since carried out a similar analysis for functional-differential equations which include these differential-difference equations as a special case [Contributions to Differential Equations **1** (1963), 401–410; MR **26** #6510; ibid. **1** (1963), 411–423; MR **26** #6511].

J. A. Nohel (Madison, Wis.)

Halanaĭ, A. [Halanay, A.] 3230
Some problems in the qualitative theory of systems with time lag. (Russian. English summary)
Qualitative methods in the theory of non-linear vibrations (Proc. Internat. Sympos. Non-linear Vibrations, Vol. II, 1961), pp. 394–408. Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963.

The present paper is a summary with some indications of the proofs of the results obtained by the author in eleven papers during the years 1957–1961 on differential equations with retarded arguments. The main theme is the existence of periodic and almost periodic solutions for equations with a small parameter. Since this paper was written, the author has written a book [*Qualitative theory of differential equations* (Romanian), Editura Acad. R. P. Romîne, Bucharest, 1963] in which the last 140 pages give a more comprehensive discussion of this subject.

J. K. Hale (Baltimore, Md.)

Shimanov, S. N. [Šimanov, S. N.] 3231
On the theory of linear differential equations with periodic coefficients and time lag.
Prikl. Mat. Meh. **27** (1963), 450–458 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1963), 674–687.

In the present paper, the author investigates the extent to which the Floquet theory is valid for linear differential equations with time lags and periodic coefficients.

The author is apparently unaware of a previous paper of A. Stokes [Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 1330–1334; MR **25** #5255] in which slightly more general results along this line have been obtained. However, the present paper sheds some new light on equations of this type because of the method of proof.

We now attempt to indicate briefly the differences in the two papers. For simplicity only, consider the equation (1) $\dot{x}(t)=A(t)x(t)+B(t)x(t-\tau)$, $x\in E^n$, $\tau>0$, A, B continuous and periodic in t of period ω. For any t_0 and any continuous function φ defined on $[t_0-\tau, t_0]$, there exists a solution $x(t, t_0, \varphi)$ of (1) with initial function φ at t_0 for all $t\geqq t_0$. Let C be the space of continuous functions taking $[-\tau, 0]$ into E^n with the uniform topology. Let $T(t, t_0)$ be the mapping of C into C which takes the function φ into the restriction of x to the interval $[t-\tau, t]$. The monodromy operator for (1) is defined to be $T(t_0+\omega, t_0)$. One shows that the spectrum of $T(t_0+\omega, t_0)$ does not depend upon t_0 and the elements of the point spectrum are called the characteristic multipliers of (1). To each characteristic multiplier ρ of (1) and any t_0, there exists a matrix $\Phi(t_0)=(\varphi_1(t_0), \cdots, \varphi_p(t_0))$ $\varphi_j(t_0)\in C$, a basis in C for the generalized eigenspace of ρ, such that $T(t, t_0)\Phi(t_0)=P(t, t_0)e^{B(t-t_0)}$, where P is periodic in t of period ω and the eigenvalues of the $p\times p$ matrix B are all equal to ρ. This is the Floquet representation corresponding to the generalized eigenspaces of the monodromy operator. In general, there is no Floquet representation for all solutions of (1), but, as both Stokes and the author prove, the Floquet representation is valid asymptotically in t. Stokes proved this by using the well-known result of the existence of a subspace of C which is invariant under $T(t_0+\omega, t_0)$ and complementary to a given generalized eigenspace of $T(t_0+\omega, t_0)$. The author shows how to actually construct this complementary subspace by means of the equation "adjoint" to (1). This allows the introduction of a coordinate system in C which reduces (1) to an ordinary differential equation which describes the solutions of (1) corresponding to a generalized eigenspace of $T(t_0+\omega, t_0)$ and a differential equation in a Banach space on the complementary subspace. This idea should lead to a better understanding of the general theory of perturbations of equations of this type as has already been the case for equations with constant coefficients (see the author [Prikl. Mat. Meh. **24** (1960), 447–457; MR **22** #9697] and the reviewer [Contributions to Differential Equations **2** (1963), 291–317; MR **27** #2700]).

J. K. Hale (Baltimore, Md.)

Blehman, I. I.; Džanelidze, G. Ju. 3232
Non-linear problems in the theory of vibro-transportation and vibro-separation. (Russian. English summary)
Applications of the methods of non-linear vibrations to the problems of physics and technology (Proc. Internat. Sympos. Non-linear Vibrations, Vol. III, 1961), pp. 41–71. Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963.

PARTIAL DIFFERENTIAL EQUATIONS

See also 3328, 3342, 3344, 3346, 3354, 3434, 3460, 3478, 3543, 3548, 3549, 3650, 3651, 3654, 3899.

Haimovici, Adolf 3233
Sur un système qui généralise le système de Cauchy-Kowalewsky. (Romanian and Russian summaries)
An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I (N.S.) **8** (1962), 33–56.

The author considers the system (1)

$$\frac{\partial u_i}{\partial x}=f_i\left(x, y, z, \cdots, u_p, v_q, \frac{\partial u_p}{\partial y}, \frac{\partial v_q}{\partial x}, \frac{\partial u_p}{\partial z}, \frac{\partial v_q}{\partial z}, \cdots\right),$$

$$\frac{\partial v_j}{\partial y}=g_j\left(x, y, z, \cdots, u_p, v_q, \frac{\partial u_p}{\partial y}, \frac{\partial v_q}{\partial x}, \frac{\partial u_p}{\partial z}, \frac{\partial v_q}{\partial z}, \cdots\right),$$

$$u_i(x_0, y, z, \cdots)=\alpha_i(y, z, \cdots),$$

$$v_j(x, y_0, z, \cdots)=\beta_j(x, z, \cdots),$$

$$(i, p=1, 2, \cdots, h;\ j, q=1, 2, \cdots, k),$$

of $h+k$ partial differential equations of first order for $h+k$ unknown functions u_i, v_j which depend on the variables $x, y, z, \cdots$. Let the f_i and g_j be bounded analytic functions of all their arguments in a neighborhood of the point $P_0(x_0, y_0, \cdots, u_{i0}, \cdots, v_{j0}, \cdots, p_{20}{}^i, \cdots, q_{10}{}^j, \cdots)$, let the functions $\alpha_i(y, z, \cdots)$, $\beta_j(x, z, \cdots)$ be analytic with respect to all their arguments in neighborhoods V_1 and V_2 of the points $p_0'(y_0, z_0, \cdots)$ and $p_0''(x_0, z_0, \cdots)$, and let the absolute values of

$$\left[\frac{\partial f_i}{\partial(\partial u_r/\partial y)}\right]_{P_0}, \left[\frac{\partial f_i}{\partial(\partial v_s/\partial x)}\right]_{P_0}, \left[\frac{\partial g_j}{\partial(\partial u_r/\partial y)}\right]_{P_0}, \left[\frac{\partial g_j}{\partial(\partial v_s/\partial x)}\right]_{P_0}$$

be less than $1/(h+k)$. Then the system (1) has an analytic solution in a neighborhood V_0 of $p_0(x_0, y_0, z_0, \cdots)$ which is unique. This is proved by showing that the coefficients of the power series expansions of the solution can be uniquely calculated, and that by comparison with majorants these expansions are convergent. The idea is the same as in the classical proof of Cauchy and Kowalewsky; the proof, however, is more complicated and is based on two theorems on matrices. In some particular cases the last of the above hypotheses can be weakened. The theory of system (1) also leads to a generalisation of the notion of characteristics. *R. F. Albrecht* (Munich)

Rosenbloom, P. C. 3234

Singular partial differential equations.

Fluid Dynamics and Applied Mathematics (*Proc. Sympos., Univ. of Maryland*, 1961), *pp.* 67–77. *Gordon and Breach, New York*, 1962.

A theory similar to the theory of regular singular points for ordinary differential equations is developed for partial differential equations of the form $tu_t = t^\alpha f(t, x, t^{-\alpha}u, t^{-\alpha}u_x)$, where x and $u(t, x)$ are elements of complex Banach spaces X and U. Under certain conditions on f and on $v_0(x)$, it is shown that there exists a unique solution of the form $u = t^\alpha v$, where v is analytic in a certain domain and $v(0, x) = v_0(x)$. A number of corollaries are presented, and application is made to the Euler-Poisson-Darboux partial differential equation. Some suggestions are made for the treatment of irregular singular points and for research in other directions. *R. C. Gilbert* (Fullerton, Calif.)

Stel'mašuk, N. T. 3235

On certain linear partial differential systems. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 166–173.

The principal result of the paper is the following theorem: The system

$$f_x = A_1\varphi_x + A_2\varphi_y + T_1 f + \theta_1\varphi + E_1, \quad f_y = A_3\varphi_x + A_4\varphi_y + T_2 f + \theta_2\varphi + E_2, \qquad \sum_1^4 |A_k| \neq 0,$$

in the unknown complex-valued functions $f(x, y)$, $\varphi(x, y)$ of class C^2 in a domain D of the real plane, where the A_k, T_j, θ_j, E_j ($k = 1, \cdots, 4$; $j = 1, 2$) are known complex-valued functions of class C^1 in D, is equivalent to a system of the form

$$\frac{\partial f}{\partial p} - \frac{\partial \varphi}{\partial q} = a_1 f + b_1\varphi + \Phi_1,$$

$$\frac{\partial f}{\partial q} = a_2 f + b_2\varphi + \Phi_2,$$

where the a_k, b_k, Φ_k, $[p, q]$ are known functions of class C^1 $[C^2]$ in D, under the following necessary and sufficient conditions, $A_1 = -A_4$, $A_1A_4 - A_2A_3 = 0$.

The study of systems of this form is motivated by the partial differential equations constituting the criterion for Fyodorov monogeneity of the function $F(x, y) = f(x, y) + \varepsilon\varphi(x, y)$ with respect to the function $P(x, y) = p(x, y) + \varepsilon q(x, y)$, where $(1, \varepsilon)$ is a basis for the Study-Clifford numbers ($\varepsilon^2 = 0$).

The special case of the system of the theorem where $a_1 = -(aq + bp)$, $a_2 = ap$, $b_1 = -ap$, $b_2 = \Phi_1 = \Phi_2 = 0$, a and b constants, is explicitly solved in terms of exponential functions. *R. F. Rinehart* (Washington, D.C.)

Haimovici, Mendel 3236

Sur l'intégration des systèmes de Pfaff complets.

Trans. Amer. Math. Soc. **111** (1964), 423–439.

On attache le nom de Mayer à une méthode bien connue d'intégration (1872) des systèmes de Pfaff complètement intégrables. Des démonstrations rigoureuses d'existence et d'unicité ont été données en particulier par W. Nikliborc (1929) et C. Carathéodory (1935), localement, et par T. Y. Thomas (1934), globalement. On y suppose que les fonctions $a_{\alpha i}$ des équations données:

$$dz_\alpha - a_{\alpha i}\, dx_i = 0 \quad (\text{où } \alpha = 1, 2, \cdots, s;\ i = 1, 2, \cdots, n)$$

sont bornées et continues ainsi que leurs dérivées premières, et satisfont à un système de relations bien connues.

Or récemment (1960), A. Haimovici a repris le problème en ne supposant plus la dérivabilité des coefficients, l'existence des solutions restant assurée, mais non plus l'unicité en général. Les conditions d'intégrabilité sont nécessaires et suffisantes, mais semblent d'un maniement difficile, car elles supposent des connaissances étendues sur les courbes intégrales du système.

Dans le présent travail, l'auteur fait des hypothèses moins générales, car elles portent seulement sur les coefficients et non sur les courbes intégrales. Il ne suppose pas l'existence des dérivées des a, ni même l'existence de dérivées généralisées. L'adjonction à l'hypothèse de l'uniforme continuité des a, dans un domaine D, d'une certaine hypothèse (H), où interviennent les expressions

$$\left|\frac{a_{\alpha l}(P_k) - a_{\alpha l}(P_0)}{h} - \frac{a_{\alpha k}(P_l) - a_{\alpha k}(P_0)}{h}\right|$$

permet de démontrer l'existence de quelque solution (pp. 424 à 437). Au cas où est réalisée une certaine hypothèse sur un système d'équations différentielles ordinaires se déduisant facilement du système donné, on peut affirmer de plus l'unicité de la solution, et, dans ce cas, on peut la prolonger comme on le fait pour les équations différentielles ordinaires. *M. Janet* (Paris)

Vološina, M. S. 3237

On the generalized Cauchy formula for a first-order elliptic system. (Russian)

Dokl. Akad. Nauk SSSR **153** (1963), 1231–1233.

Die Verfasserin untersucht ein verallgemeinertes Cauchy-Riemannsches Differentialgleichungssystem, das in der komplexen Schreibweise

$$(1) \quad \frac{\partial f}{\partial \bar z} = Af + B\bar f, \qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \qquad z \in G,$$

vorliegt. Die Koeffizienten A und B werden als analytische Funktionen der beiden komplexen Veränderlichen $z = x+iy$ und $\zeta = \bar{z}$ angenommen.

Mit Hilfe der Funktion $F = f \exp\{-\int A(z, \bar{z})\, d\bar{z}\}$ wird die Differentialgleichung (1) in die folgende überführt:

$$(2) \quad \frac{\partial F}{\partial z} = \lambda \bar{F}, \qquad \lambda = B \exp\left\{-2i \operatorname{Im} \int A(z, \bar{z})\, d\bar{z}\right\}.$$

Die Differentialgleichung (2) wird dann analog zu dem von I. N. Vekua in seinem Buch, *Verallgemeinerte analytische Funktionen* (englische und deutsche Übersetzungen sind bereits erschienen [Akademie-Verlag, Berlin, 1963; MR **28** #1312; *Generalized analytic functions*, Pergamon, London, 1962; MR **27** #321]), dargestellten Verfahren untersucht. Dabei wird gezeigt, daß die Kerne der verallgemeinerten Cauchyschen Integralformel im Falle der Gleichung (2) die folgende Gestalt besitzen:

$$(3) \quad \begin{aligned} \Omega_1(z, \zeta) &= \pi|\lambda| e^{-i \arg(z-\zeta)} H_1^{(1)}(2i|\lambda|r), \\ \Omega_2(z, \zeta) &= -\pi i \lambda H_0^{(1)}(2i|\lambda|r), \qquad r = |z-\zeta|. \end{aligned}$$

Hierbei sind die $H_i^{(k)}$ Hankelsche Funktionen. Weiter werden Formeln für das asymptotische Verhalten der Kerne (3) im Unendlichen angegeben.

Ferner wird gezeigt, daß sich jede Lösung der Gleichung (2), die im Unendlichen der Bedingung

$$(4) \quad F(z) = e^{2|\lambda|r} r^{-1/2} o(1)$$

genügt, im unbeschränkten Gebiet G, das auch mehrfach zusammenhängend sein kann, durch die verallgemeinerte Cauchysche Integralformel darstellen läßt [I. N. Vekua, op. cit.]. Darüber hinaus genügt jede dieser Lösungen noch den Bedingungen:

$$F(z) = r^{-1/2} e^{\cdot 2|\lambda|r} O(1), \qquad r = |z|;$$

$$\frac{\partial F}{\partial r} + 2|\lambda| F = e^{-2|\lambda|r} r^{-1/2} o(1).$$

W. Schmidt (Simferopol)

Lanckau, Eberhard 3238

Über eine lineare elliptische Differentialgleichung 2. Ordnung mit einem singulären Koeffizienten.

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe **12** (1963), 51–60.

This paper contains a contribution to Bergman's theory of integral operators for partial differential equations with singular lines [see also S. Bergman and R. Bojanić, Arch. Rational Mech. Anal. **10** (1962), 323–340; MR **26** #4044]. The differential equation in question is

$$(1) \quad u_{xx} + u_{yy} + [\alpha(1-\alpha)x^{-2} - 4\lambda]u = 0.$$

Solutions are expressed in terms of $z = x+iy$, $\bar{z} = x-iy$, and an analytic function $f(\zeta)$ of the complex variable ζ in the form

$$(2) \quad u(z, \bar{z}) = \int_a^z E(z, \bar{z}, \zeta) f(\zeta)\, d\zeta - f(z).$$

Using the Riemann function of (1), the author first expresses the "generating function" E in terms of a certain confluent hypergeometric series of two variables. If $\lambda = 0$, this reduces to a Legendre function, and if $\alpha = 0$, to a modified Bessel function. Once E is known, the properties of the solution (2), including the behaviour on the singular line, can be deduced from corresponding properties of the analytic function f of a single variable. The author studies also particular solutions corresponding to the choice $f_\nu(z) = \nu^{-1}(z-z_0)^\nu$, $\nu \neq 0$, $f_0(z) = \log(z-z_0)$, clarifying, in particular, the nature of the singularities of these solutions at $x+iy = z_0$.

A. Erdélyi (Pasadena, Calif.)

Roĭtberg, Ja. A. 3239

Local increase of smoothness up to the boundary for solutions of elliptic equations. (Russian)

Ukrain. Mat. Ž. **15** (1963), 444–448.

The author extends to the case of general normal boundary problems associated with an elliptic differential operator a result obtained for the case of the first (Dirichlet) boundary problem by Berezanskiĭ, Kreĭn and the author [Dokl. Akad. Nauk SSSR **148** (1963), 745–748, cf. in particular Theorem 3; MR **26** #4030]. The proof is similar.

J. Peetre (Lund)

Quilghini, Demore 3240

Su di un nuovo problema del tipo di Stefan. (English summary)

Ann. Mat. Pura Appl. (4) **62** (1963), 59–97.

Le problème étudié dans ce mémoire est le suivant: trouver les fonctions $A(x, t)$, $B(x, t)$ et $h(t)$, définies pour $t > 0$, $x > 0$, de telle manière que les conditions

$$(1) \quad D\partial^2 B/\partial x^2 = \partial B/\partial t, \quad 0 < t, \quad 0 < x < h(t), \quad h(0) = 0,$$

$$(2) \quad D\partial^2 A/\partial x^2 = \partial A/\partial t, \quad 0 < t, \quad 0 < h(t) < x,$$

$$(3) \quad \lim_{x \to 0} B(x, t) = U_1 < 0, \qquad t > 0,$$

$$(4) \quad \lim_{t \to 0} A(x, t) = U_0 \geqq 0, \qquad x > 0,$$

$$(5) \quad \lim_{x \to h(t)-} B(x, t) = \lim_{x \to h(t)+} A(x, t) = U_c(h(t)), \qquad t > 0,$$

$$(6) \quad k\left[\lim_{x \to h(t)-} \partial B/\partial x - \lim_{x \to h(t)+} \partial A/\partial x\right] = l\rho \dot{h}(t), \qquad t > 0$$

soient satisfaites. Dans ces formules, U_0, U_1, k, l, ρ sont des constantes, $U_c(x)$ est une fonction satisfaisant à la condition de Lipschitz et $U_c(0) = 0$, $\dot{h}(t)$ désigne la dérivée de $h(t)$. On a posé $D = k(c\rho)^{-1}$, c étant une constante (la chaleur spécifique). Il faut supposer encore $U_c(x) \leqq U_0$. Soit C la classe des fonctions $h(t) \geqq 0$, $h(0) = 0$, continues sur $t \geqq 0$, admettant une dérivée $\dot{h}(t)$ à l'exception d'un ensemble au plus dénombrable, telle que $0 \leqq \dot{h}(t) \leqq \Lambda t^{-1/2}$. Pour $h(t) \in C$, posons

$$\Phi = \Phi(x, t, h(\tau), \tau) = l\dot{h}(t)[2c\sqrt{(\pi D(t-\tau))}]^{-1} \exp\{-[x-h(\tau)]^2/4D(t-\tau)\},$$
$$0 < \tau < t, \quad -\infty < x < +\infty,$$
$$= 0 \text{ pour } \tau = t.$$

Soit

$$I = I(x, t, [h]) = \int_0^t \Phi(x, t, h(\tau), \tau)\, d\tau,$$

$$V(x, t, [h]) = I(x, t, [h]) - I(-x, t, [h]).$$

Soit enfin

$$U(x, t) = U_1 + \frac{2(U_0 - U_1)}{\sqrt{\pi}} \int_0^{x/2\sqrt{(Dt)}} \exp(-\eta^2)\, d\eta,$$
$$x \geqq 0, \; t > 0.$$

S'il existe une fonction $h(t) \in C$, telle que l'équation fonctionnelle (E) $U(h(t), t) + V(h(t), t, [h]) = U_c(h(t))$, $t \geq 0$, soit satisfaite, alors le problème considéré admet la solution $h(t)$, $B(x, t) = U(x, t) + V(x, t, [h])$, $0 < t$, $0 < x < h(t)$, $A(x, t) = U(x, t) + V(x, t, [h])$, $0 < t$, $h(t) < x$. La plus grande partie du mémoire est consacrée à l'étude approfondie de l'équation (E). On donne des conditions assurant l'existence et l'unicité de la solution de l'équation (E). La partie finale contient quelques résultats concernant le comportement asymptotique des solutions. *C. Corduneanu* (Iaşi)

Burčuladze, T. V. 3241

A generalization of the method of potentials for certain elliptic systems. (Russian)

Soobšč. Akad. Nauk Gruzin. SSR **30** (1963), 697–704.

Soit (S) $A(\partial/\partial x)u(x) = \omega(x)$ un système elliptique à coefficients constants, en deux variables x_1, x_2, où $A(\partial/\partial x) = [a_{ik}(\partial/\partial x)]$, $a_{ik}(\partial/\partial x) = \sum_{p+q=m} a_{pq}^{ik} \partial^m/\partial x_1^p \partial x_2^q$, $i, k = 1, 2, \cdots, n$. La fonction vectorielle $\omega(x)$ est donnée et l'on suppose que mn est pair. On construit la matrice solution fondamentale de (S) sous la forme $\Phi(x, y) = B(\partial/\partial x)\varphi(x, y)$, où $A(\partial/\partial x)B(\partial/\partial x) = [\det A(\partial/\partial x)]E$, E la matrice unité, $\varphi(x, y)$ étant déterminée par

$$\det A(\partial/\partial x)\varphi(x, y) = 0.$$

L'auteur donne une formule de représentation pour les solutions régulières de (S), à l'aide de $\Phi(x, y)$, le potentiel généralisé, formule rappelant les formules classiques de la physique mathématique. *C. Corduneanu* (Iaşi)

Akô, Kiyoshi 3242

On the equicontinuity of some class of functions.

J. Fac. Sci. Univ. Tokyo Sect. I **9**, 383–395 (1963).

The author derives estimates for derivatives of solutions of quasilinear second-order elliptic partial differential equations, assuming that the solutions are bounded in absolute value by a constant M which is not very large. The estimates and arguments are refinements of those of M. Nagumo [Osaka Math. J. **6** (1954), 207–229; MR **16**, 1116]. Essentially more general results, in which M is arbitrary, have been obtained by O. A. Ladyženskaja and N. N. Ural'ceva [(i) Uspehi Mat. Nauk **16** (1961), no. 1 (97), 19–90; MR **26** #6571; English transl., Russian Math. Surveys **16** (1961), no. 1, 17–91; (ii) Izv. Akad. Nauk SSSR Ser. Mat. **26** (1962), 5–52; ibid. **26** (1962), 753–780; ibid. **27** (1963), 161–240; (iii) Vestnik Leningrad. Univ. **18** (1963), no. 1, 10–25; MR **26** #5273]. *L. Nirenberg* (New York)

Гovmasjan, N. E. 3243

Some boundary-value problems for the Laplace equation with discontinuous boundary data. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 174–185.

Der Autor untersucht Lösbarkeit und Eindeutigkeit der Dirichlet- und Neumannschen Randwertaufgaben für die Laplacesche Differentialgleichung in einem Gebiet D des R_n, wenn die Randbedingungen in einer abgeschloßenen nirgends dichten Punktmenge Γ der Berandung S von D Singularitäten besitzen. Die Lösungen werden in der Klasse der in $\bar{D} - \Gamma$ stetigen Funktionen gesucht, die auf Γ Singularitäten bestimmter Art besitzen können. In derartigen Funktionenklassen haben die entsprechenden homogenen Randwertaufgaben in der Regel unendlich viele linear unabhängige Lösungen.

Der Autor beweist zunächst, daß im Gebiet D stets eine Lösung $U(X)$ $(X = x_1, \cdots, x_n)$ der Laplaceschen Differentialgleichung existiert, die den folgenden Bedingungen genügt: (I) $U|_{S-\Gamma} = f(X) \in M_S(m, \Gamma)$ (m ganzzahlig). Eine Funktion $f(X)$ gehört zur Klasse $M_S(\alpha, \Gamma)$, wenn sie auf $S - \Gamma$ stetig ist und außerdem der Bedingung

$$(1) \qquad \sup_{X \in S-\Gamma} |f(X)| \rho^\alpha(X, \Gamma) < \infty$$

genügt. Hierbei bezeichnet $\rho(X, \Gamma)$ den Abstand zwischen X und Γ. (II) $U(X) \in M_D(m+n-1, \Gamma)$ (siehe (1) für $X \in D$).

Der Beweis wird durch Konstruktion einer Funktion erbracht, die allen geforderten Bedingungen genügt.

Zum Beweis der Eindeutigkeit der Dirichletschen Randwertaufgabe werden im Gebiet D zwei Lösungen $U_1(X)$ und $U_2(X)$ der Laplaceschen Differentialgleichung betrachtet, die in $\bar{D} - \Gamma$ stetig sind, auf $S - \Gamma$ gleiche Werte annehmen und darüber hinaus der Bedingung

$$(2) \qquad \lim_{\varepsilon \to 0} \max_{\substack{\rho(X,\Gamma)=\varepsilon \\ X \in D}} |U_k(X)| \operatorname{mes} S_1(\rho(\zeta, \Gamma) < \varepsilon) = 0 \qquad (k = 1, 2)$$

genügen. Hierbei wurde die Berandung S von D in Form eines $(n-1)$-dimensionalen Ebenenstückes S_1 und einer glatten Fläche S_2 angenommen. Unter Anwendung des Prinzips vom Maximum auf den Betrag der Differenz beider Funktionen wird gezeigt, daß zwei derartige Funktionen überall in D identisch sind.

Weiterhin wird gezeigt, daß stets eine eindeutige Lösung der Laplaceschen Differentialgleichung der Klasse $M_D(m, \Gamma)$ $(m > n-k-2$, ganzzahlig) existiert, die auf $S - \Gamma$ verschwindet und bestimmten Grenzbedingungen bei Annäherung an Punkte von Γ genügt.

Zum Abschluß des ersten Teiles der Arbeit wird schließlich die Existenz einer Lösung der Laplaceschen Differentialgleichung im Gebiet D gezeigt, die auf $S - \Gamma$ vorgegebene Randwerte $f(X)$ annimmt.

Im zweiten Paragraphen weist der Autor analoge Lösbarkeitseigenschaften für die entsprechende Neumannsche Randwertaufgabe nach. *W. Schmidt* (Simferopol)

Wang, P. Keng Chieh 3244

Asymptotic stability of a time-delayed diffusion system.

Trans. ASME Ser. E. J. Appl. Mech. **30** (1963), 500–504.

Consider the partial differential-difference equation,

$$u_t(t, x) = Lu(t, x) + f(t, x, u(t, x), u(t-T, x), \cdots, u_{x_i}(t, x), \cdots, u_{x_i}(t-T, x), \cdots)$$

on $(0, \infty) \times \Omega$, where L is a linear elliptic operator in x for all $t > 0$ and $x = (x_1, \cdots, x_M) \in \Omega$, a bounded domain. Initial data are given on $[-T, 0] \times \Omega$ and the boundary condition $u(t, x) = 0$ on $[0, \infty) \times \partial\Omega$ is assumed. The author gives definitions, Lyapunov theorems without proofs, and examples for asymptotic stability of the solution as $t \to \infty$. *R. D. Driver* (Albuquerque, N.M.)

Pini, Bruno 3245

Sulla classe di Gevrey delle soluzioni di certe equazioni ipoellittiche.

Boll. Un. Mat. Ital. (3) **18** (1963), 260–269.

Une fonction $u(x)$, définie dans un domaine E de l'espace euclidien R^r, appartient à la classe $G_{\beta_1,\dots,\beta_r}$ de Gevrey si, pour $|q|\geq 0$, $x\in E$, $|D^q u(x)|\leq CA_1^{q_1}\cdots A_r^{q_r}q_1^{\beta_1 q_1}\cdots q_r^{\beta_r q_r}$.

L'auteur considère dans R^2 l'opérateur $P(D_x, D_y)$ hypoelliptique à coefficients constants. Sous quelques hypothèses sur le comportement asymptotique des racines $\lambda_j(s)$ de $P(-is, \lambda)$, respectivement $\mu_j(s)$ de $P(\mu, -is)$ pour $|s|\to\infty$, l'auteur démontre que chaque solution de $P(D_x, D_y)u=0$ appartient à une certaine classe de Gevrey, où β_1, β_2 dépendent de l'allure asymptotique des racines. On démontre encore: Si $P(D_x, D_y)u=0 \Rightarrow u\in G_{\alpha,\beta}$, $\alpha\geq 1$, $\beta\geq 1$, alors si $f(x, y)\in G_{\alpha',\beta'}$, $\alpha'\geq\alpha$, $\beta'\geq\beta$, $P(D_x, D_y)u=f(x, y)$, il s'ensuit que $u\in G_{\alpha',\beta'}$.

J. Nečas (Prague)

Owens, O. G. 3246

A uniqueness theorem for the Helmholtz equation.

Duke Math. J. **31** (1964), 91–98.

This paper establishes the following theorem: Let $u(x_1, x_2)$ be an everywhere twice continuously differentiable solution of

$$(1)\qquad \partial^2u/\partial x_1^2+\partial^2u/\partial x_2^2+u=0$$

satisfying the integral condition

$$(2)\qquad \lim_{R\to\infty}\int_0^{2\pi}\left|\int_0^R u(x_1, x_2)\,dr\right|d\theta=0,$$

where $x_1+ix_2=re^{i\theta}$. Then $u\equiv 0$.

The proof makes use of the mean-value theorem

$$J_0(r)u(x_1, x_2)=\frac{1}{2\pi}\int_0^{2\pi}u(x_1+r\cos\theta, x_2+r\sin\theta)\,d\theta,\qquad r\geq 0.$$

This is used to obtain integral formulas for the partial derivatives of u. These formulas and (2) are shown to imply that all the derivatives vanish at the origin. Since solutions of (1) are necessarily analytic, this implies $u\equiv 0$.

C. H. Wilcox (Madison, Wis.)

Browder, Felix E. 3247

Nonlinear elliptic problems. II.

Bull. Amer. Math. Soc. **70** (1964), 299–302.

The author continues his study of non-linear elliptic boundary problems [cf., e.g., the author, same Bull. **69** (1963), 862–874; MR **27** #6048]. In the meantime the paper of Višik [Trudy Moskov. Mat. Obšč. **12** (1963), 125–184; MR **27** #6017] appeared, where similar results were obtained by different techniques. However, there is one major difference, "namely, that the hypotheses of strong ellipticity or monotonicity which are assumed (by Višik) involve only the variation of A_α with respect to the highest-order derivatives and not the lower-order derivatives of u". In the present note are now announced, without proofs, two theorems, one in the case of boundary problems and the other in the abstract case, which are variants of previous results intended to cover this point. There are also three more theorems, seemingly unrelated, in the abstract case, one even for locally convex spaces.

J. Peetre (Lund)

Bureau, F. J.; Pellicciaro, E. J. 3248

Asymptotic behavior of the spectral matrix of the operator of elasticity.

J. Soc. Indust. Appl. Math. **10** (1962), 1–18.

Let the operator $Mu=-b^2\operatorname{grad}(\operatorname{div} v)+a^2\operatorname{rot}(\operatorname{rot} v)$ be applied to smooth vector functions $v(x)=[v_1(x), v_2(x), v_3(x)]$ which vanish on the boundary of a regular region in 3-space. Let $0<\lambda_1\leq\lambda_2\leq\cdots$ be the eigenvalues, and $v(x,\lambda_k)$ the eigenvectors of the boundary-value problem $Mv=\lambda v$. The spectral function is

$$\psi(x,\lambda)=\sum v(x,\lambda_k)\cdot v(x,\lambda_k),$$

summed over all eigenvalues $<\lambda$, and normalized by $2\psi(x,\lambda)=\psi(x,\lambda-0)+\psi(x,\lambda+0)$. F. J. Bureau [J. Math. Anal. Appl. **1** (1960), 423–483; MR **28** #343] has obtained the asymptotic behavior of the spectral function of an elliptic vector operator M, using a knowledge of the solution of the Cauchy problem for the related hyperbolic equation $\partial^2u/\partial t^2=Mu$. In the paper under review this method is applied to the specific operator M defined above. For this operator the Cauchy problem has an explicit solution, which the authors obtain by utilizing a related "potential" function. The main result is that

$$\psi(x,\lambda)\sim\frac{1}{6\pi^2}\left(\frac{2}{a^3}+\frac{1}{b^3}\right)\lambda^{3/2}.$$

Robert McKelvey (Madison, Wis.)

Agranovič, M. S. 3249

General boundary-value problems for integro-differential elliptic systems. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 495–498.

This report extends to a wider class of operators the results of the author and Dynin [same Dokl. **146** (1962), 511–514; MR **25** #4234] concerning the index of elliptic boundary-value problems. The earlier paper concerned an elliptic $p\times p$ system A of differential operators of order s (ps is assumed even) in a smooth region G in $\mathbf{R}^n$ with C^∞ boundary Γ, and a $(ps/2)\times p$ matrix B of boundary operators whose entries are differential operators of arbitrary order with singular integral operators as coefficients. The present paper also allows singular integral operators as coefficients in A. With suitable restrictions on the nature of these operators near Γ, ellipticity of the system can again be defined, and appropriate a priori estimates derived. Again the map $u\to(Au, Bu)$ has closed range in the appropriate topologies; also its null space and the complement of its range are finite-dimensional, the difference of these dimensions being the index. The results stated are then: (1) the index is invariant under homotopies; (2) index $(A, B)-$index (A, B') is the index of a system of singular integral operators on Γ; (3) if A and A' are the same on Γ, up to lower-order terms, then index $(A, B)-$index (A', B) is the index of a system of singular integral operators on $\mathbf{R}^n$; (4) if B gives Dirichlet boundary conditions, then index (A, B) is the index of a system of singular integral operators on $\mathbf{R}^n$. The last result gives sufficient conditions for the index of the Dirichlet problem to be zero: n is odd and A is a differential operator, or $n>p$.

No proofs, but fairly suggestive indications of the techniques, are given.

R. T. Seeley (Waltham, Mass.)

Cesari, L. 3250

Periodic solutions of partial differential equations. (Russian summary)

Qualitative methods in the theory of non-linear vibrations (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. II*, 1961), *pp.* 440–457. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, **1963**.

Let us consider a partial differential equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

in a strip $A = [-\infty < x < +\infty, -a \leq y \leq a]$, which degenerates on the x-axis into an ordinary differential equation $f(x, u, u_x, u_{xx}) = 0$. Suppose that both F and f are periodic in x of some period T, and that the degenerate equation has a periodic solution: $u = u_0(x)$, $u_0(x+T) = u_0(x)$, $-\infty < x < +\infty$. The following problem is discussed: can $u_0(x)$ be extended to a periodic solution $u(x, y)$ of the partial differential equation in a strip A? In the first part of this paper, the author considers a system of the form $u_{xy} = f(x, y, u, u_x, u_y)$. In the second part, a single hyperbolic equation is considered: $u_{xx} + k^2 u + g(x, y, \varepsilon) u_{xy} = \varepsilon f(x, y, u, u_x, u_y)$, $g > 0$, $\varepsilon = \text{const}$. The same questions as above are discussed, by making use of the results of the first part. Criteria of practical importance will be given in a later paper. *C. Corduneanu* (Iași)

Anučina, N. N. 3251

Difference schemes for solving the Cauchy problem for symmetric hyperbolic systems. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 247–250.

Six difference schemes are described for solving the Cauchy problem for the symmetric hyperbolic system

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{s} A^k \frac{\partial u}{\partial x_k} + Bu = f, \qquad u(x, t)|_{t=0} = u_0(x),$$

where $A^k(x, t)$ and $B(x, t)$ are real symmetric matrices of order m, and $u(x, t)$ and $f(x, t)$ are vector functions with values in R_m. All the schemes make use of rectangular grids and involve resolution of each matrix A^k into two components corresponding to its positive and negative eigenvalues. The advantage of the latter procedure is not stated. Convergence in the L_2 norm is claimed, in most instances on the assumption that certain Lipschitz or similar conditions are satisfied. There is no discussion of the merits of the six schemes or their rates of convergence, nor any comparison with schemes proposed by others (e.g., Friedrichs [Comm. Pure Appl. Math. **7** (1954), 345–392; MR **16**, 44], Lax and Richtmyer [ibid. **9** (1956), 267–293; MR **18**, 48], and Weinberger [J. Soc. Indust. Appl. Math. **7** (1959), 49–75; MR **23** #B1128]). Nor is there any mention of error bounds (Weinberger has given such bounds).

T. N. E. Greville (Madison, Wis.)

Baranovskiĭ, F. T. 3252

Differential properties of the solution of a mixed problem for a degenerate hyperbolic equation. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1963**, *no.* 6 (37), 15–24.

Dans le cylindre $D \times [0, 1]$ (D domaine borné de R^n de frontière Γ) on considère le problème suivant:

$$\varphi(t) \frac{\partial^2 u}{\partial t^2} - A(x, t; D_x) u = f(x, t),$$

$$u(x, 0) = \psi_0(x),$$

$$\frac{\partial u(x, 0)}{\partial t} = \psi_1(x),$$

$$u(x, t) \text{ nulle sur } \Gamma \times [0, 1],$$

où $A(x, t; D_x)$ est un opérateur elliptique symétrique du second ordre: $A'(x, t; \xi) \geq \lambda |\xi|^2$ pour $\xi \in R^n$ $(\lambda > 0)$, A' partie principale de A, et $\varphi(t)$ est une fonction k fois continûment dérivable avec $C_1 t^{\alpha - i} \leq \varphi^{(i)}(t) \leq C_2 t^{\alpha - i}$ $(0 < C_1 < C_2)$, $i = 0, 1, \cdots, k$.

L'auteur montre que la solution généralisée (i.e., variationnelle) de ce problème est deux fois continûment différentiable dans les cylindres $\Omega \times [0, 1]$, Ω domaine intérieur à D, lorsque les coefficients de A sont suffisamment dérivables et $f(x, t)$ ainsi que ses dérivées en t d'ordre $\leq k-2$ $(k = [\frac{1}{2}n] + 3)$ sont de carré sommable par rapport à un certain poids dans $D \times [0, 1]$. La méthode de Galerkine est utilisée pour prouver que les dérivées de u sont de carré sommable jusqu'à un certain ordre; la continuité des dérivées d'ordre ≤ 2 suit par application du théorème de Sobolev. *P. Grisvard* (Nancy)

Kuznecov, N. N. 3253

On hyperbolic systems of linear equations with discontinuous coefficients. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 299–313.

The author proves a uniqueness theorem for the generalized solution of the Cauchy problem of the hyperbolic system

$$\frac{\partial u}{\partial t} + \frac{\partial A u}{\partial x} + Bu = f(t, x),$$

where the coefficients $A = A(t, x)$, $B = B(t, x)$ are discontinuous on a finite number of certain curves, but regular in the domains between the curves. These curves intersect in a finite number of points. A similar result is given for the hyperbolic system

$$\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} + Bu = f(t, x).$$

M. Altman (Warsaw)

Veĭvoda, O. [Veyvoda, O.] 3254

On the periodic motion of a weakly non-linear string. (Russian. English summary)

Qualitative methods in the theory of non-linear vibrations (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. II*, 1961), *pp.* 120–122. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

Let

$$u_{tt} - u_{xx} = \varepsilon f(x, t, u, u_t, u_x, \varepsilon),$$

where f satisfies certain assumptions of smoothness and

$$u(0, t) = u(\pi, t) = 0,$$

$$u(x, 0) = \varphi(x), \qquad u_t(x, 0) = \psi(x).$$

If f is ω-periodic in t and $\omega/2\pi$ is irrational, then φ and ψ may always be chosen so that the motion is ω-periodic, with an amplitude of oscillation of order ε. The necessary conditions for the existence of 2π-periodic oscillations are given. The author proves the uniqueness of the solution.

The equations

$$u_{tt}-u_{xx}+au_t = g(x,t)+\varepsilon f(x,t,u,u_x,u_t,\varepsilon),$$

$$u_{tt}-[1+\varepsilon g(x,t,u,u_t,u_x)]u_{xx}=\varepsilon f(x,t,u,u_t,u_x,\varepsilon),$$

where f and g are 2π-periodic in t, may be treated in a similar way. *M. Coroi-Nedelcu* (Bucharest)

Przeworska-Rolewicz, Danuta 3255

Sur l'unique solution polyharmonique de l'équation $\sum_{k=0}^{n-1} a_k\Delta^k u=v$.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 34–39.

A generalization of a previous theorem of the author [Studia Math. **22** (1963), 337–367; MR **27** #550] on equations involving algebraic operators, and Vekua's representation for polyharmonic functions [I. N. Vekua, *New methods for solving elliptic equations* (Russian), pp. 170–186, OGIZ, Moscow, 1948; MR **11**, 598], are used to prove that the equation $\sum_{k=0}^{n-1} a_k\Delta^k u(x,y)=v(x,y)$ (a_k are complex constants) has a unique polyharmonic solution in a given domain D, if v is polyharmonic and $a_0\neq 0$.
P. C. Fife (Minneapolis, Minn.)

Nicolescu, Miron 3256

Sur un théorème de moyenne de M. Mauro Picone.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 40–44.

Polycaloric functions ($(\partial^2/\partial x^2-\partial/\partial t)^p u\equiv 0$) defined in a strip $0\leqq t\leqq\delta$ are first characterized as those functions satisfying, for all positive $h<t$, the equation $u(x,t)=\sum_0^{p-1} C_k\mu_k(u;h)(x,t)$, where $\mu_0(u;h)(x,t)$ is the weighted average, with heat kernel as weight function, of u on the line $t-h$, $\mu_k=2h^{-2}\int_0^h h'\mu_{k-1}(u;h')dh'$, $k\geqq 1$, and the C_k are constants depending only on p. Secondly, they are characterized as satisfying, for all sets of p distinct numbers $0<h_k<t$, the equation $u(x,t)=\sum_0^{p-1} D_k\mu_0(u;h_k)(x,t)$, where the D_k depend on $\{h_k\}$ and p. These results are analogues of those obtained by M. Picone [Bull. Math. Soc. Roumaine Sci. **38** (1936), no. 2, 105–112] and the author [Bull. Soc. Math. France **60** (1932), 129–151] concerning polyharmonic functions. *P. C. Fife* (Minneapolis, Minn.)

Pogorzelski, Witold 3257

Étude de la continuité des solutions du système parabolique dépendant d'un paramètre. I, II.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **32** (1962), 891–898; *ibid.* (8) **33** (1962), 22–30.

Consider the parabolic system of equations

$$(1)\quad \Psi^{(\alpha)}(u_1,\cdots,u_N) = \sum_{1\leq j\leq N} A_{\alpha j}^{k_1\cdots k_n}(X,t,\lambda)\frac{\partial^{k_1+\cdots+k_n}u_j}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}-\frac{\partial u_\alpha}{\partial t}=0,$$

$$0\leqq k_1+k_2+\cdots+k_n\leqq M,\ \alpha=1,\cdots,N,$$

where $M\geqq 2$ and $N\geqq 1$. Let $A_{\alpha j}$ be complex-valued functions of the point $X(x_1,\cdots,x_n)$ in Euclidean space E, of the real variable t and of the real parameter λ such that in the region

$$(2)\qquad \{X\in E,\ 0\leqq t\leqq T,\ -R\leqq\lambda\leqq +R\},$$

$$(3)\quad |A_{\alpha j}^{k_1\cdots k_n}(X,t,\lambda)-A_{\alpha j}^{k_1\cdots k_n}(X',t',\lambda)|\leqq \text{const}\,\{|XX'|^h+|t-t'|^{h'}\}$$

if $k_1+\cdots+k_n=M$ and

$$(4)\quad |A_{\alpha j}^{k_1\cdots k_n}(X,t,\lambda)-A_{\alpha j}^{k_1\cdots k_n}(X',t,\lambda)|\leqq \text{const}\,|XX'|^h$$

if $k_1+\cdots+k_n<M$, where $0<h,h'\leqq 1$ and $|XX'|$ is the Euclidean distance between the points X,X'. Let the coefficients $A_{\alpha j}$ be continuous in t and λ uniformly in the region (2). Also, conforming to Petrovsky's definition of a parabolic system, all of the roots ρ of the equation

$$(5)\quad \det_{\alpha,j}\left|\sum_{k_1+\cdots+k_n=M} A_{\alpha j}^{k_1\cdots k_n}(X,t,\lambda)\times(is_1)^{k_1}\cdots(is_n)^{k_n}-\delta_{\alpha j}\rho\right|=0,$$

where $\delta_{\alpha j}$ is the Kronecker delta, must satisfy

$$(6)\qquad \operatorname{Re}(\rho)<-\delta<0\qquad (\delta \text{ fixed})$$

for all values of the real variables $s_1,\cdots,s_n$ satisfying

$$(7)\qquad s_1^2+\cdots+s_n^2=1$$

and for all points in the region (2).

Let

$$(8)\qquad \Gamma_{\alpha\beta}^{(\lambda)}(X,t;Y,\tau),\qquad \alpha,\beta=1,\cdots,N,$$

denote the fundamental solution matrix of (1) which was constructed in an earlier paper [Ricerche Mat. **7** (1958), 153–185; MR **21** #4300].

In this paper, the author discusses the continuity of three solutions of (1):

$$\{U_\alpha^{(\lambda)}(X,t)\},\qquad \{J_\alpha^{(\lambda)}(X,t)\},\qquad \{V_\alpha^{(\lambda)}(X,t)\},$$

where

$$(9)\quad U_\alpha^{(\lambda)}(X,t)=\int_0^t\iiint_E \sum_{\beta=1}^N \Gamma_{\alpha\beta}^{(\lambda)}(X,t;Y,\tau)\rho_\beta(Y,\tau)\,dY\,d\tau,$$

$$(10)\quad J_\alpha^{(\lambda)}(X,t)=\iiint_E \sum_{\beta=1}^N \Gamma_{\alpha\beta}^{(\lambda)}(X,t;Y,0)f_\beta(Y)\,dY,$$

$$(11)\quad V_\alpha^{(\lambda)}(X,t)=\int_0^t\int_S\int \sum_{\beta=1}^N \Gamma_{\alpha\beta}^{(\lambda)}(X,t;Q,\tau)\varphi_\beta(Q,\tau)\,dQ\,d\tau.$$

An example of a typical theorem is Theorem 1: If the coefficients of (1) satisfy (3), (4) and (5) in the region (2), and if the components of the density $\{\rho_\beta\}$, defined in $\{Y\in E,\ 0<t\leqq T\}$, satisfy

$$(12)\qquad |\rho_\beta(Y,\tau)|<M_\rho t^{-\mu_\rho}\exp[b|YX_0|],$$

where M_ρ, b are positive constants, μ_ρ is non-negative and less than 1, and X_0 is a fixed point in E, and are integrable in all bounded measurable regions in E for $0<\tau\leqq T$, then the components of (9) and their derivatives of order $m\leqq M-1$ are defined in the region $\{X\in E,\ 0<t\leqq T,\ -R\leqq\lambda\leqq R\}$ and satisfy

$$(13)\quad |D_X^{(m)}[U_\alpha^{(\lambda)}(X,t)]-D_X^{(m)}[U_\alpha^{(\lambda')}(X,t)]|\leqq \text{const}\,M_\rho t^{1-\mu_m-\mu_\rho}\exp[b|XX_0|][\omega_A(|\lambda-\lambda'|)]^{h_\lambda},$$

where $D_X^{(m)}$ denotes a spatial derivative of order m, $\omega_A(\delta)$ is the modulus of continuity of the coefficients $A_{\alpha\beta}$ with respect to λ and is defined for all $\delta\geqq 0$ by

$$(14)\quad \omega_A(\delta)=\max_{\substack{\alpha\beta\\(k_1,\cdots,k_n)}}\ \sup_{\substack{|\lambda-\lambda'|\leq\delta\\ X\in E\\ 0<t<T}} |A_{\alpha\beta}^{k_1\cdots k_n}(X,t,\lambda)-A_{\alpha\beta}^{k_1\cdots k_n}(X,t,\lambda')|$$

for λ, λ' in the closed interval $(-R, R)$, the exponent h_λ is any positive number less than $h_1 = \min(h, Mh')$, the constant μ_m is chosen arbitrarily in the interval

$$\frac{m}{M} < \mu < 1, \tag{15}$$

the positive coefficient "const" depends on the choice of μ_m and h_λ, but not on the functions ρ_β.

J. R. Cannon (Upton, N.Y.)

Kaplan, Stanley 3258

On the growth of solutions of quasi-linear parabolic equations.

Comm. Pure Appl. Math. **16** (1963), 305–330.

The main result (Theorem 1) is a comparison theorem in which $u(x,t) \leqq v(x,t)$ for $x \in \Omega$, $0 \leqq t \leqq T$, is concluded from certain conditions, including the inequalities

$$u_t - a_{ij}(x, t, u, \nabla u)u_{ij} \leqq F(x, t, u, \nabla u),$$

$$v_t - a_{ij}(x, t, v, \nabla v)v_{ij} \geqq F(x, t, v, \nabla v),$$

where $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$, $\nabla u = (u_1, \cdots, u_n)$. The precise conditions are too long to state here, only an indication of the main type of assumptions will be given: Ω is an open connected x-set with smooth boundary $\partial\Omega$; u, v, u_t, v_t, u_i, v_i, u_{ij}, v_{ij} exist and are continuous on $\bar\Omega \times [0, T]$; (a_{ij}) is symmetric and positive definite; for $x \in \Omega$, $u(x,0) \leqq v(x,0)$; at every $x \in \partial\Omega$, either $u(x,t) \leqq v(x,t)$ or $\nabla u \cdot \eta \leqq \nabla v \cdot \eta$ where η is the outward normal to $\partial\Omega$ at x; $F(x,t,u,p) - F(x,t,v,p) \leqq \psi_M(u-v,t)$,

$$|a_{ij}(x, t, u, p) - a_{ij}(x, t, v, p)| \leqq \psi_M(u-v, t)$$

for $|u|$, $|v|$, $|p| \leqq M$ and, for every positive K and M, the ordinary initial-value problem $\phi' = K\psi_M(\phi, t)$, $\phi(0) = 0$ has the unique solution $\phi(t) \equiv 0$ in a certain general class of functions. (For related results, see Walter [Math. Z. **74** (1960), 191–208; MR **22** #8206].) Theorem 1, together with suitably constructed comparison functions, leads to uniqueness theorems, a priori estimates, results on existence and non-existence for $\Omega \times [0, \infty)$.

P. Hartman (Baltimore, Md.)

Cannon, J. R.; Douglas, Jim, Jr.; Jones, B. Frank, Jr. 3259

Determination of the diffusivity of an isotropic medium. (French, German, Italian and Russian summaries)

Internat. J. Engrg. Sci. **1** (1963), 453–455.

The determination of the diffusivity of an isotropic medium, which has a dependency upon one coordinate direction, is reduced to the solution of a simple ordinary differential equation of elliptic type. Similar problems have previously been considered in physical situations. For example, see A. Gião, J. Roulleau and R. P. Coelho [Arquivo Inst. Gulbenkian Ci. Sec. A Estud. Mat. Fís.-Mat. **1** (1963), 101–130; MR **28** #997]. The present paper is more general than that referenced.

W. F. Ames (Stanford, Calif.)

Cannon, J. R.; Jones, B. Frank, Jr. 3260

Determination of the diffusivity of an anisotropic medium. (French, German, Italian and Russian summaries)

Internat. J. Engrg. Sci. **1** (1963), 457–460.

This is a companion paper to that reviewed above [#3259]. Unlike the isotropic case for which the diffusivity depended continuously upon the boundary flow rate, the diffusivity in this case does not have that continuous dependence.

W. F. Ames (Stanford, Calif.)

Cannon, J. R. 3261

Determination of certain parameters in heat conduction problems.

J. Math. Anal. Appl. **8** (1964), 188–201.

The author treats three problems of which the following is typical: For given positive constants c, h, t_0 and functions $f(t), g(t) \in C^1([0, T])$ with $f(0) = g(0) = 0$, is there a constant κ and a solution $u(x,t)$ of the boundary-value problem (*) $u_t = \kappa u_{xx}$ for $0 < x < 1$, $0 < t \leqq T$; $u(x,0) = 0$ for $0 \leqq x \leqq 1$; $u(0,t) = f(t)$ for $0 \leqq t \leqq T$; $u(1,t) = g(t)$ for $0 \leqq t \leqq T$; and $-c\kappa u_x(0, t_0) = h$? (A solution $u(x,t)$ is required to be of class C^1 for $0 \leqq x \leqq 1$, $0 \leqq t \leqq T$.) It is shown that there is a function $F(\kappa) = F(\kappa, c, t_0)$, given explicitly in terms of integrals of the Green's functions and $f'(t)$, $g'(t)$, such that (*) has a unique solution if and only if $F(\kappa) = h$ has a unique solution $\kappa > 0$. The latter is the case if $f(t) \not\equiv 0$, $f'(t) \geqq 0$, $g'(t) \leqq 0$. More general problems have been treated by other methods by J. Douglas, Jr. and B. F. Jones, Jr. [J. Math. Mech. **11** (1962), 919–926; MR **27** #3949] and B. F. Jones, Jr. [ibid. **11** (1962), 907–918; MR **27** #3948].

P. Hartman (Baltimore, Md.)

Novruzov, A. A. 3262

The behaviour of the solution of the first boundary-value problem with zero boundary conditions for a parabolic equation. (Russian. Azerbaijani summary)

Akad. Nauk Azerbaĭdžan. SSR Dokl. **19** (1963), *no.* 10, 9–13.

The asymptotic behavior of the solution of a parabolic equation

$$\frac{\partial U}{\partial t} = \sum_{i,k=1}^{n} a_{ik}(x,t)\frac{\partial^2 U}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial U}{\partial x_i} + c(x,t)U \tag{1}$$

with prescribed initial condition $\varphi(x)$ and with zero boundary conditions is considered for the case that $c(x,t) > 0$, when the maximum principle no longer holds. Suppose that the coefficients of the equation fulfill certain conditions of smoothness and boundedness, and that

$$\sum_{i,k=1}^{n} a_{ik}(x,t)\xi_i\xi_k + \sum_{i=1}^{n} b_i(x,t)\xi_i\eta + c(x,t)\eta^2 > \alpha\left(\sum_{i=1}^{n} \xi_i^2 + \eta^2\right), \qquad \alpha > 0,$$

for arbitrary ξ_i and η. The author derives some estimates for the dependence of $\max U(x,h)$ on $\max \varphi(x)$. The constants in these estimates depend on h, on the coefficients of the equation (1), and on the measure of the region concerned.

{Unfortunately, there are a lot of misprints in the work. Note that for the case $c(x,t) \leqq c_0$, equation (1) can be turned by the substitution $U(x,t) = e^{\alpha t}V(x,t)$ into an equation for $V(x,t)$ of the same type with a negative coefficient at V, for which the use of the maximum principle can be made.}

K. Rektorys (Prague)

Èĭdel'man, S. D. 3263

Boundary-value problems for parabolic systems in a half-space. (Russian)

Dokl. Akad. Nauk SSSR **142** (1962), 812–814.

The author considers the problem:

$$\frac{\partial u}{\partial t} = \sum_{|k|=2b} A_k(-iD_x)^k u = A(iD_x)u, \qquad u(0+,x)=0,$$

(*)

$$\lim_{x_n\to 0+} \sum_{m=1}^{N} \sum_{2bq_0+|q|=r_j} B_{jm}^{(q_0q)} \frac{\partial^{q_0}}{\partial t^{q_0}}(-iD_x)^q u_m = f_j(x',t),$$

where $D_x=(\partial/\partial x_1)^{k_1}\cdots(\partial/\partial x_n)^{k_n}$, $k=(k_1,\cdots,k_n)$, $x=(x',x_n)$, $x'=(x_1,\cdots,x_{n-1})$, $-\infty<x_s<\infty$, $s=1,\cdots,n-1$, $x_n>0$, A_k are $N\times N$ constant real matrices, $u=(u_1,\cdots,u_N)$, $B_{jm}^{(q_0q)}$ are constants, and $j=1,\cdots,bN$.

Utilizing the Fourier transform in the x' variables and the Laplace transform in t, the author constructs a Green's matrix corresponding to any boundary operator satisfying the following: For any real $\sigma_1,\cdots,\sigma_{n-1}$,

$$\det\int_{\Gamma^+} B(\sigma,p)(A(\sigma)-pE)^{-1}(E,\sigma_nE,\cdots,\sigma_n^{b-1}E)\,d\sigma_n\neq 0,$$

where $\sigma=(\sigma',\sigma_n)$, $\sigma'=(\sigma_1,\cdots,\sigma_{n-1})$; $p=-\delta_1|\sigma'|^{2b}+a+ip_1$ with a, $\delta_1>0$, $-\infty<p_1<\infty$, $|\sigma'|^2=\sigma_1^2+\cdots+\sigma_{n-1}^2$; Γ^+ is a contour in the complex σ_n plane enclosing all σ_n-roots of the equation $\det(A(\sigma)-pE)=0$ with $\operatorname{Im}\sigma_n>0$;

$$B(\sigma,p) = \left\|\sum_{2bq_0+|q|=r_j} B_{jm}^{(q_0q)}p^{q_0}\sigma^q\right\|_{j=1,\cdots,bN;\ m=1,\cdots,N};$$

$$(E,\sigma_nE,\cdots,\sigma_n^{b-1}E) = \begin{pmatrix} 1 & 0 & \sigma_n & 0 & \sigma_n^{b-1} & 0\\ \ddots & & \ddots & \cdots & & \ddots\\ 0 & 1 & 0 & \sigma_n & 0 & \sigma_n^{b-1}\end{pmatrix}.$$

The author determines bounds for the Green's matrix. It then follows that the corresponding Green's operator provides a solution to (*) for quite general functions $f_j=f_j(x',t)$. No proofs are given.

This extends the work of T. Ja. Zagorskiĭ [Ukrain. Mat. Ž. **9** (1957), 252–270; MR **19**, 966]. The extension is direct in the sense that the differences in method are only technical. *R. K. Juberg* (Minneapolis, Minn.)

Melent'ev, B. V. 3264

A boundary-value problem for equations of mixed type. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 1262–1265.

The author considers the Tricomi equation (1) $yu_{xx}+u_{yy}=0$ relative to the region $\Delta=\Delta_++\Delta_-$; the region Δ_+ is in the upper half-plane $y>0$, being bounded by a smooth curve γ opening onto the x-axis at points $A(0,0)$ and $B(1,0)$ and by the segment of $y=0$ joining these points; Δ_- lies in $y<0$ and has for its boundary the segment AB and the arcs of two characteristic curves $\xi=0$ and $\eta=1$, where (2) $\xi=x-\frac{2}{3}(-y)^{3/2}$, $\eta=x+\frac{2}{3}(-y)^{3/2}$. In Δ_+, (1) is transformed into (3) $u_{\xi\xi}+u_{\eta\eta}+(1/3\eta)u_{\eta\eta}=0$ with the substitutions $\xi=x$, $\eta=\frac{2}{3}y^{3/2}$; in Δ_-, using (2), equation (1) becomes (4) $u_{\xi\eta}-\frac{1}{6}(\xi-\eta)^{-1}(u_\xi-u_\eta)=0$. The associated boundary conditions are taken to be: (a) $a\,\partial u/\partial t+b\,\partial u/\partial n+cu=f(t)$ on γ (t, n—along and normal to the tangent line on γ), (b) $u=\psi(\eta)$ on the characteristic $\xi=0$.

The author obtains conditions under which the boundary problem has a solution by considering the corresponding problem with boundary conditions $u|_{\xi=0}=0$, $u|_\gamma=\varphi(s)$. In relating the two boundary problems, $\varphi'(t)$ is shown to satisfy a singular integral equation involving the functions a, b, f. From a study of this equation the conditions cited are obtained. *J. F. Heyda* (King of Prussia, Pa.)

Smirnov, M. M. 3265

A mixed boundary-value problem for the equation $y^m\,\partial^2u/\partial x^2+\partial^2u/\partial y^2=0$. (Russian)

Sibirsk. Mat. Ž. **4** (1963), 1150–1161.

Der Exponent m in der vorgelegten Differentialgleichung bedeutet eine gerade, positive, ganze sonst beliebige Zahl. In der Halbebene $y>0$ ist die vorgelegte Differentialgleichung vom elliptischen Typus, in der Halbebene $y<0$ vom hyperbolischen Typus und längs $y=0$ ist die Gleichung parabolisch ausgeartet. D bedeutet das Gebiet, welches durch eine einfache glatte Kurve mit den Endpunkten $A(0,0)$ und $B(1,0)$ begrenzt wird, die in der oberen Halbebene $y>0$ liegt und durch die Charakteristiken $\overline{AC}$: $x-2(m+2)^{-1}(-y)^{(m+2)/2}=0$ und $\overline{BC}$: $x+2(m+2)^{-1}(-y)^{(m+2)/2}=1$ der zu Grunde gelegten Differentialgleichung. Im Bereich D wird nunmehr eine Lösung der in Rede stehenden Differentialgleichung gesucht, stetig im abgeschlossenen Gebiet $\bar D$ mit stetigen partiellen Ableitungen $\partial u/\partial x$ und $\partial u/\partial y$ in $\bar D+\sigma$, welche den Randbedingungen

$$A_s[u] \equiv y^m\frac{dy}{ds}\frac{du}{dx}-\frac{dx}{ds}\frac{du}{dy}\bigg|_\sigma = \varphi(s) \qquad (0<s<l),$$

$$u|_{AC} = \psi(x) \qquad (0\leqq x\leqq \tfrac{1}{2})$$

genügt, wobei s die Bogenlänge auf σ bezeichnet (gezählt vom Punkte $B(1,0)$ aus) und $\varphi(s)$ und $\psi(x)$ vorgegebene stetige Funktionen bezeichnen. Zur weiteren Behandlung dieses Problems (T_1) wird zunächst das Verhalten der Differentialgleichung in der hyperbolischen Halbebene studiert, in welcher die Cauchyschen Anfangsbedingungen $u(x,0)=0$, $\partial u(x,0)/\partial y=\nu(x)$ vorgeschrieben werden. Dabei ergibt sich für (T_1) das folgende Extremumprinzip: die Lösung des Problems (T_1), die auf der Charakteristik $\overline{AC}$ verschwindet, nimmt ihr Maximum oder Minimum innerhalb $\bar D$ auf σ an. Die Eindeutigkeit der Lösung des Problems folgt unmittelbar aus dem Extremumsatz.

Für die Untersuchung der Differentialgleichung in der elliptischen Halbebene ergibt sich das Problem (T_1^+): im Gebiet D_1 wird eine Lösung der Differentialgleichung gesucht, stetig im abgeschlossenen Gebiet $\bar D_1$ mit stetigen partiellen Ableitungen $\partial u/\partial x$ und $\partial u/\partial y$ in $D_1+\sigma$, die in einer Umgebung der Punkte A und B unterhalb der Ordnung $2/(m+2)$ unbegrenzt wachsen können und den Randbedingungen

$$A_s(u) \equiv y^m\frac{dy}{ds}\frac{\partial u}{\partial x}-\frac{dx}{ds}\frac{\partial u}{\partial y}\bigg|_\sigma = \varphi(s) \qquad (0<s<l),$$

$$u|_{y=0} = \tau(x) \qquad (0\leqq x\leqq 1)$$

genügen. Auch hier besteht Eindeutigkeit für die Lösung des Problems (T_1^+). Weiterhin werden im elliptischen Gebiet Ergebnisse von S. Gellerstedt [vgl. S. Gellerstedt, Ark. Mat. Astronom. Fys. **25A** (1935), no. 10], die Theorie einer geeigneten Greenschen Funktion, sowie die der

Fredholmschen Integralgleichungen herangezogen. Erwähnt sei noch das Resultat: die Funktion

$$u(x_0, y_0) = \int_0^1 \tau(x) \frac{\partial G(x, 0; x_0, y_0)}{\partial y} dx + \int_0^l \varphi(s) G(\xi, \eta; x_0, y_0)\, ds,$$

in welcher $\tau(x)$ eine auf $[0, 1]$ stetige Funktion bezeichnet und $\varphi(s)$ eine ebensolche auf $[0, l]$, stellt die Lösung des Problems $(T_1{}^+)$ im Gebiet D_1 mit Hilfe einer geeigneten Greenschen Funktion G dar. *M. Pinl* (Moscow, Idaho)

Ter-Krikorov, A. M.; Trenogin, V. A. 3266

Existence and asymptotic behaviour of solutions of "solitary wave" type for a class of non-linear elliptic equations. (Russian)

Mat. Sb. (N.S.) **62 (104)** (1963), 264–274.

The authors consider the following non-linear characteristic value problem: (1) $\Delta u + \lambda f(y, u) = 0$ for $0 < y < 1$, and $-\infty < x < \infty$, subject to the boundary conditions (2) $u(x, 0) = u(x, 1) = 0$, and (3) $\lim_{x \to \pm\infty} u(x, y) = 0$. Here λ is a real parameter and $f(y, u)$ is an entire function of u: (4) $f(y, u) = a(y)u + \Phi(y, u)$, with $\Phi(y, u) = \sum_{k=2}^{\infty} f_k(y)u^k$, where $a(y)$ is a positive function belonging to $C^2[0, 1]$.

Let λ_0 be one of the eigenvalues of the Sturm-Liouville boundary problem: $d^2v/dy^2 + \lambda a(y)v = 0$, $v(0) = v(1) = 0$, and put $\mu = \lambda_0 - \lambda$, where μ is small and positive. A crucial step in the authors' approach is to make the "stretching" $(\sqrt{\mu})x = \xi$. They next seek a formal solution of the problem in the form of a series of powers of the parameter μ. Here the concept of a generalized Jordan chain is used [V. A. Trenogin, Dokl. Akad. Nauk SSSR **140** (1961), 311–313; MR **27** #593]. The formal solution depends upon the solution of a non-linear ordinary differential equation of the second order, which can be solved explicitly. Making use of the information gained in the formal solution, the authors proceed to prove an existence theorem for the case when the Jordan chain is of length one. The general approach is similar to that used in one of the solutions of the solitary wave problem [K. O. Friedrichs and the reviewer, Comm. Pure Appl. Math. **7** (1954), 517–550; MR **16**, 413]. *D. H. Hyers* (Los Angeles, Calif.)

Zijahodžaev, M. 3267

A Somil'jan type formula for the higher-dimensional Lamé equation. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 5, 30–36.

The author gives a theorem (analogous to the Gauss divergence theorem for Laplace's equation) for the system of equations

$$\Lambda_\mu \bar{u} \equiv a \sum_{\lambda=1}^{m} \frac{\partial^2 u_\mu}{\partial x_\lambda{}^2} + b \frac{\partial}{\partial x_\mu} \sum_{\lambda=1}^{m} \frac{\partial u_\lambda}{\partial x_\lambda} = 0,$$

where a, b are constants and $\mu = 1, 2, \cdots, m$. He obtains the formal relation, for m-vectors u, v,

$$\int_Q (v_\mu \Lambda_\mu u - u_\mu \Lambda_\mu v)\, dQ = \int_Q \left(a \sum_\lambda \frac{\partial \sigma_\lambda}{\partial x_\lambda} + b \sum_\lambda \frac{\partial \delta_\lambda}{\partial x_\lambda} \right) dQ,$$

where σ_λ, δ_λ are certain functions of the components of u, v and Q is a region of m-space. By assuming v to be a solution vector, and u to satisfy certain continuity conditions, the integral $\int_Q v\Lambda_\mu u\, dQ$ is expressed in terms of a surface integral over the boundary of Q, and some constants related to the singularities of u within Q. Some applications are given to elastic problems.

F. M. Arscott (London)

Suleĭmanov, N. M. 3268

On the solutions of the Cauchy problem for countable systems of partial differential equations in the class of generalized functions. (Russian. Azerbaijani summary)

Akad. Nauk Azerbaĭdžan. SSR Trudy Inst. Mat. Meh. **2 (10)** (1963), 133–141.

The author continues his previous investigations [Akad. Nauk Azerbaĭdžan. SSR Dokl. **16** (1960), 1147–1153; MR **27** #473] concerning countable systems of partial differential equations of the form

$$\frac{\partial u(x, t)}{\partial t} = P\left(i \frac{\partial}{\partial x}\right) u(x, t),$$

where $u(x, t) = \{u_1(x, t), u_2(x, t), \cdots\}$ is a vector-function and $P(i\,\partial/\partial x)$ an infinite matrix of partial differential operators of order $\leq p$. The initial condition is $u(x, 0) = u_0(x)$. Existence of classical solutions is proved in two cases, under assumptions on the spectrum of the operator $P(s)$ and the class of initial functions.

Z. Zieleźny (Wrocław)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 3121.

Drăgilă, Pavel 3269

Les équations fonctionnelles et les fonctions presque périodiques.

C. R. Acad. Sci. Paris **258** (1964), 51–53.

The author states a number of results concerning the existence of almost periodic solutions of related pairs of differential and difference equations. For example, consider the linear differential equation with constant coefficients

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$$

and the related linear difference equation

$$f(x + nh) + a_1 f(x + (n-1)h) + \cdots + a_n f(x) = 0.$$

If the common characteristic equation, $r^n + a_1 r^{n-1} + \cdots + a_n = 0$, possesses an imaginary root, then both the differential equation and the difference equation possess an almost periodic solution. Further, if all roots of the characteristic equation are imaginary, then all solutions of the respective equations are almost periodic functions. Similar results are stated for certain non-homogeneous equations in one variable, and for Laplace's equation and other related equations in two variables.

P. E. Guenther (Cleveland, Ohio)

Gheorghiu, Octavian Em. 3270

Verallgemeinerungen einiger Funktionalgleichungen.

Enseignement Math. (2) **10** (1964), 147–151.

The paper consists of two parts. In the first, the functional equation $f(x+yg(x))=h(x)k(y)$ is solved under the supposition that f, g, h, k are 1-1 maps of the set of real numbers on itself and under the tacit assumption that h is measurable. The general solution is

$$f(x) = bcm(1+kx), \qquad g(x) = a(1+kx),$$
$$h(x) = bm(1+kx), \qquad k(x) = cm(1+kx),$$

where $m(z)=|z|^d \operatorname{sign} z\ (z\neq 0)$, $m(0)=0$, and for the otherwise arbitrary constants $abcdk\neq 0$ has to hold (in the paper z^d stands for $m(z)$ and only $abc\neq 0$ is mentioned). There are also some other solutions given, which do not fulfill the above hypotheses, but there is no statement about their generality. A more complete result under weaker conditions is to be found in a paper of E. Vincze submitted earlier to Ann. Polon. Math. but still in press.

In the second part the system of functional equations

$$g(x+y)g(x-y)+kh(x+y)h(x-y) = g(x)^2+kh(x)^2-g(y)^2-kh(y)^2,$$
$$g(x+y)h(x-y)+h(x+y)g(x-y) = 2g(x)h(x)-2g(y)h(y) \qquad (k = -1, 0, \text{ or } 1)$$

is solved by reduction to the functional equation $f(x+y)f(x-y)=f(x)^2-f(y)^2$ about which it is assumed without motivation that its general measurable hypercomplex solutions are of the same form as the complex ones found by W. H. Wilson [Bull. Amer. Math. Soc. **26** (1919/20), 303–312] and E. Vincze [Mat. Lapok **12** (1961), 18–31; MR **26** #2758]. *J. Aczél* (Gainesville, Fla.)

Daróczy, Z. 3271

Über Mittelwerte und Entropien vollständiger Wahrscheinlichkeitsverteilungen.

Acta Math. Acad. Sci. Hungar. **15** (1964), 203–210.

The principal result of the paper is this: Let Ω be the set of all functions f continuous and strictly monotonic in the half-open interval $(0, 1]$, such that $xf(x)\to 0$ as $x\to 0$. Let T be the totality of vectors $P=(p_1, \cdots, p_n)$, where $n=1, 2, \cdots$, $p_i\geq 0$, $i=1, \cdots, n$, and $\sum_1^n p_i=1$. If for f and h in Ω and every P in T we have $f^{-1}[\sum_1^n p_i f(p_i)]=h^{-1}[\sum_1^n p_i h(p_i)]$, then $h\equiv Af+B$, B and $A\neq 0$ constants, and conversely.

Application is made to the axiomatic characterization of entropies of positive order. *J. Wolfowitz* (Ithaca, N.Y.)

SEQUENCES, SERIES, SUMMABILITY

See also 3125, 3144, 3340.

Hoischen, Lothar 3272

Beiträge zur Limitierungstheorie.

Mitt. Math. Sem. Giessen, Heft 59 (1962), iii+56 *pp.*

In the first part the author considers convergence-preserving Hausdorff means (H.M.) $H=H(\mu_n)$. It is well known that the zeros of the corresponding moment function $M(z)=\int_0^1 t^z d\mu(t)$ play an essential role for the strength of $H(\mu_n)$. For example, H. R. Pitt [Proc. Cambridge Philos. Soc. **34** (1938), 510–520] proved that $|M(z)|\geq d>0$ for $\Re(z)\geq 0$ implies that $W(H(\mu_n))$ ["Wirkfeld" of $H(\mu_n)$] consists of all convergent sequences. Here the author studies H.M. $H(\mu_n)$ whose moment functions $M(z)$ have isolated zeros of arbitrary order in the right half-plane $\Re(z)\geq 0$. He starts with $M(z)=(-\alpha^{-1}(z-\alpha)/(z+1))^k$ where (1) $\Re(\alpha)>0$, (2) $\alpha\neq 0$, imaginary, (3) $\alpha=0$, and describes explicitly in all three cases the Wirkfeld. Then he is able to describe the Wirkfeld of an H.M. $H(\mu_n)$, where $M(z)$ has a zero of order k at α ($\Re(\alpha)>0$), by the Wirkfelder of

$$\tilde{M}(z) = \left(\frac{(z+1)}{(z-\alpha)}\right)^k M(z)$$

and (1) above; this allows a finite iteration of steps, leading to a direct decomposition of the Wirkfeld of $H(\mu_n)$. There are analogous results of continuous H.M., as well as an application to a class of means introduced by W. A. Hurwitz and L. L. Silvermann [Trans. Amer. Math. Soc. **18** (1917), 1–20]. In the second part it is shown that if $\sum a_n$ is $|B|$-summable and if $a_n=O(n^\rho)$, where $\rho\geq -\frac{1}{2}$, then $\sum a_n$ is also $|C_{2\rho+1}|$-summable to the same value. This is the analogous theorem for absolute summability of the well-known theorem by Hardy and Littlewood on Borel and Cesàro summability. *K. Endl* (Salt Lake City, Utah)

Pati, T. 3273

A note on the second theorem of consistency for absolute summability.

Math. Student **29** (1961), 93–100 (1962).

The author gives a simpler proof for U. C. Guha's theorem on absolute Riesz summability [J. London Math. Soc. **31** (1956), 300–311; MR **19**, 135] which asserts, under certain conditions on a function $\varphi(t)$, that $|R, \varphi(\lambda_n), \kappa|$ is stronger than $|R, \lambda_n, \kappa|$ $(\kappa>0)$. *K. Endl* (Salt Lake City, Utah)

Pati, T. 3274

The second theorem of consistency for Riesz boundedness.

Math. Student **29** (1961), 101–112 (1962).

The author studies inclusion problems for Riesz means relative to Riesz boundedness. A series is called bounded (R, λ_n, κ) if its (R, λ_n, κ)-transform is bounded. Theorem 1: Suppose κ is a positive integer. A necessary and sufficient condition that every series which is bounded (R, λ_n, κ) is also bounded $(R, \varphi(\lambda_n), \kappa)$ is that

$$\int_0^t \tau^\kappa|\varphi^{(\kappa+1)}(\tau)|\,d\tau = O(\varphi(t)) \qquad (t\to\infty).$$

Theorem 2: Suppose κ is non-integral and positive. Sufficient conditions that every series which is bounded (R, λ_n, κ) is also bounded $(R, \varphi(\lambda_n), \kappa)$ are that (1) $\varphi'(t)$ is nondecreasing for $t\geq 0$, and (2) $\int_0^t \tau^{[\kappa]+1}|\varphi^{([\kappa]+2)}(\tau)|d\tau=O(\varphi(t))$ $(t\to\infty)$. Theorems 1 and 2 are parallel results to the theorem of Kuttner [J. London Math. Soc. **26** (1951), 104–111; MR **12**, 696] and to the theorem of Hirst [Proc. London Math. Soc. (2) **33** (1932), 353–366] on Riesz summability. *K. Endl* (Salt Lake City, Utah)

Pati, T. 3275

Effectiveness of absolute summability.

Math. Student **28** (1960), 177–187 (1962).

This paper was given as an invited address to the Indian Mathematical Society. It is a survey on the results on effectiveness of absolute summability. Here, an absolute

convergence-preserving method T is called effective if for at least one sequence not of bounded variation, its T-transform is of bounded variation. Some of the central problems arising from this definition and the concept of absolute inclusion are stated. For Abel-Cesàro and Riesz methods a number of results are quoted. Some more information is given on Riesz methods, especially on the "first and second theorems of consistency". Results on ineffectiveness are discussed. An extensive bibliography is provided. *K. Endl* (Salt Lake City, Utah)

Sherif, Soraya 3276

A note on a theorem by J. Karamata.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 176–178.

The author exhibits a sequence $s_n=\sum_{i=0}^n a_i$, which is not bounded above, for which $na_n=O_L(1)$, and whose $(C,1)$-means are bounded above. This parallels a result by Karamata [Comment. Math. Helv. **25** (1951), 64–70; MR **12**, 694] concerning Abel summability.

J. Mayer-Kalkschmidt (Albuquerque, N.M.)

Delange, Hubert 3277

Théorèmes taubériens relatifs à l'intégrale de Laplace.

J. Math. Pures Appl. (9) **42** (1963), 253–309.

The function $V(t)$ is positive for $t>0$ and, for some real ρ, $V(t)=t^\rho L(t)$, where $L(\lambda t)/L(t)\to 1$ as $t\to\infty$, for every $\lambda>0$. The real or complex function $\alpha(t)$ is bounded in every finite part of $t\geqq 0$, $\int_0^\infty e^{-st}\alpha(t)\,dt$ converges to $f(s)$ in $\operatorname{Re} s>0$, and $\alpha(t)$ satisfies the Tauberian condition

$$\limsup_{t\to\infty}\,[V(t)]^{-1}\Big\{\sup_{t\leqq t'\leqq t+h}|\alpha(t')-\alpha(t)|\Big\}=\bar\omega(h)<\infty,$$

or a related but more complicated "one-sided" condition. It is then proved that $\alpha(t)=O[V(t)]$ when, for some integer $p\geqq\max[0,-\rho]$, $f^{(p)}(s)=O\{r^{-p}V(r^{-1})\phi(r)\}$ as $r=|s|\to 0$ in $\operatorname{Re} s>0$, provided that $\phi(t)$ is positive and decreasing,

$$\int_0^l \phi(t)\,dt,\qquad \int_0^l \phi(t)\log[t^{-p}V(t^{-1})]\,dt$$

both converge, and (in the case $p+\rho=0$) that $t^pV(t)$ increases for large enough t.

The conclusion can be strengthened to $\alpha(t)=o[V(t)]$ if $\bar\omega(h)=0$. The more explicit conclusion

$$\limsup|\alpha(t)/V(t)|\leqq W(+0)$$

(with analogous results in the "one-sided" case) follows if it is assumed that, for every real y, either $f^{(p)}(s)$ is bounded as $s\to iy$ in $\operatorname{Re} s>0$ or $f^{(p)}(iy+s)=O\{r^{-p}V(r^{-1})\phi_y(r)\}$ with conditions on ϕ_y similar to those above.

The results were first announced in an earlier paper [C. R. Acad. Sci. Paris **232** (1951), 589–591; MR **12**, 497] and are used here in the special case $V(t)=t^{\omega-2}$ (ω real) to derive closer estimates for $\alpha(t)$ when $f(s)$ takes the form $\sum a_j(s)s^{-\omega_j}+h(s)$ with $h(s)$, $a_j(s)$ regular near 0. Using the fact that the zeta function has no zero in $\operatorname{Re} s\geqq 1$, it is also possible to derive an expression $\sum_{j=0}^q C_j(\log\log x)^{q-j}+o[(\log\log x)^{q-1}(\log x)^{-1}]$ for the sum of the reciprocals of the positive integers up to x which are products of q distinct primes, thus sharpening a result of S. Selberg [Skr. Norske Vid.-Akad. Oslo I **1942**, no. 5; MR **6**, 57].

H. R. Pitt (Nottingham)

APPROXIMATIONS AND EXPANSIONS

See also 3162, 3283, 3320, 3527, 3528, 3529a.

Schoenberg, I. J. 3278

Spline interpolation and the higher derivatives.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 24–28.

The author considers the class of functions

$$F_m[a,b]=\{f(x)\,|\,f\in C^{m-1}[a,b],\ f^{(m-1)}\text{ absolutely continuous},\ f^{(m)}\in L^2(a,b)\}$$

and spline interpolation. Let $S_p(x|\Delta,f)$ be a spline function of degree $p-1$ such that $S_p(x_i|\Delta,f)=f(x_i)$, where the x_i are the joints of S_p and also determine the partition Δ of $[a,b]$. The main result is that $f(x)\in F_m[a,b]$ if and only if there is a constant K such that

$$(1)\qquad \int_a^b [S_{2m-1}^{(m)}(x|\Delta,f)-f(x)]^2\,dx\leqq K \text{ for all }\Delta.$$

It is shown that one need consider only equally spaced partitions of $[a,b]$. This result may be rephrased as follows. The interpolating spline $S_{2m-1}(x|\Delta,f)$ is such that

$$\lim_{\|\Delta\|\to 0}\int_a^b [S_{2m-1}^{(m)}(x|\Delta,f)-f^{(m)}(x)]^2\,dx=0$$

if and only if $f(x)\in F_m[a,b]$. It would be interesting to know if the class $F_m[a,b]$ is also characterized by

$$\lim_{\|\Delta\|\to 0}\int_a^b [S_p^{(m)}(x|\Delta,f)-f^{(m)}(x)]^2\,dx=0,\quad p\geqq m+2.$$

By use of special properties of spline functions, condition (1) may be replaced by

$$(2)\qquad h\sum_{i=0}^{n-m}(\Delta_h^m f(x_i)/h^m)^2\leqq K\quad\text{for all } n,$$

where $h=(b-a)/n$ and Δ_h^m is the mth forward difference operator with spacing h. Condition (2) for $m=1$ is due to F. Riesz [Math. Ann. **69** (1910), 449–497].

J. R. Rice (Warren, Mich.)

FOURIER ANALYSIS

See also 3136, 3170, 3269, 3324, 3329, 3526.

Kahane, Jean-Pierre; Salem, Raphael 3279

★Ensembles parfaits et séries trigonométriques.

Actualités Sci. Indust., No. 1301.

Hermann, Paris, 1963. 192 *pp.* 27 *F.*

This elegant book treats a variety of topics in the theory of trigonometric series—some fairly well known, but most of them quite recent—unified by the occurrence in each case of special classes of perfect sets. Some of the material can also be found in Zygmund's book [*Trigonometric series*, 2nd ed., Cambridge Univ. Press, New York, 1959; MR **21** #6498], but most of it has not previously been collected in one place. The first three chapters discuss the various classes of perfect sets (generalizing the Cantor set in various ways); Hausdorff measure and dimension; one-dimensional potential theory and the notion of capacity (especially capacity of order α and logarithmic capacity) of a set. Chapter 4 shows that there is a (real) trigonometric series $\sum(a_n\cos nx+b_n\sin nx)$ with $\sum(a_n^2+b_n^2)n^\beta<\infty$

$(0<\beta<1)$, diverging on the closed set E, if and only if the $1-\beta$ capacity of E is 0 (for $\beta=1$, read "logarithmic capacity"). The "if" part is an unpublished result of Beurling; the authors prove a still more general result. The authors also include a related result on radial limits, contributed by Zygmund: if $\sum(a_n^2+b_n^2)n^\beta<\infty$ $(0<\beta\leqq 1)$, E is closed, and if for all $t\in E$ we have

$$\int_0^1\left|\frac{\partial}{\partial r}\sum r^n(a_n\cos nt+b_n\sin nt)\right| = \infty,$$

then E is of zero $1-\beta$ capacity $(\beta<1)$, of zero logarithmic capacity if $\beta=1$. Chapters 5 and 6 treat sets of uniqueness and sets of multiplicity; Chapter 5 contains general results and Chapter 6 is devoted to the role of the Pisot numbers. Chapter 7 is concerned with sets of absolute convergence (N-sets); these are the sets E such that there is a trigonometric series converging absolutely on E without converging absolutely everywhere. The Denjoy-Lusin theorem says that N-sets are of measure 0. The authors prove the following generalization: if $f_k(x)$ are integrable and converge weakly (in L) to 1 and if $\sum r_kf_k(x)$ converges in a set of positive measure, with $r_k\geqq 0$, then $\sum r_k$ converges. Chapter 8 applies probabilistic methods to the problem of the asymptotic behavior of the Fourier-Stieltjes coefficients of measures carried by a set. Rather sparse perfect sets can carry measures whose Fourier transforms tend rather rapidly to zero. The chapter culminates with the most detailed information available on the span of the translations of an element of l^p $(1<p<2)$. Chapter 9 deals exhaustively with the problem of harmonic synthesis, set by Beurling and solved by Malliavin. Chapters 10 and 11 deal with properties of functions of class $A(E)$ (restrictions to E of functions with absolutely convergent Fourier series). A Carleson set E is a closed set such that every continuous function on E is represented on E by a Fourier power series $\sum\gamma_ne^{int}$ with $\sum|\gamma_n|<\infty$; a Helson set is the same thing with (two-sided) Fourier series; the principal result [Wik, Ark. Mat. **4** (1961), 209–218; MR **23** #A2707] is that these classes coincide. Chapter 12 discusses functions defined by Fourier series with gaps; the problem is to decide whether a given perfect set E carries such a function which vanishes on E but is not identically 0 [Kahane, Ann. Sci. École Norm. Sup. (3) **79** (1962), 93–150; MR **27** #4019]. A note at the end of the book states stop-press results and some open problems. {The misprint $O(1)$ for $o(1)$ in the definition of pseudofunctions on p. 163 will not confuse the thoughtful reader.}

R. P. Boas, Jr. (Evanston, Ill.)

Chen, Yung-Ming [Chen, Yung-ming] 3280
On conjugate functions.
Canad. J. Math. **15** (1963), 486–494.

Babenko [Dokl. Akad. Nauk SSSR **62** (1948), 157–160; MR **10**, 249] proved that if $|f(x)|^p|x|^\alpha\in L(-\pi,\pi)$, for $p>1$, $-1<\alpha<p-1$, then

$$\int_{-\pi}^{\pi}|\tilde f(x)|^p|x|^\alpha\,dx \leqq A\int_{-\pi}^{\pi}|f(x)|^p|x|^\alpha\,dx,$$

where $\tilde f$ is the conjugate function of $f(x)$, and A depends on p and α only. This result has been generalised by the author in a previous paper [Math. Ann. **140** (1960), 360–407; MR **22** #3777]. Concerning even and odd functions, stronger results have been obtained by K. K. Chen [Amer. J. Math. **66** (1944), 299–312; MR **5**, 262] and Flett [Proc. London Math. Soc. (3) **8** (1958), 135–148; MR **22** #3933].

In the present paper the author establishes theorems which include results of Chen and Flett. Further, the author proves that if $\tilde f(x)$ is the conjugate function of $f(x)\in L(-\pi,\pi)$, then

$$\int_{-\pi}^{\pi}\beta(x)|\tilde f(x)|\,dx \leqq K\int_{-\pi}^{\pi}\beta(x)|f(x)|\log^+\beta(x)|f(x)|\,dx+K,$$

where $\beta(x)x^{-1-\varepsilon}$ is non-decreasing and $\beta(x)$ is non-increasing. This generalises a previous result due to Zygmund [*Trigonometric series*, 2nd ed., Vol. I, p. 254, Cambridge Univ. Press, New York, 1959; MR **21** #6498] that if $|f(x)|\log^+|f(x)|\in L(-\pi,\pi)$, then $\tilde f(x)\in L(-\pi,\pi)$. Analogous theorems for Hilbert transforms have also been established.

B. N. Prasad (Allahabad)

Chen, Yung-ming 3281
Remark on uniqueness of summable trigonometric series associated with conjugate series.
Monatsh. Math. **68** (1964), 17–20.

The author observes that if a trigonometric series is Cesàro summable (of some order greater than -1) to zero, together with its conjugate series, on a set of positive measure, then all its coefficients are zero; and shows that if E is of measure zero, then there is a nontrivial trigonometric series that is Cesàro summable (of every positive order) to zero, together with its conjugate series, on E.

R. P. Boas, Jr. (Evanston, Ill.)

Tomčuk, Ju. Ja. 3282
Polynomials which are orthogonal over a given system of arcs on the unit circle. (Russian)
Dokl. Akad. Nauk SSSR **151** (1963), 55–58.

In the classical theory of polynomials orthogonal over the unit circle, the weight function is usually required to be positive a.e. Recently, N. I. Ahiezer [same Dokl. **130** (1960), 247–250; MR **22** #8263] found asymptotic expressions for the polynomials orthonormal with respect to a weight function which vanishes on a full arc. The present paper announces generalizations of these results to weight functions which vanish on a finite number of arcs.

P. L. Duren (Ann Arbor, Mich.)

Geronimus, Ja. L. 3283
The relation between the order of growth of orthonormal polynomials and their weight function. (Russian)
Mat. Sb. (N.S.) **61 (103)** (1963), 65–79.

Let $\sigma(\theta)$ be monotonically non-decreasing and bounded on $[0,2\pi]$, and let $\{\varphi_n(z)\}$ denote the corresponding orthogonal polynomials $\int_0^{2\pi}\varphi_n(e^{i\theta})\overline{\varphi_m(e^{i\theta})}\,d\sigma(\theta)=2\pi\delta_{nm}$. The author proves a number of quantitative results (tabulated at the end of the paper) which connect the rate of growth of $|\varphi_n(e^{i\theta_0})|$ at a fixed point with the behavior of $\sigma'(\theta)$ near θ_0. Roughly speaking, the faster $|\varphi_n(e^{i\theta_0})|$ grows, the more strongly $\sigma'(\theta)$ must vanish at θ_0. A sample result: Suppose $\sigma'(\theta)$ exists throughout a neighborhood of θ_0; then

$$\limsup_{n\to\infty}\left\{\frac{\log|\varphi_n(e^{i\theta_0})|}{\log n}\right\} = \infty$$

if and only if $\sigma'(\theta)$ has a zero of higher than algebraic order (in a certain precise sense) at θ_0.

P. L. Duren (Ann Arbor, Mich.)

Taberski, R. 3284
Approximation to conjugate function.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 255–260.
Let $f \in L$ be of period 2π, and let $\rho_k(\xi)$ be defined on a set E of real numbers having an accumulation point ξ_0. The results given in this paper concern the order of convergence of an operator

$$\bar{f}(x, \xi) = -\frac{2}{\pi}\int_0^\pi \psi_x(t)u(t, \xi)\,dt$$

to the conjugate function

$$\bar{f}(x) = -\frac{1}{\pi}\int_{0+}^\pi \psi_x(t)\cot\frac{t}{2}\,dt, \qquad \xi \to \xi_0,$$

where

$$\psi_x(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\}$$

and $u(t, \xi) = \sum_{k=1}^\infty \rho_k(\xi)\sin kt$ converges uniformly with respect to t in $\langle 0, \pi\rangle$. One of the author's results is that

$$(*)\quad \sup_{f\in \mathrm{Lip}\,\alpha}\left\{\max_{-\pi\le x\le\pi}|\bar{f}(x,\xi)-\bar{f}(x)|\right\}$$
$$= \frac{2^\alpha}{\pi}\Gamma(\alpha)\cos\frac{\alpha\pi}{2}\sum_{k=0}^\infty\frac{\Delta\rho_k(\xi)}{(k+\frac{1}{2})^\alpha}+O\left(\sum_{k=0}^\infty\frac{|\Delta\rho_k(\xi)|}{k+\frac{1}{2}}\right) \text{ if } \alpha < 1,$$
$$= 1-\rho_1(\xi)+O(2-3\rho_1(\xi)+\rho_3(\xi)) \text{ if } \alpha = 1,$$

as $\xi \to \xi_0$ on E. In particular, if the right-hand side of (*) tends to zero as $\xi \to \xi_0$, $\bar{f}(x, \xi) \to \bar{f}(x)$ uniformly in x.
B. N. Prasad (Allahabad)

Taberski, R. 3285
On the summability of Fourier and Bessel series of Hölder functions.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 643–647.
Denote by $H^\alpha\langle a, b\rangle$ the class of all functions f Lebesgue integrable over $\langle a, b\rangle$ and satisfying a Hölder condition $|f(t_1)-f(t_2)| \le |t_1-t_2|^\alpha$ for each $t_1, t_2 \in \langle a, b\rangle$. Let $S_n(x)$ be the nth partial sum of the Fourier series of a Lebesgue integrable function f, periodic with period 2π. In this paper the author obtains estimates for the quantities

$$\left\{\frac{1}{n+1}\sum_{k=0}^n |S_k(x)-f(x)|^p\right\}^{1/p}, \quad (1-r)\sum_{k=0}^\infty |S_k(x)-f(x)|r^k$$

if $f \in H^\alpha\langle a, b\rangle$, where $b-a=\pi$ and $0<\alpha<\frac{1}{2}$.
He further proves a theorem concerning the order of the convergence $\sigma_n^{(\nu)}(x) \rightrightarrows f(x)$ for $f \in H^\alpha\langle a, b\rangle$, where $\sigma_n^{(\nu)}(x)$ denotes the Riesz mean of order one of the Fourier-Bessel series $\sum_{m=1}^\infty d_m J_\nu(j_m x)$ of the function $f \in L\langle 0, 1\rangle$.
B. N. Prasad (Allahabad)

Ishiguro, Kazuo 3286
On circle and quasi-Hausdorff methods of summability of Fourier series.
Proc. Japan Acad. **38** (1962), 272–277.
In this paper the author, after first giving some known results regarding Lebesgue constants and Gibbs' phenomenon pertaining to the Euler and circle methods of summability, states that there are similar interesting relations between Hausdorff and quasi-Hausdorff methods of summability. He poses a few problems without proofs.
B. N. Prasad (Allahabad)

Rosenblum, Marvin 3287
Summability of Fourier series in $L^p(d\mu)$.
Trans. Amer. Math. Soc. **105** (1962), 32–42.
Let μ be a non-negative finite Borel measure on the unit circle C, and let $L^p(d\mu)$, $1 \le p < \infty$, be the Banach space of μ-measurable complex-valued functions $f(z)$, $z=e^{i\phi}$, such that

$$\|f\|_p = \left[\int |f(e^{i\phi})|^p\,d\mu(\phi)\right]^{1/p} < \infty;$$

let $H^p(d\mu)$ be the class of functions $f(z)$, $z=re^{i\phi}$, holomorphic in $0 \le r < 1$ and such that

$$\|f\|_{H,p} = \sup\left[\int |f(re^{i\phi})|^p\,d\mu(\phi)\right]^{1/p} < \infty.$$

Denoting $\mathfrak{P}$ and $\mathfrak{P}_0$ as the classes of trigonometrical polynomials $\sum c_n e^{in\phi}$ and $\sum_{n\ge0} c_n e^{in\phi}$, respectively, and $P_r(e^{i\phi})$ the Poisson kernel and $*$ the symbol of Fourier convolution, the author studies the summability of Fourier series in $L^p(d\mu)$ by characterizing the classes $\mathfrak{L}_p$ and $\mathfrak{B}_p$ of measures μ such that

$$(1)\qquad \sup\{\|P_r * f\|_p : \delta < r < 1\} \le K\|f\|_p$$

for all $f \in \mathfrak{P}_0$ and $\mathfrak{P}$, respectively. The author establishes that $\mu \in \mathfrak{L}_p$ if and only if (1) μ is absolutely continuous, $d\mu = \omega\,d\sigma$, σ being the normalized Lebesgue measure on C; $\omega = |g|$, $g \in H^1(d\sigma)$, is an outer function and

$$(2)\qquad \int P_r(e^{i(\phi-\psi)})\,|g(e^{i\phi})/g(re^{i\phi})|\,d\sigma(\phi) \le K$$

for all r, $0 \le r < 1$ and ψ is real.
For the class $\mathfrak{B}_p$ the author also establishes a number of theorems. He further establishes some sufficient local conditions for the class $\mathfrak{B}_p$ and considers the case when Abel summability is replaced by Fejér and several other types of summabilities. *B. N. Prasad* (Allahabad)

Varshney, O. P. 3288
On the absolute Nörlund summability of a Fourier series.
Math. Z. **83** (1964), 18–24.
Let p_n be a sequence of real constants, $P_n = p_0 + \cdots + p_n$. We say that s_n is summable (N, p_n) if the sequence $t_n = (1/P_n)\sum_{k=0}^n p_{n-k}s_k$ is a convergent sequence. It is absolutely summable (N, p_n) if $\sum|t_n - t_{n-1}|$ converges. Let $\varphi(t) = \frac{1}{2}[f(x+t)+f(x-t)]$ be of bounded variation on $(0, \pi)$. Then the Fourier series of $f(t)$ is absolutely summable (N, p_n) at $t=x$ if (i) $p_n \ge 0$, $P_n \to \infty$ as $n\to\infty$, (ii) $\sum|v_n - v_{n+1}| < \infty$, where $v_n = (n+1)p_n/P_n$, and (iii) $\sum_{k=n}^\infty P_n/(k+2)P_k < A$ $(n = 0, 1, 2, \cdots)$. This generalizes a theorem of Pati [J. London Math. Soc. **34** (1959), 153–160; MR **21** #2860], where (iii) was replaced by a stronger condition and $p_{n+1} \le p_n$ was required.
R. A. Askey (Madison, Wis.)

Nanda, M. M. 3289
The summability (L) of the rth derived Fourier series.
Quart. J. Math. Oxford Ser. (2) **15** (1964), 16–22.
The author has previously found [same J. (2) **13** (1962), 229–234; MR **26** #542] sufficient conditions for the logarithmic summability of the first derivative of a Fourier series. In this paper he generalizes this theorem to the rth differentiated Fourier series. *R. A. Askey* (Madison, Wis.)

Topurija, S. B. 3290

Summability of Fourier series by the $L^{(p)}$ method and by Voronoĭ's method. (Georgian. Russian summary)

Soobšč. Akad. Nauk Gruzin. SSR **32** (1963), 513–519.

A series with partial sums S_k is summable $L^{(p_n)}$ (misprinted $L_n^{(p)}$ in the text) if the means

$$(\log P_n)^{-1} \sum_{k=0}^{n} (p_k/P_k)S_k$$

converge, where $p_n > 0$, $P_n = \sum_1^n p_k \to \infty$, $p_n/P_n \to 0$. Suppose $p_k P_{k+1} - p_{k+1} P_k \geqq 0$, or $p_k P_{k+1} - p_{k+1} P_k < 0$ and $np_n = O(P_n \log P_n)$. Theorem 1: A Fourier series is $L^{(p_n)}$ summable on the Lebesgue set. Voronoĭ summability is what is usually called Nörlund summability in the Western literature. Theorem 2: Under conditions on p_n (the coefficients in the summation method) that seem to be garbled in the text, a Fourier series is Nörlund summable on the Lebesgue set. The author's main interest is in extending these results to double series. For $L^{(p_n, q_n)}$ the means are

$$\frac{1}{\log P_m \log Q_n} \sum_{i=0}^{m} \sum_{k=0}^{n} \frac{p_i q_k}{P_i Q_k} S_{i,k},$$

with obvious notation. Theorem 3: Let $\Delta(p_i/P_i) \geqq 0$, $\Delta(q_k/Q_k) \geqq 0$ (or < 0 in each case with $mp_m = O(P_m \log P_m)$, $nq_n = O(Q_n \log Q_n)$). Then the Fourier series of $f(x, y)$ is $L^{(p_n, q_n)}$ summable at points where we have

$$\lim_{h,k \to 0} \frac{1}{hk} \int_x^{x+h} \int_y^{y+k} |f(t, \tau) - f(x, y)| \, dt d\tau = 0,$$

$$\sup_{0<h<b-x} \frac{1}{h} \int_x^{x+h} \int_c^d |f(t, \tau) - f(x, y)| \, dt d\tau < \infty,$$

and the corresponding condition with x and y interchanged. It follows that there is summability almost everywhere when $f(x, y) \log^+ f(x, y) \in L$. Theorem 4: Same conclusion for double Nörlund summability, the means being defined by

$$P_{m,n}^{-1} \sum_{i=0}^{m} \sum_{k=0}^{n} P_{m-i, n-k} S_{i,k},$$

where $p_{i,k} \geqq 0$, $p_{0,0} \neq 0$, $P_{m,n}$ are the rectangular partial sums of $\sum\sum p_{i,k}$, $\sum_{k=0}^{n} p_{m,n-k}$ and $\sum_{i=0}^{m} p_{m-i,n}$ are $o(P_{m,n})$, and either $p_{i,k} - p_{i,k-1} - p_{i-1,k} + p_{i+1,k-1} \geqq 0$, or < 0 with $mn = O(P_{m,n})$. Proofs are only sketched. It seems difficult to interpret Theorem 2 in any valid way, since Hille and Tamarkin [Trans. Amer. Math. Soc. **34** (1932), 757–783] showed that when p_n decreases or satisfies some mild auxiliary conditions, the condition $\sum_{k=1}^{n} k^{-1} P_k = O(P_n)$ is necessary as well as sufficient for the Nörlund method with coefficients p_n to sum all Fourier series at all Lebesgue points. The Russian summary is nothing but an expanded form of the title. *R. P. Boas, Jr.* (Evanston, Ill.)

Arsen'ev, A. A. 3291

On the Fourier transform of a slowly decreasing function. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 251–253.

The asymptotic form, valid near zero, for the Fourier transform of a function $\omega(x) = \sum_1^N q_r^{\pm} x^{-r} + \omega_N^{\pm}(x)$ as $x \to \pm\infty$, with $\omega_N(x) = o(x^{-N})$, $\omega_N(x)x^{N-1}$ summable, is found. *J. L. B. Cooper* (Cardiff)

Igari, Satoru 3292

Correction: "On the decomposition theorems of Fourier transforms with weighted norms".

Tôhoku Math. J. (2) **15** (1963), 399.

An extended list of errata and misprints which appeared in the original article [same J. (2) **15** (1963), 6–36; MR **26** #4113].

Cooper, J. L. B. 3293

Fourier transforms and inversion formulae for L^p functions.

Proc. London Math. Soc. (3) **14** (1964), 271–298.

It is well known that if $f \in L_p(-\infty, \infty)$, $1 < p \leqq 2$, then f has a Fourier transform $\hat{f} \in L_{p'}(-\infty, \infty)$, where $p^{-1} + p'^{-1} = 1$. It is also known that the collection of Fourier transforms of L_p functions is a proper subset of $L_{p'}$, and a variety of necessary and/or sufficient conditions have been given by various authors that given a function $F \in L_{p'}$, f should exist in L_p so that $F = \hat{f}$. Often these conditions have the form

$$(*) \qquad \|f_\lambda\|_p \leqq M,$$

where

$$(**) \qquad f_\lambda(u) = \int k(u, v, \lambda) F(v) \, dv$$

and $k(u, v, \lambda)$ is some kernel, λ ranging over some directed set.

In this paper the author attempts to unify and generalize these scattered representation theories. He considers an arbitrary kernel k and examines the questions: (a) Is (*) a necessary condition that $F \in \hat{L}_p$; (b) Is (*) a sufficient condition that $F \in \hat{L}_p$; (c) If the answer to (b) is affirmative, does $f_\lambda \to f$ in some sense, where $\hat{f} = F$. Various conditions are given on k for each of the questions to be answered affirmatively. As an example, he shows that if $k(u, v, \lambda) \in L_p$ as a function of v for almost all u, and if $K_2(u, v, \lambda) \in L_p$ as a function of v for almost all u, and $K_2(u, v, \lambda) \in L_{p'}$ as a function of u for almost all v, where

$$K_2(u, v, \lambda) = \int e^{-ivt} k(u, t, \lambda) \, dt,$$

then (a) is answered affirmatively.

The author also considers kernels $k(n, v, \lambda)$, where (*) is replaced by a sequential condition, and poses and answers the same questions. He considers, too, the representation problem for Fourier coefficients, posing analogous questions and obtaining analogous answers.

The paper is well illustrated by examples, including all the known theories, and many others. *P. G. Rooney* (Toronto, Ont.)

Strzelecki, E. 3294

On a problem of interpolation by periodic and almost periodic functions.

Colloq. Math. **11** (1963), 91–99.

A sequence $\{t_n\}$ of positive numbers is said to have the property (P) or (P'), respectively, in a class $\mathscr{K}$ of sequences of real numbers, if for every $\{\varepsilon_n\} \in \mathscr{K}$ there exists a continuous periodic or almost periodic (in the sense of Bohr) function $f(t)$ $(-\infty < t < \infty)$ such that $f(t_n) = \varepsilon_n$ for $n = 1, 2, \cdots$. The following theorems are proved. Theorem 1:

Let $q_n = t_{n+1}/t_n$. Every sequence $\{t_n\}$ satisfying the conditions (1) $q_n \geqq 1+\alpha$ ($n=1, 2, \cdots; \alpha>0$), (2) if $q_n<3$, then $q_{n+1} \geqq 2(3+\beta)/(q_n-1)$ ($\beta>0$), (3) if $3 \leqq q_n \leqq 3+\beta$, then $q_{n+1} \geqq 3+\beta$, has the property (P) in the class $\mathscr{X}_2$ of all sequences taking values 0 or 1. Theorem 2: Every sequence $\{t_n\}$ for which

$$t_{n+1} \geqq (1+\alpha)t_n \qquad (n = 1, 2, \cdots; \alpha > 0)$$

has the property (P') in the class $\mathscr{X}$ of all bounded sequences. *E. Følner* (Copenhagen)

Price, J. J. 3295

On an equality involving group characters.

Proc. Amer. Math. Soc. **14** (1963), 869–874.

Let X denote a second countable compact abelian group, μ its Haar measure. Since its character group Ψ is necessarily countable, we may write $\Psi = \{\psi_j\}$ ($j=0, 1, \cdots$) with $\psi_0(x)=1$. Let Π_n be the class of functions $f(x)$ defined on X with the properties: (a) $f(x) = \sum_{j=0}^n a_j\psi_j(x)$, where the a_j are complex; (b) $f(x) \geqq 0$ if $x \in X$; (c) $\int_X f(x)d\mu(x)=1$. Define $M_n(x) = \sup f(x)$ for $f \in \Pi_n$. It is shown that $M_n(x) = M_n$, independent of x. In analogy with an extremum problem concerning polynomials [G. Szegö, *Orthogonal polynomials*, p. 181, Amer. Math. Soc., Providence, R.I., 1959; MR **21** #5029] the author shows that the inequality (1) $M_n \leqq n+1$ always holds. Using his previous results on orthonormal systems with non-negative Dirichlet kernels [Trans. Amer. Math. Soc. **100** (1961), 153–161; MR **23** #A3420] the author obtains necessary and sufficient conditions for equality to hold in (1) for a preassigned infinite sequence of indices n.

I. J. Schoenberg (Princeton, N.J.)

INTEGRAL TRANSFORMS, OPERATIONAL CALCULUS

See also 3126, 3178, 3277, 3351.

Hirschman, I. I., Jr. 3296

Variation-diminishing transformations and general orthogonal polynomials.

Canad. J. Math. **16** (1964), 98–107.

Let $\alpha(dx)$ be a finite measure defined on the Borel subsets of $[-1, 1]$, the spectrum of which is infinite. Consider the decomposition $\alpha(dx) = \alpha_s(dx) + \alpha_c(x)dx$ of α into a singular part and an absolutely continuous part, and let

$$(1) \qquad \int_{-1}^{+1} \log[\alpha_c(x)](1-x^2)^{-1/2}\,dx > -\infty.$$

With a suitable definition of "variation-diminishing multiplier" (this definition is too complex to be reproduced here), the main result of the paper may be stated as follows. If $\alpha(dx)$ satisfies the condition (1), then $M(x)$ is a variation-diminishing multiplier if and only if it is of the form $M(x) = de^{cx}\prod_k(1+a_kx)\prod_k(1-b_kx)^{-1}$, where d is real and $c \geqq 0$, $1 \geqq a_k > 0$, $1 > b_k > 0$, $\sum_k (a_k+b_k) < \infty$.

A. Edrei (London)

Haimo, Deborah Tepper 3297

Variation diminishing transformations.

Bull. Amer. Math. Soc. **70** (1964), 271–274.

Let $\mathfrak{J}(x) = 2^{\gamma-1/2}\Gamma(\gamma+1/2)x^{-1/2-\gamma}J_{\gamma-1/2}(x)$, where J is the Bessel function, and $\mu(x) = x^{2\gamma+1}/2^{\gamma+1/2}\Gamma(\gamma+3/2)$, γ a fixed positive number. Necessary and sufficient conditions are given for a function f to be represented as

$$f(x) = \int_0^\infty G(x, t)\,d\psi(t), \quad \psi(t)\uparrow,$$

where

$$G(x, t) = \int_0^\infty \frac{\mathfrak{J}(xy)\mathfrak{J}(ty)}{E(y)}\,d\mu(y), \quad 0 \leqq x, t < \infty,$$

$$E(y) = \prod_{k=1}^\infty (1+y^2/a_k^2),$$

$$0 < a_1 \leqq a_2 \leqq \cdots, \qquad \sum_{k=1}^\infty 1/a_k^2 < \infty.$$

Also, conditions for the inversion of

$$f(x) = \int_0^\infty G(x, t)\varphi(t)\,d\mu(t), \qquad 0 < x < \infty,$$

where φ is locally integrable for $t \geqq 0$, are stated. Proofs are outlined; details are to appear later.

A. E. Danese (Buffalo, N.Y.)

O'Neil, R.; Weiss, G. 3298

The Hilbert transform and rearrangement of functions.

Studia Math. **23** (1963), 189–198.

If f is a measurable function on a measure space M with measure m, then the distribution of f, $\lambda_{|f|}$, is defined by $\lambda_{|f|}(y) = m\{x \in M; |f(x)| > y\}$ for each $y>0$. With

$$f^*(t) = \inf\{y > 0; \lambda_{|f|}(y) \leqq t\},$$

$$f^{**}(s) = (1/s)\int_0^s f^*(t)\,dt, \quad H(s) = \int_0^\infty f^*(t)\sinh^{-1}(s/t)\,dt,$$

the principal result of the authors is that if $H(1)<\infty$, then the Hilbert transform

$$\tilde{f}(x) = \lim_{\varepsilon\to 0+} \frac{1}{\pi}\int_{|t-x|>\varepsilon} \frac{f(t)}{t-x}\,dt$$

exists a.e. and for each $s>0$

$$s\tilde{f}^{**}(s) \leqq \frac{2}{\pi}H(s) = \frac{2s}{\pi}\int_0^\infty \frac{f^{**}(t)}{\sqrt{(s^2+t^2)}}\,dt.$$

This result enables the authors to derive the M. Riesz inequality for the Hilbert transform of functions in L^p, $1<p<\infty$. The analogue for the conjugate function in the periodic case is considered. Extensions to singular integral operators in n-dimensional Euclidean space.

P. L. Butzer (Aachen)

Mercer, A. McD. 3299

A theory of integral transforms.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 149–154.

Wenn die Mellin-Transformierten $K_M(s)$, $H_M(s)$ von $K(x)$, $H(x)$ die Gleichung $K_M(s)H_M(1-s)=1$ erfüllen, so besteht bekanntlich formal das Paar von Transformationen

$$\bar{g}(\lambda) = \int_0^\infty g(x)K(\lambda x)\,dx, \qquad g(x) = \int_0^\infty \bar{g}(\lambda)H(\lambda x)\,d\lambda.$$

Hier hängen die Kerne nur vom Produkt der Variablen ab.

Verfasser ersetzt die Gleichung für die Mellin-Transformierten durch zwei mit einer Funktion $w(\mu, x)$ gebildete Gleichungen für K und H und erhält unter gewissen Bedingungen dasselbe Paar von Transformationen mit allgemeinen Kernen. Wenn K und H_s durch die Gleichungen

$$\int_0^\infty w(\mu, x)K(\lambda, x)\,dx = \frac{\lambda}{\lambda^2+\mu^2},$$

$$H_s(t, x) = \frac{i}{\pi}\{w(e^{is\pi/2}t, x) - w(e^{-is\pi/2}t, x)\}$$

definiert werden, so gilt

$$\lim_{s\to 1-0} \int_0^\omega f(t) \int_0^\infty K(\lambda, x)H_s(t, x)\,dx dt = \tfrac{1}{2}\{f(\lambda-0)+f(\lambda+0)\}.$$

Wenn hier die Reihenfolge der Integrationen vertauscht werden darf und der Grenzübergang $s\to 1-0$ sich unter dem Doppelintegral ausführen läßt, so erhält man mit $\lim H_s(t, x) = H(t, x)$ die Gleichung

$$\int_0^\infty K(\lambda, x)\int_0^\infty f(t)H(t, x)\,dt dx = \tfrac{1}{2}\{f(\lambda-0)+f(\lambda+0)\}$$

und damit das gewünschte Paar von Integraltransformationen. *G. Doetsch* (Freiburg)

Eliaš, Jozef 3300

Rational functions of a differentiation operator. (Slovak. Russian and English summaries)

Mat.-Fyz. Časopis Sloven. Akad. Vied **12** (1962), 263–270.

In the author's previous paper [same Časopis **8** (1958), 203–227; MR **21** #4312] an operational method for solution of a linear differential equation was introduced. Now, he proves the formulae for the operators $(s-\alpha)^{-k}$, $(s^2+\beta^2)^{-k}$, $s(s^2+\beta^2)^{-k}$, $[(s-\alpha)^2+\beta^2]^{-k}$, where $\alpha, \beta \neq 0$ are real numbers and k is a positive integer. *M. Ráb* (Brno)

INTEGRAL EQUATIONS

See also 3249, 3460, 3576, 3787, 3837.

Bancuri, R. D.; Džanašija, G. A. 3301

Equations of convolution type for the semi-axis. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 251–253.

The equation

$$\varphi(x) - \int_0^\infty k(x-y)\varphi(y)\,dy = f(x), \qquad 0 \leq x < \infty,$$

has been considered by numerous authors for various classes of functions, in particular, by Wiener and Hopf [S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. **1931**, 696–706]. The present article gives solutions, via Fourier transform and a Hilbert boundary problem, in the case that f and φ are in $L^1(0, \infty)$, k is in $L^1(-\infty, \infty)$, and the Fourier transform of k does not assume the value 1.

R. T. Seeley (Waltham, Mass.)

Mangeron, Demetrio 3302

Risolubilità e struttura delle soluzioni dei problemi al contorno non omogenei di Goursat e di Dirichlet per le equazioni integro-differenziali lineari a derivate totali d'ordine superiore.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 118–122.

The author considers two problems: Goursat's problem,

$$[D^i u(x, y)]_{x=a} = \varphi_i(y), \qquad [D^i u(x, y)]_{y=c} = \psi_i(x) \qquad (i = 0, 1, \cdots, k-1),$$

for the linear integro-differential equation

$$D^k u(A) + \lambda \sum_{i=k-1}^{0} \sum_{j=k-1}^{0} r_{ij}(A)\frac{\partial^{i+j}u(A)}{\partial x^i \partial y^j} = f(A) + \lambda \iint_S \sum_{i=0}^{k}\sum_{j=0}^{k} K_{ij}(A, B)\frac{\partial^{i+j}u(B)}{\partial \xi^i \partial \eta^j}\,dB,$$

where $Du(A) \equiv \partial^2 u(x, y)/\partial x \partial y$ is Picone's total derivative and $f(A) \equiv f(x, y)$; $K_{ij}(A, B) \equiv K_{ij}(x, y, \xi, \eta)$, $\varphi_i(y)$; $\psi_i(x)$; $r_{ij}(A) \equiv r_{ij}(x, y)$ are continuous functions defined for all x, $\xi \in [a, b]$; y, $\eta \in [c, d]$, $S = [a, b]\times[c, d]$. $u(A) \equiv u(x, y)$ is the unknown function, λ is a parameter.

Theorem I concerns the solution of the problem for (a) λ not an eigenvalue, and (b) λ an eigenvalue.

The second problem is that of Dirichlet:

$$u(A)|_R = \mu_0(x, y), \qquad D^i u(A)|_{R_1} = \mu_i(x, y) \qquad (i = 1, 2, \cdots, k-1),$$

and the same integro-differential equation as before, where R is the contour of the S domain

$$R_1 \equiv x = a, \quad c \leq y \leq d, \qquad \equiv y = c, \quad a \leq x \leq b,$$

and $\mu_i(x, y)$ $(i = 0, 1, \cdots, k-1)$ are continuous functions for all $(x, y) \in S$.

Theorem II states that for this problem an equivalent system of integral equations can be formed, and the solution of these equations can be discussed in two parts: (a) λ an eigenvalue, (b) λ not an eigenvalue.

O. Gürel (Ankara)

Mangeron, Demetrio; Krivošein, L. E. 3303

Valutazione degli errori commessi in alcuni metodi di calcolo numerico delle soluzioni di una classe di problemi al contorno per le equazioni integro-differenziali a derivate totali d'ordine superiore.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 123–129.

Dirichlet's problem is considered: $u(A)_{\mathrm{Fr}\,R} = \mu_0(x, y)$, $D^i(A)_{R_1} = \mu_i(x, y)$ $(i = 1, 2, \cdots, k-1)$ and

$$D^k u(A) + \lambda \sum_{i=k-1}^{0}\sum_{j=k-1}^{0} r_{ij}(A)\frac{\partial^{i+j}u(A)}{\partial x^i \partial y^j} = f(A) + \lambda \iint_R \sum_{i=0}^{k}\sum_{j=0}^{k} K_{ij}(A, B)\frac{\partial^{i+j}u(B)}{\partial \xi^i \partial \eta^j}\,dB,$$

where $R \equiv (a \leq x \leq b;\ c \leq y \leq d)$; $R_1 \equiv (x = a,\ c \leq y \leq d;\ y = c,\ a \leq x \leq b)$; Fr R the boundary of R; $Du(A) \equiv \partial^2 u(x, y)/\partial x \partial y$ is the total derivative in Picone's sense, $f(A) \equiv f(x, y)$; $K_{ij}(A, B) \equiv K_{ij}(x, y, \xi, \eta)$; $\varphi_i(y)$, $\psi_i(x)$, $r_{ij}(A) \equiv r_{ij}(x, y)$

are continuous functions for all values of the variables $x,\ \xi \in [a, b]$; $y,\ \eta \in [c, d]$; $R = [a, b] \times [c, d]$ is a rectangle, $u(A) \equiv u(x, y)$ is the unknown function. λ is a parameter and k is an arbitrary positive integer.

An equivalent system of two integral equations is given. Theorem I is on the evaluation of the error introduced by the approximation. In the second section of the paper the authors investigate the error due to approximation with Bernstein polynomials,

$$\bar{u}_{np}(A) = \pi(A) + \lambda \iint_R M(A, B) \sum_{\nu=0}^{n} \sum_{s=0}^{p} \varphi_{np}(\nu, s, \lambda) \times T_{n\nu}(\xi) T_{ps}(\eta)\, d\xi d\eta,$$

where the function $\bar{u}(A)$ satisfies boundary conditions and $M(A, B)$ is a function given by Mangeron in an earlier paper [#3302 above]. The Bernstein polynomials are $T_{n\nu}(x) = C_n^{\ \nu} x^\nu (1-x)^{n-\nu}$; $T_{ps}(y) = C_p^{\ s} y^s (1-y)^{p-s}$, $C_r^{\ i} = r(r-1)\cdots(r-i+1)/i!$, and $\varphi_{np}(\nu, s, \lambda)$ is equal to the solution of a given system. Also, the evaluation corresponding to the method of quadrature formula is discussed. The approximate form of the problem can be written in the following form:

$$u_{np}(A) = \pi(A) + \lambda \iint_R M(A, B)\left[\omega(B) + \lambda \sum_{i=1}^{n} \sum_{j=1}^{p} d_{ij} \mathscr{E}(B, x_i, y_j) \xi_{ij}(\lambda)\right] dB,$$

where the second term inside the brackets [ ] is the quadrature approximation of the corresponding integral. The double integral shown above can also be replaced by a corresponding quadrature. *O. Gürel* (Ankara)

FUNCTIONAL ANALYSIS

See also 3114, 3120, 3140, 3142, 3167, 3174, 3239, 3247, 3249, 3525, 3530, 3535, 3536.

Henney, Dagmar Renate; Henney, Alan Gilbert 3304
One parameter semigroups.
Math. Japon. **6** (1961/62), 39–43.

The authors use the definition of Rådström [Ark. Mat. **4** (1960), 87–97; MR **26** #3802] and study the structure of the sets of a one-parameter semigroup of non-intersecting compact subsets of a Hausdorff locally convex space. {Among misprints, the condition of compactness seems to have been omitted from the statement of Theorem 2.}

A. P. Robertson (Glasgow)

Sebastião e Silva, José 3305
Les espaces à bornés et les réunions d'espaces normés.
Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 134–137.

In this paper the author points out that the notion of a union of a family of normed spaces, which he introduced in 1946, is identical with the corresponding notion of a union of a family of normed spaces introduced in 1960 by J. Mikusiński [Studia Math. **19** (1960), 251–285; MR **22** #12378]. Furthermore, the author shows that a bounded linear space, a notion introduced by Waelbroeck [Acad. Roy. Belg. Cl. Sci. Mém. Coll. in-8° (2) **31** (1960), no. 7; MR **22** #8355], is the union of a family of normed linear spaces, and that, conversely, every union of a family of normed linear spaces is a bounded linear space in the sense of Waelbroeck.

W. A. J. Luxemburg (Pasadena, Calif.)

Gil de Lamadrid, Jesús 3306
Complementation of manifolds in topological vector spaces.
Arch. Math. **14** (1963), 59–61.

The author constructs a class of examples of topological vector spaces that can be written as the algebraic but not topological sum of two closed manifolds.

A. P. Robertson (Glasgow)

Moreau, Jean-Jacques 3307
Sur la fonction polaire d'une fonction semi-continue supérieurement.
C. R. Acad. Sci. Paris **258** (1964), 1128–1130.

In this note the author shows that two properties used in a previous note [same C. R. **257** (1963), 4117–4119; MR **28** #1476] are equivalent when the hypotheses in each are weakened as far as possible. Letting E, E' be two locally convex Hausdorff spaces dual relative to the form $\langle\ ,\ \rangle$, and f a lower semicontinuous function from E to $(0, +\infty]$, the two properties are roughly: (i) continuity of f at 0 in E, and (ii) compactness of the set $\{x' \in E' \mid \langle x, x'\rangle - f(x) \leqq k$ for all $x \in E\}$. He then shows how this simplifies his previous conclusions on sub-differentiability.

P. C. Deliyannis (Chicago, Ill.)

Lindenstrauss, Joram 3308
On operators which attain their norm.
Israel J. Math. **1** (1963), 139–148.

Bishop and Phelps [Bull. Amer. Math. Soc. **67** (1961), 97–98; MR **23** #A503] showed that if X is a Banach space and X^* its conjugate space, then the set of those f in X^* which attain on the unit ball of X the value $\|f\|$ is a dense subset of X^*. Letting $B(X, Y)$ be the set of continuous linear operators from X into another Banach space Y, the author shows that reflexivity of Y is sufficient for the set of operators which attain their norms to be dense in $B(X, Y)$. Relations are also given between this property and geometric properties of the unit ball of X or Y. *M. M. Day* (Stanford, Calif.)

Švarc, A. S. 3309
On the homotopic topology of Banach spaces. (Russian)
Dokl. Akad. Nauk SSSR **154** (1964), 61–63.

Let E, F be two Banach spaces and $A: E \to F$ a continuous Φ-operator of index k [Gohberg and Kreĭn, Uspehi Mat. Nauk **12** (1957), no. 2 (74), 43–118; MR **20** #3459]. Denote by K the unit ball in E and by L the boundary of K. A mapping $G: K \to F$ is said to be admissible if $0 \bar{\in} G(L)$ and $G - A$ is completely continuous. Two admissible mappings are said to be homotopic if they can be joined by a homotopy G_t such that G_t is admissible for each $t \in \langle 0; 1\rangle$. Denote by $S(A)$ the set of homotopy classes of admissible mappings. Theorem 1: There exists a one-to-one correspondence between $S(A)$ and the stable homotopy group $\pi_{n+k}(S^n)$, $k < n$. The next two theorems are of a similar type. In the second part of the paper, the author

considers homotopical properties of the following subsets of $L(E)$, the Banach space of all linear operators $T: E \to E$; $GL(E)$, the subset of $L(E)$ consisting of all invertible operators; $GL_c(E)$, the subset of $GL(E)$ consisting of all operators which are of the form $I+R$ where I is the identity and R is completely continuous; $R_k(E)$, the subset of $L(E)$ consisting of all Φ-operators of index k. Proofs are not given. *K. Gęba* (Gdańsk)

Zinčenko, A. I. 3310

Some approximate methods of solving equations with non-differentiable operators. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1963**, 156–161.

L'auteur considère l'équation fonctionnelle $P(x)=\Theta$ où P est un opérateur continu non-différentiel défini dans un sous-espace M d'un espace de Banach E à valeurs dans un autre E_1. L'auteur donne des méthodes approchées pour calculer des solutions de telles équations, déterminées par

$$x_{n+1} = x_n - [P_1'(x_n)]^{-1}P(x_n),$$

où P_1' est un opérateur de Fréchet approché associé.

H. Morel (Marseille)

Berman, G. H. 3311

Functors in the category of locally convex spaces. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 497–499.

Die Untersuchungen von A. S. Švarc [dieselben Dokl. **149** (1963), 44–47; MR **27** #4046] über Funktoren in Kategorien von Banachräumen werden auf Kategorien von lokalkonvexen Räumen übertragen. Außerdem wird eine Konstruktion für spezielle Funktoren angegeben. Ist λ ein vollkommener Folgenraum, so besteht $\lambda(X)$ für jeden lokalkonvexen Raum X aus allen Folgen $\mathbf{x}_n$ mit $x_n \in X$, für die mit jeder auf X stetigen Halbnorm p die Beziehung $\mathbf{p}(\mathbf{x}_n) \in \lambda$ gilt. Wenn p alle stetigen Halbnormen und M ein gewisses System $\mathfrak{M}$ von beschränkten Teilmengen des zu λ dualen Folgenraumes λ^* durchlaufen, bilden die Mengen

$$\{\mathbf{x}_n \in \lambda(X) : \sum_n |\alpha_n| p(x_n) \leq 1 \text{ für } \boldsymbol{\alpha}_n \in M\}$$

ein Fundamentalsystem von Nullumgebungen in $\lambda(X)$. Der auf diese Weise gewonnene lokalkonvexe Raum wird mit $\lambda_{\mathfrak{M}}(X)$ bezeichnet. Durch die Zuordnungen $X \to \lambda_{\mathfrak{M}}(X)$ und $\varphi(\mathbf{x}_n) \to \boldsymbol{\varphi}(\mathbf{x}_n)$ mit $\varphi \in L(X, Y)$ erhält man dann auf der Kategorie aller lokalkonvexen Räume den Funktor $\Lambda_{\mathfrak{M}}$. Die verallgemeinerten Folgenräume $\lambda(X)$ wurden bereits vom Referenten eingeführt [Math. Nachr. **25** (1963), 19–30; MR **27** #2828]. *A. Pietsch* (Berlin)

Persson, Arne 3312

A generalization of two-norm spaces.

Ark. Mat. **5**, 27–36 (1963).

This paper generalizes the results of Wiweger [Studia Math. **20** (1961), 47–68; MR **24** #A3490] and of the reviewer and Semadeni [ibid. **17** (1958), 121–140; MR **20** #6644; ibid. **18** (1959), 275–293; MR **22** #5878] to the general case of linear topological spaces. Let $\langle E, \mu, \tau\rangle$ be a triplet in which E is a linear space, τ a locally convex separated topology on E, μ another topology on E such that each τ-bounded subset of E is μ-bounded. This is called by the author a bitopological space. Then the finest locally convex topology on E, agreeing with μ on the τ-bounded subsets is denoted by μ^τ, and called the mixed topology. Then $\mu \leqq \mu^\tau$, $\mu^\tau = (\mu^\tau)^\tau$, and every τ-bounded subset of E is μ^τ-bounded. A linear map of E into a locally convex linear topological space is continuous for μ^τ if and only if its restriction to any τ-bounded set is continuous for μ; μ^τ is the only topology identical with μ on the τ-bounded sets which does have the above property. The author generalizes the notion of normality of the two-norm spaces and the Wiweger spaces: $\langle E, \mu, \tau\rangle$ is called a-normal if $\langle E, \tau\rangle$ has a fundamental system of bounded sets which are absolutely convex and μ-closed. If, moreover, this system is countable, $\langle E, \mu, \tau\rangle$ is called b-normal. If $\langle E, \tau\rangle$ has a neighborhood basis of zero composed of absolutely convex μ-closed sets, $\langle E, \mu, \tau\rangle$ is called c-normal. If $\langle E, \mu, \tau\rangle$ is b- or c-normal, then sets bounded for τ and μ^τ are identical; also $x_n \to x_0$ for μ^τ if and only if the sequence (x_n) is τ-bounded and converges to x_0 for μ. Denote by $\langle E, \mu\rangle'$ the dual of $\langle E, \mu\rangle$ endowed with its strong topology (of uniform convergence on μ-bounded sets), let β be the bornological structure associated with τ. Then for $\langle E, \mu, \tau\rangle$ the topological inclusions $\langle E, \mu\rangle' \subset \langle E, \mu^\tau\rangle' \subset \langle E, \beta\rangle'$ are valid, $\langle E, \mu^\tau\rangle'$ is complete and closed in $\langle E, \beta\rangle'$. If $\langle E, \mu, \tau\rangle$ is a-normal, $\langle E, \mu^\tau\rangle'$ is identical with the closure of $\langle E, \mu\rangle'$ in $\langle E, \beta\rangle'$. In the most interesting cases, the space $\langle E, \mu^\tau\rangle$ is not quasi-barrelled. Discussion of the cases $\langle E, \mu^\tau\rangle' = \langle E, \beta\rangle'$, $\langle E, \mu^\tau\rangle' = \langle E, \mu\rangle'$ follows. The notion of reflexivity is introduced, reflexive $\langle E, \mu^\tau\rangle$ spaces are characterized, it is also shown that a bornological space $\langle E, \beta\rangle$ is reflexive if and only if for every locally convex topology μ coarser than β, $\langle E, \mu\rangle'$ is dense in $\langle E, \beta\rangle'$. Properties of the topology μ^τ relativized to subspaces of E are discussed; in general, μ^τ-continuous linear functionals on subspaces do not have continuous extensions on the whole space. The author gives some sufficient conditions to this effect.

A. Alexiewicz (Poznań)

Dostál, Miloš 3313

Remark on the tensor algebras.

Comment. Math. Univ. Carolinae **3** (1962), *no.* 4, 3–8.

Let (E_n) be a sequence of DF spaces and M the topological direct sum $\sum_{n=1}^\infty E_n$. The author defines the projective tensor product of the sequence (E_n) to be $\sum_{p=0}^\infty \otimes^p M$; he proves that this is a DF space and a locally multiplicatively convex algebra. For a family of normed spaces, he suggests a different tensor product, defining it as a subalgebra of the algebra of continuous functions on the product of the unit balls of their duals. *A. P. Robertson* (Glasgow)

Fischer, H. R. 3314

Über eine Klasse topologischer Tensorprodukte.

Math. Ann. **150** (1963), 242–258.

In his well-known work on topological tensor products [Mem. Amer. Math. Soc. No. 16 (1955); MR **17**, 763] Grothendieck limits himself to the study of two specific topological tensor products, which are locally convex, topological-vector-space generalizations of the (least cross-norm) λ-tensor product and the (greatest cross-norm)

γ-tensor product of Banach spaces in the work of Schatten [*A theory of cross-spaces*, Princeton Univ. Press, Princeton, N.J., 1950; MR **12**, 186]. In the present paper the author studies, in the setting of locally convex topological spaces, what might roughly be regarded as corresponding to a more general class of cross-norms in the work of Schatten with Banach spaces. The topologies are introduced as follows. If E and F are locally convex topological vector spaces (the author limits himself to the real case), $\mathfrak{S}$ a family of bounded subsets of E, and $\mathfrak{T}$ a family of bounded subsets of F, he defines a bilinear mapping Φ of $E \times F$ into another locally convex space G to be $(\mathfrak{S}, \mathfrak{T})$-quasicontinuous if it is separately continuous and for each neighborhood V of $0 \in G$, each $A \in \mathfrak{S}$ and each $B \in \mathfrak{T}$, there exist neighborhoods U of $0 \in E$ and W of $0 \in F$ such that Φ maps both $U \times B$ and $A \times W$ into V. This generalizes the hypocontinuity of Bourbaki [*Espaces vectoriels topologiques*, Chap. III, Actualités Sci. Indust., No. 1229, Hermann, Paris, 1955; MR **17**, 1109]. Now the $(\mathfrak{S}, \mathfrak{T})$-topology of $E \otimes F$ is defined as the finest (strongest) locally convex topology for which the natural mapping $E \times F \rightarrow E \otimes F$ is $(\mathfrak{S}, \mathfrak{T})$-quasicontinuous. It turns out that the $(\mathfrak{S}, \mathfrak{T})$-topologies are admissible in the sense of Grothendieck, which in the cross-norm language of Schatten (for Banach spaces) corresponds to the fact that these topologies are finer (stronger) than the λ-topology and coarser (weaker) than the γ-topology, and, in particular, have the basic properties of cross-norm topologies. These new topologies include the inductive tensor-product topology of Grothendieck (which generalizes the λ-topology of Schatten), but not the projective topology (which generalizes the γ-topology), unless the spaces E and F are normed. The author investigates some of the elementary properties of what Grothendieck calls permanence in topological tensor products. These properties are expressed in theorems of the form: If E and F have property P, then so does $E \otimes F$ (or its completion) under the $(\mathfrak{S}, \mathfrak{T})$-topology. The author also discusses the generalization of his work to products of several spaces, in particular, to what he calls the tensorials, which are the tensor powers $\otimes^p E$ of E by itself p times. To these correspond the $\{\mathfrak{S}_i\}_i$-topologies, where each $\mathfrak{S}_i$ is a family of bounded subsets of E. Along with tensorials he considers the exterior powers, as well as the tensor algebras and the exterior algebras, all with the appropriate topologies.

J. Gil de Lamadrid (Minneapolis, Minn.)

Rutickiĭ, Ja. B. 3315

Scales of Orlicz spaces and interpolation theorems. (Russian)

Dokl. Akad. Nauk SSSR **149** (1963), 32–35.

The main aim of this paper is to show how some kind of scales of Orlicz spaces may be constructed. Let $M_0(u)$ and $M_1(u)$ be given N-functions with $N_0(v)$ and $N_1(v)$ their complementary N-functions [for notations and definitions see M. A. Krasnosel'skiĭ and the author, *Convex functions and Orlicz spaces*, Noordhoff, Groningen, 1961; MR **23** #A4016]. It is assumed that $M_0(u)$ is smaller in the strong sense than $M_1(u)$, i.e., that

$$\limsup_{u\to\infty} \frac{M_0(ku)}{M_1(u)} < \infty$$

holds for any $k>0$. Let $M_\tau(u)$ denote an N-function complementary to the N-function $N_\tau(v)$, inverse to the function $N_\tau^{-1}(v)=[N_0^{-1}(v)]^{1-\tau}[N_1^{-1}(v)]^\tau$, $v \geqq 0$, and $0 \leqq \tau \leqq 1$. Finally, let G denote a closed and bounded set in an n-dimensional Euclidean space. It can be proved that the spaces E_{M_τ} (the closure in $L_{M_\tau}^*(G)$ of the set of all bounded functions on G) are nested: $E_{M_{\tau_2}} \subset E_{M_{\tau_1}}$ for $0 \leqq \tau_1 < \tau_2 \leqq 1$ and that $E_{M_{\tau_2}}$ is dense in $E_{M_{\tau_1}}$ for $\tau_2 > \tau_1$. The family of spaces E_{M_τ}, $0 \leqq \tau \leqq 1$, is called a scale of Orlicz spaces including the spaces E_{M_0} and E_{M_1}. Further, if $0 \leqq \tau_1 < \tau < \tau_2 \leqq 1$, then for every $x(\cdot) \in E_{M_{\tau_2}}$ the following inequality $\|x\|_{M_\tau} \leqq \|x\|_{M_{\tau_1}}^{(\tau_2-\tau)/(\tau_2-\tau_1)} \|x\|_{M_{\tau_2}}^{(\tau-\tau_1)/(\tau_2-\tau_1)}$ holds. In other words, this theorem states that norms are logarithmically convex functions of τ. Moreover, for the scales of Orlicz spaces, some theorems concerned with interpolation of compact operators can be proved.

J. Albrycht (Poznań)

Matuszewska, W.; Orlicz, W. 3316

A note on modular spaces. VI.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 449–454.

References to the first four papers in this series of more or less independent notes may be found in the review of IV [Orlicz, same Bull. **10** (1962), 479–484; MR **26** #5396]; the fifth note is by the authors, ibid. **11** (1963), 51–54 [MR **27** #2843]. The main result of Part VI is a generalization to modular spaces of results of Matuszewska which may be found as 3 and 3.1 of same Bull. **8** (1960), 349–353 [MR **23** #A3453] and as (i) of [MR **23** #A3453].

H. Gordon (Philadelphia, Pa.)

Maĭkov, E. V. 3317

τ-continuity and τ-differentiability of a functional. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 266–269.

L'auteur introduit les nouvelles notions de τ-continuité et de τ-dérivabilité d'une fonctionnelle définie sur l'ensemble S des fonctions numériques $\xi(\tau)$, $\tau \in [0, 1]$, continues par morceaux. Fixons $\xi^0 \in S$, $t \in [0, 1]$ et un nombre réel x. Considérons $\xi \in S$ telle que $\xi(t-0)=\xi(t)=x$ et posons $\xi^\eta(\tau)=\xi(\tau)$ pour $\tau \in [t, t+\eta)$ [si $\eta<0$, alors $[t, t+\eta)$ signifie $[t+\eta, t)$], $\xi^\eta(\tau)=\xi^0(\tau)$ autrement. Si l'on a $\lim_{\eta\to 0} f(\xi^\eta) = f(\xi^0)$ indépendamment du choix de ξ, alors f est par définition τ-continue en ξ^0 par rapport à (t, x).

Pour ξ^0, ξ^η, t, x comme déjà dit, si la limite

$$\lim_{\eta\to 0+} \frac{f(\xi^\eta)-f(\xi^0)}{\eta}$$

existe et ne dépend pas de ξ, alors f est par définition τ-dérivable à droite en ξ^0 par rapport à (t, x).

L'auteur aborde l'étude du calcul différentiel basé sur lesdites notions. Il donne une représentation intégrale pour les fonctionnelles ayant leur $n$ième τ-dérivée identiquement nulle (τ-polynômes).

E. J. Akutowicz (Bologna)

Grisvard, P. 3318

Espaces intermédiaires entre espaces de Sobolev avec poids.

Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 255–296.

L'auteur développe avec des démonstrations détaillées les résultats annoncés dans une note antérieure [C. R. Acad. Sci. Paris **256** (1963), 2745–2748; MR **26** #6752].

Notations: μ est un nombre réel; U le demi-espace $x_n>0$ dans $\mathbf{R}^n$; $L_\mu{}^p(U)$ $(1<p<\infty)$ l'espace des fonctions mesurables et telles que la norme

$$\|u\|_{L_\mu{}^p(U)} = \left(\int_U |u(x)|^p x_n{}^\mu\, dx\right)^{1/p}$$

soit finie; $L^p_{\mu,\mu-p}(U)=L_\mu{}^p(U)\cap L^p_{\mu-p}(U)$; $W_\mu{}^{m,p}(U)$ (m entier >0) l'espace des fonctions $u\in L_\mu{}^p(U)$ dont les dérivées (au sens des distributions) d'ordre $\leqq m$ appartiennent à $L_\mu{}^p(U)$, muni de la norme

$$\|u\|_{W_\mu{}^{m,p}(U)} = \left(\sum_{|\alpha|\leqq m}\int_U |D^\mu u(x)|^p x_n{}^\mu\, dx\right)^{1/p};$$

$\mathring{W}_\mu{}^{1,p}(U)$ l'adhérence de $\mathscr{D}(\mathring{U})$ dans $W_\mu{}^{1,p}(U)$; $T(q,\alpha;X,Y)$ l'espace de traces de Lions [Math. Scand. **9** (1961), 147–177; MR **28** #2429] dont on rappelle rapidement la définition: X et Y étant deux espaces de Banach et $0<\theta=\alpha+1/q<1$, $W(q,\alpha;X,Y)$ est l'espace des fonctions $t\mapsto u(t)$ $(0\leqq t<\infty)$ telles que $t^\alpha u(t)\in L^q(0,\infty;X)$, $t^\alpha u'(t)\in L^q(0,\infty;Y)$ muni de la norme $\|u\|_{W(q,\alpha;X,Y)}=\max(\|t^\alpha u\|_{L^q(0,\infty;X)},\|t^\alpha u'\|_{L^q(0,\infty,Y)})$. $T(q,\alpha;X,Y)$ est alors l'image de $W(q,\alpha;X,Y)$ par $u\to u(0)$, muni de la norme quotient. On pose encore $W_\mu{}^{1-\theta,p}(U)=T(p,\alpha;W_\mu{}^{1,p}(U),L_\mu{}^p(U))$, $\theta=\alpha+1/p$, et $\mathring{W}_\mu{}^{1-\theta,p}(U)$ est l'adhérence de $\mathscr{D}(U)$ dans $W_\mu{}^{1-\theta,p}(U)$. Plus généralement, si s est réel non entier, on pose $W_\mu{}^{s,p}(U)=T(p,\alpha;W_\mu{}^{[s]+1,p}(U),W_\mu{}^{[s],p}(U))$, $\alpha+1/p=[s]+1-s$. Finalement $B^{s,p}(\mathbf{R}^k)=W^{s,p}(\mathbf{R}^k)$ (qui correspond à $\mu=0$) pour s réel positif non entier, $B^{1,p}(\mathbf{R}^k)$ désigne l'espace des $u\in L^p(\mathbf{R}^k)$ tels que $\int_0^\infty t^{-p-1}\int_{\mathbf{R}^k}|\Delta_i{}^2u(x)|^p\,dx dt$ est fini pour $i=1,\cdots,k$, où $\Delta_i{}^2u(x)=u(x_1,\cdots,x_i+t,\cdots,x_k)+u(x_1,\cdots,x_i-t,\cdots,x_k)-2u(x_1,\cdots,x_k)$ et $B^{m,p}(\mathbf{R}^k)$ (m entier $\geqq 2$) est l'espace des fonctions de $W^{m-1,p}(\mathbf{R}^k)$ dont toutes les dérivées d'ordre $m-1$ sont dans $B^{1,p}(\mathbf{R}^k)$.

Résultats: Si $\mu\geqq p-1$ ou $\mu\leqq -1$ on a $\mathring{W}_\mu{}^{1,p}(U)=W_\mu{}^{1,p}(U)$ et si $\mu>p-1$ ou $\mu\leqq -1$ on a $W_\mu{}^{1,p}(U)\subset L^p_{\mu-p}(U)$; si $\mu<p-1$ on a $\mathring{W}_\mu{}^{1,p}(U)=W_\mu{}^{1,p}(U)\cap L^p_{\mu-p}(U)$. Si $-1<\mu<p-1$, $\mathscr{D}(\bar{U})$ est dense dans $W_\mu{}^{1,p}(U)$ et $\mathscr{D}(U)$ est dense dans $T(q,\alpha;W_\mu{}^{1,p}(U),L_\mu{}^p(U))$ lorsque $1-\theta\leqq(\mu+1)/p$. Ceci généralise un résultat obtenu pour $\mu=0$ par Lions et Magenes [Ann. Scuola Norm. Sup. Pisa (3) **15** (1961), 311–326; MR **25** #4351]. Si $-1<\mu<p-1$, on a $T(q,\alpha;\mathring{W}_\mu{}^{1,p}(U),L_\mu{}^p(U))=T(q,\alpha;W_\mu{}^{1,p}(U),L_\mu{}^p(U))\cap T(q,\alpha;L^p_{\mu,\mu-p}(U),L_\mu{}^p(U))$ et de plus $T(p,\alpha;\mathring{W}_\mu{}^{1,p}(U),L_\mu{}^p(U))=W_\mu{}^{1-\theta,p}(U)$ si $1-\theta<(\mu+1)/p$ et $T(p,\alpha;\mathring{W}_\mu{}^{1,p}(U),L_\mu{}^p(U))=\mathring{W}_\mu{}^{1-\theta,p}(U)\neq W_\mu{}^{1-\theta,p}(U)$ si $1-\theta>(\mu+1)/p$. Le cas $1-\theta=(\mu+1)/p$ fournit un exemple de trois espaces de Banach (et même de Hilbert pour $p=2$) $E\subset F\subset G$, où E est fermé dans F mais $T(p,\alpha;E,G)$ n'est pas fermé dans $T(p,\alpha;F,G)$. Soit $n=1$; si $1-\theta>1/p$, les fonctions de $T(q,\alpha;W_\mu{}^{1,p}(U),L_\mu{}^p(U))$ sont continues dans U, si $0\leqq\mu<p-1$ et $(\mu+1)/p<1-\theta<1$, les fonctions de $T(q,\alpha;W_\mu{}^{1,p}(U),L_\mu{}^p(U))$ sont continues dans $\bar{U}$ et l'application $u\mapsto u(0)$ est continue. Soit toujours $-1<\mu<p-1$; si $1-\theta>(\mu+1)/p$, l'application $u\to u(x_1,\cdots,x_{n-1},0)$ de $\mathscr{D}(\bar{U})$ dans $L^p(\partial U)$ se prolonge par continuité en une application continue γ de $W_\mu{}^{1-\theta,p}(U)$ dans $L^p(\partial U)$. Finalement si $s>(\mu+1)/p$, l'opérateur γ applique $W_\mu{}^{s,p}(U)$ sur $B^{s-(\mu+1)/p,p}(\partial U)$. Pour $\mu=0$ ce résultat est dû à S. V. Uspenskiĭ [Sibirsk. Mat. Ž. **3** (1962), 418–445; MR **25** #2434].

Dans le dernier paragraphe l'auteur indique comment ses résultats se généralisent au cas où U est un ouvert borné de $\mathbf{R}^n$ dont la frontière est suffisamment régulière.

J. Horváth (Nancy)

Besov, O. V. 3319

Extension of functions with preservation of differential-difference properties in L_p. (Russian)

Dokl. Akad. Nauk SSSR **150** (1963), 963–966.

Soit $f\in L^q(0,a)$ avec une dérivée généralisée $f^{(k)}$. On pose

$$\omega_m(f^k,h)_{L^p(0,a)} = \sup_{0<t\leqq h}\left\|\sum_{\nu=0}^{m}(-1)^{m-\nu}\binom{m}{\nu}f^{(k)}(x+\nu t)\right\|_{L^p(0,a-mt)}.$$

Cette notion généralise le cas $k=0$ en faisant intervenir à la fois les différences et les dérivées. Dans le cas $k=0$, des auteurs se sont posé le problème de prolonger $\forall f\in L^p[a,b]$ par $\varphi\in L^p[a_1,b_1]$, $a_1<a<b<b_1$ de telle façon que $\varphi|[a,b]=f$ et qu'il existe une constante $C<\infty$ ne dépendant pas de f telle que $\omega_m(\varphi,h)\leqq C\omega_m(f,h)$. Des résultats ont été obtenus dans ce sens et sont mentionnés. L'auteur démontre plus généralement le théorème: Soit $1\leqq q\leqq\infty$; $1\leqq p\leqq\infty$, $0<a\leqq\infty$; alors, il existe C finie telle que $\forall f\in L^q(0,a)$, il existe $\varphi\in L^q(-a,a)$ avec $\varphi|[0,a]=f$ et

$$\|\varphi\|_{L^q(-a,a)}\leqq C\|f\|_{L^q(0,a)},$$
$$\omega_m(\varphi^{(k)},h)_{L^p[-a,a]}\leqq C\omega_m(f^{(k)},h)_{L^p(0,a)}.$$

L'auteur donne une extension au cas de plusieurs variables.

H. Morel (Marseille)

Garkavi, A. L. 3320

Approximation properties of subspaces of finite defect in the space of continuous functions. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 513–516.

Let T be a compact Hausdorff space, $C(T)$ the Banach space of all real-valued continuous functions on T with supremum norm. A subspace $\mathscr{L}^n$ of finite defect (=codimension) n in $C(T)$ is said to have property (E) [respectively, (U)] if to each element of $C(T)$ there exists at least one [at most one] nearest element in $\mathscr{L}^n$. If $\mathscr{L}^n$ has both properties, it is called a Čebyšev subspace. The reviewer [Pacific J. Math. **13** (1963), 647–655; MR **27** #6069] has characterized the Čebyšev subspaces of finite defect in $C(T)$ among those having property (E). In the paper under review, the author complements and extends this work. The main result is Theorem 1, which characterizes those $\mathscr{L}^n$ having property (E) in terms of the Borel measures μ [and their closed supports $S(\mu)$] which form the annihilator M_n of $\mathscr{L}^n$ in $C(T)^*$: L^n has property (E) if and only if (a) for each μ in M_n, $S(\mu^+)$ and $S(\mu^-)$ are disjoint, and (b) for μ and $\bar{\mu}$ in M_n, the set $S(\mu)\setminus S(\bar{\mu})$ is closed and the restriction of μ to $S(\bar{\mu})$ is absolutely continuous with respect to $\bar{\mu}$. The proof, which depends to some extent on earlier work of the author [Mat. Sb. (N.S.) **62 (104)** (1963), 104–120; MR **27** #6076], is sketched. The author announces characterizations of property (U) and the Čebyšev property, and a number of results are announced concerning the topological character of those T for which $C(T)$ admits subspaces of finite defect having property (E), or property (U), or both. An example is described which contradicts a theorem which was independently formulated and proved by both the author

[loc. cit., second part of Theorem 11] and the reviewer [loc. cit., Theorem 7]. {The proofs (as well as the error contained in each) are different. A slight change in each proof yields the result which the author states in paragraph 4.} *R. R. Phelps* (Seattle, Wash.)

Honda, Kôji 3321

A characteristic property of L_ρ-spaces ($\rho \geqq 1$). III.

Proc. Japan Acad. **39** (1963), 348–351.

Part II appeared in same Proc. **36** (1960), 123–127 [MR **22** #12381]. As a characterization of L_p-norm, the author proves: Let S be a sequentially continuous linear lattice. A norm on S is an L_p-norm for some $p \geqq 1$ if and only if

$$G(x;[p]x) = \lim_{\varepsilon\to 0} \frac{\|x+\varepsilon[p]x\|-\|x\|}{\varepsilon}$$

exists and $G(a+x;\ a)=G(a+y;\ a)$ for $a\cap x=a\cap y=0$, $\|a+x\|=\|a+y\|=1$. *H. Nakano* (Detroit, Mich.)

Nikol'skiĭ, S. M. 3322

Functions with dominant mixed derivative, satisfying a multiple Hölder condition. (Russian)

Sibirsk. Mat. Ž. **4** (1963), 1342–1364.

Dans un travail précédent [Dokl. Akad. Nauk SSSR **146** (1962), 542–545; MR **26** #630] l'auteur a introduit les classes $S_p^{(\mathbf{r})}W(R^n)$ liées aux espaces de Sobolev $W_p^{(\mathbf{r})}(R^n)$: Pour $\mathbf{r}=(r_1,\cdots,r_n)$ multi-indice à composantes entières $\geqq 0$, $S_p^{(\mathbf{r})}W(R^n)$ est l'espace des fonctions $f\in L_p(R^n)$ telles que

$$D^{\mathbf{s}}f = \frac{\partial^{s_1+\cdots+s_n}f}{\partial x_1^{s_1}\cdots\partial x_n^{s_n}} \in L_p(R^n)$$

pour tout $\mathbf{s}=(s_1,\cdots,s_n)$ tel que $s_i=0$ ou r_i. On a l'identité

$$W_p^{(\mathbf{r})}(R^n) = S_p^{(r,0,\cdots,0)}W(R^n)\cap\cdots\cap S_p^{(0,\cdots,0,r)}W(R^n).$$

Dans l'article présent l'auteur introduit une nouvelle classe $S_p^{(\mathbf{r})}H(R^n)$ liée de manière analogue aux espaces de Nikol'skiĭ $H_p^{(\mathbf{r})}(R^n)$ [Uspehi Mat. Nauk **16** (1961), no. 5 (101), 63–114; MR **26** #6757]: Pour $\mathbf{r}\in R^n$, $r_i\geqq 0$, posons $r_i=\bar r_i+\alpha_i$, $\bar r_i$ entier $\geqq 0$, $0<\alpha_i\leqq 1$ ($\bar r_i=0$ si $r_i=0$), $S_p^{(\mathbf{r})}H(R^n)$ est l'espace des $f\in L_p(R^n)$ telles que pour tout $\mathbf{s}=(s_1,\cdots,s_n)$ avec $s_i=r_i$ ou 0, les conditions suivantes soient vérifiées: (a) $D^{\boldsymbol\sigma}f$ est une fonction (localement intégrable) pour $\boldsymbol\sigma=(\bar s_1,\cdots,\bar s_n)$;

(b) $\|\Delta^{2\omega_1}_{1,h_1}\cdots\Delta^{2\omega_n}_{n,h_n}D^{\sigma}f\|_{L_p(R^n)} \leqq C^{\text{te}}|h_1|^{\alpha_1\omega_1}\cdots|h_n|^{\alpha_n\omega_n}$,

où $\omega_i=0$ [respectivement, 1] lorsque $s_i=0$ [>0], et

$$\Delta_{i,h}f(x_1,\cdots,x_n) = f(x_1,\cdots,x_i+h,\cdots,x_n)-f(x_1,\cdots,x_i,\cdots,x_n).$$

On a alors

$$H_p^{(\mathbf{r})}(R^n) = S_p^{(r,0,\cdots,0)}H(R^n)\cap\cdots\cap S_p^{(0,\cdots,0,r)}H(R^n).$$

L'auteur caractérise les fonctions de ces classes par une condition (nécessaire et suffisante) de développement en série de fonctions entières de type exponentiel. De cette caractérisation résultent divers théorèmes d'immersion, un théorème de trace pour les classes $S_p^{(\mathbf{r})}H(R^n)$ et le fait que pour une fonction donnée f et un p fixé l'ensemble des $\mathbf{r}$ tels que $f\in S_p^{(\mathbf{r})}H(R^n)$ est convexe. *P. Grisvard* (Nancy)

Dubinskiĭ, Ju. A. 3323

Some imbedding theorems in Orlicz classes. (Russian)

Dokl. Akad. Nauk SSSR **152** (1963), 529–532.

Let $W_p^{(l)}(G)$ be the space (defined by S. L. Sobolev [*Some applications of functional analysis in mathematical physics* (Russian), Izdat. Leningrad. Gos. Univ., Leningrad, 1950; MR **14**, 565; English transl., Amer. Math. Soc., Providence, R.I., 1963]) of functions on a certain region G in n-dimensional Euclidean space, having pth power summable (generalized) derivatives up to order l. Denote by $\mathring{W}_p^{(l)}$ the subspace of $W_p^{(l)}$ consisting of those functions which together with their first $l-1$ derivatives, converge in the mean to zero on the boundary of G. The author shows that for certain Orlicz spaces $\mathscr{L}_M$ (of functions on G) whose modular functions M grow faster than any polynomial, the space $\mathring{W}_p^{(l)}$ ($pl=n$) can be imbedded in $\mathscr{L}_M$. Furthermore, the imbedding operator is shown to be completely continuous and an estimate for its norm is given. Similar results are obtained for $W_p^{(l)}$ ($pl=n$). It is stated that by repeating an argument of L. Nirenberg [Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 115–162; MR **22** #823] it can be shown that, under additional hypotheses, an Orlicz space on G can be imbedded in the continuous functions on G.

R. R. Phelps (Seattle, Wash.)

Gel'fand, I. M. [Гельфанд, И. М.]; Graev, M. I. [Граев, М. И.]; Vilenkin, N. Ja. [Виленкин, Н. Я.] 3324

★Generalized functions, No. 5. Integral geometry and related problems in the theory of distributions [Обобщенные функции, Вып. 5. Интегральная геометрия и связанные с ней вопросы теории представлений].

Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1962. 656 *pp.* 1.90 *r.*

The unifying idea of this book is the application of integral geometry and of the theory of generalized functions developed in earlier volumes to problems of group representations, specifically to the representation theory of the Lorentz group, and to harmonic analysis. The book is written with great clarity, and requires little in the way of special previous knowledge of either group representation theory or integral geometry; it is also independent of the earlier volumes of the series [Gel'fand and Šilov, *Generalized functions and operations on them* (Russian), Fizmatgiz, Moscow, 1958; MR **20** #4182; *Spaces of fundamental and generalized functions* (Russian), Fizmatgiz, Moscow, 1958; MR **21** #5142a; *Some questions in the theory of differential equations* (Russian), Fizmatgiz, Moscow, 1958; MR **21** #5142b; Gel'fand and Vilenkin, *Some applications of harmonic analysis* (Russian), Fizmatgiz, Moscow, 1961; MR **26** #4173] apart from the definitions of generalized functions contained in the first of these.

Chapter I. The Radon transform of a function $f(x)$ in affine space is the integral of f over the hyperplane $(\xi,x)=p$ as a function of ξ and p. The behaviour of the transform under convolution and differentiation of the functions transformed, the reduction to it of the Fourier transform, new inversion formulae, analogues of Plancherel's formula, and theorems of Paley-Wiener type, which throughout this book are taken to mean theorems concerning the transforms of "good" functions,

usually functions infinitely differentiable and rapidly decreasing at infinity, are discussed. The results are extended to generalized functions, and a considerable number of applications, with a table of Radon transforms, are given.

Chapter II, Integral geometry in complex space. The central problem of integral geometry is held to be the deduction of the value of a function from the values of its integrals over a set of manifolds which depend analytically on parameters forming a space of the same number of dimensions as that in which the functions are defined. This problem is studied here (a) for certain 3-parameter families of lines in real affine 3-space; (b) the lines on a complex surface in complex 4-space (with a view to applications to the Lorentz group in Chapter III); (c) the Radon transform in complex affine space. In each case inversion formulae, Plancherel formulae and Paley-Wiener theorems are found.

Chapter III, Representations of the group of complex unimodular matrices of second order. D_χ, with $\chi=(n_1, n_2)$, where n_1, n_2 are complex, n_1-n_2 an integer, is the space of functions $f(z_1, z_2)$ of complex z_1, z_2 infinitely differentiable save at $(0, 0)$ such that $f(az_1, az_2)=a^{n_1-1}\bar{a}^{n_2-1}f(z_1, z_2)$, the topology being that of uniform convergence of all derivatives over compacts. χ is called an integer if n_1, n_2 are integers of same sign. Using arguments based on the study of invariant bilinear functionals on pairs of spaces D_{χ_1}, D_{χ_2}, and construction of these in terms of generalized functions, the representations T_χ, where $(T_\chi g)f(z_1, z_2)=f(\alpha z_1+\gamma z_2, \beta z_1+\delta z_2)$ in D_χ, are studied; these are irreducible if χ is not an integer, equivalent if $\chi_1=\pm\chi_2$ and χ_1 is not an integer; the isomorphisms giving the equivalences and the invariant spaces if χ is an integer are computed. The set of χ can be regarded as a Riemann surface on which T_χ has singularities at the integers. The existence of invariant positive definite bilinear forms is studied and the basic and supplementary series of representations of the Lorentz group are set out.

Chapter IV studies harmonic analysis on this group. The Fourier transform of $f(g)$ is the operator $F(\chi)=\int f(g)T_\chi(g)\,dg$ on D_χ. Its properties for rapidly decreasing, summable and L^2 functions are discussed. F is expressed as an integral transform, whose kernel is the Mellin transform of the integrals of f over the generators of $\alpha\delta-\beta\gamma=1$; this relates the inversion problem for Fourier transforms to those of integral geometry. The behaviour of F and of the corresponding kernel in translations, convolutions and differential operations is studied; inversion formulae and Plancherel type formulae are found: the latter depend only on the basic series.

Chapter V, Integral geometry in spaces of constant curvature, discusses the expression of a function in terms of its integrals over horispheres in real and imaginary Lobačevskiĭ space. Chapter VI, Harmonic analysis on homogeneous spaces connected with the Lorentz group, shows that the study of the representations of a group acting on a homogeneous space is connected with the study of integrals of functions in that space over horispheres relative to the group, i.e., orbits of points in stationary subgroups. The problem analogous to Fourier resolution is that of finding classes of functions invariant in the group. Calculations are carried out for the Lorentz group acting on: (a) a complex affine space; (b) a cone; (c) real, (d) imaginary Lobačevskiĭ space; (e) the space of pairs (z_1, z_2), $z_1\neq z_2$, of points of the complex projective line.

Chapter VII, Representations of the group of real unimodular matrices, uses methods similar to those of Chapter III. The representation spaces are D_χ with $\chi=(s, \varepsilon)$, s complex, $\varepsilon=0$ or 1, with $f(x_1, x_2)$ defined for real x_1, x_2 and satisfying $f(ax_1, ax_2)=|a|^{s-1}(\operatorname{sgn} a)^\varepsilon f(x_1, x_2)$. Integral points for which the representation is not irreducible are of the form (s, ε) with s an integer of the same parity as $\varepsilon+1$. Results, essentially due to Bargmann, concerning the representations of the group are found.

Appendices discuss generalized functions on spaces of one or more complex variables.

J. L. B. Cooper (Cardiff)

Mikusinski, J. [Mikusiński, J.]; Sikorski, R. 3325

★Théorie élémentaire des distributions.

Traduit par S. Klarsfeld. Monographies Internationales de Mathématiques Modernes, 4.

Gauthier-Villars Éditeur, Paris, 1964. vi + 108 *pp.* 20 *F.*

Translation of Parts I and II of the authors' English expository articles [Part I: Rozprawy Mat. **12** (1957); MR **20** #1214; Part II: ibid. **25** (1961); MR **23** #A4006].

Kluvánek, Igor 3326

The representation of linear transformations in integral form. (Russian. English summary)

Mat.-Fyz. Časopis Sloven. Akad. Vied **12** (1962), 241–245.

Let E be a linear lattice whose elements are real functions on a set P. Suppose that $\min(f, 1)\in E$ for every $f\in E$ and that the following condition is fulfilled: If I is a non-negative linear functional on E and if $f_n\in E$, $f_1\geqq f_2\geqq\cdots$, $f_n\to 0$, then $I(f_n)\to 0$. (E.g., E may be the family of all continuous functions with compact support on a topological space P.) Further, let X be a Banach space and let T be a linear transformation of E into X. Then there exists an X-valued measure μ, defined on a σ-ring of subsets of P, such that $T(f)=\int f\,d\mu$ $(f\in E)$ if and only if the set $\{T(f): |f|\leqq g, f\in E\}$ is relatively weakly compact in X for every $g\in E$. *J. Mařík* (Prague)

Ljance, V. È. 3327

Unbounded operators commuting with the resolution of the identity. (Russian. English summary)

Ukrain. Mat. Ž. **15** (1963), 376–384.

Let U be a unitary map $H\ni x\to Ux\in L_{\sigma,N}$ of the Hilbert space H to a direct sum of L^2 spaces, $(Ux)(\lambda)$ being the $(N=N_\lambda)$ vector $(U_\alpha x)(\lambda)$ in $L^2_{\sigma,N}$. Let U diagonalize the resolution of the identity $E(\Delta)$:

$$(UE(\Delta)x)(\lambda) = \chi_\Delta(\lambda)(Ux)(\lambda),$$

χ_Δ being the characteristic function of Δ. If A is closed with domain $D(A)$, then for UAU^{-1} to be the operator of multiplication by a σ-finite operator function it is necessary and sufficient that A commute with the $E(\Delta)$ in the following sense: (i) if $x\in D(A)$, so does $E(\Delta)x$ and $AE(\Delta)x=E(\Delta)Ax$; (ii) there is a sequence of Borel sets Δ_n such that $E(\Delta_n)\to I$ strongly and $E(\Delta_n)H\subset D(A)$. If A is a spectral operator, the corresponding multiplier is of the form $\lambda I+K(\lambda)$, where $K(\lambda)$ is a generalized nilpotent. *J. L. B. Cooper* (Cardiff)

Bočvar, D. A.; Stankevič, I. V.; Čistjakov, A. L. 3328

The symmetry of solutions in an eigenvalue problem. (Russian)

Uspehi Mat. Nauk **16** (1961), no. 3 (99), 155–158.

Let A be an operator taking functions in R_n into functions in R_n. Suppose that the domain and range of A are closed with respect to the action of the orthogonal group of R_n. Denote by G_A the subgroup of the orthogonal group consisting of all elements s satisfying the equality $sAu = Asu$ for all u in the domain of A. It is called the symmetric group of the operator A. The symmetric group G_u of a function u in R_n is defined similarly. The authors consider operators of the form $A = A_1 + A_2$: A_1 an operator symmetric with respect to the full orthogonal group of R_n and A_2 is an operator of multiplication by a function $A_2(x)$ continuous in the extended sense. Suppose u is a solution of $Au = \lambda u$ such that the set of its nodal points has an empty interior. The authors show that the symmetric group G_u of the solution is a subgroup of G_A.

Similar considerations for a Schrödinger operator were made previously by the authors with I. V. Gambarjan [Bočvar, Gambarjan, Stankevič and Čistjakov, Ž. Èksper. Teoret. Fiz. **36** (1959), 626–627] and also by M. A. Melvin [Rev. Modern Phys. **28** (1956), 18–44; MR **17**, 824].

R. K. Juberg (Minneapolis, Minn.)

Kantorovitz, Shmuel 3329

On the characterization of spectral operators.

Trans. Amer. Math. Soc. **111** (1964), 152–181.

The author establishes various conditions for a linear operator T with real spectrum to be spectral and scalar in the sense of N. Dunford. The most briefly stated of his criteria is as follows: Let T act in a reflexive space and suppose that $|e^{-2\pi i S}|$ is uniformly bounded, and that

$$[*] \qquad \left| \int_R \hat{f}(\xi) e^{-2\pi i \xi T}\, d\xi \right| \leq M |f|_\infty,$$

where the norm on the left is the operator norm, $\hat{f}$ is the Fourier transform of f, and $|f|_\infty$ refers to the sup norm. Then T is spectral and scalar.

This result is derived from a more general result wherein the Fourier integral of [*] is replaced by a corresponding singular integral which converges, formally at least, to the operator-function $f(T)$. The basic idea of the proof is to use this singular integral to define and investigate the functional calculus of the operator T.

As an application, it is shown that a certain integral operator is spectral.

The author also generalises a formula of Foguel concerning the spectral resolution of the sum of two scalar operators [cf. Ark. Mat. **3** (1958), 449–461; MR **21** #2914] and shows how the above general criterion that an operator be spectral leads to new proof of the classical spectral theorem in Hilbert space.

J. T. Schwartz (New York)

Banaschewski, B. 3330

On the derivation modules of an algebra.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 671–675.

Let A be a commutative algebra with unit 1 over a field K, and C a left module over A. A derivation of A into C is a K-linear mapping $\partial: A \to C$ such that $\partial(ab) = a\,\partial b + b\,\partial a$ for any a, b in A. Let K be the real field and let A be the algebra over K of all continuous real-valued functions of bounded variation on the closed unit interval. The author proves that: The only derivation of A into itself is the trivial mapping, $\partial f = 0$ for all f in A; but if the K-module $\mathrm{Hom}_K(A, K)$ of all K-module homomorphisms of A into K is made into an A-module by the definition $(f\phi)(g) = \phi(f \cdot g)$ for all ϕ in $\mathrm{Hom}_K(A, K)$ and f, g in A, then the mapping $\delta: A \to \mathrm{Hom}_K(A, K)$ given by $(\delta f)(g) = \int_0^1 g\, df$ is a derivation whose kernel contains only the constants. If one defines the norm of f as $\sup|f| + T(f)$, where $T(f)$ is the total variation of f on the unit interval, then A is a complete normed algebra [cf. C. E. Rickart, *General theory of Banach algebras*, p. 302, Van Nostrand, Princeton, N.J., 1960; MR **22** #5903]. The author shows that δ is a continuous mapping of A into the Banach dual of A under this norm.

{The paper contains several typographical errors. Fortunately, they are easily detected and corrected and are not likely to confuse or mislead the reader.}

L. E. Pursell (Grinnell, Iowa)

Kōmura, Yukio; Nakahara, Isamu 3331

A note on representations of algebras as subalgebras of $C(X)$ for X compact.

Kumamoto J. Sci. Ser. A **5**, 185–187 (1962).

A subset A of a real algebra $C(X)$ is regular if A contains the identity of $C(X)$ and separates points from closed sets. F. W. Anderson and R. L. Blair [Illinois J. Math. **3** (1959), 121–133; MR **20** #7214] proved that if A is a regular algebra such that every maximal ideal is real, then A is isomorphic to a regular subalgebra B of $C(X)$ for some topologically unique compact space X, and each (real) maximal ideal in B has the form $\{f: f(x) = 0\}$ for some $x \in X$. In this note a regular subalgebra of $C([0, 1])$ is exhibited such that each real maximal ideal has the form $\{f: f(x) = 0\}$, but not every maximal ideal is real. The same example shows that the converse to the analogous Theorem 4.5 of Anderson and Blair also fails.

C. W. Kohls (Syracuse, N.Y.)

Thoma, Elmar 3332

Über unitäre Darstellungen abzählbarer, diskreter Gruppen.

Math. Ann. **153** (1964), 111–138.

Let $x \to U_x$ be a unitary representation of a group G; we form the W^*-algebra A generated by $\{U_x | x \in G\}$. The best behaved case is that where A is of type I, as it was pointed out in the pioneering work of Murray and von Neumann. We shall say that a locally compact group G is of type I if every unitary representation of G generates a W^*-algebra of type I. Groups of type I are clearly of special interest and so it is always one of the central problems in the unitary representation theory of locally compact groups which groups are of type I? Various classes of type I groups have been picked up by a number of mathematicians. For them, the reader may refer to the survey of Mackey [Bull. Amer. Math. Soc. **69** (1963), 628–686; MR **27** #3745]. The present paper is an interesting contribution to the above problem in the case of discrete groups G.

The main result is as follows. Theorem 6: A countable

discrete group G is of type I if and only if G has an abelian normal subgroup of finite index. This is an affirmative solution of Kaplansky's conjecture [*Some aspects of analysis and probability*, pp. 1–34, Wiley, New York, 1958; MR **21** #286].

The difficulty of this theorem is in the "only if" part. To prove it, the author has to provide many lemmas.

S. Sakai (New Haven, Conn.)

Rolewicz, S. 3333

Example of semi-simple m-convex B_0-algebra, which is not a projective limit of semi-simple B-algebras.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 459–462.

Let X be a B_0-algebra with unit and with topology given by pseudo-norms satisfying (1) $\|x\|_i \leq \|x\|_{i+1}$ and (2) $\|xy\|_i \leq \|x\|_{i+1}\|y\|_{i+1}$. If (2) admits i instead of $i+1$ on the right side, then X is m-convex. The author constructs the example announced in the title.

D. G. Bourgin (Urbana, Ill.)

Sakai, Shôichirô 3334

On the reduction theory of von Neumann.

Bull. Amer. Math. Soc. **70** (1964), 393–398.

Von Neumann algebras have been characterized intrinsically without mention of the Hilbert space. Consequently, it is reasonable to attempt to carry out their reduction theory without reference to direct integrals of Hilbert spaces. Here, such a program is begun. For a von Neumann algebra M such that M_* is separable, the essentially bounded weakly $_*$ measurable functions from the measure space (Ω, μ) to M are shown to form a von Neumann algebra. Call it $L^\infty(M, \Omega, \mu)$, and $L^\infty(M, \Omega, \mu)_* \approx L^1(M_*, \Omega, \mu)$, the strongly integrable functions from (Ω, μ) to M_*. If M is a factor of type I_n, $n \leq \aleph_0$, then $L^\infty(M, \Omega, \mu)$ is, as expected, homogeneous of type I_n with center consisting of those functions in $L^\infty(M, \Omega, \mu)$ which are a.e. constant. Several unsolved problems are mentioned.

J. Feldman (Berkeley, Calif.)

Berezin, F. A. 3335

Letter to the editor. (Russian)

Trudy Moskov. Mat. Obšč. **12** (1963), 453–466.

In this communication the author fills in an essential gap in the argument of his earlier paper on Laplace operators on semi-simple Lie groups [same Trudy **6** (1957), 371–463; MR **19**, 867]. We shall retain the notation adopted there.

The gap occurs in 5.2 of the above paper. Here it is claimed that a certain linear functional $\tilde{\pi}_{c\xi}$, defined on a certain subalgebra of the group algebra of the complex semi-simple group G, and satisfying the characteristic differential equations of a character, can be extended to a functional $\tilde{\pi}$ on the space of all functions constant on conjugacy classes so as to satisfy the same differential equations. The present letter gives a lengthy justification of this claim.

J. M. G. Fell (Seattle, Wash.)

Cater, S. 3336

Continuous linear functionals on certain topological vector spaces.

Pacific J. Math. **13** (1963), 65–71.

Let V be a vector space of measurable functions on a measure space, including the simple functions, and suppose that V is a Hausdorff topological vector space in which a base of neighbourhoods of zero is formed by the sets $\{f \in V : \varphi(|f|) < \varepsilon\}$, where φ is a non-decreasing function mapping the non-negative real axis into itself. The author shows that if $\liminf n^{-1}\varphi(n) > 0$, there are enough continuous linear functionals to separate the points of V. If $\liminf n^{-1}\varphi(n) = 0$, the only continuous linear functionals are those that can be expressed in terms of the atoms of the measure space; hence if also the space is non-atomic, there are no nonzero continuous linear functionals on V.

A. P. Robertson (Glasgow)

Cater, S. 3337

Structure theorems for certain operator algebras.

Duke Math. J. **30** (1963), 567–577.

A result of Kaplansky [*Infinite abelian groups*, Univ. Michigan Press, Ann Arbor, Mich., 1954; MR **16**, 444] states: Let A be an operator on a complex vector space V such that for each vector z in V the linear manifold V_z spanned by all the vectors z, Az, A^2z, $\cdots$ is finite-dimensional and $\sup_{z\in V} \dim V_z < \infty$. Then A is algebraic and V is the direct sum of finite-dimensional subspaces, each reducing A. Here algebraic means "satisfies a polynomial equation". Replacing the single operator A by an algebra M of operators (they map V into V; no topology) one might make an analogous conjecture concerning the finite-dimensional character of M. The conjecture is false. The author's principal theorem considers conditions leading to the finite-dimensionality of M. Theorem: Let M be an algebra of operators on a complex vector space V such that (1) for each vector z in V, the subspace Mz is finite-dimensional and $\sup_{z\in V}(\dim Mz) < \infty$; (2) for each nonzero operator A in M there is an operator B on V (B need not be in M) for which $BM \subset M$ and BA is not nilpotent. It follows that M must be finite-dimensional. More precisely, there is a finite family of mutually orthogonal projections $\{R_i\}$ in the commutant of M for which $R_iM \subset M$ for all i, $I = \sum_i R_i$, and the contraction of M to the subspace R_iV is an n_i-fold copy of an operator algebra E_i on a finite-dimensional vector space W_i; furthermore, on each W_i a scalar product can be defined (making W_i a Hilbert space) such that $E_i * E_i \subset E_i$, where T^* denotes the operator adjoint of T. Conversely, any operator algebra with this structure must satisfy (1) and (2). The proof is spread over 6 lemmas. For example, it is established that M contains a projection P ($\neq 0, \neq I$) which is ultra-minimal (PMP consists of scalar multiples of P). Further theorems give information in case (3) for any $0 \neq A \in M$, $MA \neq 0$; and (4) each operator $A \in M$ is algebraic and $\sup_{A\in M}$ (degree A) $< \infty$. The discussion here introduces topology: Banach spaces and topological vector spaces.

E. R. Lorch (New York)

Deutsch, Nimet 3338

Espaces intermédiaires pour des espaces de fonctions holomorphes d'une variable.

C. R. Acad. Sci. Paris **258** (1964), 1686–1688.

If Ω is a domain in the plane, denote by $H(\Omega)$ the space of functions holomorphic on Ω. Suppose Ω_1 and Ω_2 are intersecting domains in C. An intermediary domain Ω is one such that $H(\Omega)$ is an interpolation space for the pair

$(H(\Omega_1), H(\Omega_2))$. It is proven that if Ω_1 and Ω_2 are discs, then every disc in $\Omega_1 \cup \Omega_2$ is an intermediary domain. Suppose that (E_1, E_2) is a pair of interpolating spaces. Then every function holomorphic on Ω_1 with values in E_1' and on Ω_2 with values in E_2' is holomorphic in these intermediary domains with values in some interpolation space. *H. Rossi* (Princeton, N.J.)

Deutsch, Nimet 3339

Espaces intermédiaires pour des espaces de fonctions holomorphes de plusieurs variables.

C. R. Acad. Sci. Paris **258** (1964), 1981–1983.

This note is a generalization to intersecting polydiscs in C^n of the results of the author's above note [#3338]. *H. Rossi* (Princeton, N.J.)

Bittner, R. 3340

Summation theorems for analytic elements.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 437–440.

The author continues his study [Studia Math. **20** (1961), 1-18; MR **25** #4316] of analytic elements (elements which are the sums of their Taylor series). He states two summability theorems involving a triangular infinite Toeplitz matrix of linear operations. There are applications to Newton's and other interpolation formulae. *A. P. Robertson* (Glasgow)

Bittner, R. 3341

Universal spaces for analytic elements.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 441–443.

For a given commutative algebra $\mathscr{B}$, "derivative" and "integral" operations are defined on the space of sequences of elements of $\mathscr{B}$; then analytic elements can be expressed as power series in the integral operation or in terms of another operation, called a "multiplier". (For definitions, see the above review [#3340] and the author's paper referred to there.) The correspondence set up between functions of the integral and of the multiplier reduces to the Laplace transformation in the usual real variable case. The author states a theorem on the uniqueness of the solution of a derivative equation. *A. P. Robertson* (Glasgow)

Ljubič, Ju. I. 3342

Density conditions on the initial manifold for the abstract Cauchy problem. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 262–265.

The author considers the Cauchy problem (1) $dx/dt = Ax$, $x(0) = x_0 \in E$, E being a Banach space and A a linear operator in E. Let I_A be the linear manifold of initial values x_0 for which there exists a smooth solution of (1). The problem posed is to determine when I_A is dense (it is not always so as indicated by an example). Let Γ be a curve in $\mathbf{C}'$ given by $\lambda = \lambda(s)$, $\lambda(s)$ piecewise smooth, satisfying: $\lambda'(s \pm 0) \neq 0$, $-\infty < s < \infty$; $\lambda(s_1) \neq \lambda(s_2)$, $s_1 \neq s_2$; $|\lambda(s)| \to \infty$ as $|s| \to \infty$; $|\lambda(s)|$ is strictly monotone near $s = \pm\infty$; and $\sup \operatorname{Re} \lambda(s) < \infty$, $-\infty < s < \infty$. Let α be such that the half-space $\operatorname{Re} \lambda < \alpha$ contains Γ and denote by Π_- the connected component of $\mathbf{C}\Gamma$ lying in this half-space, with Π_+ the other component. Let $D(A)$ be dense and A not have any spectral points on Γ or in Π_+. Then if $\|R_\lambda\| = O(1 + |\lambda|^m)$, $\lambda \in \overline{\Pi}_+$, $m \geqq 0$, where R_λ is the resolvent of A, it is asserted that I_A is dense. Another density theorem is given where this growth condition is weakened but other hypotheses are placed on A. Finally a condition that $I_A \neq \{0\}$ is given. *R. Carroll* (New Brunswick, N.J.)

Martirosjan, R. M. 3343

On the invariance of the spectrum of small perturbations of the polyharmonic operator. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 79–90.

The spectrum of the non-selfadjoint operator $T = A + UV$ is contained in the spectrum of the densely defined operator A if V is a closed operator whose domain contains the domain of A, U is a bounded operator, and $\|V(A-\lambda)^{-1}U\| \leqq 1$ whenever λ is not in the spectrum of A. Let $q(x)$ be a complex-valued function which is bounded and summable in euclidean n-space E_n, and let Δ be the Laplacian operator in $L^2(E_n)$. Integrability conditions on $q(x)$ are given which imply that $T = (-\Delta)^k + q$ is a closed operator in $L^2(E_n)$ which has the half-line $[0, \infty)$ as its spectrum. (It is assumed that $n > 3$ is odd and that $\frac{1}{2}(n+3) \leqq 2k < n$.) *L. de Branges* (Lafayette, Ind.)

Martirosjan, R. M. 3344

On the spectrum of the operator

$$-\Delta u + pLu + \int_E K(Q, P)u(P)\,dP$$

in the plane. (Russian. Armenian summary)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk **13** (1960), *no.* 1, 29–45.

The author considers the operator

$$Tu = -\Delta u + pLu + \int_E K(Q, P)u(P)\,dP,$$

where E is the complex plane, L a bounded operator in $L_2(E)$, p and K are complex-valued functions defined, respectively, on E and $E \times E$, such that p is bounded and

$$(1) \quad \int_E |p(Q)|^2\,dQ < \infty, \quad \|K\|^2 = \iint_{E\,E} |K(Q, P)|^2\,dP dQ < \infty.$$

By definition, the domain of T is the domain of the self-adjoint operator generated in $L_2(E)$ by $-\Delta$. The author proves that (i) the continuous spectrum of T coincides with the positive semi-axis, and that the other points of the spectrum are proper values and have possible accumulation points only on the positive semi-axis; (ii) the spectrum of T lies in the "interior" of the parabola $y^2 = 4\alpha(\alpha + x)$, where $\alpha = \frac{1}{2}\pi\sqrt{(M^2\|L\|^2 + \|K\|^2)}$, $\|L\|$ is the norm of L, and M is the upper bound of p; (iii) if L is a bounded operator of $L_\infty(E)$ and if $\int_E |K(Q, P)|\,dP$ is bounded in Q, is integrable on E and if so is p, then the discrete spectrum of T is bounded. These results are obtained by the use of the fact that T is a perturbation of the self-adjoint operator $-\Delta$. *C. Foiaş* (Bucharest)

Dašnic, L. S. 3345
On the closure of certain differential operators. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1959**, *no.* 6 (13), 44–47.

Let

$$l(y) = [p_0(x)y^{(n)}]^{(n)} + [p_1(x)y^{(n-1)}]^{(n-1)} + \cdots + p_n(x)y$$

be a self-adjoint differential operator, where

$$p_k(x) \in C_{n-k}[a, b], \qquad k = 0, 1, \cdots, n; \, p_0(x) \neq 0$$

on the bounded interval $[a, b]$. Denote, as usual, by L_0 the linear operator in $L_2(a, b)$ defined by l, such that its domain $\Omega_{L_0} \subset L_2(a, b)$ consists of all functions $(2n-1)$-times continuously differentiable so that $f^{(2n-1)}(x)$ is absolutely continuous, $f^{(2n)}(x) \in L_2(a, b)$ and $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, \cdots, 2n-1$. Denote by $\tilde{L}$ the operator again defined by l but whose domain $\Omega_{\tilde{L}} \subset L_2(a, b)$ consists of all functions $(m-1)$-times continuously differentiable, $f^{(m-1)}(x)$ absolutely continuous, $f^{(m)}(x) \in L_2(a, b)$, satisfying

$$(*) \quad \sum_{k=1}^{m} \alpha_{jk} f^{(k-1)}(a) + \sum_{k=1}^{m} \beta_{jk} f^{(k-1)}(b) = 0 \;\; (j = 1, 2, \cdots, m),$$

where α_{jk}, β_{jk} and m are fixed numbers, $m > 2n$. In ordinary differential equations, such boundary conditions actually occur. The aim of this note is to find the domain of the closure $\bar{\tilde{L}}$ of $\tilde{L}$, i.e., $\bar{\tilde{L}} = \tilde{L}^{**}$. The author's result can be stated as follows: $\bar{\tilde{L}} \subset L_0^*$ and its domain $\Omega_{\bar{\tilde{L}}}$ consists of all functions of $\Omega_{L^*_0}$ satisfying the conditions which are obtained from (*) by eliminating the derivatives of order $\geqq 2n$.

The proof is based on a fundamental result of M. G. Kreĭn [Mat. Sb. (N.S.) **21 (63)** (1947), 365–404; MR **9**, 515]. *C. Foiaş* (Bucharest)

Štraus, A. V. 3346
On the multiplicity of the spectrum of a self-adjoint ordinary differential operator. (Russian)
Dokl. Akad. Nauk SSSR **155** (1964), 771–774.

The study of formally selfadjoint ordinary differential operators of order greater than 2 is complicated by the fact that there is no simple alternative between regular and singular behavior at boundary points. Because of this, multiplicities of selfadjoint extensions are not obvious. The aim of the paper is to develop a more precise concept of eigenfunction expansion or spectral resolution which determines these numbers. The author does this by reformulating the problem as one for a first-order differential operator acting on vector-valued functions. The fundamental solutions of the eigenvalue equation then form a matrix whose elements are entire functions of the eigenvalue parameter. The required information is obtained immediately from this matrix.
L. de Branges (Lafayette, Ind.)

Askerov, N. G.; Kreĭn, S. G.; Laptev, G. I. 3347
On a class of non-selfadjoint boundary-value problems. (Russian)
Dokl. Akad. Nauk SSSR **155** (1964), 499–502.

Let A be a densely defined transformation in a Hilbert space H, and let T and Γ be transformations of the domain of A into a reference Hilbert space. Suppose that the restrictions of A to the kernel of T and to the kernel of Γ are selfadjoint, positive transformations having completely continuous inverses. The eigenvalue problem $Ay = \lambda y$ is considered under the side condition $\lambda Ty = \sigma \Gamma y$, where $\sigma > 0$ is fixed. The nature of the spectrum is discussed and a completeness theorem is stated for eigenvectors and supplementary vectors under an additional trace class hypothesis. The result is of interest in the study of formally selfadjoint differential operators under non-selfadjoint boundary conditions.
L. de Branges (Lafayette, Ind.)

Kemp, R. R. D. 3348
On a class of singular differential operators.
Canad. J. Math. **13** (1961), 316–330.

The author considers operators in the Banach space $L_p(I)$ $(1 \leqq p \leqq \infty)$ for a real interval $I = [a, b]$ (a or b or both may be infinite) which are generated by an ordinary differential expression $\tau = \sum_{j=0}^{n} p_j D^{n-j}$, $(D = d/dx)$. Here p_j is a complex-valued function of class C^{n-j} on I. The main new feature studied is the effect of the vanishing of the leading coefficient p_0 in the interior of I. First the domains of the minimal and maximal operators T_0, T_1 in $L_p(I)$ associated with τ are defined, and their adjoints determined in $L_q(I)$, $q = p/(p-1)$. Next it is indicated how the domain of a closed operator T satisfying $T_0 \subset T \subset T_1$ may be described by certain boundary conditions, and these latter conditions are given in terms of a certain bilinear form $\langle f, g \rangle$ for f in the domain of T_1 and g in the domain of T_0^*. Now let $N_k = \{x \mid p_j(x) = 0, j = 0, 1, \cdots, k\}$, and let N_{k0} be the interior of N_k. In this notation it is shown that $\langle f, g \rangle$ depends only on the values of f and g in the neighbourhood of the set $B = \{x \in I \mid x$ is an endpoint of I, or there exists an integer k between 0 and $n-1$ such that $x \in N_k$, $x \notin N_{k0}\}$. Thus the domain of T may be given by conditions on functions at points other than the endpoints of I. The author then discusses in detail a special class of operators, called regular operators, for which the set B above is assumed to be finite and the essential resolvent set of T_0 is non-empty. For these operators the space of boundary conditions is finite-dimensional. The case when τ is formally self-adjoint in $L_2(I)$ is considered and it is shown that if B is finite, then any maximal symmetric extension T of T_0 has a generalized resolvent which is an integral operator of Carleman type. This implies an expansion result. Finally, several examples are presented. *E. A. Coddington* (Zbl **102**, 301)

Shinbrot, Marvin 3349
A nonlinear eigenvalue problem. II.
Arch. Rational Mech. Anal. **15** (1964), 368–376.

The present work is a sequel to a previous paper of the author [Proc. Amer. Math. Soc. **14** (1963), 552–558; MR **27** #5129] to which we refer. In that paper, he considers the perturbation equation $\lambda u = Au + \lambda^{\alpha} B_{\lambda} u$, where A is a compact symmetric operator and B_{λ} is bounded—both on a Hilbert space H. The exponent α satisfies $\alpha > 1$. The result there given is that the eigenvectors u_n span a manifold of finite codimension r and that by adjoining to $\{u_n\}$ r eigenvectors of A, the resulting system is complete. This the author calls "complete with A". The present paper treats the case $0 < \alpha \leqq 1$. By a simple iterative procedure whose first step yields $\lambda u = Au + \lambda^{1+\alpha} B_{\lambda} A^{-1} u - \lambda^{2\alpha} B_{\lambda} A^{-1} B_{\lambda} u$

the present problem can be reduced to the preceding one for a new operator. Thus the author is able to prove the completeness of $\{u_n\}$ with A by hypothesizing appropriate properties for $B_\lambda A^{-1}$ or $A^{-1}B_\lambda$. This gives a new proof of some results of J. T. Schwartz [Pacific J. Math. **4** (1954), 415–458; MR **16**, 144]. For example, if L satisfies $(\mathscr{A}_2)$ (see the previous paper), and if B is bounded, then $L+B$ has an infinite sequence of eigenvalues approaching infinity, and the corresponding eigenvalues are complete with L. The author indicates four types of applications for his theorems, including hydrodynamics and ordinary differential equations. *E. R. Lorch* (New York)

Brown, Arlen; Halmos, P. R. 3350
Algebraic properties of Toeplitz operators.
J. Reine Angew. Math. **213** (1963), 89–102.

Let $e_n=z^n$ for $|z|=1$ and $n=0, \pm 1, \cdots$; let $\mathfrak{L}^2$ be the Hilbert space of functions $f\sim\sum (f, e_n)e_n$ of class L^2 with respect to normalized measure on the unit circle $|z|=1$; $\mathfrak{H}^2$ the subspace of $\mathfrak{L}^2$ consisting of the "analytic" functions f satisfying $(f, e_n)=0$ for $n<0$; P the orthogonal projection of $\mathfrak{L}^2$ onto $\mathfrak{H}^2$; W the unitary operator on $\mathfrak{L}^2$ defined by $We_n=e_{n+1}$ for $n=0, \pm 1, \cdots$; U the isometric operator on $\mathfrak{H}^2$ defined by $Ue_n=e_{n+1}$ for $n=0, 1, \cdots$. A "co-analytic" function f is an element f of $\mathfrak{L}^2$ satisfying $(f, e_n)=0$ for $n>0$. If ϕ is a bounded measurable function on $|z|=1$, the corresponding Laurent operator L_ϕ on $\mathfrak{L}^2$ is defined by $L_\phi f=\phi f$, the Toeplitz operator T_ϕ on $\mathfrak{H}^2$ by $T_\phi f=PL_\phi f=P(\phi f)$; in particular, $W=L_\phi$ and $U=T_\phi$, where $\phi=e_1$. Among other results, it is shown that a bounded operator A on $\mathfrak{L}^2$ is a Laurent operator L_ϕ if and only if $WA=AW$; a bounded operator A on $\mathfrak{H}^2$ is a Toeplitz operator [with ϕ analytic or ϕ co-analytic] if and only if $A=U^*AU$ [with $UA=AU$ or $U^*A=AU^*$]; the product $T_\phi T_\psi$ of two bounded Toeplitz operators is a Toeplitz operator if and only if either ϕ is co-analytic or ψ is analytic; if ϕ is a non-constant, bounded, measurable function, then L_ϕ is a minimal normal dilation of T_ϕ.
P. Hartman (Baltimore, Md.)

Hirschman, I. I., Jr. 3351
Extreme eigen values of Toeplitz forms associated with ultraspherical polynomials.
J. Math. Mech. **13** (1964), 249–282.

Let $W_\nu(k, \theta)=k!\, C_k{}^\nu(\cos\theta)/(2\nu)_k$, where $C_k{}^\nu$ is the ultraspherical polynomial of index ν and degree k, $\Omega_\nu(\theta)=[\sin\theta]^{2\nu}$. Then

$$\int_0^\pi W_\nu(j, \theta)W_\nu(k, \theta)\Omega_\nu(\theta)\,d\theta = \delta(j, k)/\omega_\nu(k)$$

for certain positive constants $\omega_\nu(k)$. Let $\mathbf{L}$ be the Hilbert space of complex-valued functions f defined for $k=0, 1, \cdots$ with inner product $(f|g)=\sum_{k=0}^\infty f(k)g(k)^*\omega_\nu(k)$. With a function $p(z, \theta)$ defined for $0\leqq z<\infty$, $0<\theta\leqq\pi$, which satisfies certain regularity conditions, one associates the transformations $M_n[p]$ of $\mathbf{L}$ into itself given by

$$M_n[p]f(k) = \sum_{j=0}^\infty \mathbf{p}\left(\frac{k+j}{2n}, k, j\right)f(j)\omega_\nu(j),$$

where $\mathbf{p}(z, k, j)=\int_0^\pi p(z, \theta)W_\nu(k, \theta)W_\nu(j, \theta)\Omega_\nu(\theta)\,d\theta$. Let $\lambda_{n,1}\leqq\lambda_{n,2}\leqq\cdots\leqq\lambda_{n,n+1}$ be the eigenvalues of the compression of the operator $M_n[r^*]M_n[p]M_n[r]$ to the subspace of $\mathbf{L}$ consisting of those f satisfying $f(k)=0$ for $k>n$; here p is real and $r=r(\theta)$ is continuous, vanishes at $\theta=0$ and nowhere else, and near $\theta=0$ satisfies $r(\theta)\sim\rho\theta^\alpha L(\theta)$, where ρ is a complex constant, $\alpha>0$, and L is slowly oscillating. The main result of the paper is that for fixed k as $n\to\infty$ one has $\lambda_{n,k}=|\rho|^2n^{-2\alpha}L(n^{-1})^2[\mu_k+o(1)]$, where $0<\mu_1\leqq\mu_2\leqq\cdots$ are the eigenvalues of a certain operator depending only on ν, α, and $p(z, 0)$. This is the analogue for matrices associated with ultra-spherical polynomials of a theorem of Parter [Arch. Rational Mech. Anal. **11** (1962), 244–257; MR **26** #605]. The proof depends on a perturbation theorem proved by the author, and uses certain Banach algebra properties of ultra-spherical polynomials. *H. Widom* (Ithaca, N.Y.)

CALCULUS OF VARIATIONS

See also 3896, 3897.

Gel'fand, I. M. [Гельфанд, И. М.]; Fomin, S. V. [Фомин, С. В.] 3352
★Variational calculus [Вариационное исчисление].
Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961. 228 *pp.* 0.47 *r.*

This textbook has a number of special features to recommend it: (1) A clear treatment of the differential δJ of a calculus of variations functional, as this depends on the space of curves admitted; (2) a great many explicitly worked problems; (3) a fresh approach to the fundamental du Bois-Reymond type lemmas, through differentiation of the functionals, and with some unusual lemmas. There are many physical applications, for example, to membranes and plates, there are Sturm-Liouville equations in detail; there are Haar's equations and Maxwell's equations, and control problems of Pontryagin. As to theory there is much attention paid to the difference between weak and strong extremals in various senses.
J. M. Danskin (Cambridge, Mass.)

Gelfand, I. M. [Gel'fand, I. M.]; Fomin, S. V. 3353
★Calculus of variations.
Revised English edition translated and edited by Richard A. Silverman.
Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963. vii+232 *pp.* \$10.60.

This book fills a long-standing need for a text on the calculus of variations which can be used at the advanced undergraduate level, which provides sufficient physical background, and which at the same time takes some account of 20th century mathematical developments in the subject. Through unerring good taste and elegance of style, a great amount of material has been put into a little over 200 pages.

The first three chapters include the basic ideas and necessary conditions for variational problems with fixed or variable boundary data, problems with side conditions, and so on. Chapter 4 treats the Legendre transformation, principle of least action, and the Hamilton-Jacobi equation. The connection between the Legendre transformation and the theory of conjugate convex functions is made clear. Chapter 5 is about the second variation and Jacobi's condition. Chapter 6 concerns fields and sufficient

conditions for an extremum. In Chapter 7 multiple-integral variational problems are treated. The emphasis is on deriving various famous physical principles from the principle of least action (more precisely, of stationary action) and from Noether's theorem about invariant variational problems. Chapter 8 concerns direct methods. Pontryagin's maximum principle for optimal control problems is stated (but not proved) in an appendix.

Over 125 homework problems, ranging from routine to difficult, have been prepared for the English edition. Besides that, the editor has made a considerable number of improvements in the text of the original Russian version [#3352 above], with the cooperation of the authors.

W. H. Fleming (Providence, R.I.)

Hölder, Ernst 3354

Beweise einiger Ergebnisse aus der Theorie der 2. Variation mehrfacher Extremalintegrale.

Math. Ann. **148** (1962), 214–225.

The present paper is concerned with the study of the second variation of the multiple integral $I=\int_G f(t, z, q)\,dt$, where t is m-dimensional, z is n-dimensional, and $q_\alpha{}^i=\partial z^i/\partial t^\alpha$. Hilbert space methods are used. The second variation $J(x)$ of $I(z)$ is a quadratic form in the Hilbert space whose norm $\|x\|_1$ is determined by x and its partial derivatives. The author defines regularity in terms of Gårding's inequality. This is equivalent to the assumption that J is the sum of $P(x)+K(x)$ where $K(x)$ is completely continuous and $P(x)\geq\varepsilon\|x\|_1{}^2$ $(\varepsilon>0)$. This is the Legendre condition, that is, the ellipticity condition. It is shown that $J(x)$ is of finite negative index and of finite nullity. The theory of eigenvalues is developed. It is shown that there is a finite number of negative eigenvalues and zero eigenvalues. The theory of potentials and Green's functions are developed. Hilbert space methods have the advantage that they yield a unified method of studying the second variation of variational problems. This development includes the study of elliptic partial differential equations by Hilbert methods.

M. R. Hestenes (Los Angeles, Calif.)

Belen'kii, I. M. [Belen'kii, I. M.] 3355

On an application of Hilbert's independence theorem.

Prikl. Mat. Meh. **27** (1963), 887–889 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1349–1353.

The author discusses Hilbert's integral with an application to the potential of a planar force field.

G. M. Ewing (Norman, Okla.)

Garfinkel, Boris 3356

A solution of the Goddard problem.

J. Soc. Indust. Appl. Math. Ser. A Control **1**, 349–368 (1963).

Given the equation $\dot{m}c+m\dot{V}+\frac{1}{2}C_D(V)V^2S\rho_0\exp(-X/l)+mg=0$ and end values $(X_0, V_0, m_0)=(0, 0, m_{\max})$, $m_1=m_{\min}$, the author wishes to maximize the summit altitude X_1. A formulation in 3-space (x, v, y) is obtained by setting $x=gX/c^2$, $v=V/c$, $y=\log m_0/m-v-x/\alpha$. If y and v are known, the last relation determines m. Admissible pairs (v, y) are apparently all those such that v is piecewise continuous and y piecewise smooth with $y(0)=0$, $v(0)=0$ and with side inequalities, which in effect require that m be non-increasing and that $m(t)\geq m_{\min}$. In this context, several relations with v as a divisor or involving derivatives of v or m, are disconcerting. An auxiliary Mayer-type problem with one side equality plays a central role. Conclusions for it are based on the multiplier rule and various page references to Bliss [*Lectures on the calculus of variations*, Univ. Chicago Press, Chicago, Ill., 1946; MR **8**, 212]. In order to fit the Bliss hypotheses, it is necessary to interpret v (also called u) as the derivative of a piecewise smooth function, which the author does in his relation (49) but not everywhere; e.g., his (21.3) can say, on the authority of Bliss, only that the left member is constant. Moreover, some such device as that of F. A. Valentine's dissertation [*Contributions to the calculus of variations, 1933–37*, pp. 403–447, Univ. Chicago Press, Chicago, Ill., 1937] is needed for the side inequality before the Bliss treatment is applicable. Precisely why the planar field of relations (63) exists in the large and suffices for the present problem and how the sufficiency theorem in the large of Section 14 is to be proved are not clear to the reviewer.

G. M. Ewing (Norman, Okla.)

GEOMETRY

See also 3070, 3093, 3406.

Coolidge, Julian Lowell 3357

★A history of geometrical methods.

Dover Publications, Inc., New York, 1963. xviii + 448 *pp.* $2.25.

This edition is an unabridged and unaltered republication of the work first published by the Oxford Univ. Press in 1940 [MR **2**, 113; errata, MR **2**, 419]. Table of Contents:

Book I: Synthetic Geometry. I, The Beginnings of Geometry; II, Greek Mathematics; III, Later Elementary Geometry; IV, The Non-Euclidean Geometries; V, Projective Geometry; VI, Descriptive Geometry.

Book II: Algebraic Geometry. I, The Beginnings of Algebraic Geometry; II, Extension of the System of Linear Coordinates; III, Other Systems of Point Coordinates; IV, Enumerative Geometry; V, Birational Geometry; VI, Higher Spaces and Higher Space Elements; VII, Geometrical Transformations.

Book III: Differential Geometry. I, Early Writers; II, Intrinsic Geometry and Moving Axes; III, Gauss and the Classical Theory of Surfaces; IV, Projective Differential Geometry; V, Absolute Differential Geometry.

Zarovnyĭ, V. P. 3358

An interpretation of the plane axioms of Euclidean geometry in a certain abstract group. I, II. (Russian)

Ukrain. Mat. Ž. **12** (1960), 3–12; *ibid.* **12** (1960), 244–256.

Let Φ be a group, let G_α be subgroups of Φ and let each G_α be divided in two "opposite" semi-groups, defining two opposite orderings of G_α. Call "points" the elements of Φ and "lines" the left cosets of the G_α's. In a previous paper [same Ž. **10** (1958), 351–364; MR **21** #833], the author has given a set of conditions (actually incomplete: cf. MR **21** #833) in order that the points and lines thus defined satisfy Hilbert's axioms I (incidence), II (parallels) and IV (order). He now deals with the other axioms of Hilbert.

A "system of orts" is defined as a set of points, one on each half-G_α, such that the two "orts" lying on two opposite half-G_α's are inverse of each other: this is a candidate to become a circle centered at e (the neutral element of Φ). Such a system is used to define, in the obvious way, congruence of segments and congruence of angles. The congruence of segments automatically satisfies the axioms III_1 and III_2; it satisfies III_3 if and only if Φ is abelian. To insure the validity of the angle congruence axioms III_4 and III_5, one has to impose further conditions, called "ellipticity" conditions, on the system of orts. The Archimedes axiom (V_1) is equivalent to the Archimedes axiom for the ordered groups G_α. Finally, a completeness axiom makes Φ isomorphic with the additive group of a 2-dimensional real vector space.

J. L. Tits (Bonn)

Haupt, Otto 3359

Aus der Theorie der geometrischen Ordnungen.

Jber. Deutsch. Math.-Verein. **65** (1962/63), *Abt.* 1, 148–186.

This is the first survey of "géométrie finie" since the author's report of 1948 [*Naturforschung und Medizin in Deutschland 1939–1946*, Bd. 2, pp. 197–215, Dieterich'sche, Wiesbaden, 1948; MR **11**, 539]. The central concept is that of geometric order. Let $\mathfrak{k}$ be a class of sets in a topological space R. The (point-) order of the set $M \subset R$ is roughly the least upper bound of the number of points of $M \cap K$ as K ranges through $\mathfrak{k}$. The order of a point $x \in M$ is the minimum of the orders of the intersections of M with the neighbourhoods of x in R. If M belongs to a certain class of sets in R and if the order of x is greater than the minimal order of the sets in that class, x is called $\mathfrak{k}$-singular on M.

Some of the problems of géométrie finie are purely topological: to find local or global properties of M's of given order or to determine bounds for the order of M's with certain properties. The results become stronger but lose their exclusively topological character when concepts or assumptions are introduced from direct differential geometry.

This report is divided into three chapters. In the first one, R is a plane or disk and M is a Jordan arc or curve. The elements K of $\mathfrak{k}$ are Jordan curves in R with the following property: There is a positive integer k such that there exists exactly one K through any k points sufficiently close to k points of M; two distinct K's have not more than $k-1$ points in common. A point of M is $\mathfrak{k}$-singular if its order is greater than k. The following topics might be mentioned: The cases $k=1$ and $k=2$; arcs and curves of order 2 or 3; linear and cyclic order; the two-vertex theorem. Bounds for the number of singular points; curves of order k or $k+1$. Polygons.

In Chapter 2, R is projective n-space P_n, $\mathfrak{k}$ is the class of the $(n-1)$-flats in P_n, M is an arc, a curve, or a continuum. Topics: M's of order n. Barner's generalization of the four-vertex theorem to curves in P_n. Dualization theorems. Singularities of arcs and curves of order $n+1$. Polygons in P_n, etc.

The last chapter on surfaces in P_3 and some generalizations starts with a brief exposé of Marchaud's beautiful work on surfaces of order three. Some of the other topics: The case that M is an m-dimensional manifold in P_n and $\mathfrak{k}$ is the class of $(n-m)$-flats. Generalization to compact metric spaces of a limit theorem related to Blaschke's "Auswahlsatz". *P. Scherk* (Toronto, Ont.)

Benz, Walter 3360

Elliptische Kreisbüschel als Gruppen.

Tensor (*N.S.*) **13** (1963), 232–245.

Per le notazioni, i concetti e la bibliografia sui piani di Moebius, l'autore rinvia, per l'indirizzo classico al Vol. III delle *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie*, di W. Blaschke [Springer, Berlin, 1929] e, per l'indirizzo astratto (cui il presente lavoro si riferisce) ad un suo rapporto del 1960 [Jber. Deutsch. Math.-Verein. **63** (1960), Abt. 1, 1–27; MR **22** #7012]. Questo lavoro (dedicato ad A. Kawaguchi) è diviso in due paragrafi. Nel § 1 si dimostra che i piani Σ di Moebius in senso stretto per i quali vale il teorema completo di Miquel sono caratterizzati dall'esistenza di un gruppo $G(\Sigma)$ cui si può riferire biunivocamente ogni fascio orientato di circoli (avente una data coppia ordinata A, B di punti base distinti). Tali corrispondenze σ_{AB} sono legate da due relazioni ($G1$), ($G2$) che riguardano gli elementi del gruppo $G(\Sigma)$ associati da σ_{AB}, σ_{CD}, σ_{AC}, σ_{AD} ai circoli passanti per quattro punti A, B, C, D non conciclici. Nel § 2 l'autore si occupa dei piani Σ di circoli (Kreisebene) non necessariamente di Moebius, ma dotati di un gruppo $G(\Sigma)$ tale che sia soddisfatta la ($G1$) e dimostra che in essi vale il teorema di Miquel e che $G(\Sigma)$ è abeliano. L'autore fornisce anche una descrizione algebrica di questi piani di circoli (nel caso in cui valga anche la ($G2$)) ed una interpretazione geometrica delle relazioni algebriche ($G1$) e ($G2$).

E. Morgantini (Padova)

Skopec, Z. A. 3361

An oblique mapping of a third-order surface with a double point onto the plane. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 117–121.

The mapping studied in this note is a projection of an F_3 in 3-space with a double point S from this point S onto a plane π which has three straight lines in common with F_3. In the simplest case the point S is placed into an infinite point S_∞ in a direction perpendicular to π. Together with those three lines, a circle k_2 passing through two points of each of the sides of the triangle defined by the lines, taken as intersection with π of the cylinder touching F_3 at S, defines F_3 up to contraction with respect to π. To each point $M' \in F_3$ corresponds a unique point $M \in \pi$, except to S_∞, to which corresponds the whole circle k_2. Mappings of this kind can be applied to certain practical problems in descriptive geometry.

H. Schwerdtfeger (Montreal, Que.)

Pevzner, S. L. 3362

Quadrics in an n-dimensional hyperbolic space. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **33** (1964), 15–17.

Using results of a note by Ermolaev [Dokl. Akad. Nauk SSSR **132** (1960), 257–259; MR **22** #9505] the author obtains four types of canonical equations for a quadric $x'Ax=0$ in the space ${}^1S_n$ with the absolute

$$x'Gx = -x_0^2+x_1^2+\cdots+x_n^2 = 0.$$

A complete system of invariants of the quadric $x'Ax=0$ is found to consist of the elementary divisors of the matrix $A-\lambda G$ and the signs of certain products of sums of principal minors of this matrix.
H. Schwerdtfeger (Montreal, Que.)

Semenovič, A. F. 3363
Constructions by straightedge and discrete points on oricycles in the Lobachevsky plane. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1963**, *no.* 5 (36), 101–104.
Smogorzevski (Kiev, 1948) und Nestorovič [Dokl. Akad. Nauk SSSR **66** (1949), 1047–1050; MR **11**, 50] haben gezeigt: Es lassen sich alle Konstruktionen 2. Grades der hyperbolischen Ebene ausführen, wenn man außer dem Lineal noch ein Gerät zuläßt, das entweder Grenzkreise oder eine feste Abstandslinie zu zeichnen gestattet. In der vorliegenden Arbeit zeigt Verfasser nun: Man braucht nicht zu verlangen, daß man einen ganzen Grenzkreis zeichnen kann, sondern es genügen diskrete Punkte desselben. So kann man z. B. bei Kenntnis von 4 weiteren Punkten des durch die Gerade g als Achse und den Punkt B auf b bestimmten Grenzkreises mit Hilfe des Pascalsatzes den zweiten Schnittpunkt einer Geraden c durch B mit k konstruieren. Dies ist die erste Aufgabe, die gelöst wird; weitere 16 Aufgaben betreffen Verdoppeln, Halbieren von Strecken und Winkeln, Konstruktionen von Loten, rechtwinkligen Dreiecken sowie die Bestimmung von Schnittpunkten zwischen Geraden und Kreisen. Hiermit sind dann alle Konstruktionen 2. Grades in der hyperbolischen Ebene mit den genannten Hilfsmitteln als durchführbar nachgewiesen. *W. Burau* (Hamburg)

Marchaud, André 3364
Sur les courbes et les surfaces du troisième ordre en géométrie finie.
Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 555–575.
Dans un plan projectif, on appelle "continu" tout ensemble fermé qui ne peut se décomposer en deux ensembles disjoints formés; son ordre est le nombre maximum atteint par les intersections avec une droite du plan. Les courbes du 3° ordre peuvent se ramener à quelques types: si la courbe est formée d'un seul continu, elle est somme de 2 arcs du second ordre lorsqu'elle possède ou un point nodal ou un rebroussement ou une "épine" (point avec deux demi-tangentes traversant la courbe), elle est somme de trois arcs du second ordre dans les autres cas; si elle est formée de deux continus, elle est la somme d'une ovale (pouvant se réduire à un point) et d'une courbe du type précédent sans point nodal. Cette détermination permet de démontrer le théorème de "dégénérescence" pour les surfaces du troisième ordre: "Si une surface du troisième ordre possède sept droites concourantes, c'est un cône ou la somme d'un cône et d'un ovoïde (éventuellement réductible à un point); si cet ovoïde existe, il a au plus un point commun avec le cône." *B. d'Orgeval* (Dijon)

Magari, Roberto 3365
Su certe strutture algebriche associate ai piani grafici autopolari.
Boll. Un. Mat. Ital. (3) **18** (1963), 238–251.
Es sei $\pi=(P, L; \mathcal{T})$ eine projektive Ebene (piano grafico), P bzw. L die Menge ihrer Punkte bzw. Geraden und $\mathcal{T}$ die (symmetrische) Inzidenzrelation. Unter einer Polarität in π versteht man bekanntlich einen involutorischen Isomorphismus φ von π auf die duale Ebene $\pi^*=(L, P; \mathcal{T})$: $\varphi^2=e$ (e ist der identische Automorphismus von π). Die projektive Ebene π heisst autopolar, wenn sie eine Polarität in π zulässt. Es sei nun π autopolar und φ eine Polarität in π. Verfasser führt in der Menge P eine für je zwei voneinander verschiedene Punkte $x, y\in P$ definierte Multiplikation (Verknüpfungsregel) ein: $xy=\varphi(\overline{xy})$; $\overline{xy}\in L$ ist die mit x und y inzidente Gerade. Es gilt: (1) $xy=yx$, (2) $(xy)(xz)=x$. Dies gibt die folgende Bedeutung: Unter einem projektiven Gruppoid (gruppoide grafico) versteht man eine nicht leere Menge G mit einer in ihr für je zwei voneinander verschiedene Elemente $x, y\in G$ erklärten Multiplikation mit den Eigenschaften (1) und (2). Den Inhalt der vorgelegten Arbeit bildet die Untersuchung von Beziehungen zwischen autopolaren projektiven Ebenen und projektiven Gruppoiden. Z.B. kann ein projektives Gruppoid P ohne gewisse Singularitäten zu einer autopolaren projektiven Ebene erweitert werden.
O. Borůvka (Brno)

Perry, R. L. 3366
A result on the equivalence of linear sets by finite decomposition.
J. London Math. Soc. **36** (1961), 245–253.
Let A and B be subsets of a Euclidean space. $A \underset{n}{=} B$ expresses the fact that A can be written as the union of disjoint sets A_γ ($\gamma=1, 2, \cdots, n$) and B can be written as the union of n disjoint sets B_γ ($\gamma=1, 2, \cdots, n$), so that A_γ is congruent to B_γ ($\gamma=1, 2, \cdots, n$). $A \underset{r}{=} B \underset{s}{=} C$ implies $A \underset{rs}{=} C$. For a given n, a pair r, s is called admissible if $A \underset{n}{=} B$ implies the existence of a set C satisfying $A \underset{r}{=} C \underset{s}{=} B$. The author shows that for $n\geq 3$, $2, n-1$ is not an admissible pair [see also Sierpiński, *On the congruence of sets and their equivalence by finite decomposition*, Lucknow Univ., Lucknow, 1954; MR **15**, 691; Matematiche (Catania) **10** (1955), 71–79; MR **17**, 592].
P. Erdős (Budapest)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 3141, 3359, 3393.

Grünbaum, B. 3367
Fixing systems and inner illumination.
Acta Math. Acad. Sci. Hungar. **15** (1964), 161–163.
Let K be a convex body in Euclidean n-space E^n (i.e., a compact convex set with non-empty interior); a set $A\subset \mathrm{Bd}\, K$ is called a fixing system for K (Fejes Tóth) if there exists an $\varepsilon>0$ such that $x\in E^n$, $0<|x|<\varepsilon$ implies $A\cap \mathrm{int}\,(x+K)\neq\emptyset$. A set $B\subset \mathrm{Bd}\, K$ is called an inner illuminating system of K (Soltan) if for every $x\in \mathrm{Bd}\, K$ there exists a $b\in B$ such that the segment $[b, x]$ meets $\mathrm{int}\, K$. The author proves the following. Theorem 1: For every convex body $K\subset E^n$ there exists a fixing system consisting of at most $2n$ points. Theorem 2: For every convex body $K\subset E^n$ there exists an inner illuminating system containing at most $n+1$ points.
H. Fast (Notre Dame, Ind.)

Melzak, Z. A. 3368

A property of plane sets of constant width.

Canad. Math. Bull. **6** (1963), 409–415.

The main result is: Every planar convex body K of constant width 1 may be decomposed into three subsets, each of diameter at most min $\{X(K), 3^{1/2}-X(K)\}$, where $X(K)$ is the edge-length of the largest equilateral triangle which has its vertices on the boundary of K. This generalizes a result of D. Gale [Proc. Amer. Math. Soc. **4** (1953), 222–225; MR **14**, 787]. *B. Grünbaum* (Jerusalem)

Böröczky, K.; Florian, A. 3369

Über die dichteste Kugelpackung im hyperbolischen Raum.

Acta Math. Acad. Sci. Hungar. **15** (1964), 237–245.

The authors establish the truth of Fejes Tóth's conjecture [Arch. Math. **10** (1959), 307–313; MR **21** #5169] that four spheres of radius r are packed as closely as possible when they all touch one another (so that their centers are the vertices of a regular tetrahedron of edge $2r$). It follows that the density of a packing of spheres of radius r (in hyperbolic 3-space) cannot exceed

$$d(r) = (6\alpha-\pi)(\sinh 2r-2r)/T,$$

where $\sec 2\alpha = \operatorname{sech} 2r+2$ and T is the volume of a regular tetrahedron of edge $2r$ (and dihedral angle 2α). The final section of the paper consists of a long proof that $d(r)$ is an increasing function of r, from which it follows that the density of a packing of congruent spheres (of finite or infinite radius) cannot exceed

$$\lim_{r\to\infty} d(r) = \left(1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\cdots\right)^{-1} = 0.853\cdots$$

[Fejes Tóth, Publ. Math. Debrecen **3** (1953), 158–167; MR **15**, 819; the reviewer, Acta Math. Acad. Sci. Hungar. **5** (1954), 263–274; MR **17**, 523]. The authors could have saved themselves a lot of trouble by observing that $T = 3\int_{2\alpha}^{\operatorname{arc\,sec} 3} l\,d\lambda$, where l is given by $\operatorname{sech} l = \sec\lambda - 2$. It was proved independently by Schläfli and Richmond [see the reviewer, *Non-Euclidean geometry*, 3rd ed., pp. 285–288, Univ. Toronto Press, Toronto, Ont., 1957; MR **19**, 445] that the volume T of a tetrahedron in spherical 3-space satisfies $dT = \frac{1}{2}\sum l d\lambda$ (a sum of six terms), where λ is the dihedral angle at an edge of length l. Since a regular tetrahedron in Euclidean space has dihedral angle arc sec 3, it follows that the volume of a regular tetrahedron of dihedral angle 2α in spherical space is $3\int_{\operatorname{arc\,sec} 3}^{2\alpha} l d\lambda$, where $\sec l = \sec\lambda - 2$. In hyperbolic space the formula for dT needs a minus sign, and of course sec l becomes sech l. *H. S. M. Coxeter* (Toronto, Ont.)

DIFFERENTIAL GEOMETRY

See also 3133, 3142, 3324, 3357, 3369, 3457, 3462, 3842, 3847.

Karwowski, O. 3370

Vertical lines and points of a surface.

Ann. Polon. Math. **14** (1963/64), 141–168.

In the 3-dimensional euclidean space **r** is the position vector of a point on a surface and **t** is a unit tangent vector. For a function $B(\mathbf{r}, \mathbf{t})$ P. Szymański defined a geodesic derivative in the tangent direction **l** orthogonal to **t**. The author defines a vertical direction at a point of a surface as a direction for which the geodesic derivative of the longitudinal curvature corresponding to **t** vanishes. The vertical directions are determined by a cubic equation, and it is proved that the only surfaces for which every direction is vertical at each point are the plane, the sphere and the cylinder of revolution. A vertical line is defined as a curve on a surface whose tangent is vertical at each point, and vertical lines of surfaces such as the cylindrical surface, cone and torse are discussed. A vertical point (vertex) of a surface is a point at which all the directions are vertical, and vertices on an ellipsoid are investigated. *M. Kurita* (Nagoya)

Leichtweiss, Kurt 3371

Über eine Art von Krümmungsinvarianten beliebiger Untermannigfaltigkeiten des n-dimensionalen euklidischen Raums.

Abh. Math. Sem. Univ. Hamburg **26** (1963/64), 155–190.

The author proves a generalization of the theorem that a surface of differentiable class C_3 in the 3-dimensional euclidean space which has vanishing Gaussian curvature and has no planar points is a torse. The principal curvatures $k_1, \cdots, k_m$ of an m-dimensional submanifold M_m of the n-dimensional euclidean space R_n are defined here as follows. We take a representation $A(M_m)$ of normal $(n-m)$-dimensional linear spaces of M_m in the Grassmann manifold $\tilde{G}_{n-m,m}$, and a Riemannian metric dn^2 on $A(M_m)$ is naturally defined. The principal curvatures $k_1, \cdots, k_m$ are defined as the quadratic roots of stationary values of dn^2/ds^2, where ds^2 is a metric of M_m as a submanifold of R_n. On the other hand, a p-torse $T_m{}^p$ is defined as an m-dimensional submanifold in R_n whose tangent spaces do not depend on the parameters $u_{p+1}, \cdots, u_m$, where $u_1, \cdots, u_m$ are suitably chosen parameters of the submanifold. Then the following theorems are proved. (1) For a p-torse a part $k_{p+1}, \cdots, k_m$ of the principal curvatures vanish. (2) If $k_{p+1}, \cdots, k_m$ vanish and $k_1, \cdots, k_p$ are positive, the submanifold is locally a p-torse. The special torses, such as axial and cylindrical ones, are investigated in detail in relation with principal curvatures. In conclusion, a contrast between the principal curvatures defined here and curvatures already known is explained. *M. Kurita* (Nagoya)

Vincensini, Paul 3372

Sur la déformation des réseaux cinématiquement conjugués.

J. Math. Pures Appl. (9) **42** (1963), 311–329.

Si l'on fait rouler une surface S_1 sur une surface fixe S isométriquement applicable à S_1, à chaque instant si le point de contact M décrit une trajectoire tangente à la direction Mt, la rotation se fait autour de la direction Mt_1, cinématiquement conjuguée (c.c.) de Mt. Cette relation entre directions définit une involution non dégénérée si S et S_1 ne sont pas des réglées applicables avec correspondance des génératrices; il s'endéduit les réseaux de courbes c.c. par rapport au couple SS_1, dont en particulier un également conjugué au sens ordinaire et un cinématiquement auto-conjugué. Une propriété de S est persistante (Finikoff) si les surfaces applicables sur S et

l'ayant forment une famille continue. L'auteur démontre que le triple le plus général de surfaces isométriques S, S_1, S_2 telles que les réseaux c.c. du couple SS_1 s'appliquent par isométrie sur les réseaux conjugués de S_2 s'obtiennent en prenant S_2 dans une famille de Voss et en prenant pour S et S_1 un couple conjugué de l'involution dont les éléments doubles sont deux surfaces adjointes de S_2. Le problème de la recherche de 4 surfaces isométriques telles que la déformation amenant le couple SS_1 sur le couple S_2S_3 applique tous les réseaux c.c. du (1) couple sur ceux et du (2) est ramené à l'intégration d'un système de deux équations; il y a toujours des solutions formées de couples de surfaces de Voss convenablement associées. Cas de l'héliçoïde réglé minimal. L'auteur signale le problème de la recherche de tous les couples de surfaces applicables susceptibles de déformation continue conservant tous les réseaux c.c. *B. d'Orgeval* (Dijon)

Stephanidis, N. K. 3373

Existenzfragen für Strahlensysteme.

Arch. Math. **14** (1963), 430–440.

The author reduces to differential equations the problems: Find a line congruence, given (a) the center surface and the spherical image of the principal surfaces, (b) the center surface and the spherical image of the surfaces of extremal distribution parameter, (c) the center surface and the mean curvature. He obtains existence theorems for the above problems (relative to local surfaces), and also an interpretation of the integral mean curvature of a bounded surface which serves as center surface of a congruence: the integral is equal to the segment cut off an (arbitrary) ruling by an orthogonal trajectory to a certain ruled surface derived from the ruled surface defined by the boundary of the given surface.

H. W. Guggenheimer (Minneapolis, Minn.)

Jenne, Wolfgang 3374

Eine natürliche Affingeometrie der Strahlflächen.

Math. Z. **83** (1964), 214–237.

The author treats affine properties of ruled surfaces with applications to an affine theory of curves. The fundamental transformation is a volume-preserving affine transformation (Scherung). A ruled surface $\mathfrak{F}$, which has no direction plane and whose asymptotic torse has a positively winding edge of regression (Gratlinie) $(\mathfrak{g})$, can be represented as

$$\mathfrak{x}(\sigma, \lambda) = \mathfrak{g}(\sigma) + a(\sigma)\mathfrak{g}''(\sigma) + \lambda\mathfrak{g}'(\sigma),$$

where σ is an affine arc-length of $(\mathfrak{g})$. An affine curvature $k(\sigma)$, an affine torsion $t(\sigma)$, and $a(\sigma)$ are invariants of the ruled surfaces $\mathfrak{F}$. $\mathfrak{g}'$, $\mathfrak{g}''$, $\mathfrak{g}'''$ are called associated 3-axes (begleitendes Dreibein) and are fundamental in the theory. Some properties of the invariants $k(\sigma)$, $t(\sigma)$, $a(\sigma)$ and the curves $(\mathfrak{g}')$, $(\mathfrak{g}'')$ are discussed. Next, a characteristic directrix (ausgezeichnete Leitkurve) is defined by $l(\sigma) = \mathfrak{g} + a\mathfrak{g}''$ and the three associated torses are defined by the envelopes of the planes spanned by $(\mathfrak{g}', \mathfrak{g}'')$, $(\mathfrak{g}'', \mathfrak{g}''')$, $(\mathfrak{g}''', \mathfrak{g}')$ through each point respectively. Finally, an associated ruled surface is defined by $\mathfrak{x}^{(2)}(\sigma, \lambda) = \mathfrak{g} + \lambda\mathfrak{g}''$ and its properties are given. Throughout the paper sixty theorems are stated and proved. *M. Kurita* (Nagoya)

Godeaux, Lucien 3375

Addition à la note "Surfaces dont les réglées gauches asymptotiques d'un mode appartiennent à des complexes linéaires".

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 527–528.

A supplement to an earlier paper [same Bull. (5) **49** (1963), 278–285; MR **27** #4152].

Pučkova, L. V. 3376

The Frenet formulas in polyelliptic spaces. (Russian. Azerbaijani summary)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **1963**, *no.* 4, 59–64.

Let S be a polyelliptic space, i.e., a space with a projective metric, the absolute set consisting of a set $\{Q_i, A_i, Q\}$, $i=0, \cdots, r$, where Q_i is an imaginary hypercone of second order with the vertex A_i (A_i being a subspace), Q_i is placed in A_{i-1}, Q is a non-degenerate quadric situated in A_r. For a differentiable curve in this space, the Frenet formulas are presented. *A. Švec* (Prague)

Jůza, Miloslav 3377

A one-parameter family of planes in the space S_6. (Czech. Russian and French summaries)

Mat.-Fyz. Časopis Sloven. Akad. Vied **13** (1963), 125–136.

Avons dans l'espace projectif P_6 une variété $V_{2,6}$ formée par un système monoparamétrique des plans $S_2(t) = [y_0(t), y_1(t), y_2(t)]$. Dans le cas général, on peut choisir les courbes directrices de manière que le système

$$y_0''' = ay_0'' + \sum_{j=0}^{2} b^j y_j' + \sum_{j=0}^{2} y_j, \quad y_\alpha'' = y'_{\alpha-1} + \sum_{j=0}^{2} n_\alpha{}^j y_j \qquad (\alpha = 1, 2)$$

existe. Soit $T(t_0)$ la somme des espaces tangents à tous les points du plan $[y_0(t_0), y_1(t_0), y_2(t_0)]$. Si on a $x'(t), x''(t), \cdots, x^{(k)}(t) \in T(t)$, on appelle la courbe $x = x(t)$ quasi-asymptotique d'ordre $k-1$. On démontre: Si $x(t)$ est une courbe sur $V_{2,6}$, cette courbe est quasi-asymptotique d'ordre 1 si et seulement si elle est tracée sur la surface $[y_1(t), y_2(t)]$. La seule courbe quasi-asymptotique d'ordre 2 est la courbe $y_2(t)$; elle est quasi-asymptotique d'ordre 3 si et seulement si $n_2{}^0 = -1$. Sur la variété il n'y a pas de courbes asymptotiques. Enfin, on étudie certains cas spéciaux. *A. Švec* (Prague)

Vala, Josef 3378

Über die Kongruenz W mit geradlinigen Brennflächen. (Czech. Russian and German summaries)

Mat.-Fyz. Časopis Sloven. Akad. Vied **12** (1962), 271–279.

Author's summary: "Riccatische Systeme der Kurven auf der Regelfläche ϕ, die nicht eine Torse ist, sind solche Systeme, deren vier beliebige Kurven die Erzeugenden der Fläche ϕ in vier Punkten mit konstantem Doppelverhältnis durchschneiden. Riccatisches System nennt man schichtbildend, wenn die Tangenten der Kurven dieses Systemes eine Kongruenz W mit geradlinigen Brennflächen ϕ und $\bar{\phi}$ bilden. Die Tangenten der Kurven des Riccatischen Systemes längs einer Erzeugenden p der Fläche ϕ bilden eine Regelfläche zweiter Ordnung ψ.

Die Korrespondenz P ist eine Korrespondenz der Erzeugenden der Flächen ϕ und $\bar{\phi}$ mit solcher Eigenschaft, dass die entsprechenden Geraden auf einer Fläche ψ liegen. Die Korrespondenz P ist eine projektive Abwicklung zweiter Ordnung. Es gibt Flächen ϕ, die eine solche Eigenschaft haben, dass die Korrespondenz P eine projektive Abwicklung dritter Ordnung ist." *A. Švec* (Prague)

Brauner, H. 3379

Die windschiefen Flächen konstanter konischer Krümmung.

Math. Ann. **152** (1963), 257–270.

Bei windschiefen, d. h. nicht abwickelbaren, Regelflächen des R_3 erklärt man die sogenannte konische Krümmung κ längs der Erzeugenden e als Limes des Quotienten der Winkel zweier Tangentialebenen des Richtungskegels zu dem der Erzeugenden selber, wenn man die Erzeugenden gegen e streben läßt. Bei Regelflächen mit konstantem, von 0 und ∞ verschiedenem κ ist der Richtungskegel ein Drehkegel, der auf die Öffnung $\pi/2$ normiert werden kann. Je nachdem ob der durch die Tangentialebenen in den Fernpunkten von ϕ umhüllte Torse eine Gratlinie k besitzt oder ein Drehkegel ist, spricht der Verfasser von Flächen 1. oder 2. Art. Man erhält die Flächen beider Arten durch Verschiebung der Erzeugenden einer Böschungstorse in ihren Tangentialebenen nach einem bestimmten Gesetz. Jede Fläche ϕ von konstantem κ besitzt einen Fernkreis c, und für die nähere Beschreibung erweisen sich die naheliegenden Begriffe von c-Kugeln, c-isotropen Geraden usw. als zweckmäßig. So erweist sich jede Fläche ϕ als Hüllfläche einer speziellen Schar von c-Kugeln, deren c-Mittelpunkte die sogenannte Zentralkurve von ϕ erfüllen. Das Hauptergebnis der vorliegenden Arbeit ist nun die Aufstellung aller algebraischen Flächen ϕ der Grade 3 und 4. Es ergibt sich: Die kubischen Regelflächen ϕ erster Art besitzen als Gratlinie k eine kubische Normkurve mit c als Schmiegkegelschnitt und einer Zentralkurve, die ein Kegelschnitt in einer c-isotropen Ebene ist. Bei einer kubischen Regelfläche ϕ der zweiten Art ist die Zentralkurve speziell eine Parabel einer c-isotropen Ebene. Bei den Regelflächen ϕ der Ordnung 4 ist zu unterscheiden zwischen solchen, die c als singulären Ort und solchen, die c nur als einfache Leitkurve besitzen. Bei den Flächen des ersten Typs ist die Zentralkurve ein Kegelschnitt in einer c-isotropen Ebene, bei denen des zweiten Typs eine rationale Raumkurve 4. Grades. Im einzelnen wird genau untersucht, welche Klassen von Regelflächen 3. und 4. Grades der Sturmschen Einteilung jeweils vorliegen.

W. Burau (Hamburg)

Springer, C. E. 3380

★Tensor and vector analysis. With applications to differential geometry.

The Ronald Press Co., New York, 1962. ix + 242 *pp.* $7.50.

This book provides an introduction to tensor theory, and vectors are treated as tensors of the first order. A special feature of the book is that very little knowledge is assumed on the part of the student and each concept is approached in a gradual and logical manner. The chapter headings are: (1) Coordinate transformations and mappings; (2) Loci in three-space; (3) Transformation of coordinates in space, Differentiation; (4) Tensor algebra; (5) Tensor analysis; (6) Vector analysis; (7) Vector algebra; (8) Differentiation of vectors; (9) Differentiation of tensors; (10) Scalar and vector fields; (11) Integration of vectors; (12) Geodesic and union curves.

R. S. Mishra (Allahabad)

Kawaguchi, Michiaki 3381

Jets infinitésimaux d'ordre séparé supérieur.

Proc. Japan Acad. **37** (1961), 18–22.

The author studies the structure of jets with source a product of number spaces, $R^{p_1} \times R^{p_2} \times \cdots \times R^{p_s}$.

W. F. Pohl (Stanford, Calif.)

Kawaguchi, Michiaki 3382

Une considération sur les représentants tensoriels de jets infinitésimaux.

Proc. Japan Acad. **37** (1961), 75–77.

By a kind of prolongation, a representation is found for jets which has the property that the coefficients transform linearly. *W. F. Pohl* (Stanford, Calif.)

Vagner, V. V. 3383

On Ehresmann's theory of jets. (Russian)

Dokl. Akad. Nauk SSSR **152** (1963), 17–19.

An attempt is made to generalize the Ehresmann's theory of jets in such a way that the spaces involved are without topology. *A. Švec* (Prague)

Gołąb, S. 3384

Contribution à la théorie des objets géométriques. Non-existence des objets de deuxième classe à une composante dans X_n pour $n \geqq 2$.

Tensor (*N.S.*) **14** (1963), 122–131.

The result mentioned in the title was previously proved by Kucharzewski and Kuczma [Tensor (N.S.) **10** (1960), 245–254; ibid. (N.S.) **11** (1961), 35–42; MR **24** #A3590] for linear geometric objects, and by E. Cartan and Pencow, independently, under analyticity hypotheses. Here, no assumptions are made (in the introduction or in the statement of the theorem) of any regularity of the transformation law of the one component in terms of the first- and (essential) second-order derivatives of the coordinate transformations. Yet page 125 contains a differentiability hypothesis for a function which, up to then, needed to be just one-to-one, not even continuous. The general question which is left open by the author is easily resolved under the "one-to-one" hypothesis, since then the assumption of having just one component is essentially vacuous.

A. Nijenhuis (Amsterdam)

Cenkl, Bohumil 3385

Réseaux semi-conjugués sur une surface dans l'espace à connexion projective à quatre dimensions. (Russian summary)

Czechoslovak Math. J. **13** (88) (1963), 492–506.

Quelques propriétés locales d'une surface plongée dans un espace à connexion projective à quatre dimensions ont été étudiées par M. Kimpara [Tensor (N.S.) **10** (1960), 61–72; MR **22** #12484]. Ici, on introduit la notion de la dualisation de la surface, c'est une congruence à connexion projective

au sens du référent [Ann. Polon. Math. 8 (1960), 291–322; MR 23 #A586]. Le réseau semi-conjugué est un réseau de courbes tel que les tangentes à une couche suivant une courbe de l'autre couche forment une surface développable. On prouve l'existence des surfaces qui possèdent (1) une couche de courbes asymptotiques, et un réseau semi-conjugué, (2) deux couches de courbes asymptotiques, (3) deux réseaux semi-conjugués, (4) un seul réseau semi-conjugué, (5) une seule couche de courbes asymptotiques. Enfin, on introduit l'élément linéaire projectif de la surface. *A. Švec* (Prague)

Mašanov, V. I. 3386

Invariants of ruled surfaces in a Lobachevsky space. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 131–139.

In the study (*) [same Trudy **160** (1962), 131–137; MR **27** #5187], the author constructed, by means of Cartan's method of exterior forms, a canonical frame of ruled surfaces in a Lobatchevsky space, where he related this surface to the arc length σ of the line of striction. In this paper the author relates the surface to a new invariant parameter, and characterizes geometrically the invariants of the respective canonical frame, and for these invariants determines the formulas on the basis of an analytical expression of the surface; by means of a simple relation between σ and s the author also calculates the invariants of (*), and studies the special cases when the line of striction is the principal, asymptotic or geodesic. The study of S. Gönenç [Rev. Fac. Sci. Univ. Istanbul Sér. A **20** (1955), 141–147; MR **17**, 1130] is extended to general surfaces and the author finds a condition by which the canonical frames of two ruled surfaces in a Lobatchevsky space form a fixed couple. *Z. Nádeník* (Prague)

Mašanov, V. I. 3387

On the differential geometry of ruled surfaces in a Riemannian space. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 140–153.

A canonical frame of a ruled surface in a Riemannian space was first constructed by W. Blaschke [Math. Z. **15** (1922), 309–320]. The author operates with this canonical frame, using Cartan's method of exterior forms, and indicates geometrical interpretations of the distribution parameter, determines formulas for the invariants of a surface on the basis of its analytic expression; he studies the special cases of ruled surfaces including their geometrical properties by means of their natural equations; and finally indicates the necessary and sufficient conditions (including those for the special cases) by which the frames of two ruled surfaces form a fixed couple, an analogy of Bertrand's curves [see also #3386 above]. *Z. Nádeník* (Prague)

Mašanov, V. I. 3388

On the theory of congruences of lines of a three-dimensional Riemannian space. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 154–163.

By means of Cartan's method of exterior forms the author constructs a canonical frame of congruences of lines of a three-dimensional Riemannian space, and studies the fundamental properties of these congruences (and so derives the analogies of certain fundamental congruence formulas for lines in Euclidean space), and in 15 theorems indicates the properties of special classes of these congruences (cylindrical, bi-cylindrical, normal, pseudo-normal, pseudospherical, of Ribaucour). *Z. Nádeník* (Prague)

Adati, Tyuzi 3389

On a Riemannian space admitting a field of planes.

Tensor (N.S.) **14** (1963), 60–67.

Canonical forms are found for the Riemannian spaces which admit distributions (fields of planes) with certain special properties, particularly those relating to an 'intersection' property. The case when the distribution is spanned by vector fields of various kinds is considered in detail. *A. G. Walker* (Liverpool)

Kandatu, Ayako 3390

Integrability of a field of partially null planes in a Riemannian space.

Tensor (N.S.) **14** (1963), 68–70.

If p is a field of partially null planes in a Riemannian space and p' is the conjugate field, conditions are found for the field $p+p'$ to be integrable. *A. G. Walker* (Liverpool)

Hermann, Robert 3391

Spherical compact hypersurfaces.

J. Math. Mech. **13** (1964), 237–242.

Sei N eine kompakte Hyperfläche (d.h. eine Untermannigfaltigkeit der Kodimension 1) in der riemannschen Mannigfaltigkeit M. Falls M euklidischer Raum ist, dann induziert die gausssche Abbildung auf N die zweite Fundamentalform. Wenn diese Fundamentalform nicht ausgeartet ist, dann wird dadurch auf N eine Metrik der konstanten Krümmung 1 definiert. Als Verallgemeinerung der gaussschen Abbildung betrachtet der Verfasser die Inklusion des Normalen-Bündels von N in das Tangentialbündel $T(M)$ von M. $T(M)$ trägt eine Metrik, die von der Metrik von M herkommt. Dadurch wird auch auf N eine Metrik induziert. Wir nehmen an, dass diese Metrik positiv definit ist. Das erste Ziel des Autors ist es, die Beziehung zwischen der riemannschen Krümmung K_M von M und der riemannschen Krümmung K_N von N zu studieren. Er erhält ein besonders einfaches Resultat, wenn K_M konstant ist. Als Anwendung zeigt er, dass falls $|K_M|$ konstant und genügend klein ist (der genaue Wert dieser Schranke hängt von N ab), dann liegt K_N in einem Intervall der Form $]c/4, c]$, mit $c>0$. Die universelle Überlagerung von N ist dann bekanntlich homöomorph zur Sphäre, vergleiche W. Klingenberg [Comment. Math. Helv. **35** (1961), 47–54; MR **25** #2559]. *W. Klingenberg* (Mainz)

Toponogov, V. A. 3392

A bound for the length of a closed geodesic in a compact Riemannian space of positive curvature. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 1047–1049.

Let $R_{k_0}{}^m$ be a simply-connected Riemannian manifold of m dimensions, whose sectional curvatures are bounded

below by the positive number k_0 and above by 1. Theorem 1: The length of any closed geodesic in $R_{k_0}{}^m$ is no less than 2π. This theorem has as a corollary the fact that, in such a space, any geodesic arc of length less than π is a segment. For even m the theorem is due to Klingenberg [Ann. of Math. (2) **69** (1959), 654–666; MR **21** #4445], and Klingenberg also obtained the result for odd m when $k_0 > \frac{1}{4}$. A difficulty in extending the theorem to odd dimensions has been the use of Synge's lemma in establishing a controlled homotopy of curves lying near a closed geodesic. The author introduces several new ideas in order to obtain the result, among them is (Lemma 1) an estimate involving simultaneous Jacobi fields along two geodesics and the application of A. D. Aleksandrov's isoperimetric inequality [Dokl. Akad. Nauk SSSR **47** (1945), 239–242; MR **7**, 167] to minimal surfaces. It is to be hoped that the published proof will contain details, especially in regard to the existence and regularity of the minimal surfaces used. These techniques will undoubtedly have great impact on other problems in differential geometry in the large. *L. W. Green* (Minneapolis, Minn.)

Fet, A. I. 3393

Extremal problems for surfaces of bounded Gaussian curvature. (Russian)

Dokl. Akad. Nauk SSSR **153** (1963), 292–295.

The author studies convex $(n-1)$-surfaces Γ imbedded in the Euclidean n-space E^n. If two surfaces Γ_1 and Γ_2 have the property that the Gaussian curvature k in every point x_2 of Γ_2 is less than or equal to the curvature of Γ_1 in the point x_1 having the same outer normal as Γ_2 in x_2, then the volumes V of the regions bounded by Γ_1 and Γ_2 are shown to satisfy the inequality $V(\Gamma_1) \leqq V(\Gamma_2)$. This implies some known extremal properties of the sphere. Next the author proves that the minimum [maximum] of the curvature cannot decrease [increase] by passing from Γ to its Steiner symmetrization $S(\Gamma)$ [Minkowski symmetrization $M(\Gamma)$]. Finally, let X be the class of all convex surfaces Γ of curvature $\geqq k_0$ [$\leqq k_0$], and let $T(\Gamma)$ be a functional on X such that $T(S(\Gamma)) \leqq T(\Gamma)$ [$T(M(\Gamma)) \leqq T(\Gamma)$], $\Gamma \in X$. Then T attains a minimal value on the sphere of curvature k_0. *S. Mardešić* (Zagreb)

Srivastava, T. N. 3394

A few remarks on special Kawaguchi spaces.

Tensor (*N.S.*) **15** (1964), 12–19.

S. Watanabe considered an n-dimensional special Kawaguchi space K_n in which the arc length of the curve $\xi^\kappa = \xi^\kappa(t)$ is defined by $s = \int (A_\mu \xi''^\mu + B)^{1/p}\, dt$ and introduced the connection

$$DV^\kappa = dV^\kappa + \Gamma_{\lambda\mu}{}^{*\kappa} V^\lambda d\xi^\mu + C_{\lambda\mu}^{\cdot\cdot\cdot\kappa} V^\lambda \delta\xi'^\mu = \bar{\bar{\nabla}}_\mu V^\kappa d\xi^\mu + \bar{\nabla}_\mu{}' V^\kappa \delta\xi'^\mu$$

in case $p=3$ and n is even. He defined curvature tensors in K_n as follows:

$$(\bar{\bar{\nabla}}_\lambda \bar{\bar{\nabla}}_\mu - \bar{\bar{\nabla}}_\mu \bar{\bar{\nabla}}_\lambda) V^\kappa = -(R^{*\cdot\cdot\cdot\kappa}_{\lambda\mu\nu} + k_{\lambda\mu}^{\cdot\cdot\cdot\sigma} C_{\nu\sigma}^{\cdot\cdot\cdot\kappa}) V^\nu + k_{\lambda\mu}^{\cdot\cdot\cdot\nu} \bar{\bar{\nabla}}_\nu{}' V^\kappa - 2S^{*\cdot\cdot\nu}_{\mu\lambda} \bar{\bar{\nabla}}_\nu V^\kappa,$$

$$(\bar{\bar{\nabla}}_\lambda \bar{\bar{\nabla}}_\mu{}' - \bar{\bar{\nabla}}_\mu{}' \bar{\bar{\nabla}}_\lambda) V^\kappa = -B^{*\cdot\cdot\cdot\kappa}_{\lambda\mu\nu} V^\nu + 2S^{*\cdot\cdot\nu}_{\mu\lambda} \bar{\bar{\nabla}}_\nu{}' V^\kappa + C_{\lambda\mu}^{\cdot\cdot\cdot\nu} \bar{\bar{\nabla}}_\nu V^\kappa,$$

$$(\bar{\bar{\nabla}}_\lambda{}' \bar{\bar{\nabla}}_\mu{}' - \bar{\bar{\nabla}}_\mu{}' \bar{\bar{\nabla}}_\lambda{}') V^\kappa = -P^{*\cdot\cdot\cdot\kappa}_{\lambda\mu\nu} V^\nu - 2C_{[\mu\lambda]}^{\cdot\cdot\cdot\nu} \bar{\bar{\nabla}}_\nu{}' V^\kappa.$$

The present author introduces an arbitrary symmetric tensor $g_{\lambda\mu}(\xi, \xi')$ of rank n in K_n and puts $\bar{\bar{\nabla}}_\mu g_{\kappa\lambda} = -Q_{\mu\kappa\lambda}$, $\bar{\bar{\nabla}}_\mu{}' g_{\kappa\lambda} = -\bar{Q}_{\mu\kappa\lambda}$. When $\{{}_\kappa{}^\rho{}_\mu\}$ and $\overline{\{{}_\kappa{}^\rho{}_\mu\}}$ are the Christoffel symbols based on the tensor $g_{\lambda\mu}$ with respect to ξ^σ and ξ'^σ, respectively, he shows first that $\Gamma^{*\rho}_{\kappa\mu} = \{{}_\kappa{}^\rho{}_\mu\} + Z_{\kappa\mu}^{\cdot\cdot\cdot\rho}$ and $C_{\kappa\mu}^{\cdot\cdot\cdot\rho} = \overline{\{{}_\kappa{}^\rho{}_\mu\}} + T_{\kappa\mu}^{\cdot\cdot\cdot\rho}$, where $Z_{\kappa\mu}^{\cdot\cdot\cdot\rho}$ are expressed by $Q_{\mu\kappa\lambda}$, $S^*_{\mu\kappa\lambda}$ and Γ^ν, and $T_{\kappa\mu}^{\cdot\cdot\cdot\rho}$ by $\bar{Q}_{\mu\kappa\lambda}$ and $C_{\kappa\mu}^{\cdot\cdot\cdot\rho}$. Then he proves that $R^{*\cdot\cdot\cdot\kappa}_{\lambda\mu\nu}$ can be expressed by $\{{}_\mu{}^\kappa{}_\nu\}$, $Z_{\mu\nu}^{\cdot\cdot\cdot\kappa}$ and Γ^ν, and finally that $P^{*\cdot\cdot\cdot\kappa}_{\lambda\mu\nu}$ can be expressed by $\overline{\{{}_\mu{}^\kappa{}_\nu\}}$, $T_{\mu\nu}^{\cdot\cdot\cdot\kappa}$ and $C_{\mu\nu}^{\cdot\cdot\cdot\kappa}$. These two formulae are his main results, but the analogous formula regarding the tensor $B^{*\cdot\cdot\cdot\kappa}_{\lambda\mu\nu}$ has not been considered. *T. Ohkubo* (Kumamoto)

Shamihoke, A. C. 3395

A note on a curvature tensor in a generalized Finsler space.

Tensor (*N.S.*) **15** (1964), 20–22.

By means of a non-symmetric metric tensor the author introduced a certain generalized Finsler space [Tensor (N.S.) **12** (1962), 97–109; MR **25** #4479; ibid. (N.S.) **13** (1963), 129–144; MR **27** #4188]. In the present paper a formula for the difference $R_{ijhk} - R_{hkij}$ between the corresponding components of the curvature tensor of such spaces is derived. *H. Rund* (Pretoria)

Laugwitz, Detlef 3396

Über die Erweiterung der Tensoranalysis auf Mannigfaltigkeiten unendlicher Dimension.

Tensor (*N.S.*) **13** (1963), 295–304.

One of the vital problems in founding a tensor calculus and a differential geometry in spaces of infinitely many dimensions is that of developing a suitable notation. The author has shown [Math. Z. **61** (1954), 100–118; MR **16**, 512; ibid. **61** (1954), 134–149; MR **16**, 512; see also the author and E. R. Lorch, Amer. J. Math. **78** (1956), 889–894; MR **18**, 495] that an efficient notation can be chosen, which in fact is identical with the old notation in appearance but introduces an altered meaning. For example, if H is a space (we shall assume it to be a Hilbert space) one writes for its vectors x^α, y^β and for its linear forms (elements of the dual space) l_α, m_β, etc. Thus, instead of the notation $l(x)$, one writes $l_\alpha x^\alpha$. Whereas the old notation $g_{ij}y^iz^j$ involves summation with respect to a basis, the new notation $g_{\alpha\beta}y^\alpha z^\beta$ indicates the value of the bilinear form $g_{\alpha\beta}$ on the pair y^α, z^β. The new scheme is basis-free and independent of dimension. Operations may now be made with this tensor calculus as usual with the exception of contraction (Verjüngung), where care must be exercised. For example the Kronecker delta $\delta_\beta{}^\alpha$ which represents the identity transformation ($x^\alpha = \delta_\beta{}^\alpha x^\beta$) cannot be contracted (in the finite-dimensional case contraction gives the dimension of H). Thus, only those theorems of finite-dimensional tensor calculus can be extended to general spaces which use contraction only where it can be justified. The author gives a contraction-free proof of a theorem of H. Weyl to illustrate the methods of the general calculus. This is the theorem that two Riemannian metrics on a manifold which have the same geodesics and are conformal to each other are identical up to a constant factor. The proof is carried through for Finsler spaces since this does not involve difficulties of significantly higher order. *E. R. Lorch* (New York)

GENERAL TOPOLOGY

See also 3121, 3359.

Hocking, J. G. 3397
Schwache lokale Invertierbarkeit.
Arch. Math. **15** (1964), 46–49.

Ein topologischer Raum heisst schwach invertierbar [bzw. schwach inv. im Punkt p] wenn jede kompakte echte Teilmenge [die nicht p enthält] in eine beliebige nicht-leere offene Teilmenge [die p enthält] bei einem Raum-homöomorphismus sich abbilden kann. Es wird ausgesagt, dass S schwach invertierbar ist, wenn es in jedem Punkt schwach invertierbar ist, und es wird gezeigt, dass wenn S ein Hausdorffscher Raum ist, die Menge aller Punkte in denen S schwach inv. ist, einen schwach invertierbaren Unterraum bildet. Es ist klar, dass sich "Hausdorffscher Raum" in beiden Stellen finden sollte oder in keiner. Diese Resultate sind typisch. Verfasser gibt eine ungelöste Aufgabe: Ist jedes Produkt schwach invertierbarer, nicht kompakter Räume schwach invertierbar? Man löst sie nach der Bemerkung, dass sich in eine beliebige nicht-leere offene Teilmenge in einem nicht-kompakten, schwach inv. Raume jede kompakte Teilmenge abbilden kann. *J. R. Isbell* (Princeton, N.J.)

Knight, C. J. 3398
Box topologies.
Quart. J. Math. Oxford Ser. (2) **15** (1964), 41–54.

This paper is an interesting study of box topologies on the cartesian product $X = \mathsf{X}_{a\in A} Y_a$ of a family of non-empty topological spaces indexed by a set A. If, for each a, U_a is a non-empty open subset of Y_a, then $(U_a)=\{(x_a)|x \in U_a$ for all $a\}$ is a box in X. Corresponding to a family $\mathscr{F}$ of subsets of A closed under finite intersection, the set of all boxes (U_a) such that $\{a|U_a = Y_a\} \in \mathscr{F}$ forms a topology base for X. It is shown first that in order that the product topology defined by a filter $\mathscr{F}$ on A depend on the set $\{Y_a\}$ only and not on the indexing function, $\mathscr{F}$ must consist of the complements of subsets of A of cardinal less than m for some fixed infinite m. If $m=\aleph_0$, then we have the ordinary topological product. The product spaces for all m greater than the cardinal of A are all the same, the product topology being the box topology in the usual sense. If x and y are two elements of X, let $\delta(x, y)$ denote the set $\{a|x_a \neq y_a\}$. Quotient spaces of X by equivalences defined by relations of the form 'cardinal of $\delta(x, y) < r$' for some fixed infinite r, are considered. After deriving some elementary properties of such spaces the author examines separation properties, uniform structures, path connexion, and connexion and compactness in them. A typical result is that if $m > \aleph_0$, and all but finitely many factors are T_1 and regular and if the component of Y_a containing x_a is K_a, then the component of the product space containing x is $(K_a) \cap \Sigma_x$, where Σ_x is the equivalence class of x in the equivalence relation corresponding to $r=\aleph_0$. Finally some open problems are stated. *S. Swaminathan* (Madras)

Sieber, J. L.; Pervin, W. J. 3399
Separation axioms for syntopogenous spaces.
Nederl. Akad. Wetensch. Proc. Ser. A **66** = *Indag. Math.* **25** (1963), 755–760.

This paper considers generalizations to Császár's syntopogenous spaces [*Fondements de la topologie générale*, Akadémiai Kiadó, Budapest, 1960; MR **22** #4043] of relations among various separation properties in topological spaces. Typical results, using Császár's notations and definitions, are: A syntopogenous space $[E, \mathscr{S}]$ is $\mathscr{S}$-normal if and only if for $c(A) \cap c(B) = \emptyset$, there exists an $(\mathscr{S}, \mathscr{H})$-continuous mapping f of E into R such that $f(x)=0$ for $x \in A$ and $f(x)=1$ for $x \in B$. The product syntopogenous space $[E, \mathscr{S}]$ of syntopogenous spaces $[E^\lambda, \mathscr{S}^\lambda]$ is $\mathscr{S}$-regular [$\mathscr{S}$-completely regular] if and only if each space $[E^\lambda, \mathscr{S}^\lambda]$ is $\mathscr{S}$-regular [$\mathscr{S}^\lambda$-completely regular]. *B. J. Ball* (Athens, Ga.)

Banaschewski, Bernhard 3400
On Wallman's method of compactification.
Math. Nachr. **27** (1963), 105–114.

From the author's introduction: "In Ann. of Math. (2) **39** (1938), 112–126, Wallman makes the set of ultrafilters of the lattice of all closed subsets of a T_1-space into a compact T_1 extension space of the original space and proves this to be Hausdorff if and only if the original space is normal. This method of obtaining compact Hausdorff extensions for certain spaces can be put to further uses, leading to other such extensions, if the lattice of all closed subsets is replaced by certain other collections of closed sets, as was done by Šanin [Dokl. Akad. Nauk SSSR **38** (1943), 6–9; MR **5**, 45; ibid. **38** (1943), 110–113; MR **5**, 46]. This leads, in particular, to the result that each compact Hausdorff extension given by means of a normal basis [Fan and Gottesman, Proc. Nederl. Akad. Wetensch. Ser. A **55** (1952), 504–510; MR **14**, 669] may also be obtained with the aid of the extended Wallman method." *E. Michael* (Seattle, Wash.)

Smirnov, Ju. M.; Skljarenko, E. G. 3401
Some questions in dimension theory. (Russian)
Proc. 4th All-Union Math. Congr. (Leningrad, 1961), Vol. I, pp. 219–226. *Izdat. Akad. Nauk SSSR, Leningrad,* 1963.

This is an expository article surveying the present status of some areas of dimension theory. The questions discussed include: (a) comparison of various definitions of dimension (dim, ind, Ind) in metric spaces, compact spaces and topological groups, (b) uniform dimensions and uniform imbeddings in Euclidean spaces and the Hilbert space, (c) spaces of infinite dimension, in particular, transfinite dimension, countably dimensional spaces, weakly infinite-dimensional spaces, infinite-dimensional Cantor manifolds, Tumarkin's problem concerning the existence of finite-dimensional subcompacta in infinite-dimensional compacta. 16 problems are listed together with a bibliography of 34 items.

{Reviewer's remark: Problems 1 and 2 concerning the comparison of dim and ind in metric spaces have been settled in the meantime (in the negative) by P. Roy [Bull. Amer Math. Soc. **68** (1962), 609–613; MR **25** #5495].} *S. Mardešić* (Zagreb)

Edelstein, Michael 3402
A remark on a theorem of A. F. Monna.
Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 88–89.

The author strengthens a theorem due to Monna [same Proc. **64** (1961), 89–96; MR **23** #A3477] to read: If $\{T_i\}$ is a sequence of mappings of a complete generalized (i.e., the distance between two points may be infinite) metric space (X, d) into itself satisfying: (1) there exist c and ρ ($c>0$; $0<\rho<1$) so that $d(T_ix, T_iy) \leqq \rho d(x, y)$ whenever $d(x, y)<c$; (2) $T_iT_j = T_jT_i$; (3) for some $x_0 \in X$, there exists a positive integer $N(x_0)$ such that $n > N(x_0)$ implies $d(T_{n+k}(x_n), x_n) \leqq c$ (where $x_n = T_n(x_{n-1})$), then there is a point y such that $T_{n+k}(y) = y$.

Haskell Cohen (Baton Rouge, La.)

Charatonik, J. J. 3403

Two invariants under continuity and the incomparability of fans.

Fund. Math. **53** (1963/64), 187–204.

The first part of the paper deals with hereditarily unicoherent continua C. If $X \subset C$, there exists in C exactly one continuum $I(X)$, irreducibly containing X, i.e., such that $I(X)$ contains X but no proper subcontinuum of $I(X)$ does. Let $N(C)$ be the set of all points of C at which C is not locally connected. The author now defines inductively $J^1(C) = I(N(C))$, $J^\alpha(C) = J^1(J^\beta(C))$, if $\alpha = \beta + 1$, and $J^\alpha(C) = \bigcap_{\beta<\alpha} J^\beta(C)$, if $\alpha = \lim_{\beta<\alpha} \beta$, for every ordinal $\alpha > 1$. The degree $\tau(C)$ of the non-local connectedness of a hereditarily unicoherent continuum C is defined to be the minimum ordinal α such that $J^{\alpha+1}(C) = 0$, or equal to ∞ in the case when no such α exists. Let A, B, C be hereditarily unicoherent continua. The following results are representative. If $A \subset B$, then $N(A) \subset N(B)$, $J^\alpha(A) \subset J^\alpha(B)$, and $\tau(A) \leqq \tau(B)$. If $C = A \cup B$, then $N(C) = N(A) \cup N(B)$. If $B = f(A)$, where f is a continuous mapping, then $J^\alpha(B) \subset f(J^\alpha(A))$ and $\tau(B) \leqq \tau(A)$. The last inequality is the goal in this part of the paper. It remains an open question whether, for continua C which are not hereditarily unicoherent, an analogue of the degree $\tau(C)$ can be defined such that it also would not increase under continuous mappings.

Further, the author calls a space Y uniformly arcwise connected if Y is arcwise connected and for every number $\varepsilon > 0$ there exists a positive integer k such that each arc in Y can be divided with k points into arcs all having diameters less than ε. He proves that if A is a uniformly arcwise connected continuum and B is a continuous image of A for which every pair of points $x, y \in B$ can be joined by exactly one arc in B, then B is uniformly arcwise connected.

Using the above invariants the author constructs, for every integer $n > 1$, a finite family $\mathfrak{F}_n$ consisting of n dendroids, i.e., hereditarily unicoherent and arcwise connected continua, such that no dendroid from $\mathfrak{F}_n$ can be mapped onto another dendroid from $\mathfrak{F}_n$. Moreover, each element of $\mathfrak{F}_n$ is a plane curve, and it is the union of countably many arcs such that every two of them have exactly one, and the same, point in common. {The definition of the connectedness of a space between its subsets, given on p. 195, seems to be a misunderstanding.}

A. Lelek (Wrocław)

Horne, J. G., Jr. 3404

The boundary of a one-parameter group in a semigroup.

Duke Math. J. **31** (1964), 109–117.

In a compact semigroup with identity, the closure of a one-parameter group is again a group. This is, of course, not true in the case of locally compact semigroups. If R is a one-parameter semigroup in a locally compact semigroup S, and B denotes the set $R^- - \mathring{R}$, then in many cases (for example, for Hofmann semigroups see the reviewer [Math. Z. **82** (1963), 29–36; MR **27** #2964]), B is a compact group. The author shows that B is a group if and only if it is compact. It has been an open question as to whether it is always the case that B is compact. The author answers this question in the negative with an example of a locally compact semigroup in which R is open and is homeomorphic to the reals with their natural topology. In fact, algebraically $S = R \times \{1, 2, 3, \cdots\}$, and the topology is such that $R \times \{n\}$ is open in the closed set $R \times \{n, n+1, n+2, \cdots\}$ and relative to the topology induced from this space, is homeomorphic to the reals with their natural topology.

P. S. Mostert (New Orleans, La.)

Schwarz, Štefan 3405

Probabilities on non-commutative semigroups. (Russian summary)

Czechoslovak Math. J. **13** (**88**) (1963), 372–426.

Let S be a finite semigroup, and let $\mathfrak{M}(S)$ denote the set of all non-negative real-valued functions [measures] μ on S such that $\sum_{x\in S} \mu(x) = 1$. For μ, $\nu \in \mathfrak{M}(S)$, let $\mu\nu(x) = \sum_{uv=x} \mu(u)\nu(v)$. With the obvious topology, $\mathfrak{M}(S)$ is a compact semigroup. The structure of $\mathfrak{M}(S)$ when S is a commutative group was studied by Vorob'ev [Mat. Sb. (N.S.) **34** (**76**) (1954), 89–126; MR **15**, 882] and when S is commutative by the reviewer and Zuckerman [Duke Math. J. **22** (1955), 595–615; MR **17**, 754]. The author first constructs all idempotents of $\mathfrak{M}(S)$. For a simple subsemigroup P of S, let $\bigcup_{i=1}^s \bigcup_{k=1}^r G_{ik}$ be its decomposition into disjoint groups. Let λ_{ik} be normalized Haar measure on G_{ik}, and let $\sum_{i=1}^s a_i = \sum_{k=1}^r b_k = 1$, where the a's and b's are positive. Then $\sum\sum a_ib_k\lambda_{ik}$ is an idempotent, and all idempotents have this form.

Primitive idempotents are next described, and it is shown that the kernel of $\mathfrak{M}(S)$ is the set of all primitive idempotents in $\mathfrak{M}(S)$. The subgroups of $\mathfrak{M}(S)$ are next studied. The paper closes with some limit theorems and with specializations to S that admit relative inverses or are idempotent. The semigroup $\mathfrak{M}(S)$ for infinite compact S has been studied by Glicksberg [Pacific J. Math. **9** (1959), 51–67; MR **21** #7405], Collins [Proc. Amer. Math. Soc. **13** (1962), 442–446; MR **25** #144], and Pym [Pacific J. Math. **12** (1962), 685–698; MR **26** #6298].

Edwin Hewitt (Seattle, Wash.)

Loewner, Charles 3406

On semigroups in analysis and geometry.

Bull. Amer. Math. Soc. **70** (1964), 1–15.

This is an address delivered before the Vancouver meeting of the American Mathematical Society on August 30, 1962. The author discusses some results, some already published and others to be published.

He is concerned with pseudo-semigroups and these are defined as follows. Let V be a manifold and let $\mathfrak{S}$ be a set of homeomorphisms between domains (connected open sets) in V. $\mathfrak{S}$ is called a pseudo-semigroup in V if the conditions (a) to (d) are satisfied: (a) If $f \in \mathfrak{S}$ maps O_1 onto O_2 and $g \in \mathfrak{S}$ maps the domain O_2 onto O_3, then the

composite mapping $g \circ f$, which transforms O_1 onto O_3, belongs to $\mathfrak{S}$ also; (b) If O_1 is any subdomain of the domain O of $f \in \mathfrak{S}$, then f restricted to O_1 belongs to $\mathfrak{S}$ also; (c) The identity map of V belongs to $\mathfrak{S}$; (d) If a sequence of mappings $f_n \in \mathfrak{S}$, all defined in the same domain O, converges uniformly in any compact part of O and the limit f is again a homeomorphism of O, then f belongs to $\mathfrak{S}$ also. He states that the prefix "pseudo" will be dropped and $\mathfrak{S}$ will be spoken of as a semigroup in V. If $\mathfrak{S}$ is a semigroup in V and if for every $f \in \mathfrak{S}$ the inverse also belongs to $\mathfrak{S}$, then $\mathfrak{S}$ is said to be a group in V.

The first problem that the author discusses concerns the set $\mathfrak{P}$ of all orientation-preserving projective transformations

$$\gamma = \frac{\alpha x + \beta}{\gamma x + \delta} \qquad (\alpha\delta - \beta\gamma > 0)$$

applied to intervals not containing the pole $-\delta/\gamma$. $\mathfrak{P}$ is a group in the euclidean line E_1. He is interested in finding a proper extension of $\mathfrak{P}$ to a semigroup $\mathfrak{S}$ in E_1 that is minimal in the following sense. There is no proper extension of $\mathfrak{P}$ to a semigroup contained in $\mathfrak{S}$ but not coinciding with it. This problem has not been solved, but Theorem I gives the extensions that are minimal in the set of semigroups, each of which contains a map of class C^3 not belonging to $\mathfrak{P}$.

Theorem I: There are only two such minimal extensions, say $\mathfrak{S}'$ and $\mathfrak{S}''$, one consisting of the inverses of the mappings of the other. $\mathfrak{S}'$ consists of the (real) analytic mappings of intervals of E_1 which can be analytically continued into the upper half-plane and map the latter schlicht into itself. It is conjectured that this theorem holds without any differentiability assumption.

The author then considers the possibility of generalising Theorem I to higher dimensions. He takes a convex cone C in E_n and defines an order relation by saying that $x < y$ when $y - x \in C$. A mapping f of a domain O of E_n into E_n is said to be C-monotonic if $x_1, x_2 \in O$ and $x_1 < x_2$ imply that $f(x_1) < f(x_2)$. f is locally C-monotonic if every point of O has a neighbourhood on which f is C-monotonic. $\mathfrak{G}_C$ denotes the group in E_n consisting of all homeomorphisms f between domains of E_n such that both f and its inverse are locally C-monotonic.

Special cases for C are then considered. The first of these is when $C = L_n$, the n-dimensional light cone; i.e., L_n is the set of all $(x^1, \cdots, x^n)$ satisfying the inequalities

$$(x^n)^2 - \sum_{j=1}^{n-1} (x^j)^2 \geqq 0, \qquad x^n \geqq 0.$$

Theorem II exhibits the (unique) minimal extension of $\mathfrak{G}_{L_n}$ ($n \geqq 3$) in the set of semigroups of locally L_n-monotonic transformations, each of which (semigroup) satisfies a differentiability condition.

The next case considered is where $C = P_n$, the set of all vectors $(x^1, \cdots, x^n)$ satisfying $x^1 \geqq 0, \cdots, x^n \geqq 0$. Instead of $\mathfrak{G}_{P_n}$, a finite-dimensional subgroup $\mathfrak{P}_n$ is considered and Theorem III exhibits the three minimal extensions of $\mathfrak{P}_n$ ($n \geqq 2$) in the set of semigroups of locally P_n-monotonic transformations each of which (semigroup) satisfies a differentiability condition. (When $n = 1$, $\mathfrak{P}_1 = \mathfrak{P}$ and the extension is given by Theorem I.)

The final case of a minimal extension is concerned with the cone Q_n of all negative quadratic forms in the space of all quadratic forms in n variables ($n > 1$) and the cone H_n of all non-negative Hermitian forms in the space of all Hermitian forms in n variables ($n > 1$). A similar extension theorem is obtained for $\mathfrak{G}_{Q_n}$ and $\mathfrak{G}_{H_n}$ and the minimal extension for each of them is unique.

The author then considers the following related problem. Find all closed convex cones Γ of infinitesimal transformations in the space of the hyperbolic geometry which satisfy the conditions: (a) The transformations generated by Γ map the whole space into itself; (b) Γ is finite-dimensional; (c) Γ is invariant under the group of isometries; (d) There is no nontrivial subcone of Γ of lower dimension satisfying (c). He states the solution (already published) for the two-dimensional case.

The paper also contains a discussion of monotonic mappings of higher order in E_1. *J. H. Michael* (Adelaide)

Gillman, David S. 3407

Side approximation, missing an arc.

Amer. J. Math. **85** (1963), 459–476.

R. H. Bing [Ann. of Math. (2) **77** (1963), 145–192; MR **27** #731] has shown that any 2-sphere in E^3 can be "almost" approximated from either side by a polyhedral 2-sphere. That is, Theorem (Bing): If S is a 2-sphere, U is a component of $E^3 - S$, and $\varepsilon > 0$, then there is a polyhedral 2-sphere S', a finite collection $D_1, D_2, \cdots, D_k$ of mutually exclusive disks lying in S', each of diameter less than or equal to ε, a finite collection $E_1, E_2, \cdots, E_m$ of mutually exclusive disks lying in S, each of diameter less than or equal to ε, such that (1) there is a homeomorphism of S onto S' that moves no point more than ε, (2) $S' - \sum_{i=1}^k D_i \subset U$, (3) $S \cdot S' \subset \sum_{i=1}^m E_i$. The author shows that if A is a tame arc lying on S, this theorem is true with the additional condition (4) $A \cdot \sum_{i=1}^m E_i = \emptyset$. He further shows that an arc A lying on a 2-sphere that can be approximated in such a way as to satisfy all four conditions is a tame arc. The proof that if A is tame, these are necessary conditions, is based primarily on Bing's earlier results. That of the sufficiency is complicated and global in nature and is based on recent results of Bing [Amer. J. Math. **84** (1962), 583–590; MR **26** #4331; ibid. **84** (1962), 591–599; MR **26** #4332] and the author [Trans. Amer. Math. Soc. **111** (1964), 449–456; MR **28** #5433]. It is shown as a consequence of these results, together with the cited recent work of Bing, that a 2-manifold can be pierced by a tame arc at a point p if and only if there is a tame arc lying in the 2-manifold and containing p and that the set Y of points of a 2-sphere S in E^3 at which S cannot be pierced by a tame arc is a 0-dimensional F_σ set.

L. K. Barrett (Knoxville, Tenn.)

Bing, R. H. 3408

Pushing a 2-sphere into its complement.

Michigan Math. J. **11** (1964), 33–45.

In this paper the author considers the problem of mapping a (perhaps wildly) embedded topological 2-sphere S (later a 2-manifold) in 3-space into one of its complementary domains. The author proved previously that if S can be pushed homeomorphically into either complementary domain, then S is tamely embedded [Fund. Math. **47** (1959), 105–139, Theorem 11.1; MR **21** #5954]. One of the principal techniques used in this paper is the construction of a sequence of finer and finer triangulations of S, each of which is tame. The construction of these tame triangulations of the arbitrary embedded S makes strong

use of the result, also proved by the author, that each 2-sphere contains a "tame" Sierpiński curve with largest complementary domain on S being arbitrarily small [Amer. J. Math. **84** (1962), 583–590, Theorem 1; MR **26** #4331]. First, it is shown how to push 1-skeletons of a tame triangulation into (say) the interior of S. By iteration and taking care to see that the sequence of homeomorphisms converges, the following theorem is proved. Theorem 2.1: Suppose S is a 2-sphere in E^3 and U is a component of $E^3 \setminus S$. Then for each positive number ε there exists a Cantor set C on S and a map $f: \bar{U} \to U \cup C$ such that (i) no point is moved more than ε, (ii) f is the identity on the complement in U of an ε-neighborhood of S, (iii) f is a homeomorphism on $\bar{U} \setminus f^{-1}(C)$. A similar theorem is proved for a connected 2-manifold that separates a connected 3-manifold. As an application of the former theorem the author shows that if S is an arbitrary 2-sphere in E^3 and U a component of $E^3 \setminus S$, then there exists a 0-dimension F_σ set F on S such that $U \cup F$ is 1-ULC.

In the last two sections the author considers spheres that are tame modulo Sierpiński curves and related questions. Typical results are the following. Theorem 8.2: A 2-sphere S in E^3 is tame if it is locally tame modulo a tame Sierpiński curve X in S. Theorem 9.2: Suppose S is a 2-sphere in E^3 and T_1 is a triangulation of S such that the 1-skeleton of T_1 is tame. Then for each $\varepsilon > 0$, there is a triangulation T_2 of mesh less than ε such that T_2 refines T_1 and the 1-skeleton of T_2 is tame.

O. G. Harrold (Knoxville, Tenn.)

Hempel, John 3409

A surface in S^3 is tame if it can be deformed into each complementary domain.

Trans. Amer. Math. Soc. **111** (1964), 273–287.

The author shows in the first three sections that a surface M in E^3 is tame if it can be deformed into each complementary domain by a homotopy beginning at the identity and at each subsequent stage takes M into its complement. This furnishes a positive solution in the case $n=3$ of Problem 1 of S. Eilenberg and R. L. Wilder [Amer. J. Math. **64** (1942), 613–622, p. 620; MR **4**, 87]. His method is to show that each complementary domain is 1-ULC and then apply recent results of R. H. Bing [Trans. Amer. Math. Soc. **101** (1961), 294–305; MR **24** #A1117]. (The author actually demonstrates that his hypotheses imply a certain Condition A which suffices for the application of Bing's construction.)

The author raises the following question: Is the full hypothesis of Theorem 1 required? For example, if M is a closed, connected 2-manifold in E^3 such that for each component U of $E^3 - M$ and each positive ε there is a map of M into U that moves no point as much as ε, is M necessarily tame? The author says the complementary domains are nice. (An M with the above property is already a closed, connected 2-manifold by Theorem 1 of R. L. Wilder [Fund. Math. **25** (1935), 200–208].)

In the last two sections the author considers a continuous map f of E^3 on E^3 that is a homeomorphism on a certain tame 2-manifold M. It is shown that $f(M)$ is tame if $f(E^3 \setminus M) = E^3 \setminus f(M)$. Theorem 4: Suppose C and C' are polyhedral 3-manifolds with boundary in S^3 such that C is a cube with handles and such that there is a map f of C onto C' that is a homeomorphism which takes Bd C onto Bd C'. Then C and C' are homeomorphic and $f|$Bd C can be extended to a homeomorphism of C on C'. Theorem 5: Let M be a tame torus (genus 1) in S^3, and let M' be a tame unknotted torus in S^3. Then there is a map f of S^3 on S^3 such that $f|M$ is a homeomorphism of M onto M' and such that $f(S^3 \setminus M) = S^3 \setminus M'$.

O. G. Harrold (Knoxville, Tenn.)

Haken, Wolfgang 3410

Über das Homöomorphieproblem der 3-Mannigfaltigkeiten. I.

Math. Z. **80** (1962), 89–120.

This paper describes the author's further investigations into decision procedures in problems concerned with 3-dimensional manifolds. The problem considered here is the following: Given presentations of two compact 3-dimensional manifolds (for example, by giving incidence relations between the simplexes), decide by a finite process whether they are homeomorphic or not. The author claims to have such a process in the case of a restricted class of manifolds. The class includes 3-manifolds which are formed by removing from a 3-sphere a nice open neighbourhood of a link or knot.

The methods used are a refinement of the method of normal surfaces [see the author, Acta Math. **105** (1961), 245–375; MR **25** #4519a; Math. Z. **76** (1961), 427–467; MR **25** #4519c; H. Schubert, ibid. **76** (1961), 116–148; MR **25** #4519b]. The idea is to divide up the given 3-dimensional manifold by means of normal surfaces. Only normal surfaces of a carefully restricted kind are used. The surfaces divide the 3-manifold into nice simple chunks and the homeomorphism type of the 3-manifold is determined by the nature of the chunks and the way they fit together.

Only a sketch of the methods and proofs is given here. The reviewer looks forward to reading the full details, which are promised for a later paper.

As the author points out, a fundamental problem remains untouched by the method of normal surfaces. Given a presentation of a 3-manifold, we are unable to decide whether it is a 3-sphere.

D. B. A. Epstein (Cambridge, England)

Gluck, Herman 3411

Homogeneity of certain manifolds.

Michigan Math. J. **11** (1964), 19–32.

The author calls a connected n-manifold M^n homogeneous if any two locally flat imbeddings of I^n in M^n are globally equivalent. Continuing his investigations begun with M. Brown [Brown and Gluck, Ann. of Math. (2) **79** (1964), 1–17; MR **28** #1608a; ibid. (2) **79** (1964), 18–44; MR **28** #1608b; ibid. (2) **79** (1964), 45–58; MR **28** #1608c] in which R^n, S^n and $S^{n-1} \times S^1$ were shown to be homogeneous, the author studies this property for some special classes of manifolds. For example, each manifold in the following list is homogeneous: (1) $S^n \times R^k$ if $k \neq n+1$; (2) $S^{p_1} \times S^{p_2} \times \cdots \times S^{p_r}$ if $1 = p_1 \leqq p_2 \leqq \cdots \leqq p_r$ and $p_1 + p_2 + \cdots + p_{r-1} \leqq p_r$; (3) $M^2 \times R^k$ if M^2 is closed, connected and orientable and $k \geqq 4$; and (4) $M^{n-1} \times S^1$ if $M^{n-1} \times R^1$ is homogeneous and M^{n-1} is closed.

{It should be noted that Homma's paper, which is reference [9] in the author's bibliography, has already appeared in print [Yokohama Math. J. **10** (1962), 5–10; MR **27** #4236].}

R. H. Rosen (Princeton, N.J.)

Foland, N. E.; Utz, W. R. 3412
The embedding of discrete flows in continuous flows.
Ergodic Theory (Proc. Internat. Sympos., Tulane Univ., New Orleans, La., 1961), pp. 121–134. Academic Press, New York, 1963.

The authors offer some general and historical comments on the problem indicated in the title. They then solve the problem for the simple closed curve by proving that if T is a homeomorphism of a circle X onto itself, then T is embeddable in a continuous flow on X if and only if T is orientation-preserving and some one of these three cases occurs: (a) X contains at least one fixed point under T; (b) T is periodic on X; (c) T is transitive on X in the sense that the orbit of some point of X under T is dense in X. *W. H. Gottschalk* (Middletown, Conn.)

Gottschalk, W. H. 3413
An irreversible minimal set.
Ergodic Theory (Proc. Internat. Sympos., Tulane Univ., New Orleans, La., 1961), pp. 135–150. Academic Press, New York, 1963.

The transformation group (X, T, π) is said to be reversible provided there is a homeomorphism, φ, of X onto X such that $\varphi(\pi(x, t)) = \pi(\varphi(x), t^{-1})$ for all $x \in X$, $t \in T$. Otherwise, (X, T, π) is said to be irreversible. The author gives an example of an irreversible minimal transformation group where T is the group of integers and X is a symbol space. In addition, it is shown that if T is an abelian topological group, then the universal minimal transformation group generated by T is reversible. *W. R. Utz* (Columbia, Mo.)

Štan'ko, M. 3414
Continua with the fixed-point property. (Russian)
Dokl. Akad. Nauk SSSR **154** (1964), 1291–1293.

The paper consists of two simple observations: (1) If a continuum K admits ε-mappings $g_\varepsilon: K \to T_\varepsilon$ onto a finite tree T_ε for each $\varepsilon > 0$, and $(g_\varepsilon)^{-1}(y)$ is connected for all ramification points $y \in T_\varepsilon$, then K has the fixed-point property (f.p.p.). (2) If a 1-dimensional continuum K is the union of two subcontinua K_1, K_2 having the f.p.p. and $K_1 \cap K_2$ is a tree, then K too has the f.p.p. *S. Mardešić* (Zagreb)

ALGEBRAIC TOPOLOGY

See also 2981, 3309, 3410, 3455.

Bourgin, D. G. 3415
★Modern algebraic topology.
The Macmillan Co., New York; Collier-Macmillan Ltd., London, 1963. xiii + 544 *pp.* $11.50.

This book is a large and ambitious text that covers many topics of current interest in topology, as well as many elementary topics which are of interest to students. In an undertaking of these proportions, there are bound to be many points which are treated well, as well as some which are treated less satisfactorily. Before discussing the individual merits of the different parts of the book, let us outline the general make-up of the text.

Preliminary material is located in Chapter 1 and the appendix. There are then three chapters which treat complexes, chain groups, homology and cohomology groups, both relative and absolute. The next chapter, "Manifolds and Fixed Cells", is devoted to manifolds, Poincaré duality, and the Lefschetz number. There then follows a chapter on exact sequences, which includes the usual sequence for a pair, as well as a discussion of the Mayer-Vietoris sequence. The next three chapters discuss limits, gratings, and continuity.

The material up to this point includes all the basic definitions. Next the author injects a short, but excellent chapter on those parts of homological algebra which are needed in the sequel. The following chapter contains the basic uniqueness theorem for full gratings, followed by a discussion of fixed-point indices. There is a lengthy chapter on products, which also treats H-spaces. This is followed by short chapters on groups of homeomorphisms, fibre spaces, and homotopy. Two lengthy chapters on spectral sequences and sheaves complete the book. There are many reasonable problems throughout the text.

It would appear that the author has given us a comprehensive text on topology. However, one must note that the treatment of different topics varies greatly. The author has made strong use of his right to emphasize those topics which interest him the most. Fixed points, Leray cohomology (gratings, etc.), and sheaf theory are carefully treated in considerable depth. On the other hand, in the case of fibre spaces and homotopy, the treatment seems to just skim over the surface. This unevenness, which is not apparent from the title or the preface, should be pointed out to any beginning student who uses the book.

Lack of space prevents an elaborate analysis of the virtues and vices of each section. Much of the fundamental material, as well as the material on fixed points and sheaf theory, is handled in a careful and thoroughly up-to-date manner. But there are many points which the reviewer finds disturbing. Here are a few examples.

There is much fuss about different kinds of complexes. Various kinds of cells and cell complexes are defined. We meet a simplicial complex with the following remark, from Definition 1.6, Chapter 2, "When the cells are simplices, we refer to a simplicial complex". The reviewer prefers a more complete discussion of simplicial complexes before the generalizations.

There is no need to include Poincaré duality at such an early stage, i.e., before the invariance theorem. Furthermore, Poincaré duality is given for an "n-dimensional, concrete simplicial manifold". With the techniques developed later in the book, the more general theorem would be easy.

The reviewer finds that ten chapters are too long to ask the student to wait for a proof of invariance, though the proof is actually for the cohomology algebra.

The treatment of multiplicative structures is the least adequate part of the book. The author elaborates at great length about Hopf manifolds. H-spaces are briefly mentioned as a generalization of the notion of a semigroup manifold. Products on spheres are mentioned, but there is no reference to the work of J. F. Adams on elements of Hopf invariant one. Hopf algebras are presented, but in discussing the work of Borel on Hopf algebras over perfect fields, the author never comes out and explicitly states the general structure as a tensor product of truncated polynomial algebras. The reviewer feels that these sections would be difficult and misleading for a student.

There are several minor slips and notational mysteries.

For example, we often meet such expressions as "$f^p \times F^q$" for a cross-product of two cochains. This interchanging of upper and lower case is often confusing. In the bibliography, each author is arbitrarily limited to two references. While much of the terminology is standard, the author has attempted to introduce a generic "omology". While this is reasonable and harmless, the reviewer thinks that in an area where mathematicians tend to be conservative, i.e., about the questions of terminology, this attempt will be as unsuccessful as recent efforts to introduce the term "contra-homology".

In spite of all this, the author has succeeded in gathering together a great deal of mathematics.

D. W. Kahn (New York)

Wagner, K. 3416

Beweis einer Abschwächung der Hadwiger-Vermutung.

Math. Ann. **153** (1964), 139–141.

For each positive integer n, does there exist a minimum integer $\varphi(n)$ with the property that any graph G with chromatic number $\Phi(G) \geqq \varphi(n)$ is homomorphic to the complete n-graph? Hadwiger's conjecture states that $\varphi(n) = n$. In the present paper it is shown that $\varphi(n)$ exists for every n and that $\varphi(n+1) \leqq 2\varphi(n) - 1$.

J. W. Moon (London)

Dirac, G. A. 3417

Homomorphism theorems for graphs.

Math. Ann. **153** (1964), 69–80.

Let $\lambda(n; k, i)$ denote the greatest number of edges a simple graph with n vertices can have without being homomorphic to a graph obtained from the complete k-graph by removing i edges. The value of $\lambda(n; k, i)$ is determined when $k = 5$ or 6 and $i = 1$ or 2; in addition, the extremal graphs are characterized completely. A similar result is obtained with respect to the property that the graph contains a complete 4-graph or a refinement thereof.

J. W. Moon (London)

Trent, Horace M. 3418

A decomposition theorem concerning linear graphs.

J. Math. Mech. **13** (1964), 325–328.

Es sei G ein zusammenhängender Graph ohne Schlingen, mehrfachen Kanten, Artikulationen und ohne Knotenpunkte zweiten Grades. Es sei T ein Baum in G und es sei S ein solches System von unabhängigen Kreisen, dass jeder Kreis von S eine einzige nicht zu T angehörige Kante enthält. Es sei b eine Kante von T.

Die Entfernung der Kante b und aller solcher Kanten, die nicht zu T angehören, die aber mindestens zu einem die Kante b enthaltenden Kreis von S angehören, wird vom Autor "Zerlegung des Graphen G modulo b" genannt. Eine solche Zerlegung wird vollständig genannt, wenn jede Kante des Graphen $T - \{b\}$ in dem reduzierten Graphen mindestens zu einem Kreise von S angehört.

In der früheren Arbeit [dasselbe J. **8** (1959), 827–835; MR **21** #4113] haben Auslander und der Verfasser bewiesen, dass G eine solche Kante a enthält, dass die Zerlegung des Graphen G modulo a bei gegebenem T vollständig ist. In der referierten Arbeit wird folgendes bewiesen: Wenn die Zerlegung modulo b ($b \in T$) für T nicht vollständig ist, dann existiert in G ein anderer, die Kante b enthaltender Baum, sodass ihm entsprechende Zerlegung modulo b vollständig ist. Der Beweis wird mit der Hilfe der Matrizen (wo Zeilen den Kanten und Spalten den Kreisen entsprechen) durchgeführt.

Das Lesen der Arbeit wird durch nichtgewöhnliche Terminologie und einige Ungenauigkeiten erschwert.

A. Kotzig (Bratislava)

Krötenheerdt, Otto 3419

Über einen speziellen Typ alternierender Knoten.

Math. Ann. **153** (1964), 270–284.

The author considers a class of knots that he calls rosettes (Rosettenknoten). The rosette $R_n{}^m$ might be described as a kind of alternating n lead m bight Turk's head: it has the form of a closed braid that is invariant under rotation about the axis through an angle of $2\pi/m$, it has a linking number of n with the axis, and its projection on a plane perpendicular to the axis is alternating. It is shown that $R_n{}^m$ is amphicheiral whenever n is odd. If n is even, $R_n{}^m$ is not amphicheiral whenever m is of the form $m = p^\alpha \cdot m'$, p prime, $p \equiv -1 \pmod 4$, α odd, and $p \nmid m'$. This is proved by considering the Minkowski units of the quadratic form of $R_{2j}{}^m$.

R. H. Fox (Princeton, N.J.)

Burghelea, Dan 3420

Sur les applications q-triviales.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 727–730.

The author describes a map $f: X \to Y$ as homologically [cohomologically] q-trivial with respect to the coefficient group K if $H_q(f; K) = 0$ [$H^q(f; K) = 0$]. The map f is (q, n)-homological [(q, n)-cohomological] if $X = \bigcup_{i=1}^n A_i$, A_i open in X, and $H_q(f|A_i; K) = 0$ [$H^q(f|A_i; K) = 0$], $i = 1, \cdots, n$. He then obtains generalizations and relativizations of theorems of Ganea and the reviewer [Proc. Cambridge Philos. Soc. **55** (1959), 248–256; MR **21** #4425]. However, Theorems 2.4, 2.5 and the results of Section 3 are based on an apparently incorrect relativization of the Künneth formula (Lemma 2). For example, if $f_1 = f_2$ embeds S^n in V^{n+1}, then $H^{2n+1}(f_1 \times f_2) \neq 0$, while $H^p(f_1) \otimes H^{2n+1-p}(f_2) = 0$.

P. J. Hilton (Ithaca, N.Y.)

Tynjanskiĭ, N. T. 3421

Homological properties of the distribution of sets in polyhedra and manifolds described by sequences of strong and weak homology groups. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1963**, *no.* 5 (36), 117–128.

In this paper, the author considers a compact set F belonging to a polyhedron K with $G = K - F$. Continuing the work of Sitnikov [Mat. Sb. (N.S.) **34 (76)** (1954), 3–54; MR **16**, 736], he discusses various exact sequences which relate the homology of F, G, and K. At the end of the paper, the author shows how certain simplifications are possible when K is a manifold.

D. W. Kahn (New York)

Steenrod, N. E. 3422

The cohomology algebra of a space.

Enseignement Math. (2) **7** (1961), 153–178 (1962).

In this essentially expository article, the author gives us a historical outline of the development of the notion of the

cohomology algebra of a space. He discusses the question of realizing an algebra as the cohomology algebra of a space, as well as such structures as algebras over Hopf algebras (especially the Steenrod algebra A_p). There is no mention of extraordinary cohomology, such as the various K-theories. *D. W. Kahn* (New York)

Shih, Weishu 3423

On the group $\mathscr{E}[X]$ of homotopy equivalence maps.

Bull. Amer. Math. Soc. **70** (1964), 361–365.

In the title of this note X is a CW-complex and $\mathscr{E}[X]$ is the group of homotopy classes of homotopy equivalences of X with itself. The group operation, given by composition of mappings, is non-commutative in general. The author defines a filtration of $\mathscr{E}[X]$ with reference to a Postnikov decomposition, and then describes a spectral sequence which converges to the graded group associated with the filtration. The sequence proceeds from more or less standard invariants, but even at the first stage the groups involved may be non-commutative. In case X has only two non-vanishing homotopy groups, the spectral sequence is trivial, and so a short exact sequence is obtained with $\mathscr{E}[X]$ in the middle. This note is only an outline of the theory; proofs are to be given elsewhere.

I. M. James (Oxford)

Bauèr, F. V. [Bauer, F.-W.] 3424

Universal homotopy groups. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 254–257.

Let K be a subcategory of the category of based normal spaces and let G be the category of groups. A functor $\Phi: K \to G$ is said to have property W if f is a homotopy equivalence whenever $\Phi(f)$ is an equivalence. The author offers a set of 10 axioms on the functor Φ which are asserted uniquely to determine Φ; further it is claimed that the functor Φ so specified has property W. The reviewer communicated to the author that Axioms 2 and 9 together lead to a contradiction; in reply, the author has informed the reviewer that he has suitably amended Axiom 2 in the new version of his paper, which will contain all proofs and will be published soon.

P. J. Hilton (Ithaca, N.Y.)

Ôguchi, Kunio 3425

A generalization of secondary composition and applications.

J. Fac. Sci. Univ. Tokyo Sect. I **10**, 29–79 (1963).

In earlier papers [Proc. Japan Acad. **38** (1962), 235–238; MR **26** #3064; ibid. **38** (1962), 619–620; MR **27** #766], the author published lists of generators for the 2-primary components of the homotopy groups of the classical groups, up to dimension 13. The present paper gives details, together with an exposition of the properties of the Toda bracket or toric construction. This is a useful supplement to Toda's study [*Composition methods in homotopy groups of spheres*, Princeton Univ. Press, Princeton, N.J., 1962; MR **26** #777]. Some relations between elements are established, but not all the homotopy groups under consideration are determined. However, a table has been published (without proofs) by Toda [C. R. Acad. Sci. Paris **241** (1955), 922–923; MR **17**, 395]. For some reason the author ignores the Samelson product, which one would have expected to appear in work of this kind.

I. M. James (Oxford)

Thomas, Emery 3426

Homotopy classification of maps by cohomology homomorphisms.

Trans. Amer. Math. Soc. **111** (1964), 138–151.

Suppose that a map of one space into another induces the zero homomorphism in cohomology, whatever the coefficients. Such a map is not, in general, nulhomotopic. By taking Postnikov resolutions the author arrives at a set of conditions which, when fulfilled, enable him to conclude that the map is nulhomotopic. Applications are given to the homotopy classification of maps into an H-space and hence to the stable classification of real and complex vector bundles, where there is a connection with Peterson's work [Ann. of Math. (2) **69** (1959), 414–420; MR **21** #1593]. *I. M. James* (Oxford)

Hilton, P. J. 3427

Note on H-spaces and nilpotency.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 505–509.

Consider the category of topological spaces of the based homotopy type of countable CW-complexes. Let A be a space of Lusternik-Schnirelmann category $\alpha+1$, so that $\alpha = \operatorname{cat} A$ in current terminology. Let B be a connected H-space and let $\Pi(A, B)$ denote the set of homotopy classes of maps of A into B, with binary operation inherited from the H-structure on B. It is well known that if $\alpha<3$, then $\Pi(A, B)$ is a group whose nilpotency does not exceed α, and if $\alpha<2$, then the group structure is independent of the H-structure on B. The author shows that these results remain true if α is taken to be the co-nilpotency of A, as defined by Berstein and Ganea, an invariant which may be less than cat A, and in any case, does not exceed it. *I. M. James* (Oxford)

Kahn, Donald W. 3428

A note on H-spaces and Postnikov systems of spheres.

Proc. Amer. Math. Soc. **15** (1964), 300–307.

§ 1 contains some generalities on Postnikov systems of H-spaces. §§ 2 and 3 contain some fairly obvious remarks on Postnikov decompositions of spheres and co-Moore spaces when these are not H-spaces. The statement of Theorem 2.2 seems to contain minor errors ("r" means "j", and the hypothesis that $H^m(X; Z_p)=0$ for p odd is not satisfied by $X=S^m$). Moreover, it is already known that this theorem holds without restriction on $H^*(X; Z_p)$ for p odd; this may be proved, for example, by the method of the reviewer [Topology **1** (1961), 67–72; MR **26** #5574]. *J. F. Adams* (Manchester)

Rothenberg, Melvin 3429

The J functor and the nonstable homotopy groups of the unitary groups.

Proc. Amer. Math. Soc. **15** (1964), 264–271.

Following the work of the reviewer [Proc. London Math. Soc. (3) **11** (1961), 291–310; MR **24** #A1727], J. F. Adams [Proc. Internat. Congr. Mathematicians (Stockholm, 1962),

pp. 435–441, Inst. Mittag-Leffler, Djursholm, 1963] and others, the author obtains certain information on the unstable homotopy groups of the unitary groups, that is, $\pi_i(U(m))$ for $i > 2m$. To describe these results we fix the notation. Let $J_C(X)$ denote the quotient group of $\tilde{K}(X)$ by the relation of (stable) fibre homotopy equivalence. If X is a finite complex, this is a finite group. Taking $X = P_k(C)$, complex projective space of dimension k, let $\xi \in J_C(P_k(C))$ be the element defined by the Hopf bundle. The order of ξ is known as a result of the works mentioned above. Now let $S^{2n+1} \to BU(n) \to BU(n+1)$ be the standard fibration, where $BU(n)$ denotes, as usual, the classifying space of $U(n)$. The author is concerned with the induced homomorphism $\pi_{2n+r}(S^{2n+1}) \to \pi_{2n+r}(BU(n))$. If, for any finite abelian group A and any prime p, we denote by A^p the p-primary part of A, then the main result of this paper may be formulated as follows: Let n, r be given integers, put $k = [\frac{1}{2}r]$ and suppose that $n\xi = 0$ in $J^p(P_k(C))$; then $\pi^p_{2n+r}(S^{2n+1}) \to \pi^p_{2n+r}(BU(n))$ is a monomorphism.

Since the order of ξ is explicitly known, this gives explicit information connecting the homotopy groups of spheres with the homotopy groups of $BU(n)$ (and hence of $U(n)$).

The proof of this result uses the methods of Adams and fairly standard type arguments of algebraic topology. The arguments are simple but quite ingenious. The reader should, however, be warned that the author says "Grassmannian" when he means "Stiefel-manifold" and that he uses the words "degree" and "index" interchangeably without warning. *M. F. Atiyah* (Oxford)

Berstein, Israel 3430

A note on spaces with non-associative co-multiplication.

Proc. Cambridge Philos. Soc. **60** (1964), 353–354.

Let p be an odd prime and let $\alpha \in \pi_{2p}(S^3)$ be an element of order p. Let $X = S^3 \cup_\alpha e^{2p+1}$. X is a space with a comultiplication. In this note, the author proves that X does not have a homotopy associative comultiplication.

F. P. Peterson (Cambridge, Mass.)

Arkowitz, Martin 3431

An example for homotopy groups with coefficients.

Proc. Amer. Math. Soc. **15** (1964), 136–137.

The author presents two CW-complexes X and Y whose homotopy groups are isomorphic but whose homotopy groups with coefficients Z_2 are not isomorphic.

P. J. Hilton (Ithaca, N.Y.)

Zeeman, E. C. 3432

Unknotting combinatorial balls.

Ann. of Math. (2) **78** (1963), 501–526.

Une boule [une sphère] combinatoire est un complexe équivalent à un simplexe [la frontière d'un simplexe] par un homéomorphisme linéaire par morceaux. Une paire de boules (B^p, B^q) est un couple formé d'une boule combinatoire B^p de dimension p et d'une boule combinatoire B^q de dimension q plongée dans B^p comme un sous-polyèdre, la frontière de B^q étant dans la frontière de B^p et l'intérieur dans l'intérieur. Définition analogue pour une paire de sphères (S^p, S^q). Une paire de boules (ou de sphères) est non nouée si elle est équivalente à la paire standard par un homéomorphisme linéaire par morceaux.

L'auteur montre le résultat remarquable suivant (théorème 1): Si la codimension $p-q \geqq 3$, alors toute paire de boules (B^p, B^q) est non nouée. Il en résulte facilement le théorème 2: Toute paire de sphères combinatoires est non nouée si la codimension est $\geqq 3$.

Dans le cas différentiable, le théorème 1 a été démontré par Smale [Amer. J. Math. **84** (1962), 387–399; MR **27** #2991] tout au moins si $q \geqq 5$, alors que le théorème 2 n'est plus vrai en général si $2p \leqq 3q+3$. Dans le cas topologique, Stallings a démontré le théorème 2, moyennant des conditions de régularité locale [Ann. of Math. (2) **77** (1963), 490–503; MR **26** #6946].

La démonstration du théorème 1 est délicate et exige une longue induction, mais la terminologie vivante et pittoresque de l'auteur la rend attrayante.

Le premier pas consiste à montrer que, si le théorème 1 est vrai pour des dimensions $<p$ et $<q$, et si la boule B^p peut se "collapser" sur B^q, alors la paire (B^p, B^q) est non nouée. D'après l'unicité des voisinages réguliers [J. H. C. Whitehead, Proc. London Math. Soc. (2) **45** (1939), 243–327], cela revient à montrer que B^q admet un "fibré normal" trivial dans B^p (lemme 6). Le deuxième pas est de loin le plus difficile; il s'agit de montrer que si $p-q \geqq 3$, alors B^p peut se "collapser" sur B^q. La démonstration exige un raisonnement par récurrence compliqué et occupe une dizaine de pages; l'hypothèse $p-q \geqq 3$ y intervient naturellement d'une manière essentielle (lemme 9).

L'auteur termine en montrant que si $M^q \subset M^p$ est une paire de variétés combinatoires telle que le bord de M^q est contenu dans le bord de M^p et y est localement non noué, alors M^p et M^q possèdent des "cols" compatibles.

A. Haefliger (Geneva)

TOPOLOGY AND GEOMETRY OF DIFFERENTIABLE MANIFOLDS

See also 2953, 2954, 3043, 3117, 3432.

Smeĭl, S. [Smale, S.] 3433

A survey of some recent results in differential topology. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 1 (115), 125–138.

Translation into Russian of a survey article which appeared in Bull. Amer. Math. Soc. **69** (1963), 131–145 [MR **26** #1896].

Smale, S. 3434

A structurally stable differentiable homeomorphism with an infinite number of periodic points. (Russian summary)

Qualitative methods in the theory of non-linear vibrations (Proc. Internat. Sympos. Non-linear Vibrations, Vol. II, 1961), pp. 365–366. Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963.

In this very brief note the author indicates the construction of a differentiable homeomorphism of the 2-sphere S^2 which is structurally stable and yet has periodic points of arbitrarily high period. *M. F. Atiyah* (Oxford)

Douady, Adrien 3435

Variétés à bord anguleux et voisinages tubulaires.

Séminaire Henri Cartan, 1961/62, *Exp.* 1, 11 *pp. Secrétariat mathématique, Paris,* 1964.

Douady, Adrien 3436
Théorèmes d'isotopie et de recollement.
Séminaire Henri Cartan, 1961/62, *Exp.* 2, 16 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Douady, Adrien 3437
Arrondissement des arêtes.
Séminaire Henri Cartan, 1961/62, *Exp.* 3, 25 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Morlet, Claude 3438
Le lemme de Thom et les théorèmes de plongement de Whitney. I. Les topologies des espaces d'applications.
Séminaire Henri Cartan, 1961/62, *Exp.* 4, 5 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Morlet, Claude 3439
Le lemme de Thom et les théorèmes de plongement de Whitney. II. Quelques ouverts fondamentaux des espaces d'applications.
Séminaire Henri Cartan, 1961/62, *Exp.* 5, 6 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Morlet, Claude 3440
Le lemme de Thom et les théorèmes de plongement de Whitney. III. Les théorèmes d'existence d'applications transverses.
Séminaire Henri Cartan, 1961/62, *Exp.* 6, 8 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Morlet, Claude 3441
Le lemme de Thom et les théorèmes de plongement de Whitney. IV. Les théorèmes de Whitney et de Morse.
Séminaire Henri Cartan, 1961/62, *Exp.* 7, 6 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Cerf, Jean 3442
La théorie de Smale sur le h-cobordisme des variétés.
Séminaire Henri Cartan, 1961/62, *Exp.* 11–13, 23 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Serre, Jean-Pierre 3443
Formes bilinéaires symétriques entières à discriminant ± 1.
Séminaire Henri Cartan, 1961/62, *Exp.* 14–15, 16 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Cerf, Jean 3444
La nullité de π_0 (Diff S^3). Théorèmes de fibration des espaces de plongements. Applications.
Séminaire Henri Cartan, 1962/63, *Exp.* 8, 13 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Cerf, Jean 3445
La nullité de π_0 (Diff S^3). 1. Position du problème.
Séminaire Henri Cartan, 1962/63, *Exp.* 9–10, 27 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Cerf, Jean 3446
La nullité de π_0 (Diff S^3). 2. Espaces fonctionnels liés aux décompositions d'une sphère plongée dans $\mathbf{R}^3$.
Séminaire Henri Cartan, 1962/63, *Exp.* 20, 29 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Cerf, Jean 3447
La nullité de π_0 (Diff S^3). 3. Construction d'une section pour le revêtement $\mathfrak{R}$.
Séminaire Henri Cartan, 1962/63, *Exp.* 21, 25 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Malgrange, Bernard 3448
Le théorème de préparation en géométrie différentiable. I. Position du problème.
Séminaire Henri Cartan, 1962/63, *Exp.* 11, 14 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Malgrange, Bernard 3449
Le théorème de préparation en géométrie différentiable. II. Rappels sur les fonctions différentiables.
Séminaire Henri Cartan, 1962/63, *Exp.* 12, 9 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Malgrange, Bernard 3450
Le théorème de préparation en géométrie différentiable. III. Propriétés différentiables des ensembles analytiques.
Séminaire Henri Cartan, 1962/63, *Exp.* 13, 12 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Malgrange, Bernard 3451
Le théorème de préparation en géométrie différentiable. IV. Fin de la démonstration.
Séminaire Henri Cartan, 1962/63, *Exp.* 22, 8 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Cartan, Henri 3452
Classes d'applications d'un espace dans un groupe topologique, d'après Shih Weishu.
Séminaire Henri Cartan, 1962/63, *Exp.* 6, 19 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Tamura, Itiro 3453
Classification des variétés différentiables, $(n-1)$-connexes, sans torsion, de dimension $2n+1$.
Séminaire Henri Cartan, 1962/63, *Exp.* 16–19, 27 *pp.*; *erratum*, 2 *pp.* *Secrétariat mathématique*, *Paris*, 1964.

Srinivasacharyulu, Kilambi 3454
★Sur les structures différentiables et les variations de structures complexes.
Doctoral Thesis, *Faculté des Sciences de l'Université de Paris*, *Paris*, 1962. 45 *pp.*
Cette thèse est essentiellement une énumération de théorèmes récents de topologie différentielle. Certaines conséquences sont signalées, mais quelques unes sont malheureusement fausses.

{Voir aussi Topologie et géométrie différentielle (Séminaire C. Ehresmann), Vol. IV (1962–63), Cahier 2, Inst. H. Poincaré, Paris, 1963 [MR **27** #6276].}

A. Haefliger (Geneva)

Boardman, J. M. 3455

Some embeddings of 2-spheres in 4-manifolds.

Proc. Cambridge Philos. Soc. **60** (1964), 354–356.

Let $CP(2)$ denote the complex projective plane oriented as usual, and γ the standard generator of $H_2(CP(2); Z) \cong Z$. It follows from work of Milnor and Kervaire [Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 1651–1657; MR **24** #A2968] that 3γ cannot be represented in $CP(2)$ by a differentiably embedded 2-sphere.

The author shows that if P_1 and P_2 are two copies of $CP(2)$ and γ_1 and γ_2 the corresponding generators of $H_2(P_1; Z)$ and $H_2(P_2; Z)$, then $3\gamma_1$ can be represented in $P_1 \# P_2$ (the standard connected sum) by a differentiably embedded 2-sphere. The fundamental group of the complement of this 2-sphere is Z_3, and this is the smallest group possible. *H. R. Gluck* (Cambridge, Mass.)

Sasao, Seiya 3456

A property of certain differentiable manifolds.

Proc. Japan Acad. **39** (1963), 413–414.

Let M be a compact oriented differentiable manifold without boundary such that (i) M is $(n-1)$-connected, (ii) $\dim M = 2n+1$, where n is even. The following result on the Stiefel-Whitney number $\omega = W_n \cdot W_{n+1}[M]$ is proved. Theorem: (1) $\omega = 0 \Leftrightarrow H_n(M) \cong F + T + T$, (2) $\omega \neq 0 \Leftrightarrow H_n(M) \cong F + T + T + Z_2$, where F, T are respectively a free and a torsion group. The connectivity assumptions imply that $\omega = 0$ if and only if M is cobordant to zero, so that the latter property is shown to be determined by $H_n(M)$.

A similar theorem was proved by C. T. C. Wall [Trans. Amer. Math. Soc. **103** (1962), 421–433; MR **25** #2621] for closed oriented 5-manifolds M such that $H_1(M)$ is torsion-free and $H_2(M)$ finite. *R. Brown* (Liverpool)

Adler, A. W. 3457

On spherical characteristic cohomology.

Trans. Amer. Math. Soc. **108** (1963), 240–250.

The differential-geometric approach to characteristic classes goes as follows: Let G be a (connected) Lie group and $G \to P \to X$ a principal G-bundle over a manifold X. Now there exists a universal G-bundle $G \to P_G \to B_G$ and a bundle mapping $f: P \to P_G$. Denote by $\mathfrak{g}$ the Lie algebra of G and $I_{\mathfrak{g}}$ the ring of (Ad G)-invariant polynomials on $\mathfrak{g}$. Then, if Ω is a ($\mathfrak{g}$-valued) curvature in $P_G \to B_G$, $\lambda(f^*(\Omega)) = f^*(\lambda(\Omega))$ is a closed scalar form on X, and these things give the characteristic classes (under de Rham) of $G \to P \to X$. This seems to be essentially the whole story in general, but there are certainly refinements possible by specializing. One of these considered in this paper is where $G \subset O(n)$ and X has a torsionless G-structure with principal bundle P (example: $G = U(n/2)$ and X is Kählerian, or maybe almost-Kählerian). Then P_G may be considered as a bundle over a suitable high-dimensional sphere into which f isometrically embeds X. The author then considers the scalar forms $\lambda(f^*(\Omega'))$ on X, where Ω' is a suitable piece of Ω. The torsion-free condition gives $d(f^*(\lambda(\Omega'))) = 0$, and the resulting classes are called the spherical characteristic cohomology (these are generally real classes).

The author then discusses some general facts (e.g., independence of the embedding) and also mentions certain special properties (e.g., the fundamental form of a Kähler metric is a spherical form).

Phillip A. Griffiths (Berkeley, Calif.)

Clifton, Yeaton H.; Smith, J. Wolfgang 3458

The Euler class as an obstruction in the theory of foliations.

Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 949–954.

The authors have previously [same Proc. **47** (1961), 190–195] introduced a category extending the category of topological spaces; the objects are spaces provided with certain sheaves of germs of mappings into themselves. One purpose of this notion is to provide an appropriate structure to serve as the "quotient" of a manifold by a foliation. Here this structure is exploited to define an obstruction to embedding a foliation in one in which the leaves are of higher dimension.

In order to do this, the authors extend the notion of vector-bundle over a space to one of vector-bundle over objects of the larger category. A foliation of a manifold gives rise to a vector bundle over the "quotient" of the manifold by the foliation, and an extension of the foliation is equivalent to a cross-section of this vector bundle.

Finally, an obstruction theory for such cross-sections is developed, using an adaptation of singular cohomology to the extended category. *A. Heller* (Urbana, Ill.)

Huebsch, William; Morse, Marston 3459

The bowl theorem and a model nondegenerate function.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 49–51.

Le "bowl theorem" de M. Morse est d'une importance exceptionnelle pour l'étude des variétés M, car il permet, quand on a une fonction f définie sur M, d'en réduire d'une façon tout à fait naturelle le nombre des points critiques. Le théorème en question, sous des hypothèses de régularité convenables, mais d'ailleurs très générales, affirme que, si parmi les points critiques il y en a un w d'indice $k-1$ et un autre $\bar{w}$ d'indice k, il est possible d'inclure ces deux points dans un ouvert convenable $N \subset M$ et de construire une fonction f^* sur M, régulière, qui coincide avec f sur $M - N$. La démonstration de ce précieux théorème, qui permettra de rendre beaucoup plus aisés de nombreux problèmes et donnera un sens plus complet à nombre de résultats de la théorie de Morse, est donnée en deux parties. Dans la première, f^* sera définie sur un sous-ensemble $\gamma \subset M$ qui contient w et $\bar{w}$; la seconde permettra de définir f^* hors de γ. La première partie de cette démonstration est le but de la note en question. La méthode employée est celle des trajectoires.

E. Baiada (Modena)

Bojarski, B. 3460

On the index problem for systems of singular integral equations.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 653–655.

The author gives a formula for the index of an elliptic system of singular integral equations on a compact

smooth manifold M_n which is the boundary of a domain in R^{n+1}. This formula is in fact a very special case (modulo some standard algebraic topology) of the general formula given by the reviewer and I. M. Singer [Bull. Amer. Math. Soc. **69** (1963), 422–433; MR **28** #626]. Moreover, in the author's formula an unknown constant K appears which depends a priori on M, and the author asks about the nature of this dependence. From the general formula referred to above one finds easily that $K = \pm 1/(n-1)!$ (the sign depends on the orientation conventions), and so is independent of M.

Finally it should be pointed out that a key step in the author's proof is the fact (Lemma 3) that the index of a system of rank N vanishes if $N < n$. The authority given for this is the paper of the reviewer and Singer [loc. cit.]. Unfortunately, this result is deduced in that paper from the general index formula. This means that the author's proof of the index formula, in the special case considered here, rests fundamentally on the general index formula.

M. F. Atiyah (Oxford)

Rosenberg, Harold 3461

A generalization of Morse-Smale inequalities.

Bull. Amer. Math. Soc. **70** (1964), 422–427.

Let X be a C^∞ vector field on a closed C^∞ manifold M^n. A periodic i-surface β of X is a C^∞ submanifold of M, homeomorphic to the i-dimensional torus, to which X is tangent (if $i=0$, β is a critical point of X, and if $i=1$, β is a closed orbit of X). There is a notion of simplicity for periodic i-surfaces which generalizes the notion of non-degeneracy of a critical point, and an integer j $(0 \leq j < n-i)$, the index of β, is associated to each simple periodic i-surface β. The stable [unstable] manifold of β is the union of all orbits of X whose ω [α] limit set lies in β. If $\beta_1, \cdots, \beta_\nu$ are periodic surfaces of X, say that there is a "cycle of orbits among the β_i" if there is a sequence $i = i_1, \cdots, i_k = i$ such that the unstable manifold of β_j meets the stable manifold of β_{j+1} $(j = 1, \cdots, k-1)$. Let R_i denote the ith Betti number of M with respect to a field of coefficients, A_j^i the number of simple periodic i-surfaces of index $j-i$, and let $M_q = \sum_{k=0}^n \sum_{i=0}^k \binom{k}{i} A_{q+i}^k$. Then the author's main result is Theorem 1: Assume X has a finite number of disjoint, simple, periodic surfaces $\beta_1, \cdots, \beta_\nu$ satisfying: (1) The α [ω] limit set of each orbit of X is included in some β_i. (2) There is no cycle of orbits among the β_i. Then

$$\sum_{q=0}^{k} (-1)^{k-q} M_q \leq \sum_{q=0}^{k} (-1)^{k-q} R_q, \quad 0 \leq k < n,$$

$$\sum_{q=0}^{n} (-1)^q M_q = \sum_{q=0}^{n} (-1)^q R_q.$$

This generalizes the famous Morse inequalities (taking X to be the gradient of a function with non-degenerate critical points) and also S. Smale's generalization of the Morse inequalities [same Bull. **66** (1960), 43–49; MR **22** #8519].

R. S. Palais (Waltham, Mass.)

Hermann, Robert 3462

Geometric aspects of potential theory in symmetric spaces. III.

Math. Ann. **153** (1964), 384–394.

Part II appeared in same Ann. **151** (1963), 143–149 [MR **27** #5276]. The author first extends the results of his previous papers on bounded symmetric domains [ibid. **148** (1962), 349–366; MR **26** #6995] to general non-compact Riemannian symmetric spaces by means of the following Theorem A: Let G be a connected semi-simple non-compact Lie group with trivial center. Let K be a maximal compact subgroup of G. Let A and N be, respectively, the connected abelian and unipotent Lie groups that occur in the Iwasawa decomposition $G = N \cdot A \cdot K$ for G. Let K_0 be the subgroup of elements of K that commute with the elements of A. Let $S = N \cdot A \cdot K_0$, $M = G/K$, $B = G/S$. Let db be a non-singular C^∞ measure (an everywhere non-zero differential form of highest degree) on B which is invariant under K and such that its integral over B is 1. (From the Iwasawa decomposition, K acts transitively on B, hence such a measure exists and is unique.) Let $K(x, b)$ be the function on $M \times B$ defined as follows: If x_0 is the identity coset of $M = G/K$, if $x = gx_0$ with $g \in G$, then $K(x, b) = J_{g^{-1}}(b)$, where J_g is the function on B which is the Jacobian determinant of the transformation on B defined by g with respect to the form db. Let $\mathfrak{G}$, $\mathfrak{K}$, $\mathfrak{H}$, etc., be the Lie algebras of G, K, H, etc., considered, when no mention made otherwise, as Lie algebras of vector fields on B via the action of G on B. Then: (a) The number of orbits of S acting on G/S (the number of double cosets of G by S) is equal to the order of the Weyl group of the symmetric space G/K (the Bruhat-Harish-Chandra double coset lemma). Further, each orbit is the stable manifold of the gradient function of a function on G/S with non-degenerate critical points. (b) Any differential operator on B which is invariant under G must be identically zero. (c) Let Δ be the differential operator on M induced by the Casimir element of the enveloping associative algebra of $\mathfrak{G}$. Then, for each continuous function $f(b)$ on B, the function

$$x \to \int K(x, b) f(b)\, db$$

is a solution of $\Delta = 0$. If $g(t)$, $0 < t < \infty$, is a non-singular one-parameter group of transvections of the symmetric space M, then the limit as $t \to \infty$ of $\int K(g(t) \cdot x_0, b) f(b)\, db$ is equal to the value of f at the (unique) attractor point for the vector field on B which is the infinitesimal generator of the one-parameter group of transformations on B defined by $g(t)$.

The second result of the author gives away one of the trade secrets of those who work on symmetric domains. Theorem B: Let G be a connected simple non-compact Lie group with trivial center, let K be a maximal compact subgroup of G which has a non-discrete center, and let G_u be a compact real form of G_c, the complexification of G. It is well known that G_u can be chosen so that K is a symmetric subgroup of G_u. Let $D = G/K$, $M = G_u/K$ and consider, as usual, D as equivariantly imbedded as an open subset of M. M and D are symmetric Kähler manifolds of, respectively, non-negative and non-positive curvature. Let r be the rank of these dual symmetric spaces, let D^r be the product of r copies of the unit disk in the complex plane, and let S^r be the product of r copies of the 2-sphere. D^r and S^r are naturally provided with symmetric Kähler metrics of, respectively, non-positive and non-negative curvature, and, since they are dual symmetric spaces, D^r can be imbedded as an open subset of S^r. Then D^r and S^r can be isomorphically imbedded as

totally geodesic sub-manifolds of, respectively, D and M, so that: (a) The following diagram of inclusion maps is commutative:

$$\begin{array}{ccc} D & \rightarrow & M \\ \uparrow & & \uparrow \\ D^r & \rightarrow & S^r \end{array}$$

(b) $K \cdot D^r = D$ and $K \cdot S^r = M$, i.e., every point of D and M can, respectively, be thrown into D^r and S^r by the action of the isotropy subgroup at one point.

Phillip A. Griffiths (Berkeley, Calif.)

Hermann, Robert 3463

Complex domains and homogeneous spaces.

J. Math. Mech. **13** (1964), 243–247.

The author gives a new proof of a theorem by E. Cartan and Harish-Chandra that a symmetric hermitian space of non-compact type is realized as a symmetric bounded domain. Let M be a connected C^∞-manifold. We say that the exponential map relative to a point $p \in M$ is defined at a (C^∞-) vector field X on M when an integral curve $\sigma(X, t)$ of X with $\sigma(X, 0) = p$ can be extended to $t = 1$. Let $E(M, p)$ be the set of vector fields on M at which the exponential map is defined. If H is a finite-dimensional vector space of vector fields on M, $E(M, p) \cap H$ is open (Theorem 2.1). Let $\text{Exp}: E(M, p) \cap H \to M$ be the map which assigns $\sigma(X, 1)$ to X in $E(M, p) \cap H$.

The following general theorems are fundamental for the author's approach to the theorem mentioned above: Exp is diffeomorphic provided that M is simply connected, that $E(M, p) \cap H$ is bounded and that Exp is of maximal rank everywhere (Theorem 2.4 and Corollary); If M admits a complex structure J and if H consists of vector fields which preserve J infinitesimally, then Exp is complex analytic (Theorem 2.2). *J. Hano* (St. Louis, Mo.)

Hepp, Klaus 3464

Klassische komplexe Liesche Gruppen und kovariante analytische Funktionen.

Math. Ann. **152** (1963), 149–158.

Let V be a complex vector space and $G \subset GL(V)$ a classical complex Lie group. Set $W = (\otimes^r V) \otimes (\otimes^s V^*)$. An invariant is a polynomial Q on W such that $Q(g \cdot \zeta) = Q(\zeta)$ for all $g \in G$ and $\zeta = (\xi_1 \otimes \cdots \otimes \xi_r \otimes \eta_1 \otimes \cdots \otimes \eta_s) \in W$. Clearly the invariants form a ring I, which is finitely generated in the cases under consideration. Let $P_1, \cdots, P_N$ be generators of I (example: $G = GL(V)$, $N = rs$ and $P_{ij}(\zeta) = \langle \xi_i, \eta_j \rangle$ for $1 \leq i \leq r$; $1 \leq j \leq s$); then there is defined $P \cdot W \to \mathbf{C}^N$ by $P = (P_1, \cdots, P_N)$. Furthermore, the relations for P are finitely generated; thus $P(W)$ is an algebraic set $\hat{I} \subset \mathbf{C}^N$. Let now $D \subset V$ be a domain such that $P^{-1}(P(D)) = D$. Then the main theorem is that $P(D)$ is a normal algebraic subset of $\mathbf{C}^N$ and each holomorphic function f on D such that $f(g \cdot \zeta) = f(\zeta)$ is of the form $f = \hat{f} \cdot P$ for some holomorphic function on $P(D)$.

A main lemma in proving this result is the following. Let $\hat{x} \in \hat{I}$. Then there exists $x \in W$ and an open neighborhood $U \supset \{x\}$ such that $P(x) = \hat{x}$ and $P(U)$ is a neighborhood of $\hat{x}$ in $\hat{I}$. This suggests that the fibres of P should be reasonable. *Phillip A. Griffiths* (Berkeley, Calif.)

Tondeur, Philippe 3465

Fast-komplexe Strukturen auf einer Lieschen Gruppe.

Comment. Math. Helv. **38** (1963), 14–25.

This paper is a study of invariant almost-complex structures on a Lie group. Let G be a Lie group with Lie algebra $\mathfrak{g}$; a bi-invariant almost-complex structure is given by a $J \in \text{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g})$ satisfying $J^2 = -I$. Thus $J[x, y] = [x, Jy]$ for all $x, y \in \mathfrak{g}$. The author observes that such a structure is integrable. (Proof: The torsion tensor (or Nijenhuis tensor or obstruction to integrating the structure) at the identity of G is given by $T(x, y) = [Jx, Jy] - J[Jx, y] - J[x, Jy] - [x, y] = 0$ for $x, y \in \mathfrak{g}$.) Some other facts concerning such structures are obtained.

Phillip A. Griffiths (Berkeley, Calif.)

PROBABILITY

See also 3020, 3139, 3271, 3499, 3508, 3514, 3555, 3826.

Onicescu, O. 3466

Considérations sur les problèmes actuels de la théorie des probabilités.

Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (*N.S.*) **4 (52)** (1961), *no.* 1–2, 77–89 [1963].

The author reviews the progress made during the last twenty five years and speculates about the likely course of future developments.

D. G. Kendall (Cambridge, England)

Hobby, Charles; Pyke, Ronald 3467

Remarks on the equivalence principle in fluctuation theory.

Math. Scand. **12** (1963), 19–24.

The authors give a new constructive proof of the so-called equivalence principle of fluctuation theory. In its simplest form this principle is a lemma concerning the partial sums of the $n!$ permutations of n real numbers: The number of permutations for which exactly k of the n partial sums are positive is the same as the number of permutations for which the kth partial sum exceeds all its predecessors and is at least as large as the subsequent partial sums. *F. L. Spitzer* (Ithaca, N.Y.)

Csáki, E.; Vincze, I. 3468

On some combinatorial relations concerning the symmetric random walk.

Acta Sci. Math. (*Szeged*) **24** (1963), 231–235.

Consider a tied-down symmetric walk of length $2n$. This is probabilistically equivalent to a randomly selected sequence $Z = (Z_1, Z_2, \cdots, Z_{2n})$ from the set of all $\binom{2n}{n}$ sequences of n $+1$'s and n -1's. Set $S_i = Z_1 + \cdots + Z_i$ and $M = \max(S_1, S_2, \cdots, S_{2n})$. Let J equal the index of the first maximum of the partial sums. Define λ (the number of half waves) and γ (half of the number of "positive" sums) by stating that $\lambda - 1$ equals the number of subscripts i for which $S_i = 0$ and $S_{i-1}S_{i+1} = -1$, while 2γ equals the number of subscripts i for which either $S_i > 0$ or $S_i = 0$, $S_{i-1} = 1$. Set λ' equal to the number of subscripts i for which $S_{i-1} = 0$ and $S_i = 1$, and let π equal the number

of strictly positive partial sums. The authors derive the following two equivalence relations:

$$P[M = m] = 2^{-1}\{P[\lambda = m]+P[\lambda = m+1]\},$$
$$P[M = m, J = j] = P[\lambda' = m, \pi = j].$$

The method of proof is to exhibit a suitable 1-1 correspondence between the sets of paths. As pointed out by the authors, a similar construction to that used to prove the second relationship mentioned above is used by Hobby and the reviewer [#3467 above].

R. Pyke (Seattle, Wash.)

Berman, Simeon M. 3469

Limiting distribution of the maximum of a diffusion process.

Ann. Math. Statist. **35** (1964), 319–329.

Suppose that $X(t)$, $t \geqq 0$, is a stationary strong Markov process with continuous paths and state interval $S = (r_1, r_2)$, $-\infty \leqq r_1 < r_2 \leqq \infty$, and that all points of S are accessible from each other. To continue his previous work [same Ann. **33** (1962), 894–908; MR **25** #5535], the author studies the limiting behaviour of $Z(t) = \max(X(s): 0 \leqq s \leqq t)$ and allied variables. $Z(t)$ being asymptotically the maximum of identically distributed independent variables, the fundamental theorems of Gnedenko and Feller, as well as the author's previous results, are applicable. Let T_0, T_1', T_1, T_2', $\cdots$ be successive times of $X(t)$ alternately passing through x_1, x_2, define $m(x_1, x_2) = E(T_1 - T_0)$, $X = \max(X(s): T_0 < s \leqq T_1)$, $N(t) = \max(k: T_k \leqq t)$, and put $G(x; x_1, x_2) = P(X \leqq x)$. Then the main results are as follows. If $X(t)$ is recurrent and $m(x_1, x_2) < \infty$, then the only limiting distribution of a normalized variable $(Z(t) - \beta(t))/\alpha(t)$, $\alpha(t) > 0$, is one of three known types, say $\Phi(x)$; this happens if and only if $G(x; x_1, x_2)$ is in the attraction domain of $\{\Phi(x)\}^{m(x_1,x_2)}$ for some x_1, x_2, this condition being independent of the choice of x_1, x_2. There exists $\sigma(t) > 0$ such that $N(t)/\sigma(t)$ has a limiting distribution if and only if $m(x_1, x_2) = \infty$ and

$$p(s, x, E) = \int_0^\infty e^{-st}P(t, x, E)\,dt$$

satisfies the condition that

$$\lim_{s\to 0+} h(s^{-1})s^\alpha p(s, x, E) = \delta,$$

where $P(t, x, E)$ is the transition probability of $X(t)$, $h(x)$, $x > 0$, is a slowly varying function, α, δ are positive constants, and E is a proper sub-interval of S; in this case the limiting distribution function is Mittag-Leffler's function with index α. The author obtains a condition of stability for $Z(t)$, its limiting distribution for non-recurrent $X(t)$, and that of an occupation time for an interval. A related subject was studied by G. F. Newell [J. Math. Mech. **11** (1962), 481–496; MR **25** #3559].

G. Maruyama (Fukuoka)

Beck, Anatole 3470

On the strong law of large numbers.

Ergodic Theory (*Proc. Internat. Sympos., Tulane Univ., New Orleans, La.*, 1961), *pp.* 21–53. *Academic Press, New York*, 1963.

This is primarily an expository paper on the strong law of large numbers for sequences of independent random variables defined on a separable Banach space.

R. Pyke (Seattle, Wash.)

Survila, P. 3471

The remainder term in the asymptotic expansion of the density. (Russian. Lithuanian and German summaries)

Litovsk. Mat. Sb. **2** (1962), *no.* 2, 233–251.

The usual theorems on the estimation of the remainder term in the asymptotic expansion of the distribution function $F_n(x)$ of the sum of n independent random variables give estimates of the form $p_n(x) - \phi(x) - P(x) = O(n^\alpha)$ which are uniform in $x \in (-\infty, \infty)$; here $p_n(x)$ is the density of $F_n(x)$ (assumed to exist), $\phi(x)$ is the Gaussian density and $P(x)$ is a finite sum of polynomials. The author proves a theorem in which the estimate, no longer uniform in x, has the form $O(n^\alpha(1+|x|)^{-s})$, where s is the order of the highest absolute moment β_s of the component distributions assumed to exist, and the constant in O involves s, β_2 and β_s.

K. Balagangadharan (Bombay)

Iosifesku, M. [Iosifescu, M.]; Teodoresku, R. [Theodorescu, R.] 3472

Properties of chains with complete connections. (Russian)

Ukrain. Mat. Ž. **16** (1964), 93–99.

In the present paper the investigations of two previous notes [Com. Acad. R. P. Romîne **11** (1961), 1451–1453; MR **24** #A2996; ibid. **12** (1962), 295–297; MR **26** #1929] are extended to the case of an arbitrary set of states. The following notations are used: $(\Omega, \mathscr{K}, P)$ a probability space; $(X, \mathscr{F})$ a measurable space; $\mathfrak{M}$ the set of all probability measures defined on $\mathscr{F}$ and $(T_x^{(n)})_{x\in X}$ a family of applications of $\mathfrak{M}$ into $\mathfrak{M}$ for every $n \in N^*$. The sequence of random variables $(\xi_n)_{n\in N^*}$, $N^* = \{1, 2, \cdots\}$, defined on Ω and with values in X forms a chain with complete connections if the conditional probability

$$p_{x_1\cdots x_n}(A) = P(\xi_{n+1}(\omega) \in A \mid \xi_j, 1 \leqq j \leqq n)_{\xi_j(\omega) = x_j,\ 1\leqq j\leqq n}, \quad A \in \mathscr{F},$$

is of the form

$$p_{x_1\cdots x_n} = T_{x_n}^{(n)}p_{x_1\cdots x_{n-1}}, \quad p_{x_1} = T_{x_1}^{(1)}p,$$

where $p(A) = P(\xi_1(\omega) \in A)$, $A \in \mathscr{F}$, for any $(x_1, \cdots, x_n) \in X^{(n)}$ and $n \in N^*$. For such chains an existence theorem, as well as an ergodic theorem (for the stationary case), is proved.

O. Onicescu (Bucharest)

Walldin, K. E. 3473

Stochastic processes with stationary and independent increments. I. (French summary)

Rev. Inst. Internat. Statist. **30** (1962), 293–335.

The author studies stochastic processes with stationary independent increments, especially those (class A_+) having positive increments. His ultimate objective will be a combination of the usual inventory models with queueing theory. In this paper he surveys and adds to the general theory as a preparation. Special attention is paid (for an A_+ process ξ_t) to the first time the process surpasses a

value x and to the amount of "overshoot". The asymptotic distribution of the overshoot is studied, as is also the asymptotic distribution of $\xi_t-[\xi_t]$. The "key renewal theorem" of W. L. Smith [Proc. Roy. Soc. Edinburgh Sect. A **64** (1954), 9–48; MR **15**, 722] is generalised to A_+-processes. *D. G. Kendall* (Cambridge, England)

Sinaĭ, Ja. G. 3474

Higher-order spectral measures of ergodic stationary processes. (Russian. English summary)

Teor. Verojatnost. i Primenen. **8** (1963), 463–470.

The author gives necessary conditions [see also #3475] for ergodicity of stationary processes belonging to the Fortet-Blanc-Lapierre class. They are used to obtain new ergodic and uniqueness theorems for such processes.

M. Loève (Berkeley, Calif.)

Širjaev, A. N. 3475

Ergodicity conditions for stationary processes in terms of higher-order moments. (Russian. English summary)

Teor. Verojatnost. i Primenen. **8** (1963), 470–473.

The author considers stationary processes $\xi(t)=\int e^{it\lambda}X(d\lambda)$ belonging to the Fortet-Blanc-Lapierre class, with all moments finite and all spectral moments

$$\int_\Lambda EX(d\lambda_1)\cdots X(d\lambda_n)$$

of bounded variation. He gives necessary [see also #3474] and sufficient conditions for ergodicity of such processes in terms of these moments. Known conditions for ergodicity of Gaussian processes follow.

M. Loève (Berkeley, Calif.)

Stratonovič, R. L. 3476

A new form of representing stochastic integrals and equations. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 1, 3–12.

The author considers a diffusion $x(t)$, $t\in T=[a_0,b_0]$, satisfying the following conditions:

$$\lim_{h\downarrow 0} E\{[x(t+h)-x(t)]/h\,|\,x(t)=\xi\}=a(\xi,t),$$

$$\lim_{h\downarrow 0} E\{[x(t+h)-x(t)]^2/h\,|\,x(t)=\xi\}=b(\xi,t),$$

$$\lim_{h\downarrow 0} E\{|x(t+h)-x(t)|>\delta\,|\,x(t)=\xi\}=0,\quad \delta>0,$$

where a and b are continuous (or piecewise continuous) in both arguments and also $\partial b(x,t)/\partial x$ is continuous. Let $\Phi(x,t)$ be continuous in t on T with continuous derivative $\partial\Phi(x,t)/\partial x$ and satisfy the following conditions

$$\int_{a_0}^{b_0} E\{\Phi(x(t),t)a(x(t),t)\}\,dt<\infty$$

and

$$\int_{a_0}^{b_0} E\{|\Phi(x(t),t)|^2 b(x(t),t)\}\,dt<\infty.$$

For this setup the author defines a symmetrized stochastic integral:

$$\int_{a_0}^{b_0}\Phi(x(t),t)\,dx(t)=\mathop{\text{l.i.m.}}_{\Delta\to 0}\sum_{j=1}^{N-1}\Phi(\tfrac12[x(t_j)+x(t_{j+1})],t_j)[x(t_{j+1})-x(t_j)],$$

where

$$a_0=t_1^{(\Delta)}<t_2^{(\Delta)}<\cdots<t_N^{(\Delta)}=b_0 \text{ and } \Delta=\max_j(t_{j+1}-t_j).$$

Its relation to the stochastic integral of Itô is

$$\int_{a_0}^{b_0}\Phi[x(t),t]\,dx(t)=(\text{Itô})\int_{a_0}^{b_0}\Phi[x(t),t]\,dx(t)+\frac12\int_{a_0}^{b_0}\frac{\partial\Phi}{\partial x}[x(t),t]b[x(t),t]\,dt.$$

For Brownian motion, then,

$$\int_{a_0}^{b_0}[x(t)-x(a_0)]\,dx(t)=\tfrac12[x(b_0)-x(a_0)]^2,$$

whereas

$$(\text{Itô})\int_{a_0}^{b_0}[x(t)-x(a_0)]\,dx(t)=\tfrac12[x(b_0)-x(a_0)]^2-\tfrac12(b_0-a_0).$$

The author then generalizes these definitions and relations to n-dimensional vector-valued processes $\mathbf{x}(t)=\{x_i(t),\ i=1,\cdots,n\}$. With suitable smoothness requirements on functions $\Psi(\mathbf{x},t)$ and $\Phi_\alpha(\mathbf{x},t)$ $(\alpha=1,\cdots,n)$ he considers the process

$$z_\alpha(t)=\int_{a_0}^t\Psi(\mathbf{x},t)\,dt+\int_{a_0}^t\Phi_\alpha(\mathbf{x},t)\,dx_\alpha,$$

and calculates the corresponding n-dimensional $a(\cdot,t)$ and $b(\cdot,t)$. He discusses a special 2-dimensional case of the form

$$x(t)=x(a_0)+\int_{a_0}^t m[x(t),t]\,dt+\int_{a_0}^t\sigma[x(t),t]\,dy(t).$$

The rest of the paper is devoted to showing the effect of changes of variable when one uses this definition of integral: transformation equations analogous to those of elementary calculus. In the concluding section is a discussion of an invariant form for the infinitesimal generator equation. *J. Chover* (Stanford, Calif.)

Kolomiec, V. G. 3477

Parametric random effect on linear and non-linear oscillating systems. (Russian)

Ukrain. Mat. Ž. **15** (1963), 199–205.

The author examines the equation

$$\frac{d^2x}{dt^2}+2\delta[1+\xi(t)]\frac{dx}{dt}+\omega^2[1+\xi(t)]=0$$

describing an oscillator under stationary Gaussian random disturbances $\xi(t)$. The application of approximation methods leads finally to the expression

$$x(t)=a(t)\cos(\omega t+\theta(t)),$$

where $\theta(t)$, $\ln a(t)$ are Wiener processes, the latter with a

local shift, whose sign determines the stability of the system. The oscillator with a small non-linearity is treated similarly. *Petr Mandl* (Prague)

Feldman, Jacob 3478

On the Schrödinger and heat equations for nonnegative potentials.

Trans. Amer. Math. Soc. **108** (1963), 251–264.

In the paper the equation

$$(1)\qquad \frac{1}{\sigma}\frac{\partial u(x,t)}{\partial t} = (\Delta - V(x))u(x,t)$$

for measurable non-negative $V(x)$ is studied. $V(x)$ may assume infinite values. It is shown that the Wiener integral

$$(2)\qquad T_V{}^t f(x) = E\left\{\exp\left[-\int_0^t V(\xi_s+x)\,ds\right]f(\xi_t+x)\right\}$$

defines a strongly continuous semigroup in the space $\mathscr{L}_p{}^V$, $1 \leq p < \infty$, which is the $\mathscr{L}_p$ space on the set

$$\left\{x : P\left\{\lim_{t\to 0}\exp\left[-\int_0^t V(\xi_s+x)\,ds\right]\right\} = 1\right\}.$$

The generator A_V of $T_V{}^t$ is a self-adjoint non-negative operator in the space $\mathscr{L}_2{}^V$, and it is possible to define $T_V{}^\zeta = e^{-\zeta A_V}$ for complex ζ with non-negative real part. The case ζ purely imaginary corresponds to purely imaginary σ, for which (1) is the Schrödinger equation. $V(x)$ is called Riemann-approximable if, for almost every x, $P\{V(\xi_s+x)$ is Riemann-integrable on $(0,t)\}=1$. Then $T_V{}^\zeta$ is the limit of $\prod_j e^{-\zeta\tau_j V}e^{\zeta\tau_j\Delta}$, where $\tau_j>0$, $\sum\tau_j=1$, $\max|\tau_j|\to 0$. It is also shown that (2) may be expressed by means of a Green's function. *Petr Mandl* (Prague)

Matthes, Klaus 3479

Stationäre zufällige Punktfolgen. I.

Jber. Deutsch. Math.-Verein. **66** (1963/64), *Abt.* 1, 66–79.

The theory of labelled random point sequences (l.r.p.s.) is the study of a particular probability space $[M_K, \mathfrak{M}_{\mathfrak{R}}, P]$ which is a general model for problems arising in the queueing theory, renewal theory, etc. The elements Φ of the basic space M_K are countable sets of pairs $[x,k]$, where x is a real number representing the time when the event of type k has occurred. k takes values from a measurable space $[K,\mathfrak{R}]$. There are no two points with the same time-coordinate in Φ, and the time-coordinates of points in Φ do not possess any limit point. $\mathfrak{M}_{\mathfrak{R}}$ is the smallest σ-algebra with the property that for every $L\in\mathfrak{R}$ and every time interval the number of events of the type belonging to L is a measurable function. The concept of a stationary, ergodic, mixing l.r.p.s. is introduced. In the main part of the paper the theory of a Palm distribution, representing the distribution of Φ under the condition that an event has occurred at time 0 is developed, and a formula relating the Palm distribution to the probability P is given. As an important special case l.r.p.s. of Poisson type are examined.

Petr Mandl (Prague)

Matthes, Kl. [Matthes, Klaus] 3480

Unbeschränkt teilbare Verteilungsgesetze stationärer zufälliger Punktfolgen.

Wiss. Z. Hochsch. Elektrotech. Ilmenau **9** (1963), 235–238.

A point process is defined as a probability triple $(M,\mathfrak{M},P)$, where M is the set of all countable subsets of the real line which have no finite accumulation points, $\mathfrak{M}$ is the minimal σ-algebra of subsets of M such that for each bounded interval the function from M to the non-negative integers defined by $\Phi\to\text{card}\,(\Phi\cap I)$ is measurable, and P is a probability measure on M. The convolution $P_1 * P_2$ of measures P_1 and P_2 on M is defined by

$$P_1 * P_2(\text{card}\,\Phi\cap I = n) = \sum_{k=0}^{n} P_1(\text{card}\,(\Phi\cap I)=k)P_2(\text{card}\,(\Phi\cap I)=n-k)$$

for all bounded intervals I and all non-negative integers n; the convolution of more than two measures is defined similarly. The process $(M,\mathfrak{M},P)$ is said to be stationary if P is shift-invariant, and infinitely divisible if there exists a set $P_{n,r}$ $(n=1,2,\cdots;r=1,2,\cdots,n)$ of measures on $\mathfrak{M}$ such that $P_{n,1}*P_{n,2}*\cdots*P_{n,n}\to P$ as $n\to\infty$. In this paper, a characterization of stationary infinitely divisible point processes is given. The author also defines regular stationary point processes as processes such that for every set $I_1,\cdots,I_s$ of bounded intervals the joint distribution of card $(\Phi\cap I_1),\cdots,$ card $(\Phi\cap I_s)$ conditional on $\Phi\cap((-n-m,-m)\cup(m,n+m))=\emptyset$ tends to the corresponding unconditional distribution as n and m approach ∞, and singular stationary point processes as processes such that

$$P(\Phi\cap(-t,t)\neq\emptyset\,|\,\Phi\cap[(-t-m,-t)\cup(t,t+m)]=\emptyset)\to 0 \quad\text{as } m\to\infty.$$

It is proved that every stationary infinitely divisible point process is the superposition (convolution) of a regular stationary infinitely divisible component and a singular stationary infinitely divisible component. Finally, the author shows that any regular stationary infinitely divisible point process is a Poisson clustering process [cf. Bartlett, J. Roy. Statist. Soc. Ser. B **25** (1963), 264–296], and conversely. *P. M. Lee* (Cambridge, England)

Urbanik, K. 3481

Generalized convolutions.

Studia Math. **23** (1963/64), 217–245.

This paper consists of a detailed exposition, with proofs, of results announced earlier [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 165–168; MR **27** #3026] on the properties of certain binary operations in the class of probability measures on the non-negative real line. *J. F. C. Kingman* (Cambridge, England)

Orey, Steven 3482

Potential kernels for recurrent Markov chains.

J. Math. Anal. Appl. **8** (1964), 104–132.

Let P be the transition matrix of a recurrent indecomposable Markov chain with stationary measure α. The author studies possible potential operators (matrices) for such a chain. If A is such an operator, potentials are functions g on the state space of the form $g=Af$ with f a function with finite support. The function (charge) f is called a null charge if $\alpha f=0$. The author requires that for every null charge (i) $(P-I)Af=f$ and (ii) Af is bounded. It is shown that for every P there is at least one operator A with properties (i) and (ii), and a method is given for

characterizing the most general such operator. These operators include the potential operator used by Spitzer in developing a potential theory for recurrent sums of independent random variables [Spitzer, J. Math. Mech. **11** (1962), 593–614; MR **25** #2655] and also the operator used by Kemeny and the reviewer [J. Math. Anal. Appl. **3** (1961), 196–260; MR **25** #3563] for a general class of recurrent chains. The latter authors considered primarily potentials with null charges. It is shown here that many of their results have analogues for any A satisfying (i) and (ii). Spitzer developed a potential theory using non-negative charges, and the author also investigates conditions on A that lead to an interesting theory for these potentials. The author makes use of various Martin boundaries which he develops for recurrent chains. These are Martin boundaries for transient chains naturally associated with the recurrent chain, for example, the chain obtained by stopping the process when it hits state i. A boundary of this type was introduced independently by Kemeny and the reviewer [Trans. Amer. Math. Soc. **106** (1963), 495–520; MR **26** #1925].

J. L. Snell (Hanover, N.H.)

Blumenthal, R. M.; Getoor, R. K. 3483

Additive functionals of Markov processes in duality.

Trans. Amer. Math. Soc. **112** (1964), 131–163.

An additive functional of a stochastic process is a Borel function A of the time $t \geqq 0$ and the sample path $w: s \to x(s)$, depending for each $t \geqq 0$ upon $x(s): s \leqq t$ alone and such that (a) $0 = A(0+)$, (b) $A(t\pm) = A(t) < \infty$, and (c) $A(t) = A(s) + A(t-s, w_s^+)$ $(s \leqq t)$, w_s^+ being the shifted path $t \to x(t+s)$. A is non-negative or positive according as $0 \leqq A$ or $0 < A$ for $t > 0$. The reviewer and H. Tanaka [Mem. College Sci. Univ. Kyoto Ser. A Math. **33** (1960/61), 479–506; MR **24** #A1147] proved that a d-dimensional diffusion with the same hitting probabilities as a standard d-dimensional Brownian motion b can be expressed as $b(A^{-1})$, A^{-1} being the inverse function of a positive additive functional of the Brownian path; in addition, they proved that the non-negative additive functionals can be placed in one-to-one correspondence with a class of so-called smooth measures e so that formally

$$A(t) = \int_{R_d} \frac{\text{measure } (s : b(s) \in da)e(da)}{2da}$$

with A positive if and only if e is positive on fine open subsets of R_d. The authors and the reviewer [Illinois J. Math. **6** (1962), 402–420; MR **25** #5550] then obtained this linking together of motions with the same hitting probabilities, via time substitutions based upon additive functionals, for a wide class of processes that begin afresh at stopping times. Here the connection between additive functionals and an appropriate class of smooth measures is proved for a little narrower class of motions; the principal additional condition is that the motion admit a positive super-stable mass distribution e:

$$\int e(da)P_t(a, db) \leqq e(db).$$

Under the conditions imposed, a nice dual or backward motions exists, and it is now proved that, after slight technical precautions, time substitutions based upon additive functionals do not spoil this duality. Roughly speaking, if the motion y has the same hitting probabilities as x, then the smooth measure corresponding to the additive functional that gives rise to the time substitution between them is also smooth for the motion x^* dual to x and so gives rise to an additive functional for x^*; the associated time substitution maps x^* into a motion y^* with the same hitting probabilities, and y^* is dual to y.

H. P. McKean, Jr. (New York)

Courrège, Philippe; de Sam Lazaro, José 3484

Sous-martingales de classe (D) et processus croissants intégrables.

C. R. Acad. Sci. Paris **258** (1964), 1995–1997.

P. A. Meyer showed that the analogue of Doob's decomposition of a discrete parameter submartingale (into a martingale plus an increasing chain) holds for a wide class of continuous-parameter submartingales [Illinois J. Math. **6** (1962), 193–205; MR **28** #2576; ibid. **7** (1963), 1–17; MR **26** #1927]. The authors now give conditions to ensure that the increasing part is the integral of a positive density.

H. P. McKean, Jr. (New York)

Breny, H. 3485

Cheminements conditionnels de chaînes de Markov absorbantes.

Ann. Soc. Sci. Bruxelles Sér. I **76** (1962), 81–87.

From an absorbing Markov chain with a finite number of states a second one is obtained by replacing the given probability space by a conditional one, the condition being that the original process ends up in a given absorbing state. The new transition matrix is computed. The same result (but presented in less detail) is in Kemeny and Snell [*Finite Markov chains*, p. 64, Van Nostrand, Princeton, N.J., 1960; MR **22** #5998]. The example has square matrices with five rows by definition and six after computation.

J. Th. Runnenburg (Muiderberg)

Stratton, Howard H., Jr.; Tucker, Howard G. 3486

Limit distributions of a branching stochastic process.

Ann. Math. Statist. **35** (1964), 557–565.

Consider a population of particles which is growing according to a branching process. Let $X_N(t)$ be the number of particles at time t if there were N at time zero. Traditional limit theorems for branching processes have referred to the behavior of $X_N(t)$ for large t. The authors' idea in the present work is to hold t fixed and let $N \to \infty$. Simultaneously, the parameters governing the growth of the population change in a manner similar to that employed in the Poisson approximation to the binomial distribution. It is then shown that the process $\{X_N(t) - N\}$ converges in distribution to another process which has independent increments, and whose characteristic function is given.

P. E. Ney (Ithaca, N.Y.)

Jiřina, Miloslav 3487

Harmonisable solutions of ordinary differential equations with random coefficients and random right-hand side. (Russian. English summary)

Czechoslovak Math. J. **13** (88) (1963), 360–371.

The author continues his study of generalized random distributions (depending on a given sub-σ-field $\mathfrak{Z}$ of the base σ-field), begun in his article [same J. **12** (87) (1962),

457–474; MR 26 #3098; in the review, read $\mathfrak{Z}$ for ζ]. There he considered "$\mathfrak{Z}$-stationary" random distributions $x(\varphi)$, $y(\varphi)$ and conditions under which a stochastic equation $\sum_{m=0}^{r} \alpha_m x^{(m)}(\varphi) = y(\varphi)$ has a "$\mathfrak{Z}$-stationary solution" $x(\varphi)$ given a "$\mathfrak{Z}$-stationary" $y(\varphi)$. In the present paper, in analogy with the usual notion of harmonizability, he defines "$\mathfrak{Z}$-harmonizability" for random distributions $x(\varphi)$, and gives conditions on the α_m and the spectral measure of a given $\mathfrak{Z}$-harmonizable $y(\varphi)$ in order that the above equation have a $\mathfrak{Z}$-harmonizable solution $x(\varphi)$.

J. Chover (Stanford, Calif.)

Ibragimov, I. A. 3488

An estimate for the spectral function of a stationary Gaussian process. (Russian. English summary)

Teor. Verojatnost. i Primenen. **8** (1963), 391–430.

Let $\{x_n\}$ be a real stationary Gaussian process with spectral function $F(\lambda)$ and spectral density $f(\lambda)$. The author considers the estimate $F_N^*(\lambda) = \int_0^\pi I_N(\lambda)\,d\lambda$ for $F(\lambda)$, where $I_N(\lambda) = (2\pi N)^{-1}|\sum_1^N x_k e^{i\lambda k}|^2$. He studies first the asymptotic behaviour of $E\int_{-\pi}^{\pi} \varphi(\lambda) I_N(\lambda)\,d\lambda$ ($\varphi(\lambda)$ a bounded function) and

$$R_N(\lambda, \mu) = E[(F_N^*(\lambda) - EF_N^*(\lambda))(F_N^*(\mu) - EF_N^*(\mu))].$$

Define $\zeta_N(\lambda) = N^{1/2}(F_N^*(\lambda) - F(\lambda))$. The author then proves that if $F(\lambda)$ is strictly increasing and $\int_{-\pi}^{\pi} f^2(\lambda)\,d\lambda < \infty$, then the k-vector $(\zeta_N(\lambda_1), \cdots, \zeta_N(\lambda_k))$ has asymptotically, as $N \to \infty$, a normal distribution with mean zero and correlation-matrix $(2\pi \int_0^{\min(\lambda_i, \lambda_j)} f^2(t)\,dt)$. Using the well-known convergence theorems of Prohorov, he now shows that if P_N and P are the measures induced on $C[0, \pi]$, respectively, by $\zeta_N(\lambda)$ and a Gaussian process $\zeta(\lambda)$ with $\zeta(0) = 0$, $E\zeta(\lambda) \equiv 0$ and $E\zeta(\lambda)\zeta(\mu) = 2\pi \int_0^{\min(\lambda, \mu)} f^2(t)\,dt$, $0 \leq \lambda, \mu \leq \pi$, then P_N converges to P weakly; this is valid if $\int_a^b f(t)\,dt > 0$ for every $(a, b) \subset [-\pi, \pi]$ and

$$\int_{-\pi}^{\pi} (f(t))^{2+\delta}\,dt < \infty$$

for some $\delta > 0$. Finally, the author shows that the results carry over to the continuous-parameter case.

K. Balagangadharan (Bombay)

Neuts, Marcel F. 3489

Absorption probabilities for a random walk between a reflecting and an absorbing barrier.

Bull. Soc. Math. Belg. **15** (1963), 253–258.

In this paper the author, using elementary recursion relations, investigates the probability that a particle, initially at i, is absorbed at 0 at time t in a general stationary random walk on the integers $\{0, 1, \cdots, b\}$ when 0 is an absorbing barrier and b is a (semi-) reflecting barrier.

S. C. Port (Santa Monica, Calif.)

Takács, Lajos 3490

The distribution of majority times in a ballot.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 118–121 (1963).

In this sequel to a previous paper [same Z. **1** (1962), 154–158; MR **26** #3131] further probabilities are calculated concerning the number of times one candidate leads over another during the successive stages of a ballot. The combinatorial proofs are based on lemmas which are relevant also to fluctuation theory, order statistics, and the theory of queues. The author also gives a generalization to processes with independent increments.

F. L. Spitzer (Ithaca, N.Y.)

Takács, Lajos 3491

The distribution of the virtual waiting time for a single-server queue with Poisson input and general service times.

Operations Res. **11** (1963), 261–264.

An elementary and direct method for the determination of the distribution function described in the title of this paper is given, based on a lemma connecting partial sums of identically distributed independent service times with ordered independent drawings from a uniform distribution. Such a nice result might have had a less concise proof. One would have liked to see the equilibrium distribution derived from this result.

J. Th. Runnenburg (Muiderberg)

Jewell, William S. 3492

Multiple entries in traffic.

J. Soc. Indust. Appl. Math. **11** (1963), 872–885.

A traffic stream has independent, identically distributed gaps. A second traffic stream tries to merge with it. The author assumes that in any gap of duration t the number of entries from the second stream is a random variable with specified distribution. Various of the more elementary aspects of this renewal problem are described such as the number of entries in any given time and the time to the nth entry. Various specific cases are then compared.

G. Newell (Providence, R.I.)

Franken, Peter 3493

Approximation durch Poissonsche Prozesse.

Math. Nachr. **26** (1963), 101–114.

This note deals with the approximation of the finite-dimensional distributions of a process which is the sum of n independent random point-sequences by the finite-dimensional distributions of the Poisson process. Let η be the sum of n independent m-component random vectors with non-negative components, and P_n the distribution of η; furthermore, let ψ be the m-variate Poisson distribution which is the product of m univariate Poisson distributions. The basic bounds on the discrepancy between P_n and ψ for fixed m, n are derived by elementary methods. These bounds are applied to estimation of the error term in the m-variate Poisson convergence theorem for sums of independent vectors. The second half of this paper is concerned with the special case in which the individual random point-sequences are realized as the renewal points of a standard renewal process. In this case, the error bound is expressed in terms of the "life-time" distribution and the renewal function.

S. M. Berman (New York)

STATISTICS

See also 3490, 3882.

Bol'šev, L. N. 3494

On the random numbers of M. Kadyrov in the statistical tables of J. Janko. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 152–154.

Author's summary: "The table of random numbers in J. Janko's statistical tables is a sample from the table of M. Kadyrov's random numbers. The χ^2-test was applied to this sample to test the deviations of the sample frequency distribution from the corresponding theoretical equiprobability distribution. Statistical analysis shows that these deviations are significant at the 0.05% significance level."

Hannan, J.; Harkness, W. 3495
Normal approximation to the distribution of two independent binomials, conditional on fixed sum.
Ann. Math. Statist. **34** (1963), 1593–1595.

Using an approximation due to Feller [*An introduction to probability theory and its applications*, Vol. I, 2nd ed., p. 182, Wiley, New York, 1957; MR **19**, 466], the authors prove a theorem, too long to repeat here, about the approximation referred to in the title.

D. M. Sandelius (Åmål)

Pearson, E. S. 3496
Some problems arising in approximating to probability distributions, using moments.
Biometrika **50** (1963), 95–112.

From the author's introduction: "In the history of the development of statistical distribution theory there have been many instances where it has been possible to determine the sampling moments of the distribution of a statistic, without any immediate prospect of deriving the mathematical distribution itself in explicit form. Insofar as there may be a number of alternative mathematical forms which could be used to approximate the unknown true distribution, the question arises as to how to select between them. Insofar as a distribution can be represented by a Gram-Charlier or Fisher-Cornish type of expansion we might expect in theory that agreement in moments would lead to agreement in probability integrals, but it is well known that questions of the convergence of such expansions arise in the case of distributions which are far from normal. In the following paper it is proposed to draw together several hitherto unpublished investigations, some dating back a number of years, which bear on these points. In particular, we shall: (a) Consider the proportionate contributions, arising from different parts of the parent frequency, to each of the first six moments of certain selected distributions. (b) Make a comparison of the distribution functions of three leptokurtic distributions, namely (i) the Pearson type IV, (ii) the non-central t, and (iii) Johnson's S_U, when their first four moments have identical values. (c) Apply some of the conclusions drawn from the studies (a) and (b) to the problem of determining significance points for the moment ratio statistics used in testing for departure from normality." {For Johnson's S_U see N. L. Johnson [Biometrika **36** (1949), 149–176; MR **11**, 527].} *D. M. Sandelius* (Åmål)

Haldane, J. B. S.; Jayakar, S. D. 3497
The distribution of extremal and nearly extremal values in samples from a normal distribution.
Biometrika **50** (1963), 89–94.

R. A. Fisher and L. H. C. Tippett [Proc. Cambridge Philos. Soc. **24** (1928), 180–190] stated that the first asymptotic probability $\Phi_{(x)}$ of the largest value

$$(1) \qquad \Phi_{(x)} = \exp[-e^{-y}]; \quad y = \alpha(x-u),$$

can be used for normal variables only for samples of size $n = 10^{12}$, while graphical procedures have shown that except for the tail the approximation is not too bad even for $n = 100$. The two statements are compatible because Fisher and Tippett based their argument on numerical values of the moment quotients which depend strongly on the extremes of the extremes. Since the distribution of the largest value taken from an exponential distribution converges quickly to (1), a quadratic function of a normal variable will approach (1) more quickly than a linear one. The authors use an average v of the largest normal value defined by

$$(2) \qquad v^2 \exp[v^2] = n^2/2\pi,$$

and give a table of v as a function of n. This is easier to calculate than the mode u of the largest normal value. The difference between the two is negligible when $n \geqq 1000$. The reduced largest normal value y is then defined by

$$(3) \qquad x^2 = v^2 - 2v^{-2} + 7v^{-4} + 2(1 - v^{-2} + 3v^{-4})y.$$

This transformation leads to a distribution of the largest value which approaches (1) much more quickly than the linear transformation used by Fisher and Tippett, which may be written $x = v + y/v$. When v is large, x^2 approaches v^2. The expected values and the variance of x^2 are given in terms of v^2. In addition, the numerical values of the first few reduced cumulants are presented.

All formulae are very useful because they contain the error terms not stated by Fisher and Gumbel [the reviewer, *Statistics of extremes*, Columbia Univ. Press, New York, 1958; MR **20** #2826]. The transformation (3) is used also to obtain the asymptotic distribution of the mth largest normal value in a sample of size n. The asymptotic density function

$$\varphi(y) = \Gamma^{-1}(m)\exp[-my - e^{-y}]$$

holds for $m \leqq 3$ and $n \leqq 1000$ with an error of 0.01. In a sample of size 10^6, the rank may be $\leqq 50$. For the normal distribution, this approximation is more accurate and simpler than the reviewer's general expression for the mth extreme value for an initial distribution of exponential type. From the moment generating function $M(t) = \Gamma(m-t)/\Gamma(m)$ the authors derive the exact values of the first four cumulants and show how they approach the normal values as n increases. The article ends on the note: "It was worthwhile to prove that the Fisher and Tippetts distribution (1) is a good deal more useful than they believed". *E. J. Gumbel* (New York)

Borges, Rudolf; Pfanzagl, Johann 3498
A characterization of the one parameter exponential family of distributions by monotonicity of likelihood ratios.
Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 111–117 (1963).

Let $\mathfrak{P}$ be a set of probability measures P_θ and $P_0 \notin \mathfrak{P}$. Let P_0 and each P_θ be mutually absolutely continuous. If the family $\mathfrak{P}^{(n)}$ of n-fold product measures of $\mathfrak{P}$ has monotone likelihood ratios with respect to the product measure $P_0{}^{(n)}$ for each n, then $\mathfrak{P}$ is an exponential family.

The densities p_θ with respect to P_0 are given for each θ by $c(\theta)h \exp(a(\theta)g)$, P_0 a.e., where h and g do not depend on θ and where $a(\theta) \geqq 0$. This theorem is, of course, closely related to the theory of sufficient statistics (see also E. B. Dynkin [Uspehi Mat. Nauk **6** (1951), no. 1 (41), 68–90; MR **12**, 839]). *L. Schmetterer* (Vienna)

Blum, Marvin 3499

On the central limit theorem for correlated random variables.

Proc. IEEE **52** (1964), 308–309.

The author proves two central limit theorems for dependent random variables. The first is for m-dependent random variables, and the second for a sequence of random variables which are almost m-dependent for m sufficiently large. There are several incorrect statements in this paper. It should also be pointed out that the results contained in this paper have been in the literature for at least ten years. *J. R. Blum* (Albuquerque, N.M.)

Kakeshita, Shin'ichi 3500

Generalized efficient estimates and its attainable parametric functions.

Bull. Math. Statist. **10** (1961/62), *no.* 3–4, 5–15.

It is well known that A. Bhattacharyya [Sankhyā **8** (1946), 1–14; MR **8**, 524] has generalized the inequality of Cramér and Rao by using differentiability assumptions up to the order $N \geqq 1$. The case $N = 1$ gives the result of Cramér and Rao. An estimate for which equality holds in Bhattacharyya's inequality is called N-efficient. The N-efficient estimates are characterized. Assuming that the distribution of a population is of exponential type, the author characterizes also the functions of the parameter which are estimable by N-efficient estimates.

L. Schmetterer (Vienna)

Geary, R. C. 3501

Statistical efficiency of different methods of estimation of coefficients in a very simple system of equations.

Proc. Roy. Irish Acad. Sect. A **61** (1961), 67–76.

The author is concerned with the comparison of asymptotic efficiencies of different estimators. In particular, he considers the methods of estimation known as maximum likelihood, double least squares, and reduced form. He applies these methods to a particular system of structural equations, which he chooses to be an extremely simple recursive system of linear equations. (He declines to treat limited information since it produces the identical estimates as maximum likelihood on his chosen illustrative system.) He compares pairs of estimators by examining the differences of their asymptotic variances.

The results depend on numerous parameters so that a general comparison is not possible, but observations are made with regard to special circumstances favorable to particular estimators. The superiority of maximum likelihood is, however, apparent. The author asserts the need for further investigation of these questions.

Hanan Rubin (Melville, N.Y.)

Ross, John; Weitzman, R. A. 3502

The twenty-seven per cent rule.

Ann. Math. Statist. **35** (1964), 214–221.

Given a sample of size n from a bivariate normal distribution with zero means, unit variances, and correlation ρ, it is desired to estimate ρ on the basis of the numbers of observations n_1, n_2, n_3 and n_4 falling into the four regions $[x>h, y>0]$, $[x<-h, y>0]$, $[x>h, y<0]$ and $[x<-h, y<0]$. Mosteller [same Ann. **17** (1946), 377–408; MR **8**, 477] and others have shown that when $\rho = 0$, the maximum likelihood estimate $\tilde{\rho}$ of ρ based on n_1, n_2, n_3, n_4 has minimum variance when h is chosen so that $P\{X>h\} \cong .27$. This paper investigates the asymptotic variance of $\tilde{\rho}$ for arbitrary values of ρ. It is seen that the optimal choice of h does not vary much, provided $|\rho| < .7$, say. If, however, the cost of an observation stems only from observing the Y variate, so that the cost of the above procedure is proportional to $n_1+n_2+n_3+n_4$ (by not observing Y when $|X|<h$), then it is preferable to choose larger values of h (the authors suggest $P\{X>h\} = .10$ or $.05$) and the resulting procedure is more efficient than the customary one based on the product moment of a completely random sample. Tables are included.

T. S. Ferguson (Los Angeles, Calif.)

Chakraborty, P. N. 3503

On a method of estimating birth and death rates from several agencies.

Calcutta Statist. Assoc. Bull. **12** (1963), 106–112.

Let k agencies independently record "events", e.g., births or deaths in a given area. The ith agency is assumed to have a probability p_i of recording correctly each "event", and a multinomial model is used: the "events" represent independent trials with probability $p_1p_2\cdots p_k$ that all will record correctly, etc. An estimator of the number of "events" is proposed, generalizing that proposed for $k=2$ by Chandra Sekar and Deming [J. Amer. Statist. Assoc. **44** (1949), 101–115]: $\hat{N} = (\prod_{i=1}^{k} n_i/C_{12\cdots k})^{1/(k-1)}$, where n_i is the number of events recorded by the ith agency and $C_{12\cdots k}$ is the number of events, each of which was recorded by all agencies. The author observes $\hat{N}$ is consistent, computes an asymptotic formula for the variance of $\hat{N}$ when $k=3$, and finds that, for $k=2$, $\hat{N}$ coincides asymptotically with the maximum likelihood estimator. Results of a sampling experiment using random numbers are reported. *H. D. Brunk* (Columbia, Mo.)

Hendricks, Walter A. 3504

Estimation of the probability that an observation will fall into a specified class.

J. Amer. Statist. Assoc. **59** (1964), 225–232.

Author's summary: "When observations in a random sample of n are classified into k categories with n_i falling into a given class, the usual estimate of the corresponding probability is n_i/n. When a priori information about the distribution of the probabilities is available, more precise estimates can be derived from data in any one sample of n. When the a priori distribution is not specified completely, but its general form can be inferred, the parameters of that distribution can be estimated from the average of the n_i/n and their estimated sampling variances. The computations are analogous to those that arise in regression theory with the bivariate normal frequency distribution when $Y = X + e$ and the expected value of X for a given Y is estimated from the regression of X on Y.

The parameters of the distribution of X have to be estimated from the observed distribution of Y and the sampling errors in the individual values of those observations."
M. Atiqullah (Dacca)

van Eeden, Constance 3505
Note on the consistency of some distribution-free tests for dispersion.
J. Amer. Statist. Assoc. **59** (1964), 105–119.
Two-sample rank tests for dispersion proposed by (1) Sukhatme, (2) Ansari and Bradley, and (3) Mood are studied in their two-sided forms under the condition that both random variables, X and Y, have median zero. A necessary and sufficient condition for consistency is given in terms of a parameter D which is the expected value of the (suitably centered and normed) test statistic; if $D=0$, the test is not consistent, and if $D\neq 0$, its sign determines whether the test will judge X or Y to have the greater dispersion. For each of the tests it is shown that if neither X nor Y has a symmetric distribution, then it can happen that D defined for (X, Y) and D defined for $(-X, Y)$ may have opposite signs. Since X and $-X$ have equal dispersion, this is an unsatisfactory property. The paper concludes with some remarks about tied observations.
L. E. Moses (Stanford, Calif.)

Kokan, A. R. 3506
A note on the stability of the estimates of standard errors of the ordinary mean estimate and the ratio estimate.
Calcutta Statist. Assoc. Bull. **12** (1963), 149–158.
The object is to compare variability of the "usual statistics" used to estimate variances of estimates of a population mean. The estimates of a mean considered are the sample mean and "ratio estimate", respectively. The comparison is made for several continuous populations (bivariate normal and lognormal) and for some numerical data.
M. Dwass (Evanston, Ill.)

Kokan, A. R. 3507
Optimum allocation in multivariate surveys.
J. Roy. Statist. Soc. Ser. A **126** (1963), 557–565.
This paper discusses the problem of optimum allocation in sample surveys when several characters are under study. The author states, p. 558, that the purpose of his paper is "to develop a particular optimality criterion" and to apply it to various sampling procedures.

The criterion presented is essentially the following: minimize the cost of the survey subject to the condition that the variance V_j, $j=1,\cdots,p$, of the jth character does not exceed a positive quantity v_j.

This criterion was in fact suggested by the reviewer in 1949, as stated in his book [*Sampling in Sweden*, p. 200, Almqvist & Wiksell, Stockholm, 1957; MR **23** #A2267]. The author develops, in a very useful way, the mathematics of the non-linear programming methods which are the natural tools to apply when adopting the particular optimality criterion under discussion. Furthermore, he presents some realistic applications, which bear out very clearly the usefulness of the approach.
T. Dalenius (Stockholm)

Golosov, Ju. I.; Tempel'man, A. A. 3508
The likelihood ratio for the hypothesis about the trend in certain Gaussian processes. (Russian)
Dokl. Akad. Nauk SSSR **153** (1963), 1242–1244.
Let $\xi(t)$, $t\in[a,b]$ be a Gaussian process with zero mean values, and let $m(t)$ be a fixed function. Let P_0 and P_1 be probability measures corresponding to $\xi(t)$ and $\xi(t)+m(t)$, respectively. The authors investigate conditions concerning $m(t)$ under which $P_1\ll P_0$, and if so, explicit formulas for dP_1/dP_0. They solve the problem for three cases: (I) $\xi(t)$ is a wide-sense Markovian process, (II) $\xi(t)$ has independent increments and (III) $\xi(t)$ is stationary and $-\infty<t<\infty$.
J. Hájek (Prague)

Turner, Malcolm E.; Monroe, Robert J.; Homer, Louis D. 3509
Generalized kinetic regression analysis: Hypergeometric kinetics.
Biometrics **19** (1963), 406–428.
Matrix hypergeometric functions are applied to the simultaneous regression of several variables with respect to time, assuming that observations corresponding to different times are independent.
C. Villegas (Montevideo)

Glasser, Gerald J. 3510
Relationships between the mean difference and other measures of variation.
Metron **21** (1961), 176–180.
The relationship between Gini's coefficient of mean difference Δ and other measures of dispersion is considered. Distribution-free inequalities are developed for the ratios of Δ to the other measures of dispersion, and the corresponding exact values are listed for several population models.
C. Villegas (Montevideo)

Olkin, Ingram; Rubin, Herman 3511
Multivariate beta distributions and independence properties of the Wishart distribution.
Ann. Math. Statist. **35** (1964), 261–269.
The following result is well known for the univariate case. Let X and Y be two independent random variables each of which is distributed as a chi-square. Then the random variables X/Y and $X/(X+Y)$ have, respectively, F- and beta distributions.

In the present paper the authors obtain some interesting generalizations of these distributions in the multivariate case. Some problems connected with the independence of statistics in the multivariate case are also studied.
R. G. Laha (Washington, D.C.)

Uranisi, Hisao 3512
Symmetric designs for exploring response surfaces.
Bull. Math. Statist. **10** (1961/62), *no.* 3–4, 17–30.
The author considers the problem of obtaining experimental plans for response surfaces that can be approximated by a second-degree polynomial in k variables. The plan consists of N treatment combinations $x_1',\cdots,x_N'$, where $x_u'=(x_{1u},x_{2u},\cdots,x_{ku})$ and $-1\leq x_{iu}\leq 1$. A k-dimensional design of order 2 for which the following

condition is satisfied is defined to be a symmetric design of order 2: For any integers $\alpha_i \geqq 0$ such that $\sum_{i=1}^{k} \alpha_i \leqq 4$,

$$N^{-1} \sum_{u=1}^{N} x_{1u}^{\alpha_1} x_{2u}^{\alpha_2} \cdots x_{ku}^{\alpha_k} = \lambda(\alpha_1, \alpha_2, \cdots, \alpha_k), \quad \text{if all the } \alpha_i \text{ are even}, \qquad = 0, \text{ otherwise},$$

where

$$\lambda(2, 0, 0, \cdots, 0) = \lambda(0, 2, 0, \cdots, 0) = \cdots = \lambda(0, 0, \cdots, 0, 2) = \lambda_2,$$
$$\lambda(2, 2, 0, \cdots, 0) = \lambda(2, 0, 2, 0, \cdots, 0) = \cdots = \lambda(0, 0, \cdots, 0, 2, 2) = \lambda_3,$$
$$\lambda(4, 0, 0, \cdots, 0) = \lambda(0, 4, 0, \cdots, 0) = \cdots = \lambda(0, 0, \cdots, 0, 4) = \lambda_4.$$

It is shown that, in order for all the coefficients of a symmetric design of order 2 to be estimable, $\lambda_2 \geqq \lambda_4 > \lambda_3 > 0$ and $\lambda_4 + (k-1)\lambda_3 - k\lambda_2^2 > 0$. It is also shown that the minimum value of the variance function of a symmetric design of order 2 is attained when $\lambda_4 = \lambda_2$ and hence that $x_{iu} = +1, -1$, or 0 for all i and u. Two theorems are presented regarding the min-max value of the variance function. The author then deduces the second-order symmetric design which minimizes the expected value of the variance function, assuming the uniform distribution over the sphere $|t| = \rho$. A theorem concerning the alias matrix and the bias due to third-degree terms that might be included in the true response surface is then presented. The author concludes the paper with a comparison of two three-factor experimental plans with $N = 15$.

S. Addelman (Durham, N.C.)

Gupta, Shanti S.; Sobel, Milton 3513

On selecting a subset containing the population with the smallest variance.

Biometrika **49** (1962), 495–507.

The paper describes a multiple decision approach to the problem of selecting a subset, from k given normal populations, which includes the "best" population, where "best" is defined as the population with the smallest population variance. The population variances are assumed to be unknown and the means may be either known or unknown. A procedure R is defined which selects a subset which is never empty, small in size, and yet large enough to guarantee with pre-assigned probability that it includes the best population. It is shown that the infimum of the probability of correct selection using R, for the case where the k sample variances have a common number of degrees of freedom, is identical with the probability integral of the ratio of the minimum of $(k-1)$ independent chi-squares to another independent chi-square. [The distribution theory is given in the paper below (#3514).] Two generalizations are considered: unequal degrees of freedom and selection of more than one best population.

D. Teichroew (Stanford, Calif.)

Gupta, Shanti S.; Sobel, Milton 3514

On the smallest of several correlated F statistics.

Biometrika **49** (1962), 509–523.

This paper gives the distribution theory needed for the application given in the paper above [#3513]. Exact expressions, as well as approximations and bounds, for computing the probability integral, percentage points and moments are derived for the smallest of p correlated F statistics, each of which has the same chi-square in the denominator and such that all $(p+1)$ chi-squares are independent.

D. Teichroew (Stanford, Calif.)

Kulldorff, G. 3515

Some problems of optimum allocation for sampling on two occasions. (French summary)

Rev. Inst. Internat. Statist. **31** (1963), 24–57.

The author solves completely some interesting problems about optimum allocation of observations to be made on two successive occasions from a changing population.

L. Törnqvist (Helsinki)

Doss, S. A. D. C. 3516

On the efficiency of BAN estimates of the parameters of normal populations based on truncated samples.

Calcutta Statist. Assoc. Bull. **12** (1963), 159–167.

Let $X(\mu, \sigma^2, a)$ denote a random variable following a normal distribution with mean μ and variance σ^2, subject to the condition $X(\mu, \sigma^2, a) > a$, where a is known. A number n of independent observations on $X(\mu, \sigma^2, a)$ are to be used to obtain: (i) $\hat{\mu}(a)$, a BAN estimate of μ, where σ^2 is known, and (ii) $\hat{\sigma}^2(a)$, a BAN estimate of σ^2, where μ is known. It is shown that the asymptotic variance of $\hat{\mu}(-\infty)$ is always less than that of $\hat{\mu}(a)$ for $a > -\infty$. On the other hand, the asymptotic variance of $\hat{\sigma}^2(-\infty)$ is less or greater than that of $\hat{\sigma}^2(a)$ depending on whether $a < \mu$ or $a > \mu$.

J. Neyman (Berkeley, Calif.)

Stange, K. 3517

Vergleich der Genauigkeit von zwei Verfahren der Probenahme bei Verkehrszählungen.

Metrika **6** (1963), 114–132.

It has been known for quite a time that, in stratified sampling for estimating a proportion, with equal cost per sample unit in all strata, the relative reduction in variance when optimum allocation is used instead of proportional allocation is slight, even if the stratum proportions vary considerably [cf. W. G. Cochran, *Sampling techniques*, p. 92, Wiley, New York, 1953; MR **14**, 887]. In the present paper, which deals with stratified road traffic sampling for the estimation of the total number of vehicles which are travelling between two given districts, the strata being the roads connecting these districts, proportional allocation is shown to be only slightly better than equal allocation.

D. M. Sandelius (Åmål)

Theil, H.; Boot, J. C. G. 3518

The final form of econometric equation systems. (French summary)

Rev. Inst. Internat. Statist. **30** (1962), 136–152.

Given a system of simultaneous stochastic difference equations, the change (shock) of an exogenous variable at one period leads to subsequent changes in the endogenous variables. In several illustrations using Klein's Model I, effects which are oscillatory and damped in time are obtained. The approach is generalized for multiple lags in the structural equations and involves the reduced and

final forms of the equations. Of central importance in the study of the effects is a certain characteristic root. The asymptotic variance of the estimate of this root is derived.

H. Chernoff (Stanford, Calif.)

D'Ortenzio, Remo J. 3519

Introductory statistics and sampling concepts applied to radar evaluation.

RCA Rev. **25** (1964), 116–147.

Author's summary: "The analysis of any type of digital data requires a working knowledge of statistics and sampling theory. This paper presents some fundamentals that are particularly useful in analyzing radar data. Included are definitions, least-mean-squares curve-fitting equations, sampling and bandwidth considerations, data smoothing, and basic power spectrum concepts. Portions of the contents are also applicable to many nonradar situations."

Godart, O. 3520

Sur la statistique du vent.

Ann. Soc. Sci. Bruxelles Sér. I **75** (1961), 128–147.

After a presentation of the practical problem of finding the distribution of wind, the author compares this distribution with the theoretical distribution as given by Brooks, that is, a bivariate normal distribution where the two variables have the same variance and are independent. After a χ^2-analysis he finds highly significant deviation from this theoretical law.

At high levels the friction plays a minor role and the wind can fairly well be determined as the geostrophic wind, defined by a direct balance between the pressure gradient and Coriolis acceleration. Thereafter, the deviation of the actual wind from this theoretical wind is considered, and for levels of 300-800 meters the distribution of this deviation is approximated by a bivariate normal distribution of the above-mentioned kind. For the surface winds, however, higher-order terms must be included and the distribution is described by a Gram-Charlier series in two dimensions.

E. Lyttkens (Uppsala)

NUMERICAL METHODS

See also 3048, 3148, 3218, 3251, 3303, 3555.

★Computational mathematics and technology 3521
[Вычислительная математика и техника].

Proceedings of aspirants in the Cybernetics Institute of the Academy of Sciences of the Ukrainian SSR.

Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1962. 179 *pp.* 0.49 *r.*

Sweeney, Dura W. 3522

On the computation of Euler's constant.

Math. Comp. **17** (1963), 170–178.

Euler's constant, γ, has been computed to 3566 decimal places by means of an asymptotic expansion derived from the exponential integral, viz.,

$$\gamma \cong x - \ln x - \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} - \cdots - \frac{e^{-x}}{x}\left(1 - \frac{1}{x} + \frac{2!}{x^2} - \cdots\right).$$

As a byproduct of this computation, ln 2 has been evaluated to 3683 decimal places. These calculations required about one hour of machine time on an IBM 7094.

T. Erber (Brussels)

Jagerman, David L. 3523

The autocorrelation function of a sequence uniformly distributed modulo 1.

Ann. Math. Statist. **34** (1963), 1243–1252.

Let $\rho(x) = (1/2) - \{x\}$ and let x_j $(j = 1(1)\infty)$ be a sequence of real numbers. For the integer τ the autocorrelation function of the sequence x_j is taken to be

$$\psi(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \rho(x_j)\rho(x_{j+\tau}).$$

The author proves that for the sequence $x_j = \alpha j^2$, where α is irrational, $\psi(\tau)$ vanishes for all positive integers τ and suggests that $\{\alpha j^2\}$ would make a suitable random number generator. The simpler sequence $x_j = \alpha j$, on the other hand, is less suitable since its autocorrelation function does not vanish identically. In fact, $\psi(\tau) = (1/12) - \int_0^{\alpha\tau} \rho(u)\,du$ so that $-1/24 < \psi(\tau) < 1/12$. The sequence $x_j = j^\sigma$ $(0 < \sigma < 1)$, like the two previous examples, is uniformly distributed mod 1. Nevertheless, for this sequence $\psi(\tau) = 1/12$ for $\tau \geqq 0$.

{For practical applications to digital computers, the reviewer wonders whether it might be well to point out that irrational numbers do not exist. For a machine with 35 binary digits in its word the irrational $\alpha = (1/2) + 2^{-50}\sqrt{2}$ would be taken as $\alpha = 1/2$ and then would give highly correlated results for $x_j = \alpha j$ or even αj^2. In practice α would be of the form $N\cdot 2^{-35}$, where N is an integer. The sequence $\{\alpha j\}$ would then be formed by repeated additions of N without attention to overflow. The sequence $\{\alpha j^2\}$ would be given by $x_j \equiv 2x_{j-1} - x_{j-2} + 2N \pmod{2^{35}}$. Similar schemes of Fibonacci type, though inexpensive, have not proved satisfactory.}

D. H. Lehmer (Berkeley, Calif.)

Ulanovskiĭ, V. P. [Улановский, В. П.]; Hovanskiĭ, G. S. [Хованский, Г. С.] 3524

★The interpolation of tabulated functions of several variables by numerical and nomographic representation [Интерполирование табличных функций многих переменных средствами численного и номографического представления].

Vyčisl. Centr Akad. Nauk SSSR, Moscow, 1963. 75 *pp.* 0.24 *r.*

This small book discusses sequential linear interpolation of functions of two and more variables which are tabulated at points of a rectangular mesh. For functions of three variables this involves an interpolation formula of the form

$$f(x, y, z) \simeq Axyz + Bxy + Cyz + Dxz + Ex + Fy + Gz + H$$

for a region $x_i \leqq x \leqq x_{i+1}$, $y_j \leqq y \leqq y_{j+1}$, $z_k \leqq z \leqq z_{k+1}$ which has vertices at points for which the function is tabulated. The first of the three chapters derives the basic formulas and gives a geometrical discussion of them. The second and third chapters discuss nomograms and how they can be used to carry out the interpolation. The methods described are mostly of interest for hand calculations.

A. H. Stroud (Lawrence, Kans.)

Meinardus, Günter 3525

Über eine Verallgemeinerung einer Ungleichung von L. V. Kantorowitsch.

Numer. Math. **5** (1963), 14–23.

Es sei $\mathfrak{B}$ ein Banach-Raum und A ein auf $\mathfrak{B}$ erklärter linearer beschränkter Operator, der eine auf $\mathfrak{B}$ definierte Inverse besitzt. Bei festem y sei x^* die Lösung der Gleichung $Ax=y$. Der Verfasser betrachtet Iterationsverfahren der Form

$$x_{n+1} = x_n + \sum_{\nu=1}^{p} \alpha_{\nu,n} A^{\nu-1}(Ax_n - y)$$

zur Berechnung von x^*. Z.B. sind darin bekannte Gradientenverfahren enthalten [siehe L. V. Kantorovič, Uspehi Mat. Nauk **3** (1948), no. 6 (28), 89–185; MR **10**, 380; M. Š. Birman, ibid. **5** (1950), no. 3 (37), 152–155; MR **12**, 32; B. A. Samokiš, ibid. **12** (1957), no. 1 (73), 238–240; MR **19**, 322].

Der Verfasser nennt verschiedene Möglichkeiten zur Wahl der $\alpha_{\nu,n}$ und gibt für diese Fehlerabschätzungen der Art

$$\|x_{n+1} - x^*\| \leqq L_p(A)[1 - L_p(A)]^{-1}\|x_{n+1} - x_n\| \cdot C$$

an (mit $C=$ const., z.B. $C=1$).

Ist A ein positiv-definiter selbstadjungierter Operator in einem Hilbert-Raum, so kann man $L_p(A)$ durch

$$H_p(A) = \mathrm{Max}\{|P_p(\lambda)| : \lambda \in \sigma(A)\}$$

abschätzen. Dabei ist $\sigma(A)$ das Spektrum von A, und $P_p(\lambda)$ bedeutet dasjenige Polynom vom Grade $\leqq p$ mit $P_p(0)=1$, welches auf $\sigma(A)$ die beste Tschebyscheff-Approximation der Nullfunktion darstellt. Ist $\sigma(A) \subset [m, M]$, so gilt

$$(1) \qquad H_p(A) \leqq \left[T_p\left(\frac{M+m}{M-m}\right)\right]^{-1} < 1$$

mit dem p-ten Tschebyscheffschen Polynom T_p.

Als Verallgemeinerung einer Ungleichung von Kantorovič beweist der Verfasser

$$(2) \quad 0 \leqq D_p(x, Ax, \cdots, A^p x) \leqq H_p^2(A) \cdot D_{p-1}(Ax, A^2x, \cdots, A^p x)\|x\|^2,$$

wobei D_p, D_{p-1} Gramsche Determinanten bedeuten. (Die Ungleichung von Kantorovič erhält man hieraus, wenn man im Fall $p=1$ (1) benutzt.) Aus (1) und (2) kann man eine Abschätzung von M/m erhalten. Beim Gradientenverfahren ist $C^2 = M/m$ in der Fehlerabschätzung.

Zwei numerische Beispiele erläutern die Methode zur Fehlerabschätzung. *J. Schröder* (Cologne)

Šteĭnberg, A. S. 3526

On the effective construction of best trigonometric approximations. (Russian. French summary)

Ukrain. Mat. Ž. **15** (1963), 173–184.

En partant d'une note de G. Hornecker [C. R. Acad. Sci. Paris **246** (1958), 43–46; MR **20** #2833] relative à la construction effective du polynome de meilleure approximation d'ordre n, au sens de Tchebycheff, d'une fonction bornée et continue sur un segment fini, l'auteur généralise ce procédé aux fonctions développable en série trigonométriques assez rapidement convergente. Le procédé est illustré par deux exemples. *M. Tomić* (Belgrade)

Bellman, Richard; Kalaba, Robert; Kotkin, Bella 3527

Polynomial approximation—a new computational technique in dynamic programming: Allocation processes.

Math. Comp. **17** (1963), 155–161.

From the authors' introduction: "In this series of papers, we wish to present a number of applications of a new, simple and quite powerful method, that of polynomial approximation. We shall begin with a discussion of the allocation process posed in the foregoing paragraphs [maximize $\sum_i^N g_i(x_i)$ when $\sum_i^N x_i = x$, $x_i \geqq 0$ and maximize $\sum_i^N g_i(x_i, y_i)$ when $\sum_i^N x_i = x$, $\sum_i^N y_i = y$, $x_i, y_i \geqq 0$] and continue, in subsequent papers, with a treatment of realistic trajectory and guidance processes. In a separate series of papers we shall apply this fundamental attack upon dimensionality to the solution of a number of the equations of mathematical physics." *M. L. Balinski* (Princeton, N.J.)

Remez, E. Ja. 3528

On the construction of Chebyshev approximations of linear fractional and certain related types. (Russian. French summary)

Ukrain. Mat. Ž. **15** (1963), 400–411.

This paper discusses a practical method for carrying out the method of sequential trials for the problem of finding $z^0 \equiv (z_1^0, \cdots, z_n^0)$ to minimize

$$\max_{i=1,\cdots,N} |\psi_i(z^0) - f_i|,$$

where the f_i are given quantities and

$$\psi_i(z) = \mu_i \omega[R_i(z)], \quad i = 1, \cdots, N,$$

$$R_i(z) \equiv \frac{k_{i1}z_1 + \cdots + k_{ip}z_p}{k_{i,p+1}z_{p+1} + \cdots + k_{in}z_n},$$

where $\mu_i > 0$ and $\omega(u)$ is a strictly monotone and continuous function of u. The problem is restricted to a set where the denominators of the R_i are positive. *A. H. Stroud* (Lawrence, Kans.)

Gloden, R.-F. 3529a

★Recherche de la meilleure approximation pour l'évaluation d'une fonction donnée. Approximation polynomiale au sens de Tchebycheff, développement asymptotique et application aux fonctions de Bessel.

EUR 282.f.

Communauté Européenne de l'Énergie Atomique—EURATOM, Centre Commun de Recherche Nucléaire, Établissement d'Ispra—Italie, Centre de traitement des informations scientifiques—CETIS, Brussels, 1963. 41 *pp.* *FB.* 60.00.

Gloden, R.-F. 3529b

★Approximation des fonctions de Bessel. Optimisation des programmes correspondants.

EUR 282.f. addendum.

Communauté Européenne de l'Énergie Atomique—EURATOM, Centre Commun de Recherche Nucléaire, Établissement d'Ispra—Italie, Centre de traitement de l'information scientifique—CETIS, Brussels, 1964. 14 *pp.* *FB.* 40.00.

These two papers apply the Remes algorithm to the computation of best approximations for the Bessel functions. The coefficients of 12-decimal place approximations to several Bessel functions are given. There is a detailed analysis of the problem of constructing a computer subroutine for Bessel functions valid for the interval $[0, \infty)$.
J. R. Rice (Warren, Mich.)

Liht, M. K. 3530
Conditionality in the problem of the minimum of a quadratic functional. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 979–987.
The author deals with the problem of finding the minimum of a quadratic functional $\|Ax-h\|^2$, where A is, in general, a linear operator from a Banach space to a Hilbert one. In many practical cases this problem reduces to the solution of the Euler-Lagrange equation $A^*Ax-A^*h=0$. The author shows that such replacement gives a deterioration of the conditionality. He suggests use of the biorthogonalization method to get a better conditionality.
M. Altman (Warsaw)

Huzino, Seiiti 3531
On the convergence of some linear stationary iterative processes of second degree.
Mem. Fac. Sci. Kyushu Univ. Ser. A **17** (1963), 202–208.
For solving $Ax=d$, the iterations $x^{(1)}=x^{(0)}$, $x^{(k+1)}=Lx^{(k)}+Mx^{(k-1)}+Nd$, and $\xi^{(k+1)}=Lx^{(k)}+Mx^{(k-1)}+Nd$, $x^{(k+1)}=\xi^{(k+1)}+\omega(\xi^{(k+1)}-x^{(k)})$ are considered, where $L+M+NA=I$. If $\tau(X)$ represents the spectral norm (largest singular value) of X, then the first iteration converges when $\tau(L)+\tau(M)<1$; if, in addition,

$$\tau(M) \leqq \tau(L)(1-\tau(L))/2$$

and $0<\omega<\tau(L)/(2-\tau(L))$, then the second also converges, and more rapidly.
A. S. Householder (Oak Ridge, Tenn.)

La Budde, C. Donald 3532
A new algorithm for diagonalizing a real symmetric matrix.
Math. Comp. **18** (1964), 118–123.
The author describes an iterative method for computing the eigenvalues of a real symmetric matrix using orthogonal similarity transformations. In each step two orthogonal transformations are performed. The matrix of the first is of the form $I-2ww^T$ $(w^Tw=1)$, and w is chosen so as to annihilate elements a_{1i} $(i=3,\cdots,n)$ as in the first step of Householder's method. The second is a plane rotation designed to annihilate a_{12} but at the expense of making the a_{1i} $(i=3,\cdots,n)$ non-zero. It is shown that, with the correct choice of signs, $\sum_{i\neq 1}(a_{1i})^2\to 0$, and a_{11} tends monotonically to an eigenvalue. The remaining eigenvalues are those of the symmetric matrix of order $n-1$ in the bottom right-hand corner.
James H. Wilkinson (Teddington)

Dufour, H. M. 3533
Résolution des systèmes linéaires par la méthode des résidus conjugués.
Bull. Géodésique (N.S.) No. 71 (1964), 65–87.
In the method of conjugate gradients, each correction is the linear combination of previous residuals that minimizes the next residual in a certain norm. For the equations $Nx=k$, with N positive definite, the norm is given by $\|r\|^2=r*N^{-1}r$. It is here proposed to use the norm given by $\|r\|^2=r*r$. Experience with this and related methods is discussed. *A. S. Householder* (Oak Ridge, Tenn.)

Rutishauser, H. 3534
On Jacobi rotation patterns.
Proc. Sympos. Appl. Math., Vol. XV, pp. 219–239. *Amer. Math. Soc., Providence, R.I.*, 1963.
The author describes a number of algorithms based on similarity transformations using (a) plane rotations of Jacobi type, (b) matrices of the form $I-2ww^T$ $(w^Tw=1)$ (Householder). The method of Givens is described first in the orthodox form, and then in a modified form which leads to a convenient method for reducing a large symmetric matrix of order mn to a symmetric band of width $2m+1$ working with blocks of m^2 elements. This is followed by variants of the classical methods of deflation used to establish the Schur canonical form. The case of deflation of band symmetric matrices with preservation of band form is discussed with some reference to numerical stability. Particularly interesting are algorithms for reducing band symmetric matrices to tri-diagonal form without introducing any significant number of elements outside the band. Finally, the QR transformation is discussed and some applications to continued fractions. It is convenient to have all these algorithms assembled together.
James H. Wilkinson (Teddington)

Ul'm, S. 3535
On an interpolation analogue of the gradient method. (Russian. Estonian and German summaries)
Eesti NSV Tead. Akad. Toimetised Füüs.-Mat. Tehn. Tead. Seer. **12** (1963), 238–243.
The author defines an iterative procedure which is similar to a gradient method and is based on an interpolation analogue of the gradient of a functional. He applies his method to approximate solutions of nonlinear and linear operator equations in Hilbert space and gives also sufficient convergence conditions.
M. Altman (Warsaw)

Maloň, Stanislav 3536
On nonlinear numerical iteration processes.
Comment. Math. Univ. Carolinae **3** (1962), *no.* 3, 14–22.
Let Y be a Banach space, F its closed subset and K a contractive operator defined on F. Then the equation $y=Ky$ can be solved numerically by the iterative process $y_{n+1}=Ky_n$ $(n=0,1,2,\cdots)$. The author shows how such an iterative process is realized in digital computation. According to Kantorovič [Uspehi Mat. Nauk **3** (1948), no. 6 (28), 89–185; MR **10**, 380], the author supposes that, in digital computation, the iterative process is carried on by a certain contractive operator $\bar{K}$ in a certain Banach space $\bar{Y}$ isomorphic with Y; $\bar{K}$ is an approximation of K. The author supposes Y is mapped into $\bar{Y}$ by a linear bounded operator φ and shows that, under the natural conditions concerning the domains, the iterative process by $\bar{K}$ in $\bar{Y}$ yields an approximate solution of the original

equation $y=Ky$ if $\|\varphi\|\leq 1$ and $\|\varphi K-\bar{K}\varphi\|$ is small. Next, following the reviewer's paper [J. Sci. Hiroshima Univ. Ser. A **19** (1956), 479–489; MR **19**, 1081], the author considers the effect of round-off errors. Following the reviewer's method, the author obtains a slightly more general result. Last, summing up the above results, the author estimates the quantity $\|\varphi y^*-\tilde{y}_n\|$, where y^* is a solution of $y=Ky$ and $\tilde{y}_n$ is an nth approximation obtained numerically in $\bar{Y}$. From this result, the author refers to the state of numerical convergence in the sense of the reviewer.

No concrete example is illustrated. To the reviewer, it seems more practical to consider $\bar{Y}$ homomorphic with Y and to get the proper error estimate, namely, to estimate the quantity $\sup\|y^*-\varphi^{-1}y_n\|$, where sup is taken over the set $\varphi^{-1}\tilde{y}_n$. Of course, this could be done by a slight modification of the author's theory.

M. Urabe (Madison, Wis.)

Buchanan, Mary Louise 3537

A necessary and sufficient condition for stability of difference schemes for initial value problems.

J. Soc. Indust. Appl. Math. **11** (1963), 919–935.

Let F denote a family of matrices A with n rows and n columns. Essentially the author derives necessary and sufficient algebraic conditions such that all powers A^ν ($\nu=1,2,\cdots;A\in F$) are uniformly bounded. The author transforms every $A\in F$ to triangular form $B=(b_{ij})$ with help of a unitary matrix in such a way that the eigenvalues κ_i appear in a certain order on the main diagonal. Then the interesting fact is that the above-mentioned algebraic conditions can be stated explicitly, namely: $|\kappa_i|\leq 1$; $|b_{ij}|\leq W, j>i$; $|b_{ij}|\leq L(2-(|\kappa_i|+|\kappa_j|)+|\kappa_i-\kappa_j|)$; L, W independent of A.

H. O. Kreiss (New York)

Stadnikova, N. A. 3538

On the character of the convergence in the Ritz method. (Russian)

Vyčisl. Mat. **7** (1961), 187–190.

The author proves that for an arbitrary $\varepsilon>0$ there is an n_0 such that $|u_n-u|\leq\varepsilon\|f\|$ for $n>n_0$ and all f, where u_n is Ritz's approximate solution of the equation $Au=f$ with self-adjoint, positive definite, linear operator defined on a dense set of a Hilbert space and having a completely continuous inverse A^{-1}; $|u|^2=(Au,u)$.

M. Altman (Warsaw)

Ionescu, D. V. 3539

Généralisation d'une formule de dérivation numérique de V. N. Fadeeva.

Ann. Polon. Math. **14** (1963/64), 169–181.

Let $x_1,\cdots,x_n$ be equally spaced points and let $f(x)\in C^{n+1}[x_1,x_2]$. This paper derives a numerical differentiation formula

$$\Delta^{n-1}f(x_1) = A_1f'(x_1)+\cdots+A_nf'(x_n)+R,$$

where

$$\Delta^{n-1}f(x_1) = f(x_n)-C^1_{n-1}f(x_{n-1})+\cdots+(-1)^{n-1}C^{n-1}_{n-1}f(x_1),$$

and where the remainder R is zero if $f(x)$ is a polynomial of degree $\leq n$.

A. H. Stroud (Lawrence, Kans.)

Laurent, Pierre-Jean 3540

Formules de quadrature approchée sur domaines rectangulaires convergentes pour toute fonction intégrable Riemann.

C. R. Acad. Sci. Paris **258** (1964), 798–801.

Integration formulas for approximating the value of the Riemann integral of a function of any finite number of real variables are considered. The author points out that a Richardson extrapolation procedure applied to such formulas yields a succession of approximations which converges rather quickly to the integral for any such Riemann-integrable function.

W. J. Kotzé (Montreal, Que.)

Zaĭdenberg, E. D. 3541

On the third method of statistical linearization of a class of non-linear differential equations. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 195–200.

Author's summary: "The application of the statistical linearization method to the solution of a class of nonlinear differential equations is considered. The use of two known methods of the statistical linearization is shown to yield a solution error equal to 20%. The third method of the statistical linearization providing the solution error equal to $\sim 0.1\%$ is proposed. The accuracy of the method is illustrated with examples."

Konoval'cev, I. V. 3542

On the stability of linear difference equations. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **2** (1962), 983–996.

The author considers initial-value problems for linear homogeneous difference equations with variable coefficients, $u_{m+k}=a_1(m)u_{m+k-1}+\cdots+a_k(m)u_m$, $m=1,2,\cdots$; $u_i=\tilde{u}_i$, $i=1,\cdots,k$, and gives sufficient conditions for stability, that is, for the boundedness of all solutions when $m\to\infty$. The conditions are expressed in terms of the eigenvalues of the difference equation, that is, the roots $\lambda_i^{(m)}$, $i=1,\cdots,k$, of the equation $\lambda^k-a_1(m)\lambda^{k-1}-\cdots-a_k(m)=0$. The results are applied to the numerical solutions of initial-value problems for linear homogeneous ordinary differential equations with variable coefficients; such a differential equation is approximated by a difference equation by replacing derivatives by finite difference quotients corresponding to a mesh width h. In this case sufficient conditions for stability of the difference equation and estimates for its solutions are given which depend on the behaviour of the corresponding eigenvalues for small values of h.

V. Thomée (Göteborg)

Molčanov, I. N. 3543

Methods of solving second-order elliptic equations, economizing the memory of the computer. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 720–729.

The author considers finite-difference methods of solving the boundary-value problem for the elliptic differential equation of the form

$$\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(q\frac{\partial u}{\partial y}\right)-su = f$$

in an arbitrary lattice domain by using the simplest five-point stencil. For the solution of the corresponding

system of equations he suggests simple iterative methods which allow one to memorize simultaneously the values of the required function not in all lattice nodes but in a part of them. These methods are convenient for carrying out the calculations on a computer. *M. Altman* (Warsaw)

Griffin, D. S.; Varga, R. S. 3544
Numerical solution of plane elasticity problems.
J. Soc. Indust. Appl. Math. **11** (1963), 1046–1062.

This paper considers the solution of the biharmonic equation in two dimensions, with conditions specifying the value and normal derivative of the required stress function at all points of a closed curved boundary. Finite-difference equations are derived by the technique of integrating over mesh regions whose boundaries lie midway between the mesh lines which intersect the boundary at vertices of an approximating polygonal boundary. The technique, which for uniform mesh spacings gives the standard 13-point formula, is described in detail for general mesh points and for boundary points, where integration is performed over partial mesh regions. The boundary conditions enter naturally, there is no need to use "fictitious" points, and it is proved that the resulting matrix is symmetric and positive definite.

The equations are solved by a two-line block cyclic Chebyshev semi-iterative method, and good results are obtained with a small number of mesh points, and no great amount of computation, for a typical problem in stress analysis. *L. Fox* (Oxford)

Douglas, J., Jr.; Kellogg, R. B.; Varga, R. S. 3545
Alternating direction iteration methods for n space variables.
Math. Comp. **17** (1963), 279–282.

The authors investigate the convergence of single-parameter alternating-direction iterative methods for linear systems of the form

$$(X_1+X_2+\cdots+X_n)z=f \quad (n\geqq 2),$$

where each matrix X_j is Hermitian and positive-definite. For $n=2$ the alternating direction methods specialize to the methods of Peaceman-Rachford, D'yakonov, and Kellogg. Convergence is proved without commutativity assumptions on the X_j, although the acceleration parameter ρ is assumed to satisfy $\rho>(n-2)b/2$, where b is an upper bound for the largest eigenvalues of any X_j. Because of this restriction on ρ, the authors doubt that the convergence of the iterative procedures, in the case $n>2$, can be very rapid. *M. Lees* (Pasadena, Calif.)

Greenberg, H. J. 3546
Extended initial-value problems and their numerical solution.
Progress in Applied Mechanics, pp. 25–40. *Macmillan, New York*, 1963.

This expository paper presents a novel approach to the construction of multi-step difference methods for the numerical solution of initial-value problems for partial differential equations. A rough description of the method can be given as follows: Let $Lu(x,t)=0$ be a single-step difference approximation to an initial-value problem over a grid G. Assume that u has been computed for the time-lines $t=t_0,t_1,\cdots,t_{q-1}$. A grid $G^*\subset G$ is constructed so that G is obtained from G^* by refinement only in the x-direction. A q-step difference equation $L^*u^*(x,t)=0$ on G^* is determined in such a way that $u^*(x,t_q)=u(x,t_q)$ on G^*. Once G^* and L^* have been determined, it follows that the stability of L^* on G^* is equivalent to the stability of L on G, and the truncation error associated with L^* is expressible in terms of the truncation error associated with L. The method is examined in detail for the one-dimensional heat and wave equations. *M. Lees* (Pasadena, Calif.)

Mysovskih, I. P. 3547
On the construction of cubature formulae for the simplest regions. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 3–14.

This paper gives some new cubature formulas. These are: (1) Formulas for the circle $x^2+y^2\leqq 1$ which have degree $4p-1$ and use $4(p^2-p+1)$ points, $p=1,2,\cdots$. (2) A special formula for the circle of degree 15 using 44 points. (3) Formulas similar to those in (1) for a square. (4) Formulas for a square of degree $4p+1$ using $4p^2+2p+1$ points, $p=1,2,\cdots$. (5) Formulas for an n-dimensional sphere ($n\geqq 3$) of degree 7 using $2n(2n^2-3n+7)/3$ points. *A. H. Stroud* (Lawrence, Kans.)

Strang, Gilbert 3548
Wiener-Hopf difference equations.
J. Math. Mech. **13** (1964), 85–96.

Consider the following initial boundary value problem in the quarter-space $t\geqq 0$, $x\geqq 0$:

$$(*)\quad \partial u/\partial t+\partial u/\partial x=0;\quad u=u_0(x) \text{ for } t=0, x>0;$$
$$u=0 \text{ for } x=0,\ t\geqq 0.$$

Introduce the difference approximation:

$$(**)\quad \sum b_jv(x+jh,(n+1)k,h)=\sum c_jv(x+jh,nk,h);\quad |j|\leqq M;\quad x=h,2h,\cdots;$$
$$v(x,0,h)=u_0(x);\quad v=0 \text{ for } x\leqq 0$$

and let $c(\theta)$, $b(\theta)$, $r(\theta)$ denote the functions $c(\theta)=\sum c_je^{ij\theta}$, $b(\theta)=\sum b_je^{ij\theta}$, $r(\theta)=c(\theta)/b(\theta)$. Then the author proves the following interesting theorem: (**) is stable and has a unique solution if and only if for all real θ: (1) $b(\theta)\neq 0$, (2) $\int_{-\pi}^{\pi} d(\arg b(\theta))=0$, (3) $|r(\theta)|\leqq 1$. The author shows further that a similar result holds for equations (*), (**) with variable coefficients. *H. O. Kreiss* (New York)

Lax, Peter D. 3549
Survey of stability of different schemes for solving initial value problems for hyperbolic equations.
Proc. Sympos. Appl. Math., Vol. XV, pp. 251–258. *Amer. Math. Soc., Providence, R.I.*, 1963.

This is an expository paper on difference approximations for the initial-value problem for partial differential equations. The author announces also some new results concerning the stability of some second-order accurate schemes. The following interesting matrix theorem should especially be mentioned: Let C be a square matrix of order n with $|(Cu,u)|\leqq 1$ for all unit vectors u. Then there is a constant K depending on n only such that $\|C^n\|\leqq K$, $n=0,1,2,\cdots$. *H. O. Kreiss* (New York)

Aziz, A. K.; Hubbard, B. E. 3550

Bounds on the truncation error by finite differences for the Goursat problem.

Math. Comp. **18** (1964), 19–35.

The purpose of this paper is to give maximum-norm estimates for the difference of the solutions of the Goursat problem $u_{xy}+au_x+bu_y+cu=f$, $u(x,0)=\phi(x)$, $u(0,y)=\psi(y)$, $\phi(0)=\psi(0)$, $0\leq x\leq 1$, $0\leq y\leq 1$, and the discrete counterpart obtained by replacing for grid-points $(x,y)=(mh,nh)$, u_{xy} by $h^{-2}[v(x+h,y+h)+v(x,y)-v(x+h,y)-v(x,y+h)]$, u_x by $(2h)^{-1}[v(x+h,y+h)+v(x+h,y)-v(x,y+h)-v(x,y)]$ and similarly for u_y, and a,b,c,u by their averages over the four points (x,y), $(x+h,y)$, $(x,y+h)$, $(x+h,y+h)$, and finally taking f at $(x+\frac{1}{2}h, y+\frac{1}{2}h)$. Essential for the proof is a representation of the solution of the discrete problem by means of a discrete analogue of Riemann's function. *V. Thomée* (Göteborg)

Bramble, James H. 3551

Fourth-order finite difference analogues of the Dirichlet problem for Poisson's equation in three and four dimensions.

Math. Comp. **17** (1963), 217–222.

One method of deriving error bounds for the standard difference approximations to the Dirichlet problem for Poisson's equation makes use of the fact that the associated matrix representing the linear system of difference equations is of positive type. In a previous paper [Numer. Math. **4** (1962), 313–327; MR **26** #7157] the author and B. E. Hubbard showed that the condition of positive type may be replaced by the conditions of interior positivity, strict diagonal dominance, and connectedness of the interior of the mesh. This paper applies the latter criteria to the derivation of error bounds for fourth-order approximations (in three and four dimensions) for which the associated matrix is not of positive type.

M. Lees (Pasadena, Calif.)

Reutter, Fritz 3552

★Untersuchungen über die praktische Verwendbarkeit einiger Verfahren der angewandten Mathematik, insbesondere der graphischen Analysis, sowie Entwicklung weiterer Verfahren für bestimmte Anwendungsaufgaben.

Forschungsberichte des Landes Nordrhein-Westfalen, Nr. 1003.

Westdeutscher Verlag, Cologne-Opladen, 1961. 99 *pp.* *DM* 32.10.

In a number of previous publications the author has been active in applying nomography to functions of a complex variable, the representation of elliptic functions, conformal transformation, etc. [Z. Angew. Math. Mech. **40** (1960), 75–93; MR **22** #5137; ibid. **41** (1961), 54–65; MR **22** #10199; ibid. **40** (1960), 433–448; MR **23** #B604; ibid. **40** (1960), 529–541; MR **23** #B605; *Die nomographische Darstellung von Funktionen einer komplexen Veränderlichen und damit in Zusammenhang stehende Fragen der praktischen Mathematik*, Westdeutscher Verlag, Cologne, 1960; MR **23** #B1682]. This report extends his previous results by representing a class of pseudo-analytic functions (restricted systems of pairs of functions of two variables) nomographically, and by treating the nomographic representation of derivatives and indefinite integrals of elliptic functions. There are examples of applications to a variety of problems in applied mathematics; comparisons are also made with other methods of solution.

J. G. L. Michel (Teddington)

COMPUTING MACHINES

See also 2976, 3521, 3648, 3885.

★Annual Review in Automatic Programming. Vol. 3. 3553

Edited by Richard Goodman, assisted by R. M. Paine, C. Strachey, M. Woodger. A Pergamon Press Book.

The Macmillan Co., New York, 1963. viii + 360 *pp.*

It is difficult to review in a reasonable space a book having such a diverse, yet specialized, subject matter, so that a list of the chapters and their authors is perhaps the best that can be done: The description of computing processes: Some observations on automatic programming and ALGOL 60, by M. Woodger. Generalized ALGOL, by A. van Wijngaarden. On the design of machine independent programming languages, by E. W. Dijkstra. The use of recursive procedures in ALGOL 60, by H. Rutishauser. JOVIAL—A programming language for real-line command systems, by C. J. Shaw. Towards an ALGOL translator, by B. Higman. A multi-pass translation scheme for ALGOL 60, by E. N. Hawkins and D. H. R. Huxtable. The structure and use of the syntax directed compiler, by E. T. Irons. The compiler compiler, by R. A. Brooker, I. R. MacCallum, D. Morris and J. S. Rohl. Progress in some commercial source languages, by A. D'Agapeyeff, H. D. Baecker and B. J. Gibbens. Rapidwrite, by E. Humby. "File processing" in SEAL, by K. W. Clark.

Unfortunately, a remark of one of the authors is all too appropriately applied to much of the material: "All of this sounds most impressive, yet it is really nothing but disguising, by pompous terminology, a triviality as a scientific theory." However, in fairness it must be said that the book represents a good deal of hard thought in a difficult, diffuse field of knowledge and is, in spite of many defects, a definite contribution to its field.

R. W. Hamming (Murray Hill, N.J.)

Antidze, Dž. G. 3554

A device for searching for words in the memory of a machine. (Georgian. Russian summary)

Soobšč. Akad. Nauk Gruzin. SSR **32** (1963), 265–270.

A flow diagram for the machine mentioned in the title (essentially the ZU-I or ZU-II) is given.

GENERAL APPLIED MATHEMATICS

Margenau, Henry; Murphy, George Moseley (Editors) 3555

★The mathematics of physics and chemistry. Volume Two.

D. Van Nostrand Co., Inc., Princeton-Toronto-London-New York, 1964. v + 786 *pp.* $15.00.

Vol. 1 (1943) was reviewed in MR **4**, 268. This monumental

work cannot be considered as a straightforward continuation of the editors' well-known and popular textbook of the same title. While the old book was a systematic survey of various mathematical techniques and used physical examples only as illustrations, the present volume is rather an anthology of special topics in advanced theoretical physics and develops, in general, mathematical concepts and methods alongside the concrete problems treated. By no means are these remarks meant to devaluate the usefulness of the book. On the other hand, as is the case with most anthologies, the style, level of sophistication, and way of approach vary very much from contribution to contribution.

The first chapter (D. Mintzer) is an introduction to the transport theory of gases. The principal topic of the elaborations is the Boltzmann equation, simple methods for its solution, and a few illustrative applications.

Chapter 2 (A. S. Householder) is devoted to numerical analysis, the term being taken to represent the art and science of digital computation. The linear algebraic problem, including the topics of characteristic roots and vectors, is followed by a shorter survey of non-linear equations and systems, and the survey concludes with a discussion of approximations and remainders.

Chapter 3 (E. M. Hofstetter) is a slightly unconventional introduction to the theory of stationary random processes. Among other things, it discusses distribution functions, the theorem of averages, the concept of moments, and the characteristic function. The method employed is that of time averaging rather than ensemble averaging. As examples, the shot noise process and the Gaussian process are discussed, and the survey ends with comments on the harmonic analysis of random processes.

Chapter 4 (R. G. Gallager) is on information theory, the term being used in the sense of Shannon's classical work. The central topic of the discussion is the celebrated coding theorem.

Chapter 5 (T. L. Saaty) is an introduction, mainly through specific examples, to the fascinating topic of operations analysis. It discusses various methods and the structure of operations.

Chapter 6 (N. Minorsky) is on non-linear problems in physics and engineering. It is divided into two main parts: (1) topological methods, and (2) analytical methods. The remainder of the article discusses applications of a few important types of non-linear oscillatory phenomena, and the survey terminates with a short introduction to relaxation oscillations and non-analytic non-linearities.

Chapter 7 (H. Margenau) is essentially an amendment to some topics treated in the old book. The first part of this article is a survey of angular momentum operators, and the modern treatment of connected problems. The second part is a brief review of the transformation theory of quantum mechanics, including an introduction to the density matrix.

Chapter 8 (W. Band) is entitled "The Mathematical Formalism of Quantum Statistics". The first part is an introduction to Hilbert space theory and the Dirac ket and bra formalism of quantum theory, including the many-body problem in terms of creation and annihilation operators and the Fock representation. This discussion is followed by a review of ensembles and the density matrix.

The next three chapters (S. S. Schweber) give an introduction to relativistic quantum theory and quantum field theory. They are based on lectures which the author gave on advanced quantum mechanics at M.I.T. in 1961–62. The treatment puts physical concepts in the foreground and is extremely clear.

The last chapter (J. O. Dimmock and R. G. Wheeler) discusses symmetry properties of magnetic crystals. From the mathematical point of view, the highlight of the paper is a discussion of representations of non-unitary groups. Special attention is paid to the 32 point groups. This chapter is a rather specialized contribution.

P. Roman (Boston, Mass.)

★Interstellar communication. 3556

A collection of reprints and original contributions. Edited by A. G. W. Cameron.

W. A. Benjamin, Inc., New York-Amsterdam, 1963. xiv + 320 *pp.* $8.50.

Once in a while an active research physicist should be permitted to relax for a few days and engage himself in reading material which is peripheral, not only with respect to his own field of research, but also with respect to present-day science in general. If this is the desire of anyone, he could do no better than to skim through the book here reviewed. This is an anthology of 32 papers and essays written by nineteen different specialists—specialists in physics, astrophysics, communication theory, chemistry, biological sciences, etc. Twenty nine of the contributions have been already published in journals. Apart from well-founded reviews concerning well-established facts in many fields, the reader will also find a substantial amount of exciting, yet scientific, speculations concerning planetary systems, possibility of life on them, and suggestions for communicating with intelligent beings within our Galaxy. *P. Roman* (Boston, Mass.)

MECHANICS OF PARTICLES AND SYSTEMS

See also 3209, 3221, 3254, 3355, 3893.

Cattaneo, Carlo 3557

Sulla struttura locale delle equazioni dinamiche di un sistema anolonomo.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 396–402.

The author has introduced in other papers [Nuovo Cimento (10) **10** (1958), 318–337; MR **21** #1889; Ann. Mat. Pura Appl. (4) **48** (1959), 361–386; MR **23** #A2172] a differential operator in Riemann geometry, called "derivazione trasversa", and applies this concept here to the configuration space S of a mechanical system in which metric is defined by the kinetic energy function. A non-holonomic condition (which is supposed to be linear and independent of the time) is given by a vector in S. If one makes use of the introduced notion, the Lagrangian equations for a non-holonomic system have the same form as those for a holonomic system, and if the system is conservative, the same is true for the Hamilton equations.

O. Bottema (Delft)

Sedláček, Z. 3558

Equations of motion of particle in circular accelerator with general field.

Czechoslovak J. Phys. **14** (1964), 14–20.

Author's summary: "Lagrange equations of motion are

derived for a particle in a circular accelerator with arbitrarily spatially variable guiding magnetic field describing the motion of a particle by means of dimensionless deviations of the particle from a circle as the reference curve. The author also derives linearized equations of motion (so-called equations of perturbations used in stability investigations according to the Ljapunov method of the first approximation). The equations are given in closed form and are thus quite exact."

Pisarenko, G. S. 3559
Application of the methods of asymptotic expansions in powers of a small parameter for investigating oscillations of mechanical systems with dissipation of energy in the material. (Russian. English summary)
Applications of the methods of non-linear vibrations to the problems of physics and technology (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. III*, 1961), *pp.* 347–375. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

Author's summary: "A method is given for calculating the oscillations of a mechanical system in the case where energy dissipation in the material of the elastic element is taken into account. Asymptotic expansions in powers of a small parameter are used. The first approximation is shown to be correct within great accuracy when examining a class of weakly nonlinear problems."

Tzénoff, I. [Cenov, I. V.] 3560
Sur le principe de Lagrange. (Russian summary)
C. R. Acad. Bulgare Sci. **16** (1963), 345–348.

L'auteur a donné [mêmes C. R. **16** (1963), 225–228; MR **28** #1804] une formulation des équations de la mécanique dont il ajoute ici une application. Le principe de Lagrange s'exprime ainsi: si l'on passe du mouvement réel à un mouvement voisin, les deux mouvements ayant la même position initiale et finale, et ayant la même énergie complète, alors la variation asynchrone de l'action de Lagrange est nulle. *O. Bottema* (Delft)

Tzénoff, I. [Cenov, I. V.] 3561
Sur le principe de la moindre action. Principe de Maupertuis. (Russian summary)
C. R. Acad. Bulgare Sci. **16** (1963), 461–464.

Suite de l'article précédent [#3560]. L'auteur dérive, partant de ses équations, le principe de la moindre action, qui détermine la trajectoire d'un système matériel indépendamment du temps. *O. Bottema* (Delft)

Nikodem, Hansjörg 3562
Ein elastischer Stoss als lineares Dreikörperproblem.
Acta Phys. Austriaca **17** (1963/64), 136–146.

Three elastic bodies of equal mass, A, B and C, lie in a line; B and C touch, while A approaches from the left with speed v. The usual assumption is that, after the collision, A and B remain at rest, touching each other, while C moves off to the right with speed v. This paper demonstrates that this usual solution is wrong, under very general assumptions about the potential of the elastic force between A and B, and between B and C. The usual solution does hold if B and C do not touch each other initially, but are separated far enough so that the collision process becomes a sequence of separate two-body collisions. *J. M. Blatt* (Kensington)

Arkhangelskii, Iu. A. [Arhangel'skiĭ, Ju. A.] 3563
On the algebraic and single-valued integrals in the problem of motion of a rigid body in the Newtonian force field.
Prikl. Mat. Meh. **27** (1963), 697–698 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1059–1062.

Consider the problem of motion of a heavy rigid body about a fixed point in the Newtonian central force field, assuming that the fixed point is at the distance R from the center of attraction. Further assume that R is large in comparison with the dimensions of the body, and expand the force function $U(\alpha, \beta, \gamma)$ in powers of $1/R$. Let $U^{(n)}(\alpha, \beta, \gamma)$ be the expression of U in which all terms of the order higher than n in $1/R$ are truncated. With this $U^{(n)}$ in mind, the approximate equations of motion are written down. For $n=0$ they reduce to those of the conventional problem of motion in a uniform gravitational field.

There is a well-known theorem to the effect that for $n=0$ the equations of motion assume a fourth algebraic integral in those and only those cases (of Euler, Lagrange and Kovalevskaya) in which the general solutions for $p, q, r; \alpha, \beta, \gamma$ are single-valued functions of the time t in the whole complex t-plane. In a previous paper the author [Prikl. Mat. Meh. **26** (1962), 568–570; MR **26** #4524] showed that for $n=1$ a fourth algebraic integral exists in two cases which are analogous to those of Euler and Lagrange, and that a search for other cases with single-valued integrals does not lead to any new cases different from the above two.

In the present paper the author shows that, for $n=2$ and a body for which $U^{(2)}=U^{(2)}(\gamma)$, the existence of a fourth algebraic integral does not imply its single-valuedness. Similarly for $n>2$ and for a body for which $U^{(n)}=U^{(n)}(\gamma)$. Consequently, for $n\geqq 2$ the above theorem does not apply.

{The author's name in items 1 and 8 of the bibliography should be Beletskiĭ, and not Veletskiĭ. In addition, the translation is faulty in saying that $U^{(n)}$ is an expression like U in which all terms of order R^{-n} and higher are truncated.} *E. Leimanis* (Vancouver, B.C.)

Apykhtin, N. G. [Apyhtin, N. G.] 3564
Permanent axes of rotation of a rigid body with a fixed point when the integrals of D. N. Goriachev exist.
Prikl. Mat. Meh. **27** (1963), 894–898 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1361–1369.

In this paper, there are determined permanent axes of rotation of a rigid body under the action of forces for which certain integrals of motion exist. This work is closely related to that of V. V. Rumjancev on the stability of permanent rotations of a heavy rigid body [Prikl. Mat. Meh. **20** (1956), 51–66; MR **19**, 77]. *H. P. Thielman* (Alexandria, Va.)

Bautin, N. N. 3565
The theory of point transformations and the dynamical theory of clocks. (Russian. English summary)
Qualitative methods in the theory of non-linear vibrations (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. II*, 1961), *pp.* 29–54. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

Für eine Anzahl von vereinfachten Antriebsmodellen für

eine Uhr werden allgemeine Überlegungen zur theoretischen Bestimmung der Schwingungsformen angestellt. Zur analytischen Berechnung des Uhrenverhaltens wird das Verfahren der Punkt-Transformation (auch stroboskopische Methode) verwendet; die geometrische Interpretation geschieht wie üblich im Phasenraum.
K. Magnus (Stuttgart)

Grobov, V. A. 3566
Non-linear problems of turbine rotor dynamics. (Russian. English summary)
Applications of the methods of non-linear vibrations to the problems of physics and technology (Proc. Internat. Sympos. Non-linear Vibrations, Vol. III, 1961), *pp.* 106–119. *Izdat. Akad. Nauk Ukrain. SSR, Kiev,* 1963.
Author's summary: "The results of the study of nonstationary vibrations of turbine shafts are given. The author takes into account nonlinear elasticity of supports, gyroscopic effects of disks situated on the shaft, forces of external and internal friction and some other factors which affect rotor dynamics. The problems are solved by means of the asymptotic methods of N. M. Krylov, N. N. Bogoliubov and Yu. A. Mitropolsky which permit finding relatively simple design schemes for determining the amplitudes and phases of stationary and nonstationary vibrations. This permits constructing more realistic schemes of design objects than is usually done. The possibility of using asymptotic methods for the investigation of the stability of the stationary movement of shafts is shown." *C. S. Coleman* (Baltimore, Md.)

Shanks, Daniel 3567
A duality relation in exterior ballistics.
SIAM Rev. **6** (1964), 54–56.
One of the classical idealizations of ballistics is the one-dimensional trajectory of a missile acted upon solely by a drag proportional to the nth power of velocity, the equation $\ddot{s}=-a\dot{s}^n$. The author presents the solutions, $s_n(t)$—the two most famous values of n, 1 and 2, being special cases—and points out the striking similarity of the inverse functions, $t_n(s)$, to the functions $s_{3-n}(t)$. By taking the origin and the units of s and t such that $s_0=t_0=0$, $\dot{s}_0=\ddot{s}_0=1$, he shows that there is indeed the relation $s_n(t)=-t_{3-n}(-s)$. The situation is one of those paradoxes which are due to our not having a common name for the reciprocal of the velocity. If $y=1/v$, the equation $dv/dt=-av^n$ maps into $dy/ds=+ay^{3-n}$.
S. J. Zaroodny (Aberdeen, Md.)

ELASTICITY, PLASTICITY

See also 3267, 3544.

Bressan, Aldo 3568
Cinematica dei sistemi continui in relatività generale.
Ann. Mat. Pura Appl. (4) **62** (1963), 99–148.
Herglotz tried in 1911 to develop a theory of elasticity in (special) relativity. Not much work was done on the subject later, the main difficulty being to carry over into space-time the concept of deformation because no natural state seems to exist relative to which the strain may be defined. Referring to recent work of Synge and Rayner, the present interesting memoir develops a new theory of elasticity in general relativity, which is not easy to summarize, but which may be characterized by the terms axiomatic, abstract (i.e., non-physical) and general (including, for instance, viscoelasticity). The author succeeds in obtaining extensions which are analogous to the classical equations. *O. Bottema* (Delft)

Mindlin, R. D. 3569
Micro-structure in linear elasticity.
Arch. Rational Mech. Anal. **16** (1964), 51–78.
The author deals with a linear theory of the three-dimensional elastic continuum, including the idea of the unit cell. Assuming that a micro-volume is embedded in each material particle, the kinetic and potential energies and variational equations of motion are deduced. Classical Cauchy stresses, relative stresses and double-stresses are introduced in the theory. On the basis of the constitutive law, the displacement equations of motion are given. Micro-vibrations and very long wave-length approximations are considered in detail. In particular, Cosserat's couple-stress theory and Toupin's generalization are refound. A solution is given in the case of the concentrated force for the approximate equations of equilibrium.
P. P. Teodorescu (Bucharest)

Ghez, R.; Piuz, F. 3570
A generalised surface stress.
Phys. Lett. **4** (1963), 275–276.
Using methods of tensor calculus and differential geometry, the authors show how a Gibbs dividing surface, representing the physical boundary, can be associated with surface stresses. The expression for the total elastic energy of any volume V containing a Gibbs dividing surface is derived, and in the isotropic case it is shown to reduce to the well-known expression used in classical thermodynamics. *P. D. S. Verma* (Kharagpur)

Ghez, Richard; Piuz, François 3571
Sur l'énergie superficielle et l'état de contrainte en surface.
C. R. Acad. Sci. Paris **257** (1963), 2795–2796.
This is an appendix to the authors' previous article [#3570]. *P. D. S. Verma* (Kharagpur)

Hill, R. 3572
New derivations of some elastic extremum principles.
Progress in Applied Mechanics, pp. 99–106. *Macmillan, New York,* 1963.
New derivations are presented for the variational principles of Hashin and Shtrikman [J. Mech. Phys. Solids **10** (1962), 335–342; MR **26** #4549]. These principles are applicable to the linear theory of elasticity for inhomogeneous anisotropic materials. It is shown that the results of Hashin and Shtrikman are derivable from the classical principles of minimum potential and complementary energy. The method of derivation shows that the classical inequalities are stronger for a given choice of approximating field. Some practical advantages of the theorems of Hashin and Shtrikman over the classical principles are briefly discussed. *R. L. Fosdick* (Chicago, Ill.)

Förster, W. 3573

Biorthonormierte Funktionen zur Lösung allgemeiner Randwertaufgaben der Elastizitätstheorie.

Ing.-Arch. **33** (1963/64), 162–172.

Author's summary: "Die vorliegende Arbeit hat ihren Ausgangspunkt in den bekannten Trefftzschen Gleichungen für eine allgemeine dritte Randwertaufgabe der Elastizitätstheorie. Da die Koeffizientenmatrix des Systems symmetrisch und nichtsingulär ist, gelingt es, einen Weg zur Berechnung biorthonormaler Ansatzfunktionen aus sonst beliebig gewählten partikulären Integralen der homogenen Grundgleichungen anzugeben. Die Matrix der Trefftzschen Gleichungen geht damit in die Einheitsmatrix über. Unter der Voraussetzung stetiger Berandung der Körperoberflächen, der Existenz der Lösung gewisser Nebenprobleme und gleichmäßiger Konvergenz der Reihen kann bewiesen werden, daß der Trefftzsche Ansatz gegen die Werte der strengen Lösung an der Oberfläche konvergiert."

Biot, M. A. 3574

Continuum theory of stability of an embedded layer in finite elasticity under initial stress.

Quart. J. Mech. Appl. Math. **17** (1964), 17–22.

Author's summary: "The writer's theory of elasticity under initial stress is applied to the problem of buckling of a thick elastic slab embedded in an elastic medium of infinite extent. The initial stressed state of the system is one of homogeneous finite strain, and perfect adherence is assumed at the interface of the slab and the embedding medium. The characteristic equation is solved numerically and the relation between the stability and wave length parameters is plotted for various values of the ratio of the rigidities of the two media. It is shown that for vanishing wave length the buckling degenerates into an interfacial instability in analogy with Stoneley waves. By viscoelastic correspondence the present result is also an exact solution for two viscoelastic media whose operators and initial effective compressive stresses differ only by the same constant factor." *R. L. Fosdick* (Chicago, Ill.)

Singh, Avtar; Puri, Pratap 3575

Stresses in a cylinder rotating with variable angular velocity. (Polish. Russian and English summaries)

Rozprawy Inż. **11** (1963), 449–462.

Authors' summary: "This paper is concerned with the problem of stress distribution in an infinite circular cylinder rotating about its axis with variable angular velocity. The particular cases considered are those when $\bar\omega$, the angular velocity, is proportional to (1) $\cos st$ (2) $\cos^2 st$ and (3) e^{-st}, where s is a given constant and t is the time. In cases (1) and (2) the angular velocity becomes zero first at $t=\pi/2s$, but whereas the direction of rotation is reversed in the former case, the latter preserves the same direction after the lapse of this time. This cycle is then repeated. The solution for case (2) cannot be obtained from case (1) by simple superposition because of the occurrence of $\bar\omega^2$ in the first equation of motion. In case (3) the angular velocity is taken to vanish exponentially with time. The stresses and displacements are found in all cases in closed form. As far as the authors' knowledge is concerned, the problems treated are of new type where the inertia terms in equations of motion are also included."

Popov, G. Ia. [Popov, G. Ja.] 3576

Some properties of classical polynomials and their application to contact problems.

Prikl. Mat. Meh. **27** (1963), 821–832 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1255–1271.

A "contact problem" involves finding the linearized displacements of an elastic half-space when its otherwise free surface is statically loaded in a specified way. The physical motivation for the author's work comes from two-dimensional and axisymmetric contact problems in which the contact area is connected and its width or diameter is $2a$.

The author gives two methods for solving the singular integral equation arising from the above contact problems. One is a Wiener-Hopf technique which reduces the problem to iterated quadratures, and will not be reviewed further. The other is based on the following observation: Let

$$W_\mu{}^\nu(x,y)=\int_0^\infty s^\nu J_\mu(sx)J_\mu(sy)\,ds$$

(J_μ a Bessel function). Let $\omega=(1-\nu)/2$. For any function $\psi(\tau)$ defined on the open unit interval let

$$(L^*_{\lambda\mu\nu}\psi)(t)=a^{1+\nu}\int_0^1\frac{(t\tau)^\lambda W_\mu{}^\nu(at,a\tau)\psi(\tau)}{\tau^{2\lambda-1}(1-\tau^2)^\omega}\,d\tau.$$

The author shows that if $0\leq\nu<1$ and $-1<\mu<\infty$, then the mth eigenfunction ($m=0,1,2,\cdots$) of $L^*_{\lambda\mu\nu}$ is $t^{\lambda+\mu}P_m{}^{(\mu,-\omega)}(1-2t^2)$, where $P_m{}^{(\mu,-\omega)}$ is a Jacobi polynomial. The corresponding eigenvalue is

$$\mu_m=\frac{(-1)^m2^{\nu-1}\pi\Gamma(1+m+\mu-\omega)}{\sin\pi\omega m!\Gamma(1+m+\mu)\Gamma(\omega-m)}.$$

Special and limiting cases of this result exhibit Legendre, Gegenbauer, and Chebyshev polynomials as eigenfunctions of singular integral operators with singularities $|t-\tau|^{-\nu}$ or $\ln|t-\tau|$.

The integral operators of the contact problems have the same singularity as L^*, so the author writes the kernel of the contact problem as the kernel of L^*, plus an infinite sum of simple monomials $t^j\tau^k$. If the sum is truncated after any finite number of terms, the resulting approximation to the contact problem is exactly soluble.

The author observes that his techniques also yield solutions of certain singular integral equations of the second kind. *G. E. Backus* (La Jolla, Calif.)

Iurchenko, S. I. [Jurčenko, S. I.] 3577

The mixed oscillation problem for an infinite plate of unit width.

Prikl. Mat. Meh. **27** (1963), 951–956 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1459–1467.

Approximate descriptions of periodic oscillations are obtained, and the error in the approximations is estimated. *T. R. Kane* (Stanford, Calif.)

Ichino, Ichiro; Takahashi, Hiroshi 3578

Theory of nonsymmetrical bending state for spherical shell.

Bull. JSME **7** (1964), 28–35.

The fundamental differential equation of the title problem is derived in terms of stress function and axial displacement components. Homogeneous solutions are then

introduced assuming that the stresses and displacements are uni-valued. Numerical results are worked out for a hemispherical shell with a tangential load at the apex.

H. D. Conway (Ithaca, N.Y.)

Wainwright, W. L. 3579

On a nonlinear theory of elastic shells. (French, German, Italian and Russian summaries)

Internat. J. Engrg. Sci. **1** (1963), 339–358.

The author derives a series expansion for the strain energy function in powers of certain strain invariants. A further expansion in powers of the coordinate normal to the shell middle surface in conjunction with the use of the Love-Kirchhoff hypothesis allows an integration through the thickness of the shell to be performed explicitly. The strains are assumed to be small, that is, the geometrically linear strain-displacement relations are used, and dependence upon the third strain invariant is depressed. The results are explicit constitutive relations for a thin elastic shell with nonlinear material properties. As examples, the author presents the first-order linear form of his theory (which coincides with the Flugge-Lur'e-Byrne theory) and the additional terms present in the second-order theory. Comparison is made with the theory of Zerna.

J. L. Sanders, Jr. (Cambridge, Mass.)

Krall, Giulio; Caligo, Domenico 3580

Ha influenza la flessione sul λ_{cr} critico di una volta cilindrica?

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 340–344.

Continuando una serie di note sul moltiplicatore critico [stessi Atti (8) **30** (1961), 131–139, 315–322, 421–428, 608–617; ibid. (8) **31** (1961), 9–16; MR **24** #B1365], gli autori dimostrano che, in campo elastico, la sovrapposizione di uno stato flessionale non ha influenza sul moltiplicatore λ_{cr} di un regime estensionale di equilibrio per un'asta, per un arco nel proprio piano o per una volta cilindrica generica (in particolare autoportante). Con questa occasione si fanno qualche osservazioni interessanti sulle caratteristiche della deformazione flessionale.

P. P. Teodorescu (Bucharest)

Einspruch, Norman G.; Manning, Robert J. 3581

Third-order elastic moduli of anisotropic solids.

J. Appl. Phys. **35** (1964), 560–567.

The displacement fields for a crystal subjected to hydrostatic pressure, compression along the X_3-axis, etc., are determined for cubic, hexagonal, tetragonal and rhombic crystals by application of linear elasticity theory. A harmonic disturbance is then superimposed upon the displacement field for hydrostatic pressure (say) and the wave velocity is then computed using the stress strain law appropriate to second-order elasticity theory. This gives an expression for a linear combination of the third-order elastic constants. By varying the initial displacement field and the type of wave motion superimposed, a sufficient number of linear combinations of third-order constants may be obtained so as to enable one to compute each third-order elastic constant. This program is carried out for the high-symmetry hexagonal, high-symmetry tetragonal, low-symmetry cubic and rhombic crystal classes.

G. F. Smith (New Haven, Conn.)

Barenblatt, G. I.; Cherepanov, G. P. [Čerepanov, G. P.] 3582

On the equilibrium and propagation of cracks in an anisotropic medium.

Prikl. Mat. Meh. **25** (1961), 46–55 (*Russian*); *translated as J. Appl. Math. Mech.* **25** (1961), 61–74.

Some problems of equilibrium and propagation of rectilinear cracks in a two-dimensional anisotropic medium are studied, under conditions of plane strain.

B. A. Boley (New York)

Craggs, J. W. 3583

Dislocations as sources of fracture.

J. Mech. Phys. Solids **11** (1963), 249–253.

It is observed that the infinite energy associated with Volterra dislocations can be avoided if it is assumed that every dislocation is accompanied by an incipient crack. The application of a modified form of the Griffith energy criterion then leads to estimates of the fracture strength of a material containing dislocations.

B. A. Boley (New York)

Chou, Y. T. 3584

Planar stress field of a dislocation in an anisotropic plate.

J. Appl. Phys. **34** (1963), 3608–3614.

The elastic field of a straight dislocation lying in the mid-plane of an anisotropic plate is calculated by building up a solution with the required displacement discontinuity from elementary solutions which vary sinusoidally parallel to the plane of the plate and exponentially perpendicular to it. The material is assumed to be orthotropic with two axes in the plane of the plate and the third parallel to the dislocation line. Only the stresses acting across the mid-plane ("planar stress field") are considered, but they are all that are needed in certain calculations (dislocation energies, equilibrium of dislocation arrays, stacking fault energy calculations). The stresses are found in the form of an integral which is evaluated explicitly for a screw dislocation but which can only be calculated numerically for an edge dislocation with its Burgers vector in or perpendicular to the plane of the plate. The degree of departure from the corresponding results for an isotropic medium depends on one (for the screw) or two (for the edge) characteristic ratios of elastic constants. The effect of varying these ratios is discussed qualitatively.

J. D. Eshelby (Birmingham)

Bartlett, C. C. 3585

The vibration and buckling of a circular plate clamped on part of its boundary and simply supported on the remainder.

Quart. J. Mech. Appl. Math. **16** (1963), 431–440.

Using techniques developed in an earlier paper by the author and B. Noble [Appl. Sci. Res. B **9** (1963), 403–419], the author computes bounds for the lowest eigenvalues in two problems associated with a circular elastic plate. The plate is assumed to be clamped on part of the boundary and simply supported on the remaining portion. The two problems considered are (a) the buckling problem, and (b) the vibration problem. In each case the problem is

reduced to one of solving a set of dual series equations. Numerical results are given.

L. E. Payne (College Park, Md.)

Chao, C. C.; Pao, Yih-Hsing 3586

On the flexural motions of plates at the cut-off frequency.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 22–24.

Mindlin's equations for flexural motions of plates are used to study the reflection of waves at the free edge of a semi-infinite plate. The case in which the incident wave has a frequency equal to that of the lowest mode of thickness-shear vibration is examined in detail.

T. R. Kane (Stanford, Calif.)

Galustjan, S. B. 3587

Solution of a problem on singular transverse vibrations of a rod of variable cross-section by means of the generalized Green's function. (Russian. Azerbaijani summary)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **1963**, *no.* 5, 3–14.

The eigenvalue problem for the corresponding fourth-order differential equation is considered. The edges of the rod are assumed to be elastically clamped and supported. The problem is reduced to the solution of an infinite system of homogeneous algebraic equations.

M. Sokolowski (Warsaw)

Nadeeva, R. I. 3588

Solution of the problem of torsional impact by the method of successive approximations. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 1, 65–68.

In the differential equations and the boundary conditions non-linear terms appear due to large deflections of the shell; inertia forces acting tangentially to the shell surface are neglected; only symmetric forms of vibrations are considered. Applying a number of simplifications the author arrives at an approximate expression for critical values of the amplitudes of compressive forces.

M. Sokolowski (Warsaw)

Šuležko, L. F. 3589

Investigation of the dynamic stability of a cylindrical shell subject to high-frequency axial periodic forces. (Russian)

Ukrain. Mat. Ž. **15** (1963), 338–343.

In an infinite elastic medium a uniformly distributed torque is instantaneously applied at the surface of an infinite cylindrical cavity; the assumed approximate solution of the torsional wave equation has the form of rational functions. Three consecutive approximations are found and compared with each other.

M. Sokolowski (Warsaw)

Gutin, L. Ya. [Gutin, L. Ja.] 3590

The radiation into an elastic medium from a piston vibrating in an infinite elastic screen.

Akust. Ž. **9** (1963), 314–323 (*Russian*); *translated as Soviet Physics Acoust.* **9** (1964), 256–262.

The author solves the following dynamic boundary-value problem for a half-space $z>0$. The surface $z=0$ is subject to the boundary conditions

$$u_z = e^{i\omega t}, \quad 0 < r < a,$$
$$u_z = 0, \quad r > a,$$
$$\tau_{rz} = 0, \quad \text{all } r,$$

and the resultant steady-state problem is solved. Initially, the problem is set up for a finite cylinder $r<b$ and the surface $r=b$ is defined absolutely rigid and smooth. {The reviewer interprets this as meaning $u_r=\tau_{rz}=0$ for $r=b$.} The solution for the semi-infinite problem is obtained from the limit $b\to\infty$.

The results obtained are expressed in the form of a complex radiation impedance Z defined by $W=\frac{1}{2}Z\omega^2$, where W is the mean power dissipated by the "piston" source; limiting formulae for W are calculated for low and high frequencies.

The artificial nature of the boundary conditions on $z=0$ entails discontinuities in u_z for $z=0$ and $r=a$ which in turn leads to infinite values of the real part of Z for all ω. This difficulty is removed in a second calculation in which the free surface implacement boundary conditions are modified in the form

$$u_z = e^{i\omega t}(1-r^2/a^2), \quad 0 < r < a,$$
$$= 0, \quad r > a.$$

Finally, asymptotic displacements functions are derived for distances remote from the source.

{The reviewer considers that the curious boundary conditions employed on $z=0$ prevent applicability to transient half-space problems of greatest interest in which the regions outside the source are stress-free rather than subject to zero normal displacement.}

S. C. Hunter (Sevenoaks)

Handelman, G. H. 3591

Surface waves over a slightly curved elastic half-space.

Progress in Applied Mechanics, pp. 43–54. *Macmillan, New York*, 1963.

The author investigates the problem of Rayleigh surface waves on a slightly curved elastic half-space. The surface of the slightly curved half-space is represented by $y=\varepsilon f(x)$, where y is the vertical coordinate, ε is a small parameter, and $f(x)$ is the shape of the surface. Following a method of Eringen and the reviewer [J. Appl. Mech. **26** (1959), 491–498; MR **22** #1182], expressions are derived for the displacement field throughout the half-space with corrections up to the first-order terms in ε. The displacement at the surface is then given special attention. Through use of Lamb's asymptotic expansion for the displacements at the surface of a flat half-space, the author is able to show that the first-order correction terms consist of three types of waves, namely, the ordinary surface wave, a type of cylindrical shear wave, and a type of cylindrical dilatation wave.

J. C. Samuels (Lafayette, Ind.)

Dunkin, J. W.; Eringen, A. C. 3592

On the propagation of waves in an electromagnetic elastic solid. (French, German, Italian and Russian summaries)

Internat. J. Engrg. Sci. **1** (1963), 461–495.

The authors start with the usual linearized equations of electromagnetic and elastic interactions in solid conductors. As has been shown by a number of authors, the coupling is weak under laboratory conditions. The formal solution is written for a vibrating elastic conducting plate in an external static magnetic field. The attenuation factors for the first two symmetric and antisymmetric modes are obtained as a perturbation of the formal solution with the usual plate velocities in these modes arising as the zero-order solution. *L. Knopoff* (Pasadena, Calif.)

Ignaczak, Józef 3593

Rayleigh waves in a non-homogeneous isotropic elastic semi-space. I. (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 341–346.

The paper deals with the problem of Rayleigh waves in a non-homogeneous isotropic elastic half-space, $z \geqq 0$. The body is assumed to be in a state of plane strain. Pure stress equations of motion [the author, same Arch. **15** (1963), 225–234] are used. Assuming that the Lamé constants, λ and μ, are functions of z only, the stresses are obtained in terms of a function which satisfies a fourth-order ordinary differential equation with variable coefficients. This is solved in closed form for the case that μ is constant and for a particular range of Poisson's ratio. *M. Hayes* (Newcastle upon Tyne)

Huston, Ronald L. 3594

Wave propagation in rotating elastic media.

AIAA J. **2** (1964), 575–576.

The author examines the effect of rigid-body rotation on propagation velocities of amplitude and phase waves in an unbounded homogeneous isotropic elastic medium. *M. Hayes* (Newcastle upon Tyne)

Amiel, René 3595

Pouvoir rotatoire des milieux capillaires.

C. R. Acad. Sci. Paris **258** (1964), 1709–1711.

The material considered is an isotropic material with an internal energy which is a quadratic in the elements of the infinitesimal strain tensor e_{ij} and its space derivatives $e_{ij,k}$. The internal energy is therefore a linear combination of eight listed invariants under orthogonal transformations of e_{ij} and $e_{ij,k}$.

The equations of motion of the medium are stated and consideration given to plane transverse waves propagating through the material. It is shown that there is a solution in which the plane of polarisation of the waves rotates, the angle of rotation being proportional to the distance travelled by the wave in the direction of propagation. *A. J. M. Spencer* (Nottingham)

Nariboli, G. A.; Nyayadhish, V. B. 3596

One-dimensional thermo-elastic wave.

Quart. J. Mech. Appl. Math. **16** (1963), 473–482.

The authors consider a one-dimensional wave-propagation problem for a half-space, according to the coupled thermo-elastic theory. No mention is made of the closely related papers by Boley and Tolins [J. Appl. Mech. **29** (1962), 637–646; MR **26** #4570] and by the reviewer and Breuer [Österreich. Ing.-Arch. **16** (1961/62), 349–368]. *R. Muki* (Tokyo)

Sackman, J. L. 3597

Uniformly progressing surface pressure on a viscoelastic half plane.

Proc. 4th U.S. Nat. Congr. Appl. Mech. (Univ. California, Berkeley, Calif., 1962), Vol. 2, pp. 1067–1074. *Amer. Soc. Mech. Engrs., New York,* 1962.

This paper considers the steady motion of a uniform step pressure on the surface of a linear viscoelastic half-space. The speed of the load is assumed to be greater than the velocities of irrotational and equivoluminal waves in the medium. The analysis is restricted to a material which is elastic in bulk behavior, and exhibits standard-linear-solid behavior in shear. The jumps in the values and first derivatives of the stress components at the wave fronts are determined, and numerical results based on a simple approximate solution are presented. *F. J. Lockett* (Providence, R.I.)

Müller, Karl-Heinz 3598

Über elastische Wellen in porösen Medien. (English summary)

Z. Angew. Math. Phys. **14** (1963), 372–376.

Author's summary: "The expansion process of elastic waves in porous media is described by a system of differential equations which is a generalisation of the wave equation for homogeneous continua. For the plane problem four parameters allow comprehension of the interactions of longitudinal and transversal waves due to the porosity."

Tao, L. N. 3599

The associated elastic problems in dynamic visco-elasticity.

Quart. Appl. Math. **21** (1963/64), 215–222.

By introduction of an extra independent variable τ, additional to the space variables and the time t, and by replacement of t by τ in some of the governing equations, adjunct problems are defined. The Laplace transform of the variable τ applied to one adjunct problem gives one of the two associated elastic problems; an adjoint problem is the other. {The author's claim that the associated elastic problems enable solutions to be obtained, which are unobtainable by present methods, is not substantiated by examples.} *D. R. Bland* (Manchester)

Rogers, Tryfan G.; Pipkin, Allen C. 3600

Asymmetric relaxation and compliance matrices in linear viscoelasticity. (German summary)

Z. Angew. Math. Phys. **14** (1963), 334–343.

For an anisotropic linear viscoelastic material there are at most thirty-six independent relaxation moduli. The authors first find the relations between these moduli imposed by various material symmetries and then find what further relations are implied by a thermodynamic theory. They suggest that these further relations could form the basis of experimental tests of the thermodynamic theory. A hypothetical experiment is constructed. {Reviewer's comment. That the material is viscoelastic is not essential to the material symmetry arguments, which also apply in anisotropic elasticity.} *D. R. Bland* (Manchester)

Nordgren, R. P.; Naghdi, P. M. 3601

Finite twisting and expansion of a hole in a rigid/plastic plate.

Trans. ASME Ser. E. J. Appl. Mech. **30** (1963), 605–612.

The paper treats the axisymmetric finite twisting and expansion of an annular plate in plane stress by circumferential shear together with normal pressure on the inner boundary. Tresca's yield function and flow rule is adopted, and isotropic hardening is permitted. A partial comparison with the solution for Mises's yield function is included. Numerical results for plate thickening and boundary displacement are given. *R. Hill* (Cambridge, England)

Goodier, J. N. 3602

Instabilities of plastic solids in sustained flow.

Progress in Applied Mechanics, pp. 221–233. *Macmillan, New York,* 1963.

The general field equations and boundary conditions for small increments of stress and displacement are formulated for a workhardening Prandtl-Reuss material undergoing continued flow with finite changes in geometry. The equations are then specialized for a rigid/plastic rectangular slab, instantaneously under uniaxial compression, with its lateral faces free and its ends in contact with smooth rigid compression-plates. Eigenmodes of incremental deformation are sought under the corresponding homogeneous boundary conditions, in the context of Shanley buckling. Symmetric and antisymmetric wave-like modes are found. The problem is treated as one of generalized plane stress; there is of course an entirely analogous (and slightly more rigorous) analysis in plane strain.

R. Hill (Cambridge, England)

Kirakosjan, R. M. 3603

A problem for a conical shell of revolution with nonlinear creep. (Russian. Armenian summary)

Akad. Nauk Armjan. SSR Dokl. **37** (1963), 151–156.

L'auteur étudie, dans la théorie de membrane, le fluage d'une plaque mince conique de rotation, encastrée au bout et soumise à une traction (compression) au long de la génératrice; la surface normale de chargement varie peu avec le temps. On utilise la théorie nonlinéaire de l'hérédité. Le problème est réduit à la détermination d'une fonction d'intégration, pour laquelle on obtient une équation intégrale nonlinéaire.

En remplaçant les tensions tangentielles par leur valeur maximale (approchée), dans le cas où les tensions annulaires sont plus grandes que les tensions transversales instantanées (de compression), on obtient une solution sous forme finie. *P. P. Teodorescu* (Bucharest)

Ziegler, Hans 3604

Methoden der Plastizitätstheorie in der Schneemechanik. (English summary)

Z. Angew. Math. Phys. **14** (1963), 713–737.

Several aspects of snow avalanches are considered by approximating the snow as a perfectly plastic solid. The case of the infinite layer, as well as the uniform thickness layer with rigid restraining walls, is considered. As far as one can anticipate the applicability of the idealized material for snow, the results are within reasonable expectations. Undoubtably snow has rheological properties more complex than those of a perfectly plastic solid which would introduce many sophistications to the problems considered here. *P. R. Paslay* (Houston, Tex.)

Ting, Thomas C. T. 3605

The plastic deformation of a cantilever beam with strain-rate sensitivity under impulsive loading.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 38–42.

The title problem is considered for a rigid-plastic material and a certain stress-strain-rate relationship. Two solutions are obtained by numerical methods. In one of these solutions the momentum in the plastic region is neglected and in the other it is included in an approximate manner. The results for the two solutions are compared and found to give very similar answers. *G. Eason* (Glasgow)

Fox, N. 3606

Stresses associated with a moving line source of heat.

Quart. J. Mech. Appl. Math. **17** (1964), 85–89.

The temperature distribution and associated thermal stress field is found for the problem of a uniform line source of heat moving through an infinite medium at a constant velocity v perpendicular to its length. Previous solutions to this problem failed to reduce to the solution for a stationary source as $v \to 0$, and give in fact unbounded stresses throughout in this limiting case. This unsatisfactory feature is removed here by the addition of a suitable complementary solution. *B. A. Boley* (New York)

Florence, A. L.; Goodier, J. N. 3607

The linear thermoelastic problem of uniform heat flow disturbed by a penny-shaped insulated crack. (French, German, Italian and Russian summaries)

Internat. J. Engrg. Sci. **1** (1963), 533–540.

The authors consider the axisymmetric linear thermoelastic problem of a uniform heat flow in an infinite elastic solid, the flow being disturbed by the presence of an insulated penny-shaped crack whose faces are stress-free. By using Hankel transforms they reduce the problem to solvable dual integral equations and so determine the temperature field and the stresses in the solid. Numerical results are given for the stress components in the plane of the crack and comparison is made with the stresses in the corresponding problems of a uniform heat flow disturbed by spherical and spheroidal cavities in a solid.

W. D. Collins (Manchester)

Martin, Charles J.; Payton, Robert G. 3608

Thermoelastic stresses in a slab.

J. Math. Mech. **13** (1964), 1–30.

This paper is concerned with a two-dimensional steady state thermoelastic problem of an infinite slab with constant width of which one edge is rigidly fixed and the other is held stress-free. The problem is solved by means of the Fourier transform and numerical results are presented for a certain particular distribution of temperature.

R. Muki (Tokyo)

Achenbach, Jan D. 3609

Approximate transient solutions for the coupled equations of thermoelasticity.

J. Acoust. Soc. Amer. **36** (1964), 10–18.

The author considers an approximate transient solution of the coupled thermoelastic equations. In the heat conduction equation, the time derivative of the dilatation is replaced by the δ-function term that travels with the wavefront. In this way, though the equations become formally uncoupled, most of the coupling effect is retained. The inhomogeneous equations thus obtained are solved in terms of tabulated functions and the solution is found to be fairly accurate when compared to the existing exact numerical solutions for one-dimensional problems. The procedure is then applied to a three-dimensional problem where the temperature and displacement response of an infinite medium under the action of a suddenly applied force is considered. *P. Choudhury* (Howrah)

FLUID MECHANICS, ACOUSTICS

See also 3259, 3260, 3814, 3869.

Capodanno, Pierre 3610

Sur les forces aérodynamiques exercées par un fluide parfait incompressible, en mouvement irrotationnel, sur une grille.

C. R. Acad. Sci. Paris **258** (1964), 1146–1147.

The author uses the method of Malavard, Siestrunck and Germain to derive a conformal transformation of the grid into a circle. A formula is given for determining the moment of the aerodynamic forces on an element of the grid. It is shown that when the direction of the free stream is varied, a metacentric parabola can be derived. *A. W. Babister* (Glasgow)

Parhomovskiĭ, S. I. 3611

The force on a piecewise smooth profile in a symmetric stream. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1963**, *no.* 5 (36), 89–96.

The author attacks the problem of the calculation of the drag of a piecewise smooth contour in a symmetrical stream flow of an incompressible fluid. The body in question is rotationally symmetrical and the problem which the author intends to solve is to calculate the horizontal, vertical and the circumferential component forces acting upon the contour in question. The technique of solving the problem is that of complex variables and conformal mapping. The complex potential w is represented in the standard form $w=\varphi+i\psi$ and $w=v_1w_1+v_2w_2+\omega w_3$ $(w_k=\varphi_k+i\psi_k)$, where the velocity vector in the vertical meridional (x, y)-plane has two components, $\mathbf{V}(v_1, v_2)$, and ω refers to the circumferential direction. The meridional $(x+iy)=z$-plane is transformed conformally into the $(\xi+i\eta)=\zeta$-plane, where the meridional cross-section of the body is represented as a segment of a straight line located on the horizontal axis (ξ). This is analogous to the well-known conformal transformation of a circle into a horizontal segment in the theory of a two-dimensional incompressible fluid flow. The transformation equations from $(x, iy)\leftrightarrow(\xi, i\eta)$ must be found in each particular case. Care must be taken of the singular points where $dw_k/dz=\infty$. To obtain general formulas for the horizontal, vertical and circumferential component forces, the author assumes at first the general expressions for the transformation equations like $z(\zeta)=(\zeta-1)^r f_1(\zeta)$, etc. By using the formula for the pressure $p=-\rho\varphi$, $\rho=$ density, the potential function φ, which is very simple in the ζ-plane (uniform flow), transformed back into the z-plane gives the pressure distribution on the body in question. As a particular case the author calculates the forces acting on a cone. *M. Z. v. Krzywoblocki* (E. Lansing, Mich.)

Couchet, Gérard 3612

Sur un cas d'intégration par quadratures du mouvement d'un profil au sein d'un fluide parfait incompressible en mouvement irrotationnel.

C. R. Acad. Sci. Paris **258** (1964), 1722–1724.

Le torseur des efforts aérodynamiques agissant sur un profil étant connu par les travaux antérieures de l'auteur [*Mouvement plans d'un fluide en présence d'un profil mobile*, Mémor. Sci. Math., No. 135, Gauthier-Villars, Paris, 1956; MR **19**, 702], celui-ci intègre les équations du mouvement du profil dans un cas particulier. *J. P. Guiraud* (Paris)

Ter-Krikorov, A. M. 3613

Théorie exacte des ondes longues stationnaires dans un liquide hétérogène.

J. Mécanique **2** (1963), 351–376.

Existence of long waves for the general case of stationary flow of an ideal incompressible fluid, including vorticity and heterogeneity, is considered. The problem is formulated as an eigenvalue problem of a non-linear second-order (elliptic) differential equation. An approximate solution for long waves is shown to be obtainable by an asymptotic process. Then existence of a solution is demonstrated, and an estimate for the accuracy of the approximate solution is presented. The estimate, being independent of the wavelength, is of general validity. *K. Forster* (Los Angeles, Calif.)

Moran, John P. 3614

Image solution for vertical motion of a point source towards a free surface.

J. Fluid Mech. **18** (1964), 315–320.

It is shown that the image solution for the potential representing the vertical constant-speed rise of a constant-strength source towards a horizontal free surface can be simply expressed by a vertical distribution of sources whose strength depends upon the Froude number. The vertical trail of sources extends from the image point of the submerged source upward to infinity while their strength decays exponentially. The solution is extended to apply to the case in which there is a lower density fluid above the free boundary. *E. V. Laitone* (Berkeley, Calif.)

Wu, T. Yao-tsu; Wang, D. P. 3615

A wake model for free-streamline flow theory. II. Cavity flows past obstacles of arbitrary profile.

J. Fluid Mech. **18** (1964), 65–93.

In an earlier paper [J. Fluid Mech. **13** (1962), 161–181; MR **26** #957] Wu proposed a wake model and developed a theory for cavity flow past an oblique flat plate. The theory is now generalized to deal with cavity flows past arbitrary profiles, and methods of solution, with examples, are presented. The earlier wake model which consists of a near wake at constant pressure, and a far wake, in which the pressure eventually reaches the free stream value, is adopted. The points of detachment from the profile are supposed known.

The theory is developed first for a polygonal profile. A parametric variable t is introduced such that the expression $f(t)$ for the complex potential takes a simple form which, for the flat-plate problem, becomes identical to that for the complex velocity $w(t)$. An equation for $w(t)$ is then derived. The physical plane $z(t)$ follows directly from $f(t)$ and $w(t)$. The formulation is completed by writing down the equations relating the parameters defining the shape of the profile to the corresponding parameters in the t-plane. The theory is then generalized to deal with arbitrary profiles by applying a limiting process to the polygonal profiles.

Because of the difficulty of solving the equations directly, various iterative schemes are proposed for approximate solution. Examples are calculated which demonstrate satisfactory convergence and fair agreement with experimental observations.

M. G. Hall (Farnborough)

Voitsenia, V. S. [Voĭcenja, V. S.] 3616

On the oscillations of a body above the interface between two liquids.

Prikl. Mat. Meh. **27** (1963), 910–917 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1389–1401.

The linear water wave theory is applied to solve the plane motion of gravity waves radiated outwards from an oscillating body immersed in a lighter liquid of finite depth, which has an upper free surface and a lower interface, supported underneath by another heavier liquid of infinite depth. By a distribution of singularities of unknown strength over the body surface of arbitrary shape, a linear integral equation of Fredholm type is obtained, for which a method of solution by successive approximations has been devised. The average hydrodynamic forces and moments are derived and evaluated.

T. Yao-tsu Wu (Pasadena, Calif.)

Illingworth, Charles R. 3617

Fluctuating flow at small Reynolds number. (German summary)

Z. Angew. Math. Phys. **14** (1963), 681–694.

Kaplun [J. Math. Mech. **6** (1957), 595–603; MR **19**, 1005; also Kaplun and Lagerstrom, ibid. **6** (1957), 585–593; MR **19**, 1004] has suggested that flow past an obstacle at small Reynolds number can be analyzed to first approximation by an inner flow obeying Stokes's equations and an outer flow obeying Oseen's equations. These two flows are then matched together by making the inner boundary conditions of the outer flow agree with the outer boundary conditions of the inner flow. This now well-known method has been applied very successfully to a number of steady-flow problems.

In this paper this method is used to study a fluctuating flow at small Reynolds number past a sphere and a cylinder. The uniform free stream is assumed to fluctuate about a steady mean value with a small amplitude and at a specific frequency. It concludes that the effect of fluctuations on the drag is not very large.

L. N. Tao (Chicago, Ill.)

Moiseev, N. N. 3618

Mathematical methods of investigating non-linear oscillations of a fluid. (Russian. English summary)

Applications of the methods of non-linear vibrations to the problems of physics and technology (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. III,* 1961), *pp.* 275–285. *Izdat. Akad. Nauk Ukrain. SSR, Kiev,* 1963.

Author's summary: "The paper deals with some problems in the theory of nonlinear oscillations of a fluid mass under gravitational forces. In particular, the following three problems have been considered: the problem of free oscillations of a bounded mass of heavy liquid with gravitational forces, the problem of its resonance oscillations, and the methods of asymptotic analysis in the problem of linear oscillations of a viscous fluid. The investigation methods generalize the classical methods of nonlinear mechanics and are of a formal character."

J. N. Newman (Washington, D.C.)

Taylor, T. D.; Acrivos, Andreas 3619

On the deformation and drag of a falling viscous drop at low Reynolds number.

J. Fluid Mech. 18 (1964), 466–476.

The classical solution to the problem of a liquid drop falling (or a gas bubble rising) in a viscous liquid leads to a curious result: If the inertia of both fluids is completely neglected, the radial component of the stress-vector jump is uniform over the surface of a spherical drop, so that the spherical shape is a permissible equilibrium configuration. As is physically evident, however, fluid inertia tends to modify the shape. At low Weber numbers (i.e., with interfacial tension a dominant effect), this tendency is somewhat offset by capillary forces. Since attempts to improve the Stokes drag law—even for a solid sphere—by perturbation expansion in the Reynolds number lead to the Whitehead paradox, these attempts a fortiori are fruitless for calculating the deformation of a falling drop.

During the past decade, however, a technique has been developed for determining liquid inertia effects on an obstacle in low Reynolds number flow: An inner, or "Stokes" expansion in the neighborhood of the obstacle is analytically matched to an outer or "Oseen" expansion valid far away. The present paper appears to be the first application of this method to the flow-field about a deformable obstacle, viz., the falling drop.

The drop configuration is assumed to be a small perturbation of the spherical shape. This assumption is self-consistent for small values of the Weber number, the drop taking up an oblate spheroidal shape; the parameters describing the oblate spheroid are surprisingly insensitive to the viscosity-ratio of the two phases. For moderate values of the Weber number, the assumption of small departure from the spherical ceases to be self-consistent; nevertheless, it is interesting to note that the drop shape tends toward the experimentally observed spherical cap.

W. Langlois (San Jose, Calif.)

Dzhaugashtin, K. E. [Džaugaštin, K. E.] 3620

Some problems in the theory of the boundary layer of a conducting liquid.

Ž. Tehn. Fiz. **33** (1963), 843–850 (*Russian*); *translated as Soviet Physics Tech. Phys.* **8** (1964), 634–638.

The paper gives the solution of several problems on the steady-state laminar motion of an electrically conducting incompressible liquid in the boundary layer of a flat plate. Two types of boundary layers are considered, viz., those in which the current flows along the plate in the direction of the flow or perpendicular to this direction. Two cases of thermal boundary conditions are considered separately: (A) the thermally insulated plate, and (B) the plate with a constant surface temperature. The case of a uniform stream flowing over a porous plate is also discussed. Finally, a generalisation is given of Stepanov's transformation for the flow of an incompressible conducting liquid and of a compressible gas in a magnetic field, which makes it possible to construct similar solutions for flows in magnetohydrodynamic boundary layers of solids of revolution. *J. N. Kapur* (Kanpur)

Kennedy, Ernest D. 3621

Application of a new method of approximation in the solution of ordinary differential equations to the Blasius equation.

Trans. ASME Ser. E. J. Appl. Mech. **31** (1964), 112–114.

An approximate solution, in closed form, of Blasius's equation $f''' + 2ff'' = 0$ with $f(0) = f'(0) = 0$, $f'(\infty) = 1$, is obtained as follows. The related equation $f_0''' + 2(f_0 f_0'' + f_0'^2) = 0$ has a simple solution in Airy integrals which satisfies the condition $f'(0) = 0$. The quantity f in Blasius's equation is then replaced by f_0 and the resulting equation integrated directly. The result, for wall shear, is only about 6 percent too low.

The novelty claimed for this method is that, instead of linearizing a difficult equation or omitting terms from it, an addition is made which results in an equation that is exactly integrable. Now the author's result may be regarded as the first approximation in a method of solution by successive approximations. It happens that for Blasius's equation such a process converges rapidly even if the zeroth approximation bears little resemblance to the final result, as is true of f_0 in this case. In general, there is no check of the accuracy of the first approximation without checking the convergence of the whole process. The author's suggestion that his method might be usefully applied elsewhere should therefore be treated with caution. *M. G. Hall* (Farnborough)

Gallagher, A. P.; Mercer, A. McD. 3622

On the behaviour of small disturbances in plane Couette flow. II. The higher eigenvalues.

J. Fluid Mech. **18** (1964), 350–352.

The numerical analysis of the authors' earlier paper [same J. **13** (1962), 91–100; MR **25** #874], which gave only the first eigenvalue for the title problem, is extended to obtain the higher eigenvalues. Their present results are in agreement with the earlier results of Southwell and Chitty [Philos. Trans. Roy. Soc. London Ser. A **229** (1930), 205–253] but not with those of Grohne [Z. Angew. Math. Mech. **34** (1954), 344–357; MR **16**, 478]. *J. W. Miles* (Canberra)

Betchov, R. 3623

A simplified theory of magnetohydrodynamic isotropic turbulence.

J. Fluid Mech. **17** (1963), 33–51.

The author here attempts to describe some of the basic properties of hydromagnetic turbulence by means of a simple, qualitative theory involving only a few integral properties of the motion. In the case of ordinary turbulence, the theory involves only the time-dependence of the mean kinetic energy, the mean square vorticity, and the usual skewness factor. All fourth-order cumulants are neglected and a special hypothesis is introduced to evaluate the dissipative terms. These approximations lead to a system of three first-order non-linear differential equations which effectively depend on only one parameter. The permissible values of this parameter can be severely restricted by the simple physical requirement that the kinetic energy remain positive but is otherwise arbitrary, and must be estimated from experimental results. In the case of hydromagnetic turbulence, both the number of governing equations and the number of parameters are greatly increased. These equations are integrated for several typical cases to illustrate the various mechanisms of energy transfer. *W. H. Reid* (Chicago, Ill.)

Legendre, Robert 3624

Tourbillons en cornets des ailes delta.

C. R. Acad. Sci. Paris **257** (1963), 3814–3817.

This note gives a mathematical representation of near-sonic flow in the neighbourhood of the apex of a slender wing with conical vortex sheets, based on the physical model of M. Roy. The problem is reduced to the solution of a non-linear integro-differential equation. *A. W. Babister* (Glasgow)

Roy, Maurice 3625

Remarques sur la note précédente.

C. R. Acad. Sci. Paris **257** (1963), 3817.

This note gives the author's comments on Legendre's mathematical representation of the flow past a delta wing [#3624]. Attention is drawn to the conception of source-vortices. *A. W. Babister* (Glasgow)

Salwen, Harold; Grosch, Chester E.; Ziering, Sigi 3626

Extension of the Mott-Smith method for a one-dimensional shock wave.

Phys. Fluids **7** (1964), 180–189.

Authors' summary: "A method has been developed for adding an arbitrary number of additional terms to the two-term Mott-Smith distribution function for a one-dimensional shock wave and a calculation has been carried out, for a monatomic gas of Maxwellian molecules, with a three term distribution function. For weak shocks, the calculated reciprocal shock thicknesses are within 2% of the Navier-Stokes results. This is a substantial improvement over the Mott-Smith results which, for the same force law, are 15 to 30% too low. For strong shocks, our reciprocal shock thicknesses are below the Mott-Smith results for v_x^2 and for v_x^3 moments."

Baidedaev, A. [Baĭdedaev, A.]; Senkevich, A. A. [Senkevič, A. A.] 3627
Vibrational relaxation in gases.
Akust. Ž. **9** (1963), 279–282 (*Russian*); *translated as Soviet Physics Acoust.* **9** (1964), 229–231.
Authors' summary: "A generalized gas kinetic equation with coefficients expressed in terms of the distribution function and effective cross-sections is applied to the investigation of vibrational relaxation in gases. The usual reaction equations are obtained, along with a reaction equation for the simultaneous excitation and deactivation of both colliding molecules."

Le Fur, Bernard 3628
Critère de transition pour l'écoulement dans un tuyau circulaire d'une classe de liquides non newtoniens.
C. R. Acad. Sci. Paris **258** (1964), 2482–2485.
Author's summary: "Une loi rhéologique simple permet de représenter le comportement d'un grand nombre de liquides non newtoniens et de calculer les pertes de charge dans des tuyaux circulaires. L'utilisation d'un critère empirique de transition conduit à des relations entre deux nombres sans dimensions, relations assez bien vérifiées par l'expérience."

Coleman, Bernard D.; Markovitz, Hershel 3629
Normal stress effects in second-order fluids.
J. Appl. Phys. **35** (1964), 1–9.
The theory of simple fluids with fading memory has been developed in a series of papers by Noll and Coleman; full references are given in the paper under review. Briefly, such fluids are characterised by the properties that the present stress is a functional of the past strain, that all local configurations are equivalent in response, and that, in a precisely defined sense, deformations which occurred in the distant past have less effect on the present stress than deformations which occurred in the recent past. In the limit of very slow flows, the constitutive equation for such a fluid reduces to that for a Newtonian fluid; in the limit of small deformations, the equations of infinitesimal linear viscoelasticity are obtained.

The second-order theory for slow flows of an incompressible fluid involves two material constants in addition to the Newtonian viscosity. It is shown that one of these constants is simply related to the shear-relaxation modulus of infinitesimal viscoelasticity. A number of experimental methods of determining the second-order constants are suggested; these are based on known solutions for a number of simple steady flows.

A. J. M. Spencer (Nottingham)

Crupi, Giovanni 3630
Su particolari onde cilindriche della magneto-idrodinamica.
Atti Sem. Mat. Fis. Univ. Modena **10** (1960/61), 1–10.
The motion in a fluid cylinder with finite electric conductivity and an exterior magnetic field parallel to its axis is considered. The fluid is assumed to be incompressible. The new feature of the present investigation is that the term representing the displacement current is retained when Maxwell's equations are used. Otherwise, the fundamental set of equations describing the field and the equations of the motion is set up in the usual way. Cylindrical coordinates are introduced and a differential equation of the fourth order for the velocity v is derived. The only surviving velocity component is orthogonal to the cylinder axis and the radius of the cylinder, and the magnetic induction added by the motion is parallel to the velocity. It may be pointed out that the first member of the first equation of system (7) should be $-(c/\mu)\,\partial b/\partial z$ instead of $-(c/\mu)(1/r)\partial(rb)/\partial z$. This error is accidental; the corresponding equation in system (11) is correct, so the error has nothing to do with the results.

The second part of the paper deals with a particular solution of the fourth-order differential equation for the velocity. As a trial the form $v = A\varphi(r)\exp(i\omega t + \alpha z)$ is used, and it is found that $\varphi(r) = J_1(kr)$, where $J_1(x)$ is the Bessel function of the first order. Expressions for the additional variable components of the magnetic induction and the components of the electric displacement are also given.

E. Lyttkens (Uppsala)

Magneto-fluid dynamics. 3631
Progr. Theoret. Phys. Suppl. No. 24 (1962), iii + 193 *pp.*
Le but de cette publication est de donner une vue d'ensemble des plus récents travaux effectués en magnétodynamique des fluides, et plus particulièrement de ceux qui sont faits au Japon. La division en chapitres, chacun traité par un auteur différent, correspond aux diverses directions de recherches actuelles. Principaux sujets traités: (1) (I. Imai) Equations fondamentales. Théorie des mouvements à champ aligné (avec la méthode du fluide fictif). Equations des ondes. Méthodes d'approximation par linéarisation: les différents cas limites obtenus en faisant varier les viscosités cinématique et magnétique sont passés en revue. L'approximation de Stokes ($R\to 0$, $R_m\to 0$) fait l'objet d'une étude particulière, avec application au mouvement d'une sphère. (2) (H. Hasimoto) Fluide incompressible. Revue des solutions exactes (en particulier problème de Rayleigh) et des problèmes réductibles. Etude des sillages magnétodynamiques, avec application aux cas des nombres de Reynolds faibles et forts. (3) (K. Kusukawa) Courant de Hartmann. Ecoulements dans les canaux. Etude de l'action de l'effet Hall. (4) (K. Kusukawa) Revue de travaux récents sur les écoulements non visqueux autour d'un profil mince, à 2 et 3 dimensions, dans le cadre de la théorie linéarisée. Cas des mouvements non-stationnaires. (5) (A. Sakurai) Etude de la propagation des chocs cylindriques. Courte note sur les chocs stationnaires. (6) (T. Tatsumi) Effets d'un champ magnétique sur la stabilité des mouvements laminaires. L'effet de stabilisation d'un champ magnétique uniforme sur la turbulence est démontré. Turbulence homogène et isotrope.

L'intérêt de ce recueil est évident, les divers articles donnant une mise au point claire des sujets traités. Nombreuses références bibliographiques.

R. Thibault (Sotteville-lès-Rouen)

Agostinelli, Cataldo 3632
Un teorema sul flusso di energia nel moto di un fluido di alta conduttività elettrica in cui si genera un campo magnetico.
Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 471–474.

La plus grande partie de cette note est consacrée à l'établissement de l'équation qui donne la variation de l'énergie totale à l'intérieur d'une surface fermée fixe, dans le cadre de la magnétodynamique des fluides non dissipatifs. Dans le cas d'un mouvement périodique, il en résulte que le flux de l'énergie totale à travers la surface au cours d'une période est égal au flux des pressions magnétique et hydrodynamique.

R. Thibault (Sotteville-lès-Rouen)

Carstoiu, John 3633

Sur l'effet du courant de déplacement de Maxwell dans la propagation des ondes magnétohydrodynamiques.

C. R. Acad. Sci. Paris **258** (1964), 1728–1731.

A discussion of the magneto-dynamic equations for a fluid of a single component, neglecting the viscosity and the finiteness of the conductivity, but taking account of the displacement current. A homogeneous static magnetic field $\mathbf{H}_0$ is assumed. In the case of slow movements of the fluid the influence of possible deviations of the dielectric constant and magnetic permeability from their free-space values proves to disappear. The general linearized equations lead to simple wave equations for the longitudinal (direction of $\mathbf{H}_0$) component j_z of the current density and ω_z of the vorticity of the fluid. The corresponding transverse components are less simple, but their equations involve the possibility of wave propagation along the lines of force of $\mathbf{H}_0$, associated with a diffusion around these lines. The concise computations confirm the properties of limiting cases discussed earlier.

H. Bremmer (Eindhoven)

Dragoş, Lazare 3634

L'écoulement d'un gaz conducteur en présence des profils minces.

C. R. Acad. Sci. Paris **256** (1963), 4158–4161.

On détermine l'écoulement d'un gaz fortement ionisé et compressible autour des profils minces, en présence d'un champ magnétique homogène de même orientation que la vitesse de l'écoulement.

J. Naze (Bergen)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 3207, 3519, 3558, 3872.

Kneisly, John A., II 3635

Local curvature of wavefronts in an optical system.

J. Opt. Soc. Amer. **54** (1964), 229–235.

Formulas are derived for tracing the local curvatures of wavefronts through an optical system. Steps considered include transition, refraction, reflection and also transmission or reflection by a diffraction grating.

E. W. Marchand (Rochester, N.Y.)

Pavageau, James 3636

Pouvoirs de réflexion et de transmission d'un réseau diélectrique à profil sinusoïdal et de pas inférieur à la longueur d'onde.

C. R. Acad. Sci. Paris **258** (1964), 2987–2990.

A study is made of the reflection and transmission generated by a plane wave impinging on a surface of the form $z=h\cos(\mu x+\nu y)$ separating two media. Of special interest is the case in which the sinusoidal surface has a period substantially less than the wavelength of the incident light.

E. W. Marchand (Rochester, N.Y.)

Pavageau, James 3637

Pouvoirs de réflexion et de transmission d'un dioptre imparfaitement poli.

C. R. Acad. Sci. Paris **258** (1964), 3449–3451.

A study is made of the reflection and transmission properties of a surface separating two media in cases where the surface has irregularities considered small with respect to the wavelength of the light. The method is to regard the surface as the resultant of sinusoidal components, the behavior of the sinusoidal type having been investigated in a previous paper [see #3636].

E. W. Marchand (Rochester, N.Y.)

Ko, H. C. 3638

The entropy radiation of a mixture of incoherent beams.

Proc. Phys. Soc. **82** (1963), 1070–1072.

A mixture of incoherent light beams is described in terms of entropy radiation, a concept originally due to Planck.

I. R. Senitzky (Fort Monmouth, N.J.)

Boillat, Guy 3639

Des ondes électrodynamiques.

C. R. Acad. Sci. Paris **258** (1964), 2487–2490.

The possible nonlinear wave motions ("shock waves") in nonlinear electromagnetic theory are investigated. One keeps the linear equations expressing that $F_{\mu\nu}$ ($\mu,\nu=1,\cdots,4$) is the curl of a 4-potential but replaces the other set of Maxwell equations by a nonlinear set derived from a Lagrangian $\mathscr{L}$ which is an arbitrary function of the two independent invariants $(\mathbf{H}^2-\mathbf{E}^2)/2$ and $\mathbf{E}\cdot\mathbf{H}$. The results are: (1) standing waves are impossible; (2) if a certain expression in the second derivatives of $\mathscr{L}$ vanishes, there exist running waves with the fundamental velocity 1 in any direction; (3) otherwise such waves can propagate in only two directions, making the same angle with the Poynting vector. In this last case, it can happen that the propagation velocity along a direction making an angle different from $\pi/2$ with the Poynting vector is different for the two senses of propagation along this line.

R. Ingraham (University Park, N.M.)

Leibowitz, M. A.; Ackerberg, R. C. 3640

The vibration of a conducting wire in a magnetic field.

Quart. J. Mech. Appl. Math. **16** (1963), 507–519.

Authors' summary: "The vibration of a perfectly flexible, stretched, electrically conducting wire in a magnetic field, perpendicular to its plane of motion, is considered. When the wire forms part of a closed circuit, its movement in the field induces a current which interacts with the field so as to inhibit the motion. This effect is investigated quantitatively. The integro-differential equation for the displacement of the wire from equilibrium is derived, and it is shown that the damping caused by the field is weaker than the familiar frictional damping. The eigenfunctions of the integro-differential equation are found when both

ends of the wire are fixed. A solution to the initial value problem is obtained when the displacement and its rate of change at time $t=0$ are prescribed. Finally, the influence of the field on the electrical impedance of the wire is discussed."

Keller, Herbert B.; Sachs, David 3641

Calculations of the conductivity of a medium containing cylindrical inclusions.

J. Appl. Phys. **35** (1964), 537–538.

Authors' summary: "Finite difference calculations of the conductivity of an infinite medium containing a square array of perfectly conducting cylindrical inclusions are reported. For dense packing the results agree with and yield an estimate of the next term in an expansion derived by J. B. Keller. For sparse and intermediate packing the calculations are in excellent agreement with a formula derived by Rayleigh and improved by Runge."

Petit, Roger 3642

Diffraction d'une onde plane monochromatique par un réseau métallique.

C. R. Acad. Sci. Paris **258** (1964), 1429–1432.

Der Autor überträgt das von ihm entwickelte Verfahren [dieselben C. R. **257** (1963), 2018–2021; MR **27** #5489] zur Berechnung des reflektierten E^r und transmittierten E^t elektromagnetischen Feldes einer einfallenden ebenen Welle an einem Gitter aus ideal leitenden Streifen, die durch $y=f(x)$ gegeben sind, auf solche mit endlicher Leitfähigkeit. Er setzt daher in der Wellengleichung $(\Delta+k^2N^2)E=0$ für die z-Komponente des elektrischen Feldvektors den Brechungsindex N komplex $=\nu+i\cdot\chi$ voraus. Für die Teilfelder E^r und E^t wird ein Fourier-Reihenansatz der Form

$$(*)\qquad \exp(jkx\sin\alpha)\cdot\sum_{p=-\infty}^{\infty} g_p(y)\cdot\exp(jp2\pi x/d)$$

gemacht, wenn d die Gitterkonstante ist. Die Funktionen $g_p(y)$ ergeben sich durch Einsetzen von (*) in die Wellengleichung zu $g_p(y)=c_p\cdot\exp[\sqrt{((k\sin\alpha+2\pi p/d)^2-k^2N^2)}y]$. Aus der Forderung des stetigen Durchgangs der Tangentialkomponenten des elektrischen und magnetischen Feldvektors an $y=f(x)$, welche Bedingung auf der Projektion auf die x-Achse erfüllt wird, ergibt sich ein unendliches Gleichungssystem zur Berechnung der Fourier-Koeffizienten c_p. Das System wird mittels des E. Schmidtschen Orthogonalisierungsverfahrens gelöst.

E. Meister (Riegelsberg)

Zauderer, Erich 3643

Wave propagation around a convex cylinder.

J. Math. Mech. **13** (1964), 171–186.

The diffraction of an incident two-dimensional wave u_i by an open, smooth strictly convex infinite cylinder gives rise to a total field u satisfying $(\Delta+k^2)u=0$ and $u=0$ on the cylinder boundary. Suppose u_i satisfies the same equation as u and has an asymptotic expansion, as $k\to\infty$, $u_i\sim\exp(ik\psi)\sum_{n=0}^{\infty}(ik)^{-n}v_n(x,z)$. The infinite domain is subdivided into the illuminated region, the shadow region and a boundary layer on the cylinder. In the illuminated region (away from the shadow boundary) the method of Keller [Proc. Sympos. Appl. Math., Vol. 8, pp. 27–52, McGraw-Hill, New York, 1958; MR **20** #640] gives the complete asymptotic expansion for u as $u=u_i+u_r$, where u_r is the reflected field. In the shadow region the approach of Wu and Seshardi [Harvard Univ. Cruft Lab. Sci. Rep. No. 22 (1958)], in principle, provides the asymptotic expansion of u providing the boundary layer of the cylinder is treated by the variable stretching method of the boundary-layer theory.

A special analysis is required near the point of diffraction which can be considered as the intersection of two boundary layers, the shadow layer and the diffracting object layer. The basic content of this paper is that of treating the neighborhood of the point of diffraction as a boundary layer, thus allowing the determination of all unknown constants in the other domains of the field by a matching technique. *W. F. Ames* (Stanford, Calif.)

Zauderer, Erich 3644

Wave propagation around a smooth object.

J. Math. Mech. **13** (1964), 187–199.

This paper is related to the author's article [see #3643 above] in general problem statement. Its immediate concern is the extension of the Wu and Seshardi [Harvard Univ. Cruft Lab. Sci. Rep. No. 22 (1958)] asymptotic method to the solution of the diffraction problem of three dimensions. Since Keller's theory of diffraction [Proc. Sympos. Appl. Math., Vol. 8, pp. 27–52, McGraw-Hill, New York, 1958; MR **20** #640] gives, correctly, the leading term in the asymptotic development of several exactly solvable three-dimensional problems, the expansion of the field u for the problem $(\Delta+k^2)u=0$, $u=0$ on the smooth boundary, is assumed to have the same form. The correct form of the asymptotic expansion is in powers of $k^{-1/3}$. *W. F. Ames* (Stanford, Calif.)

Akhiezer, I. A. [Ahiezer, I. A.] 3645

Reflection of electromagnetic waves from a plasma.

Ž. Tehn. Fiz. **33** (1963), 935–942 (*Russian*); *translated as Soviet Physics Tech. Phys.* **8** (1964), 699–703.

If an electromagnetic wave is sent into a plasma, the back-scattered radiation provides information on the dielectric inhomogeneities of the ionized gas (fluctuations in charge density, turbulence, etc.). The author develops a relatively simple theory of the phenomenon for a semi-infinite plasma having a plane boundary face. Special attention is paid to the effect of the boundary on the back-scattered radiation. The results are expressed in terms of the correlation functions of the electron density, and are largely of technical, but not of mathematical, interest. *E. L. Hill* (Minneapolis, Minn.)

Keller, Joseph B. 3646

The field of an antenna near the center of a large circular disk.

J. Soc. Indust. Appl. Math. **11** (1963), 1110–1112.

The far field of an infinitesimal dipole antenna, situated at the centre of a circular disk, in a direction perpendicular to it, is calculated with the aid of the geometrical theory of diffraction developed earlier by the author. The total field then appears as the sum of a geometric series, the individual terms of which can be ascribed to contributions that have been diffracted any number of times around the

edges of the screen in some meridional plane. The corresponding approximation which only takes account of single or double diffractions of this type is identical with an expression obtained by Tang [same J. **10** (1962), 695–708; MR **27** #5487] with the aid of a much more complicated method, applying a Wiener-Hopf procedure.
H. Bremmer (Eindhoven)

Moon, Parry; Spencer, Domina Eberle 3647
A new mathematical representation of alternating currents.
Tensor (N.S.) **14** (1963), 110–121.
The paper starts with a survey of different methods (introduced in the theory for alternating currents in electrical networks) used when describing electromotive forces, currents, impedances and power in terms of complex quantities. Difficulties arise from the fact that time-harmonic conditions involve simultaneous contributions proportional to $\cos(\omega t)$ and $\sin(\omega t)$, respectively, or even (as in the case of the instantaneous power) constant contributions and components of the first harmonic frequency 2ω. The authors combine the coefficients of these various contributions into quantities called "holors" which can be compared with vectors and tensors. For instance, the voltage and current holor behave as a contravariant and covariant vector (of two elements), the impedance holor Z_{ij} and the admittance holor Y^{ij} as tensors with four elements, the instantaneous power as a covariant vector of three elements; the connection of the latter with the voltage and current holors depends on a new holor $\gamma^{kl}_{\cdot\cdot j}$ with twelve elements. The most important relations obtained are illustrated with the aid of graphical representations. *H. Bremmer* (Eindhoven)

Clarke, S.; Krikorian, A.; Rausen, J. 3648
Computing the N best loopless paths in a network.
J. Soc. Indust. Appl. Math. **11** (1963), 1096–1102.
This paper deals particularly with the establishment of an algorithm for finding N best proper paths (shortest loopless paths) between a pair of nodes in a connected graph (nonoriented) when N is larger than the number of minimal paths of the nodes. A path p is represented in general as a sum $p=r(p)+s(p)$ of two paths, called the "root" and the "spur" respectively, such that all links in $r(p)$ are basic and the first link of $s(p)$ is the first nonbasic link. A link is "basic" if it belongs to the set of minimal paths, and a path whose spur is proper is "admissible". Then, the algorithm consists of the following steps: (a) choose all minimal paths, A_0, (b) construct a set of all admissible detours of the minimal paths, A_1, and separate the proper paths from the improper ones, (c) construct a set of all admissible detours of the proper paths generated by Step (b), A_2, and separate the proper paths from the improper ones. This process continues until the number of proper paths generated is equal to or greater than N. (d) Form a set S consisting of N shortest proper paths from $A_0, A_1, A_2, \cdots$, and also form a set T of all improper admissible paths generated of length less than M, where M is the maximum length of paths in S. Then, the authors propose criteria for determining any of these improper paths in T as well as the paths in S, which may have proper detours of length less than M in order to replace some of the longer paths in S. Also discussed are the efficient construction and testing of admissible detours of given paths, as well as the results of the algorithm tested by a computer.
W. H. Kim (New York)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 3609.

Norminton, E. J.; Blackwell, J. H. 3649
Transient heat flow from constant temperature spheroids and the thin circular disk.
Quart. J. Mech. Appl. Math. **17** (1964), 65–72.
Authors' summary: "Formal solutions for the transient heat flow from constant-temperature prolate and oblate spheroids have been obtained as series in descending powers of the time, useful for large time calculations. The expansion for the oblate case yields, in the limit, the result for the practically important case of a thin circular disk, for which the steady-state solution is well known. Numerical calculations for the disk are included."
W. F. Ames (Stanford, Calif.)

Cannon, J. R. 3650
The solution of the heat equation subject to the specification of energy.
Quart. Appl. Math. **21** (1963), 155–160.
The initial temperature of a one-dimensional homogeneous heat conductor is given. It is shown that the temperature distribution is determined uniquely if one of the boundary conditions is replaced by the specification at all times of the total heat energy in a part of the conductor. For both the finite and semi-infinite medium, the problem can be reduced to the solution of a Volterra integral equation of the second kind. *F. J. Lockett* (Providence, R.I.)

Hrustalev, A. F. 3651
On a boundary-value problem for the Poisson equation. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1963**, no. 5 (36), 129–132.
A steady-state two-dimensional heat conduction problem is solved for the infinite strip $0\leq x\leq b$. Radiation into a medium at zero temperature takes place across the boundary $x=0$, and across the boundary $x=b$, $0<y<\infty$. The remainder of the latter boundary is maintained at constant temperature, and a heat source distribution which depends only on the coordinate x is also considered.
F. J. Lockett (Providence, R.I.)

Pattle, R. E.; Monaghan, J. 3652
The calculation of time-lag in thermal and electrical conduction, and in diffusion.
Quart. J. Mech. Appl. Math. **17** (1964), 73–79.
From the authors' summary: "Formulae for time-lag are derived applicable to certain electrical networks, and to phenomena mathematically analogous to heat conduction, provided that the properties of the system do not vary with temperature or its analogue. The relation of the

time-lag to the time-constants of exponential decay is discussed. Examples of the formulae are given."
S. D. Nigam (Kharagpur)

Jackson, T. A. S. 3653
Note on calculation of time-lag.
Quart. J. Mech. Appl. Math. **17** (1964), 81–83.
Starting with the differential equation governing diffusion or heat conduction, the problem of the preceding review [#3652] is solved for large time using the Laplace transform method. It should be noted that the parameters of the medium are functions of location only.
S. D. Nigam (Kharagpur)

Reid, Walter P. 3654
Linear heat flow in a graded slab.
Contributions to Differential Equations **3** (1964), 57–63.
Write $L_s[U(s, t)] = [p(s)U_s(s, t)]_s + h(s)U(s, t)$ and $M_i[U] = \beta_i U(a_i, t) - \gamma_i U_s(a_i, t)$, $i = 1, 2$, where β_i and γ_i are constants. Let the heat equation in the form $L_s[U] + \Omega(s, t) = g(s)U_t$, $a_1 < s < a_2$, $t > 0$, be accompanied by conditions $U(s, 0) = F(s)$, $M_i[U] = \psi_i(t)$. The author describes a procedure for obtaining solutions of such boundary-value problems in the form $U = \sum_{n=0}^{\infty} Q_n(t) R_n(s)$, where R_n are eigenfunctions of the Sturm-Liouville problem $L_s[R(s)] + \xi^2 g(s) R(s) = 0$, $M_i[R] = 0$. The functions $Q_n(t)$, the Sturm-Liouville transform of $U(r, t)$, satisfy a problem in ordinary differential equations of first order. The author treats special cases in which h is constant and p and g are linear functions, where the functions R_n and Q_n are found explicitly. *R. V. Churchill* (Ann Arbor, Mich.)

QUANTUM MECHANICS

See also 2955, 3116, 3555, 3558, 3802, 3803, 3857.

Yaris, Robert 3655
Resolvent operator formulation of stationary state perturbation theory.
J. Chem. Phys. **40** (1964), 1891–1897.
Author's summary: "By starting with an exact operator equation and using different methods of expanding the resolvent operator, the Schrödinger, Wigner-Brillouin, similarity transformation, gauge transformation, and first-order perturbation iteration method, perturbation expansions are generated in a rigorous and straightforward manner. It is also shown how additional perturbation and perturbation iteration methods can be generated."

Delves, L. M. 3656
Variation principles for an arbitrary operator. II. Off-diagonal elements and scattering states.
Nuclear Phys. **45** (1963), 313–320.
Author's summary: "In a previous paper [Nuclear Phys. **41** (1963), 497–503; MR **26** #7317] a variation principle was given for the diagonal matrix elements $\langle\alpha|W|\alpha\rangle$ of an arbitrary Hermitian operator W. The method is here extended to give variation principles for the off-diagonal elements $\langle\alpha|W|\beta\rangle$, separate principles being given for the real and imaginary parts of the matrix element. The results apply directly to bound state solutions of the Schrödinger equation; the extension to scattering states representing arbitrarily involved collisions is also given, and variation principles derived for matrix elements between two scattering states and between scattering and bound states." *J. McKenna* (Murray Hill, N.J.)

Vetchinkin, S. I. [Vetčinkin, S. I.] 3657
Determination of optimal functions in approximating matrix elements.
Dokl. Akad. Nauk SSSR **147** (1962), 1328–1331 (*Russian*); *translated as Soviet Physics Dokl.* **7** (1963), 1132–1134.
The author notes that if ψ_A and ψ_B are approximate wave functions for a given Hamiltonian determined by a variational principle with respect to the energy, then $D_{AB} = \int \psi_A{}^* D \psi_B \, d\tau$ may be a very poor approximation to the matrix element of an arbitrary operator D. He then proceeds to construct trial wave functions which better approximate the matrix elements of D.
J. McKenna (Murray Hill, N.J.)

Konstantinov, O. V.; Perel', V. I. 3658
Coherence of states in the scattering of modulated light.
Ž. Èksper. Teoret. Fiz. **45** (1963), 279–284 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 195–198.
An analysis is made of the resonance scattering of amplitude modulated light by a three-level system in which the upper two levels are closely spaced. The method is that of semi-classical, time-dependent perturbation theory with Weisskopf-Wigner damping coefficients. It is shown that a resonance exists between the modulation frequency and the upper level separation.
I. R. Senitzky (Fort Monmouth, N.J.)

Bevensee, R. M. 3659
Quantum electrodynamics of the stimulated emission of radiation. I.
J. Mathematical Phys. **5** (1964), 308–324.
This article begins with a summary of the work of the reviewer on induced and spontaneous emission in a coherent field which employs perturbation theory to describe the short-time development of interacting molecules and electromagnetic field in a resonant cavity. The results of this work are then iterated to obtain a description of the long-time behavior of the combined system by means of second-order coupled differential equations for the field energy and molecular energy (or population). These differential equations are applied to several problems, one being the self-modulation of a solid-state laser.

The above differential equations seem to be, essentially, coupled rate equations, and suffer from the same weakness as rate equations, namely, the neglect of correlation effects between molecules and field. An interaction in which these effects are important cannot be described in terms of energies only. The correlation is dropped, of course, in the iteration process, since the initial state is an uncorrelated one. These coupled equations are, however, an improvement over those rate equations—frequently

used—in which the energy of only one of the two interacting systems, molecules and field, is considered to vary.
I. R. Senitzky (Fort Monmouth, N.J.)

Średniawa, Bronisław; Weyssenhoff, Jan 3660
On the approximate applicability of the Schrödinger equation to non-isolated systems.
Acta Phys. Polon. **23** (1963), 177–188.
The authors consider the problem of two almost isolated systems coupled by a weak interaction and satisfying the time-independent Schrödinger equation

$$(*) \qquad [H_A+H_B+\lambda V-E]\psi = 0.$$

They show that with suitable restrictions on H_B, and for sufficiently small λ, (*) can be reduced to a time-dependent Schrödinger equation for the system A. The conditions correspond to (B) being of macroscopic dimensions.
L. M. Delves (Kensington)

Glasser, M. L. 3661
Summation over Feynman histories: Charged particle in a uniform magnetic field.
Phys. Rev. (2) **133** (1964), B831–B834.
Author's summary: "Using a particular parametrization of paths, the nonrelativistic propagator for a charged particle in a uniform magnetic field is derived by the Feynman method of summation over histories. It is shown that this sum is independent of the parametrization as long as the classical path is included. The result is used to obtain the density matrix for the system."
L. M. Delves (Kensington)

Amai, Saburo 3662
Theory of measurement in quantum mechanics. Destruction of interference.
Progr. Theoret. Phys. **30** (1963), 550–562.
The author asserts that a new supplementary assumption should be attached to quantum mechanics in order to explain the destruction of interference which brings about the so-called "reduction of a wave packet". After a somewhat long discussion of several examples, he concludes that it is necessary to assume that there are limits in the precision of measurements of coordinates and momenta of individual particles. *H. Wakita* (Hiroshima)

Sasakawa, Tatuya 3663a
New method for solving eigenvalue problems.
J. Mathematical Phys. **4** (1963), 970–992.

Sasakawa, Tatuya 3663b
Determinantal method in perturbation theory.
J. Mathematical Phys. **5** (1964), 379–382.

This is a lengthy paper manipulating the perturbation formulae used by physicists in computing operator eigenvalues. Its actual mathematical content, if any, is very far from clear, although the second companion paper affords some clarification. The author claims increased computational effectiveness from his formulae.
F. H. Brownell (Seattle, Wash.)

Bazley, Norman W.; Fox, David W. 3664
Error bounds for expectation values.
Rev. Modern Phys. **35** (1963), 712–716.
This is a brief survey of various methods for estimating the expectation value $(B\psi, \psi)$ of a self-adjoint operator B in a state ψ, where ψ is a normalized eigenvector of the Hamiltonian H. If $B=H$, the Rayleigh-Ritz method or its various generalizations can be used to estimate $E=(H\psi, \psi)$. Let φ be an approximate eigenvector determined by such a method; then $\|\psi-\varphi\|$ can be estimated. If B is any bounded operator, $(B\psi, \psi)$ is approximated by $(B\varphi, \varphi)$ with an error which can be estimated in terms of $\|\varphi-\psi\|$ and B. If B is unbounded, $(B\varphi, \varphi)$ is expected to give an approximation of $(B\psi, \psi)$ only when B is bounded relative to H. *T. Kato* (Berkeley, Calif.)

Jordan, T. F.; Macfarlane, A. J.; Sudarshan, E. C. G. 3665
Hamiltonian model of Lorentz invariant particle interactions.
Phys. Rev. (2) **133** (1964), B487–B496.
This paper contains an explicit solution (apparently the first non-trivial solution ever published) of Dirac's problem [P. A. M. Dirac, Rev. Modern Phys. **21** (1949), 392–399; MR **11**, 409], which is to exhibit the ten generators of infinitesimal Lorentz transformations for a system of two interacting particles in quantum mechanics. Some explicit solutions of this problem were already known [L. Foldy, Phys. Rev. (2) **122** (1961), 275–288], but they were physically unacceptable because the interaction between the particles did not vanish in asymptotic states, i.e., when the particles were infinitely distant from each other.
The authors start from the well-known generators of Lorentz transformations for two free (non-interacting) particles and perform a unitary transformation which does not affect, of course, the commutation relations of the generators. For instance, the free Hamiltonian H_0 becomes $H=\Omega_+H_0\Omega_+{}^*$, where Ω_+ is unitary and time-independent. The authors then construct another unitary operator Ω_- such that $H=\Omega_-H_0\Omega_-{}^*$ and show that $\exp[-iHt]\Omega_\pm\Psi\to\exp[-iH_0t]\Psi$ for $t\to\mp\infty$, for any Heisenberg state Ψ. This means that the Schrödinger states of the new theory are asymptotically free particle states. The interaction is therefore physically acceptable.
The authors further show that the total momentum and angular momentum operators are the same as for a system of two free particles. This ensures familiar transformation properties under space translations and space rotations. The scattering amplitude is a manifestly invariant function of the particle momentum variables, and can be made to have a variety of analyticity properties by a suitable choice of the arbitrary form factors which occur in the model. *A. Peres* (Haifa)

Fuščič, V. I. 3666
Analytic properties of the production amplitudes in a one-particle approximation as functions of two variables. (Russian)
Ukrain. Mat. Ž. **15** (1963), 227–232.
By means of the Jost-Lehmann-Dyson integral representation, the author studies analytic properties of the production amplitude for $\pi+n\to n+\pi+\pi$ as a function of the invariant momentum transfers $(p_1-p_3)^2$ and $(p_2-p_4)^2$

between the two nucleons and between two of the mesons. In the single-particle approximation he obtains a domain of analyticity significantly larger than that of Ascoli [Nuovo Cimento (10) **18** (1960), 754–769; MR **24** #B279].
J. M. Cook (Argonne, Ill.)

Lomont, J. S.; Moses, H. E. 3667
The assignment of wave functions to energy densities and probability densities. (Italian summary)
Nuovo Cimento (10) **30** (1963), 1291–1297.

In many problems involving wave propagation the squares of the absolute values of the wave functions, rather than the wave functions themselves, are the physically observable quantities. It is sometimes of interest to determine whether and how these physically observable quantities determine the wave function uniquely. This paper provides the following particular answer for the special case of one-dimensional wave motion: Let $f(x)$ denote the value of a given one-dimensional wave function at fixed time (say $t=0$) and $f_a(x)$ the truncated function $\eta(x-a)f(x)$, where $\eta(x)=1$ for $x>0$ and 0 for $x<0$. Let $F_a(x)$ denote the Fourier transform of $f_a(x)$. Then the values of $|f_a(x)|$ and $|F_a(x)|$ for all a determine the value of $f(x)$ up to a constant multiple of absolute value one.

The paper then discusses the interpretation of this result in classical radiation and elementary quantum mechanical systems. *R. T. Prosser* (Lexington, Mass.)

Jánossy, L. 3668
Zum hydrodynamischen Modell der Quantenmechanik.
Z. Physik **169** (1962), 79–89.

The hydrodynamical model of quantum mechanics is defined as that in which a quantum system is studied through the functions $P=\psi^*\psi$ and $\mathbf{v}=(\hbar/m)\,\mathrm{Im}\,\psi^*\nabla\psi$. This paper investigates, under certain conditions, the possibility of defining the problem entirely through these functions, thus carrying somewhat further the well-known results in which the differential equations satisfied by them are derived from the Schrödinger equation. Subject to only one simple and not particularly restrictive condition (involving quantization and conservation of the vortex strength), these functions as determined in the course of time by their own differential equations are shown to determine the wave function and therefore all observable results in the cases of (a) time-independent scalar potential, and (b) electromagnetic forces expressed through a given scalar and vector potential. The density and velocity distributions undergo oscillations with frequencies equal to the differences between the eigenfrequencies of the Schrödinger equation; this appears to provide a physical picture of the origin of the frequencies found in emission. It is also stated, although no proof is given in this paper, that correct selection rules correlated with dipole, octupole, etc., modes can be derived.
A. Siegel (Boston, Mass.)

Boguš, A. A.; Fedorov, F. I. 3669
Properties of the Duffin-Kummer matrices. (Russian)
Dokl. Akad. Nauk BSSR **6** (1962), 81–85.

The properties of the Duffin-Kummer matrices are analyzed from a general point of view and useful formulae for traces of their products are obtained.
J. Zak (Cambridge, Mass.)

Andrade e Silva, Joao; Fer, Francis; Leruste, Philippe; Lochak, Georges 3670
Quantification, stationnarité et non-linéarité.
Cahiers de Phys. **15** (1961), 210–224.

The notions of stationarity and quantization, suitably defined for a probabilistic process, are discussed in their mutual relation. A tentative outline of a nonlinear theory is given. *F. Herbut* (SA **64A** #15690)

Berezin, F. A.; Pohil, G. P.; Finkel'berg, V. M. 3671
The Schrödinger equation for a system of one-dimensional particles with point interaction. (Russian. English summary)
Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 1, 21–28.

Authors' summary: "The solution of the Schrödinger equation for n one-dimensional particles with point interaction is obtained in explicit form. The ψ-functions of the scattering theory are found and their completeness is proved. The scattering operator is constructed and its eigenfunctions together with scattering phases are found."

Robinson, Peter D.; Hirschfelder, Joseph O. 3672
Generalized momentum operators in quantum mechanics.
J. Mathematical Phys. **4** (1963), 338–347.

The quantum-mechanical definition of the generalized momentum operator P_i conjugate to a generalized coordinate q_i is discussed in connection with the requirement of "hermiticity". The usual formally self-adjoint expression for P may not be hermitian when $q_i(x,y,z)$ behave unfavourably near the points where q_i takes one of its extreme values q_i (e.g., $\theta=0$ or π in the parabolic coordinate). It is pointed out that the hermiticity is recovered by the addition of some delta function having support in $q_i=q_i$. {Reviewer's remark: This addition may be replaced by a sort of boundary condition on the set $q_i=q_i$.} No investigation is made on the self-adjointness of P. The paper also contains an application to hypervirial operators and an Appendix by Charles J. Goebel, in which the same form for P is deduced in a different way.
S. T. Kuroda (Tokyo)

Robinson, Peter D. 3673
On the nonorthogonality of generalized momentum eigenfunctions in quantum mechanics.
J. Mathematical Phys. **4** (1963), 348–353.

In connection with the study of eigenfunctions of a generalized momentum operator discussed in the paper reviewed above [#3672], the author asserts that "the orthogonality requirement is not a necessary condition to be met in order that a consistent expansion be possible". The following example may illustrate the author's viewpoint. Any L^2 function $f(x)$ in a finite interval can be expanded as

$$f(x) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} a(\lambda)\exp(i\lambda x)\,d\lambda,$$

where $a(\lambda)$ is the Fourier transform of f extended as 0 outside of the interval, even though the set of "eigenfunctions" $\{\exp(i\lambda x),\ -\infty<\lambda<\infty\}$ of the operator $-d^2/dx^2$ does not satisfy the orthogonality requirement in the interval. *S. T. Kuroda* (Tokyo)

Mandelbrojt, Jacques; Mandelbrojt, Szolem 3674
Sur les "coupures" en mécanique quantique.
C. R. Acad. Sci. Paris **258** (1964), 1173–1176.

The authors consider a conventional, non-relativistic Schrödinger equation for a bound S-state. As is well known, the behaviour at infinity of the Fourier transform of the radial wave function depends on the singularity at the origin of the potential. In a problem where one is forced to make a numerical integration of the Schrödinger equation, a standard procedure is to calculate the normalization integral over a finite interval and to put this number equal to one. The paper under review here discusses the accuracy of such a procedure carried out in momentum space. The authors show that if one uses a finite interval δ for normalization and thereby obtains a value A for the normalization constant, these two numbers must be related by an inequality of the form $\delta > \Delta(A)$, where $\Delta(A)$ is an expression which can be explicitly constructed. It depends on the asymptotic behaviour of the wave function of the bound state but does not (explicitly) involve the singularity at the origin of the potential. *G. Källén* (College Park, Md.)

Rosen, S. P. 3675
Some properties of triangular representations of $SU(3)$.
J. Mathematical Phys. **5** (1964), 289–293.

Irreducible representations of $SU(3)$ are symmetric traceless tensors with p upper indices and q lower indices. Generally, the direct product of an irreducible representation $T^{i_1\cdots i_p}_{j_1\cdots j_q}$ with its conjugate $\bar{T}^{j_1\cdots j_q}_{i_1\cdots i_p}$ contains the adjoint representation twice:

$$\sum_{i_1\cdots i_{p-1}} \sum_{j_1\cdots j_q} \bar{T}^{j_1\cdots j_q}_{i_1\cdots i_{p-1}a} T^{i_1\cdots i_{p-1}b}_{j_1\cdots j_q},$$

$$\sum_{i_1\cdots i_p} \sum_{j_1\cdots j_{q-1}} \bar{T}^{j_1\cdots j_{q-1}b}_{i_1\cdots i_p} T^{i_1\cdots i_p}_{j_1\cdots j_{q-1}a}.$$

Triangular representations have $p=0$ or $q=0$. Evidently, for them this direct product contains the adjoint representation but once. This fact is significant in the eightfold model of strong interaction symmetry [M. Gell-Mann, California Inst. Tech. Synchrotron Lab. Rep. No. 20 (1961) (unpublished); Y. Ne'eman, Nuclear Phys. **26** (1961), 222–229; MR **23** #B2856] where elementary particles form multiplets transforming under representations of $SU(3)$. In the work under review, a new proof of this "theorem" is given, and its physical consequences are explored. *S. L. Glashow* (Berkeley, Calif.)

Rosendorff, S.; Tani, S. 3676
New series for phase shift in potential scattering.
Phys. Rev. (2) **131** (1963), 396–406.

Authors' summary: "A new series for the phase shift has been derived for the Schrödinger, Klein-Gordon, and Dirac equations. This series converges faster than the Born series for the tangent of the phase shift. This is so because the sum of the first n terms in the new series includes exactly all the terms up to the $2(2^n-1)$th order in the Born series. Under a condition which is tantamount to the fact that the phase shift cannot be larger than 63°, the series converges absolutely. At high energies, the series can be analytically continued with respect to the strength of the potential beyond such a limit. It is shown that the high-energy limit of the phase shift is given by its first Born approximation and that the difference between even and non-even potentials is reflected in the respective phase shifts to all orders." *H. M. Nussenzveig* (Rio de Janeiro)

Brehm, J. J.; Sucher, J. 3677
Dispersion theory methods for pion-deuteron elastic scattering. I.
Ann. Physics **25** (1963), 1–47.

Authors' summary: "The problem of low energy pion-deuteron elastic scattering is undertaken using the methods of dispersion theory. Complex singularities do not occur if the momentum transfer is used as the variable in a dispersion relation with the energy taken to be fixed and real. Anomalous discontinuities are developed by analytic continuation in the deuteron mass. Subsequent analytic continuation in the energy provides the representation of the physical amplitude. A sample calculation is made, ignoring spin. The contributions of pion-nucleon scattering are estimated by assuming the dominance of the $T=\frac{3}{2}$ resonance in the energy channel and of the ABC resonance in the momentum transfer channel. The angular distribution is computed at 85 Mev and some agreement with experimental data is obtained."

Brehm, J. J. 3678
Dispersion theory methods for pion-deuteron elastic scattering. II.
Ann. Physics **25** (1963), 221–248.

In an earlier paper the author and Sucher [#3677] proposed a representation for the pion-deuteron scattering amplitude. It is based on the analyticity properties of some of the simplest Feynman graphs, and takes the form of a dispersion relation in the momentum transfer t with the squared energy s fixed at a real, physical value. To simplify the analysis, all particles are given zero spin. The representation is meant as a substitute for the Mandelstam representation, the latter being invalid in this case because of complex singularities. Certain anomalous thresholds that occur are regarded as crucial. The present paper is concerned with the determination of discontinuities over the cuts connecting anomalous and normal thresholds. The technique is to begin with values for the deuteron mass and the variable s such that there are no anomalous thresholds and discontinuities are given by physical unitarity. The initial value for s is such as to make it a physical momentum transfer for each of the crossed channels. Analytic continuation in the deuteron mass and in s finally leads to expressions for the anomalous discontinuities. This procedure was sketched in #3677 above, and the author fills in the details. *R. L. Warnock* (Chicago, Ill.)

Falk, David S. 3679
Equivalence of the Brysk approximation and the determinantal method.
Phys. Rev. (2) **129** (1963), 2340–2341.

Recently, Brysk [same Rev. (2) **126** (1962), 1589–1595; MR **25** #4791] obtained for a spherically symmetric potential a new formula for the tangent of the phase shifts. In the article under review the author shows that

Brysk's result is obtained from the lowest-order approximation in the determinantal method based on the Fredholm solution for an integral equation. An advantage of this derivation, besides being more transparent than Brysk's derivation, is that it provides a systematic method of generating convergent higher-order approximations. *S. Rosendorff* (Haifa)

Adawi, I. 3680

Incoming waves in final states in matrix elements for scattering problems.

Amer. J. Phys. **32** (1964), 211–215.

Author's summary: "The use of modified plane waves with the spherical incoming waves modification as final states in matrix elements used in calculating scattering cross sections is discussed from the standpoint of two techniques. The first method makes use of general properties of Green's functions for outgoing and incoming waves and their reciprocity relations. The second method is based on expanding Green's function in terms of modified plane waves with either incoming or outgoing spherical wave modification."

★**Quantum scattering theory. Selected papers.** 3681

Edited by Marc Ross. Introduction by Roger G. Newton.

Indiana University Press, Bloomington, Ind., 1963. xvi+307 *pp.*

This highly welcome and neatly printed book is a collection of reprints of papers on the fundamental questions of scattering theory. The first is a translation of Brenig and Haag, "General quantum theory of collision processes", which appeared originally in German [Fortschr. Physik **7** (1959), 183–242; MR **21** #6978] and is relatively unknown in this country. The second is Møller's classical paper, "General properties of the characteristic matrix in the theory of elementary particles" [Danske Vid. Selsk. Mat.-Fys. Medd. **23** (1945), no. 1]. Then follows Newton's thorough article "Analytic properties of radial wave functions" [J. Mathematical Phys. **1** (1960), 319–347; errata, ibid. **1** (1960), 452; MR **22** #6490]. The last two contributions of both great historical and practical interest are the reprints of Lippmann and Schwinger "Variational principles for scattering processes" [Phys. Rev. (2) **79** (1950), 469–480; MR **12**, 570], and Gell-Mann and Goldberger "The formal theory of scattering" [ibid. (2) **91** (1953), 398–408; MR **15**, 382]. The volume is preceded by a short orientation to the subject, written by R. G. Newton, which also contains an extremely useful list of relevant material for further reading.

P. Roman (Boston, Mass.)

Cunningham, J. 3682

The number of independent variables for an n-particle collision.

Austral. J. Phys. **16** (1963), 586–587.

The problem is to count the number of independent variables necessary to determine all of the scalar products (p_i, p_j), where p_i is the four-momentum of a particle in an n-body collision. A solution is supplied by elementary linear algebra. *R. L. Warnock* (Chicago, Ill.)

Curtiss, C. F.; Powers, Robert S., Jr. 3683

An expansion of binary collision phase shifts in powers of h.

J. Chem. Phys. **40** (1964), 2145–2150.

Authors' summary: "The phase shifts in binary collisions are expressed as series in increasing powers of Planck's constant, h. The series begins with a term in h^{-1} and contains all odd powers. Recursion relations are developed which lead to an expression for an arbitrary term in the series. The results apply to an arbitrary interaction potential function, for which any odd number of classical turning points may arise."

Gronskiĭ, V. K. 3684

Scattering of a polarized particle with a spin 3/2 Coulomb center. (Russian)

Dokl. Akad. Nauk BSSR **7** (1963), 598–601.

Matrix elements for the scattering of polarized particles with spin $\frac{3}{2}$ in the case of varying absolute value of the component of the spin have been calculated. The particular case of scattering of polarized particles with spin $\frac{3}{2}$ in a Coulomb field is considered in detail.

J. Zak (Cambridge, Mass.)

Sooy, Walter R. 3685

Approximate calculations of scattering phase shifts.

J. Mathematical Phys. **5** (1964), 147–154.

This paper considers an approximate method for calculating the scattering phase shifts for a non-relativistic Schrödinger equation with a spherically symmetric potential. The method involves the use of a selected change of variables which transforms the given Schrödinger equation into an equation for which the phase shifts are known approximately. The author suggests several appropriate changes of variable which depend upon such well-known approximation schemes as those of W.K.B. and Born; however, it is not clear under what conditions this procedure is an improvement over the well-known schemes. *I. Kay* (Ann Arbor, Mich.)

Nishida, Yoshihiko; Shimauchi, Masaaki; Tanaka, Hiromi 3686

An approximation method for high-energy potential scattering.

Nuclear Phys. **50** (1964), 403–416.

Authors' summary: "A simple approximation method is proposed to investigate the scattering of high-energy Schrödinger particles by some spherically symmetrical potentials which have no sharp boundaries but whose shapes do not deviate much from a square well form. The method is applicable when the strength of the potential is substantially smaller than the incident energy and k_0R is greater than unity, where k_0 is the wave number of the incident particle and R is a rough measure of the range of the potential. The procedure consists in making use of the Hart-Montroll internal wave function for the square well potential which is equivalent to the original potential in the least-square sense as a trial function on solving an integral equation for the scattering wave function. Then, it is shown that the scattered amplitude can be written in a similar form to that of the second Born approximation. As examples, the method is applied to the scattering by

Gaussian and harmonic well potentials which may be complex. In these cases, the scattered amplitudes can be written in tractable forms and furthermore it is expected that they can be used with good accuracy for a fairly wide range of scattering angles including those around $(k_0R)^{-1/2}$ in contrast with the Schiff or Glauber approximation. Finally, numerical illustrations are given and the results are compared with those of the exact method and other approximations, and thereby the validity and usefulness of the present method are confirmed."

Amado, R. D. 3687

Soluble problems in the scattering from compound systems.

Phys. Rev. (2) **132** (1963), 485–494.

Author's summary: "The problem of making an exact theory of the scattering of particles from composite targets is attacked by introducing elementary particles to represent the composite systems. It is shown that if the only couplings are those between the particle representing the compound and its constituents, soluble linear integral equations, reducible to the Lippmann-Schwinger equation, can be written for the scattering of one of the constituents of the composite system. As examples, an exactly soluble three-dimensional three-body problem based on nucleon-deuteron scattering and an exactly soluble three-dimensional model of deuteron stripping are presented. Each can be reduced to exact optical models. It is proven that these equations have solutions even when the singular limit which corresponds to an exact resemblance between the elementary and composite system is taken. The method for extending the equations to three-body problems with local interactions and the relation of the equations presented here to high-energy diffraction properties of amplitudes is discussed."

S. Gartenhaus (Lafayette, Ind.)

Ferretti, B. 3688

On the possibility of a macroscopically causal quantum-relativistic theory.

Nuovo Cimento (10) **27** (1963), 1503–1505.

The problem of local measurements in quantum field theory is analyzed. It is stated (full details are postponed to a future publication [Atti Accad. Sci. Ist. Bologna]) that for free fields a suitable set of quasi-local operators can be found through which the interaction with sources and measuring devices can be defined so that the theory is Lorentz-invariant and macroscopically (i.e., asymptotically) causal. The same conclusions are shown for a theory in which two scalar fields interact through a non-local bilinear interaction. *E. R. Caianiello* (Naples)

Sukhanov, A. D. [Suhanov, A. D.] 3689

On the charged vector theory of Lee and Yang.

Ž. Èksper. Teoret. Fiz. **44** (1963), 2087–2089 (*Russian. English summary*); *translated as Soviet Physics JETP* **17** (1963), 1404–1405.

Author's summary: "The charged vector meson theory of Lee and Yang (ξ limiting process) is derived within the framework of the Lagrangian formalism and can thus be treated (for finite values of ξ) as a more consistent variant of the Pauli-Villars regularization method. An interpretation of the two postulates of Lee and Yang employed in going to the limit $\xi \to 0$ is presented. It is found that, while the second postulate can be formulated in accordance with the usual concepts of quantum field theory, the validity of the first postulate is highly questionable, since it allows without any justification the exclusion of the most important terms in non-renormalizable theories."

E. R. Caianiello (Naples)

Peres, A. 3690

Spurious nature of ultraviolet divergences. (Italian summary)

Nuovo Cimento (10) **28** (1963), 78–89.

The following recipe is given to compute the (otherwise divergent) integrals which appear in the perturbative solutions of quantum field theory: (a) regularize the integrals by some method (e.g., that of Pauli-Villars) to make them uniformly convergent in the external variables (e.v.); (b) take enough derivatives with respect to the e.v. to make the resulting integrals uniformly convergent even without regularization; (c) remove the regularization; (d) integrate back with respect to the e.v.

In the last stage undetermined constants appear; the author claims that a suitable choice of these constants reproduces the results usually obtained by renormalization techniques.

A few applications of the recipe are given, and it is argued that the undetermined constants may be related to terms in the Lagrangian which are of order higher than 1 in the coupling constants. *E. R. Caianiello* (Naples)

Berezin, F. A. 3691

Canonical transformations in the second quantization representation. (Russian)

Dokl. Akad. Nauk SSSR **150** (1963), 959–962.

Let the fermion (boson) field variables be given by linear mappings $\hat{p}$ and $\hat{q}$ from the Hilbert space L into an irreducible set of operators satisfying the anticommutation (commutation) relations on antisymmetric (symmetric) Fock space **H**. The orthogonal (augmented symplectic) transformations Υ of L (of a manifold D dense in L) induce canonical transformations $\mathbf{A}_\Upsilon$ of the field variables. $\mathbf{A}_\Upsilon$ is said to be proper if and only if there exists a unitary U_Υ on **H** such that $\mathbf{A}_\Upsilon(\cdot) = U_\Upsilon(\cdot)U_\Upsilon^{-1}$. A linear operator $\hat{A}$ on **H** is said to sustain $\mathbf{A}_\Upsilon$ if and only if the strong limit, depending only on Υ, of $U_{\Upsilon_n}\hat{A}U_{\Upsilon_n}^{-1}f$ exists for all $\Upsilon_n \to \Upsilon$ such that f and $U_{\Upsilon_n}^{-1}f$ are in the domain of $\hat{A}$.

Three theorems are stated without proof: (1) Every Υ is the strong limit of Υ_n belonging to proper $\mathbf{A}_{\Upsilon_n}$. (2) In the fermion case, embed L in its Clifford algebra [I. E. Segal, Ann. of Math. (2) **63** (1956), 160–175; MR **17**, 1114]. Form the inner product by means of its central state, and adjoin all operators which are the limit of a sequence which converges both in that norm and in the strong topology of operators on **H**. Then the resulting pre-Hilbert space contains a bounded $\hat{A}$ if and only if it sustains all $\mathbf{A}_\Upsilon$. (3) In the boson case $\hat{A}$ can be made to correspond to a Wigner-Moyal quasi-probability function $\mathfrak{A}_{\hat{A}}(p, q)$ on classical phase space. Suppose $\mathfrak{A}_{\hat{A}}$ to be square-integrable with respect to some measure quasi-invariant under a particular $\mathbf{A}_\Upsilon$. Then $\hat{A}$ sustains $\mathbf{A}_\Upsilon$, and $\mathfrak{A}_{\mathbf{A}_\Upsilon(\hat{A})}(p, q) = \mathfrak{A}_{\hat{A}}(\Upsilon(p, q))$. *J. M. Cook* (Argonne, Ill.)

Weier, Joseph 3692
Une contribution au formalisme quantique de M. Daniel Kastler.
C. R. Acad. Sci. Paris **256** (1963), 2303–2304.

Let a spinor field over L (real four-dimensional Lorentz space) be represented by a contravariant non-homogeneous antisymmetric tensor over C_2 (complex two-dimensional space) [D. Kastler, *Introduction à l'électrodynamique quantique*, Dunod, Paris, 1961; MR **26** #2202]. In the note under review, one obtains in this representation the commutation relation between the operator div (and rot) and a set of operators β_λ, $\lambda = 1, \cdots, 4$. The β_λ's satisfy the commutation relations $\beta_\lambda\beta_\mu + \beta_\mu\beta_\lambda = 2\delta_{\lambda\mu}$ and are obtained as a linear form of any antisymmetric contravariant tensor over C_2.
E. R. Caianiello (Naples)

Schwinger, Julian 3693
Quantized gravitational field. II.
Phys. Rev. (2) **132** (1963), 1317–1321.

In an earlier paper [same Rev. (2) **130** (1963), 1253–1258; MR **27** #6520] the author identified canonical variables for the gravitational field and verified Euclidean invariance of the quantized theory. But he found that the implicit dependence of the energy density upon the canonical variables through the constraint equations presented an obstacle to verification of Lorentz invariance. In the paper under review he constructs a Lorentz-invariant formulation for the quantized gravitational field coupled to fields of spin 0 and 1 by applying a technique whereby canonical operator variables are combined with mathematical parameters of a functional-transformation group [the author, Nuovo Cimento (10) **30** (1963), 278–291].
P. W. Higgs (Edinburgh)

Sommerfield, Charles M. 3694
On the definition of currents and the action principle in field theories of one spatial dimension.
Ann. Physics **26** (1964), 1–43.

From the author's summary: "The properties of a model relativistic field theory in one space dimension are investigated. The model consists of a massless Fermion field whose current is coupled to itself and also to a vector Boson field with mass. It is found at an early stage that the usual definition of the Fermion current as the simple juxtaposition of the Fermion field and its conjugate is ambiguous and does not lead to a relativistic two-vector. A more precise definition of the current which remedies this situation is obtained by taking the average of the limits of a nonlocal product of the Fermion fields as the coordinates of those fields approach one another from a spacelike and the orthogonal timelike direction. The original use of the action principle in solving the model is re-examined in the light of these developments and a corrected version of the Lagrange function which contains operators at one time only is derived. Since the corrected Lagrange function was written down only after complete knowledge of the solution was at hand, the question of whether it is possible to know the Lagrange function without such knowledge is discussed. It is found that in the present case the answer is no, unless the current appearing in the field equations is defined in a manner different from the one used. This discussion is extended briefly to more general models."
I. Bialynicki-Birula (Warsaw)

Scarfone, Leonard M. 3695
Transition probabilities for the forced quantum oscillator.
Amer. J. Phys. **32** (1964), 158–162.

The transition probabilities between unperturbed energy eigenstates of a single harmonic oscillator forced by a given external "current" $\propto F(t)$ are worked out, using the reduction technique for S-matrix elements of Lehmann, Symanzik, and Zimmermann [Nuovo Cimento (10) **1** (1955), 205–225; MR **17**, 219], which employs only Heisenberg picture fields and states. This solution is useful per se and also has some pedagogical value as an illustration of field-theoretic techniques. However, it is a serious misconception to think that one can bypass the problem of renormalization by using the LSZ reduction technique in field theory, as the author states as one of the advantages of this method. If the Heisenberg equations of motion are used in the reduction, then they must be the equations for the renormalized fields, i.e., the renormalization problem—the determination of the correct counter terms with the correct coefficients—must already have been solved. This is because it is the renormalized, not the original, Heisenberg fields which have as their (weak) time limits the in- and out-fields. The LSZ method can dispense with the problem of renormalization only if the "vacuum τ functions" are given independently, which is never the case in practice. {In his treatment the author uses the original interaction term (second term on the right in Equation (2)) as the correct renormalized interaction. Anent this, as far as this reviewer knows, the correct renormalization theory of a quantized system in interaction with a general external field is unknown; all versions of the existing theory are equivalent to demanding the stability of one quantum state [S. S. Schweber, *An introduction to relativistic quantum field theory*, Section 16, Row, Peterson, Evanston, Ill., 1961; MR **23** #B841], but this obviously can not be the criterion when the external field can create or absorb real quanta.}
R. Ingraham (University Park, N.M.)

Preziosi, B. 3696
Finite-part integrals: A generalization of Hadamard's definition. (Italian summary)
Nuovo Cimento (10) **31** (1964), 187–202.

The author is interested in the particular version of the concept of the "finite part" of an integral which was introduced by M. Riesz [Acta Math. **81** (1949), 1–223; MR **10**, 713]. An integral of the form $g(\alpha) = \int f(x, \alpha)\,dx$, where $f(x, \alpha)$ is an analytic function of α regular in a certain domain, normally defines an analytic function $g(\alpha)$ regular in the same domain of α. However, it may happen that $g(\alpha)$ has an analytic continuation to a larger domain where the original integral does not exist. This analytic continuation is the author's "finite part" of the quantity $\int f(x)\,dx = \int f(x, 0)\,dx$. The uniqueness properties of this definition (and various refinements of it) are discussed in some detail and illustrated with several examples. The procedure turns out not to be unique. This is, however, considered an advantage by the author who wants to apply his formalism to the renormalization technique developed by Caianiello [Nuovo Cimento (10) **13** (1959), 637–661] where no explicit use is made of counter terms. It appears to the reviewer, that the counter terms describing, e.g., the self-mass of a particle

have a very definite physical significance. This is illustrated by the mass difference between the neutral and the charged π-mesons. Therefore, it is not clear how an attempt to avoid all counter terms is to be justified on physical grounds. *G. Källén* (College Park, Md.)

Fairlie, D. B. 3697
An inequality in Feynman integrands.
Proc. Cambridge Philos. Soc. **59** (1963), 157–160.

A particular term in the expansion in perturbation theory of the scattering amplitude of four particles can be represented in parametric form

$$\int_0^\alpha \frac{d\alpha}{C^2} \exp \frac{iD(\alpha)}{C(\alpha)},$$

where D is given in terms of the usual invariants s, t, u and the external and internal masses, M_i and m_j, respectively, by

$$D = \sum_{i=1}^4 K_i(\alpha)M_i^2 + f(\alpha)s + g(\alpha)t + h(\alpha)u - \sum_j m_j^2 C(\alpha).$$

By means of graphical techniques the following inequalities among the coefficients of D, conjectured by Nakanishi, are proved:

$$K_1K_4 \geqq fg,$$
$$K_2K_3 \geqq fg,$$

for the domain $\alpha_i \geqq 0$. *A. O. Barut* (Boulder, Colo.)

Ezawa, Hiroshi 3698
A note on the Van Hove-Miyatake catastrophe.
Progr. Theoret. Phys. **30** (1963), 545–549.

Eigenstates of the total Hamiltonian of quantum field theory are examined. The model used is the neutral scalar field interacting with a fixed source. The field is contained in a box and calculations are performed in the separable Hilbert space $\mathfrak{H}$ (Fock space), where the free Hamiltonian is definable. When the high momentum components of the interaction Hamiltonian are cut off for $|\mathbf{k}|$ larger than K, the eigenstate of the total Hamiltonian is obtained by means of the well-known unitary transformation. The author examines the limits of eigenvectors as $K\to\infty$ and (1) discusses the condition for the form of the source such that the limits exist and the total Hamiltonian is definable as an operator in $\mathfrak{H}$, and (2) when the limits do not exist, i.e., when the so-called van Hove-Miyatake catastrophe occurs, he indicates its mathematical implications. *Y. Kato* (Kobe)

Araki, Huzihiro 3699
von Neumann algebras of local observables for free scalar field.
J. Mathematical Phys. **5** (1964), 1–13.

In the recent work on the axiomatic formulation of quantum field theory von Neumann algebras are associated with certain distinguished subsets of the space-time M. The postulates of the theory require that these algebras satisfy Lorentz covariance, completeness, locality, and (Einstein and primitive) causality conditions [Haag and Schroer, same J. **3** (1962), 248–256; MR **25** #1834]. The author investigates the von Neumann algebras of the free scalar field. Let B be an open subset of M; let $R(B)$ be the von Neumann algebra generated by the spectral projections of $A(h) = \int A(x)h(x)\,d^4x$, where $A(x)$ are field operators, and $h(x)$ infinitely differentiable real functions with support in B. Theorems 1 and 2 show that $R(B)$ satisfies the properties stated above. The duality theorem $R(B') = R(B)'$, where $B' = \{x : (x-y)^2 < 0,\ y \in B\}$, is valid for a restricted class of regions B. A counterexample for the general case of the duality theorem is given. It is shown that $R(B)$, for certain regions B, is not of Type I. *S. Sankaran* (London)

Liu, Yi-Cheng [Liu, I-Ch'en]; Todorov, I. T. 3700
Singularities of some Feynman graphs.
Nuclear Phys. **50** (1964), 273–280.

The explicit determination of the complex singularities of higher-order perturbation diagrams is a very difficult task. The position of such singularities is of particular importance in checking the validity of the Mandelstam representation. The authors consider a parametrization of the Landau surface of singularities which permits a simplification in this determination. In particular, they consider the 'envelope' diagram, which has been considered a possible counter-example, and find this is not so. They also determine the singularities of the 'opened envelope' diagram. *John G. Taylor* (Cambridge, England)

Borgardt, A. A. 3701
Relativistically invariant transformations in wave function space.
Ž. Èksper. Teoret. Fiz. **45** (1963), 116–122 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 86–89.

The operator formalisms of quantized field theory admit the introduction of symmetry transformations which are not associated with transformations of the space-time reference system. Yet these symmetries, which work directly in state space, have associated systems of invariants which presumably express conservation properties of the physical system which is under discussion. The author investigates certain of these schemes which are associated with the operator algebras of Dirac and of Kemmer. In this way he obtains generalizations of certain transformation groups which have appeared earlier in boson field theory. *E. L. Hill* (Minneapolis, Minn.)

Borgardt, A. A. 3702
Transverse and longitudinal states of boson fields and dibaric particles.
Ž. Èksper. Teoret. Fiz. **45** (1963), 123–127 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 90–92.

The concept of transversality and longitudinality in relativistic field theory is studied in the light of new ideas on the interpretation of the Kemmer algebra [see #3701 above]. By introduction of reducible representations, equations for a boson field having two mass states are given without the use of higher-order differential operators. The resulting formalism is similar to earlier work on the intermediate boson in weak interaction theory. *E. L. Hill* (Minneapolis, Minn.)

Remler, Edward A. 3703
Scattering of Yang-Mills quanta.
Phys. Rev. (2) **133** (1964), B1267–B1270.
Author's summary: "Dirac spinor quanta interacting via a Yang-Mills field are considered and lowest order perturbation theoretic transition matrix elements are given for the elementary scattering processes. All expressions are invariant under the full gauge group. All processes exhibit infinite forward scattering as expected from the masslessness and charge of the Yang-Mills field quanta. A helicity conservation law holds when Yang-Mills quanta scatter each other."

Wichmann, Eyvind H.; Crichton, James H. 3704
Cluster decomposition properties of the S matrix.
Phys. Rev. (2) **132** (1963), 2788–2799.
Let S_{nm} denote the part of the S-matrix which describes the probability amplitudes for a transition from an initial m-particle configuration to a final n-particle configuration. If we arrange the m initial particles in r far separated "clusters" containing, respectively, $m_1, m_2, \cdots$ particles ($m = m_1 + m_2 + \cdots + m_r$) and do likewise with the final particles, then it is evident that for such a configuration one should have $S_{nm} = \prod_{i=1}^{r} S_{n_i m_i}$, because the total process in that case consists of r independent (disconnected) processes. This consideration leads to a decomposition of the S-matrix in momentum space of the form

$$S_{nm}(p_1' \cdots p_n'; p_1 \cdots p_n) = \sum \prod_i S^{(T)}_{n_i m_i},$$

where the sum extends over all possible subdivisions of the n arguments p' and the m arguments p into "clusters" and where each factor $S^{(T)}$ contains exactly one energy-momentum δ-function. This structure (which applies to the S-matrix elements as well as to many other hierarchies of functions appearing in quantum field theory or statistical mechanics) is called the "cluster decomposition property". The quantities $S^{(T)}$ will be called here "truncated S-matrix elements".

The authors point out the necessity for this structure on physical grounds and emphasize that while it is easy to write down unitary Lorentz-invariant S-matrices which do not have the cluster structure, it is a non-trivial problem to satisfy the cluster structure together with unitarity and Lorentz invariance. They further point out that the cluster structure is only the first of a number of conditions which the S-matrix must satisfy due to the fact that it makes sense to talk at least in a rough way about the position of an event in space-time. The algorithm relating $S^{(T)}$ and S is also discussed.

R. Haag (Urbana, Ill.)

Penzlin, F. 3705
Die Methode der Feynmanschen Graphen.
Fortschr. Physik **11** (1963), 357–420.
Der Verfasser legt eine Ausarbeitung von Vorlesungen vor, die er z.T. gemeinsam mit Petzold in Marburg und Heidelberg gehalten hat. Ziel dieser Vorlesungen war es, eine elementare Einführung in die Methode Feynmans bei der Behandlung von Wechselwirkungen zwischen Elementarteilchen zu geben. Dabei soll lediglich die Kenntnis der Schrödingerschen Wellenmechanik vorausgesetzt werden. Insbesondere wünscht Verfasser jede Verwendung von Hilfsmitteln der Feldquantisierung zu vermeiden. Er stützt sich dabei vor allem auf die Originalveröffentlichungen Feynmans. Die Darstellung ist sehr ausführlich und im wesentlichen relativ leicht verständlich. Die Methode wird zunächst am Beispiel der Quantenelektrodynamik ausführlich entwickelt. Es wird besonders behandelt: Bewegung von Elektronen in äußeren elektromagnetischen Feldern, Rutherford'sche Streuung, Paarerzeugung, Bremsstrahlung, Möller- und Bhabhastreuung, anomales magnetisches Moment und Selbstenergie des Elektrons, Renormierung bzgl. Masse und Ladung, Regularisierung. Auch über die Infrarotdivergenzen findet sich ein kurzer Abschnitt. Schließlich wird am Schlußabschnitt kurz gezeigt, wie sich diese Technik auch auf starke Wechselwirkungen anwenden läßt.

Insgesamt kann man sagen, daß es sich um eine gelungene, wirklich leicht verständliche Einführung in die genannte Methode handelt. Es ist in der Tat erstaunlich, wieviel man ohne die Verwendung der vollständigen Feldquantisierung erreichen kann. Allerdings scheint es dem Referent, daß man nicht ganz ohne einige Hilfsmittel der Feldquantisierung auskommt. Z.B. läßt sich sonst die Antikommutativität der Diracschen Spinoren kaum verstehen. *G. Heber* (Leipzig)

Kirzhnits, D. A. [Kiržnic, D. A.] 3706
Field theory with nonlocal interaction. II. The dynamical apparatus of the theory.
Ž. Èksper. Teoret. Fiz. **45** (1963), 143–154 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 103–110.
The author treats various questions in a quantum field theory of a pseudoscalar meson coupled to a Dirac field with the usual trilinear contact interaction term replaced by a nonlocal interaction characterized by the form factor $F(x_1, x_2, x_3)$. Renormalization is effected by the method of counterterms in the interaction Lagrangian; he asserts that it can be carried out (i.e., the ingoing one-particle states can be made stable) by a suitable choice of scalar, constant renormalization parameters Z_i ($i = 1, 2, 3$), δM, and $\delta\mu^2$. A generalized S-operator is defined by replacing the ingoing fields in the usual exponential functional by the interacting fields and "ordering with respect to the charge" [the author, same Ž. **41** (1961), 551–559; MR **25** #2811]. It is unitary, relativistically invariant (provided the form factor is a Lorentz form invariant, as is tacitly assumed), and can be proved to reduce to the usual S-operator in the local limit. However, its physical unicity is unclear, that is, its relevance to the scattering implied by the original nonlocal Lagrangian. "Corrected" interacting fields $\mathbf{\Phi}(x)$, etc., are defined with the aid of variational derivatives of this S with respect to ingoing fields. The weak limits $\phi_{\text{out}}(x) \equiv \text{Lim}_{t \to +\infty} \tilde{\mathbf{\Phi}}(x)$, etc., are then guaranteed to be related in the desired way: $\phi_{\text{out}}(x) = S^+ \phi(x) S$ to the ingoing fields $\phi(x)$, etc., whereas this was not necessarily true for the solutions $\mathbf{\phi}(x)$, etc., for the original nonlocal Lagrangian. Other questions are discussed, such as the form of the total four-momentum, the total charge, etc.; the introduction of the Interaction and Schrödinger Pictures alongside the original Heisenberg Picture; and the Lehmann spectral representation of the renormalized complete propagators. The author gives several examples of simple nonlocal theories in which the solutions can be chosen to be either causal or free from additional degrees of freedom, but not both. The nonlocal

theory considered here is free from difficulties with negative energies and does not possess additional degrees of freedom, requiring initial conditions in excess of those needed in local field theory, for any choice of $F(x_1, x_2, x_3)$; in return, it is acausal, violating both microcausality and the Bogoliubov form [N. N. Bogoliubov and D. V. Shirkov, *Introduction to the theory of quantized fields*, § 17.5, Interscience, New York, 1959; MR **22** #1349] of macrocausality.

It should be noted that it has not been proved (1) that a form-invariant form factor can be found which eliminates all ultraviolet divergences from S; (2) that the definition of the interacting fields here given, in particular, the "interpolating" fields $\phi(x, \sigma)$, Equation (30), is free from the Haag Theorem difficulty [R. Haag, Nuovo Cimento (10) **25** (1962), 287–299; MR **26** #3449]; and (3) that renormalization in this nonlocal theory can be effected by means of the scalar and momentum-independent parameters $\{Z_i\}$, δM, and $\delta\mu^2$.

R. Ingraham (University Park, N.M.)

Drummond, I. T. 3707

Singularities of Feynman amplitudes. (Italian summary)

Nuovo Cimento (10) **29** (1963), 720–741.

The author investigates in detail the analytic structure of some perturbation theory amplitudes corresponding to single- and double-loop diagrams. The method adopted is to regard the amplitudes as integrations over internal invariants and then use the standard pinch technique. For single-loop diagrams a condition, equivalent to the positive α condition, is derived for singularity in the physical limit. For double-loop diagrams it is shown that the usual pinch conditions do not always give rise to singularity, and a careful discussion is given of some specific examples.

The reviewer [Nuovo Cimento (10) **27** (1963), 952–959; MR **26** #7338] has given a discussion complementary in part to the author's paper, based on pinch conditions in momentum space. The author states (p. 739) that a remarkable feature of one mechanism for generating mixed singularities is that it involves only Landau curves. In the reviewer's paper (p. 956) it is shown that in a similar case, working with loop momenta, the mechanism involves either Landau curves touching at infinity or a Landau and a non-Landau curve touching at a finite point; the point of view depends on the order of integration and in the basic product space of internal loop momenta correspond to only some of these momenta being infinite.

It is also possible to understand from this point of view why spurious singularities arise. For the two-loop diagram, let S_i be singularity surfaces in $k \times k'$. On integrating over k, we are left with the k'-integration and, apart from the original S_i, Landau singularity surfaces L_{ij}, L_{ijk}, etc., are present in k'-space, L_{12} arising from the touching of S_1, S_2 to trap the k-contour. Singularities of the final amplitude will be caused by L_{12} touching S_3, etc., in k'-space, but not by L_{12} touching, say, L_{13} as this corresponds only to S_1 touching S_2, S_3 at different points in k-space and gives no k'-pinch. This is exactly the condition derived by the author (equations (74), (75)) and a consideration of the singularity configuration in the $(k \times k')$-space clarifies why this does not correspond to a genuine pinch situation. *M. Fowler* (College Park, Md.)

Klauder, John R. 3708

Continuous-representation theory. III. On functional quantization of classical systems.

J. Mathematical Phys. **5** (1964), 177–187.

Parts I and II appeared in same J. **4** (1963), 1055–1058 [MR **27** #2279]; ibid. **4** (1963), 1058–1073 [MR **27** #2280]. Author's summary: "The form of Schrödinger's equation in a continuous representation is indicated for general systems and analyzed in detail for elementary Bose and Fermi systems for which illustrative solutions are given. For any system, a natural continuous representation exists in which state vectors are expressed as continuous, bounded functions of the corresponding classical variables. The natural continuous representation is generated by a suitable set $\mathfrak{S}$ of unit vectors labeled by classical variables for which, for the system in question, the quantum action functional restricted to the domain $\mathfrak{S}$ is equivalent to the classical action. When a classical action is viewed in this manner it contains considerable information about the quantum system. Augmenting the classical action with some physical significance of its variables, we prove that the classical theory virtually determines the quantum theory for the Bose system, while it uniquely determines the quantum theory for the Fermi system."

H. Wakita (Hiroshima)

Calogero, F. 3709

Perturbation theory and nonanalyticity in the coupling constant in a field-theoretical model. (Italian summary)

Nuovo Cimento (10) **30** (1963), 916–930.

A Hamiltonian is given whose exact spectrum contains a branch that is inaccessible in the perturbative solution.

G. Barton (Brighton)

Ramakrishnan, Alladi; Venkatesan, K.; Devanathan, V. 3710

A note on the use of Wick's theorem.

J. Math. Anal. Appl. **8** (1964), 345–349.

Using the concept of "typical sequence of events" introduced in a previous paper [A. Ramakrishnan, T. K. Radha, and R. Thunga, same J. **4** (1962), 494–526; MR **27** #5520] it is shown that considerable simplification can be obtained in the use of Wick's theorem.

P. Roman (Boston, Mass.)

Blokhintsev, D. I. [Blohincev, D. I.] 3711

Geometric optics of elementary particles. (Italian summary)

Nuovo Cimento (10) **30** (1963), 1094–1099.

Geometric optical approximation for two-body scattering in quantum field theory and the convergence of this approximation process, in the high-energy limit, are briefly discussed. *S. Azuma* (Fukusima)

Goldstone, Jeffrey; Salam, Abdus; Weinberg, Steven 3712

Broken symmetries.

Phys. Rev. (2) **127** (1962), 965–970.

In this paper formal proofs are given of Goldstone's conjecture which states that if there is a continuous symmetry transformation under which the Lagrangian is

invariant, then either the vacuum state is also invariant under the transformation, or there must exist spinless particles of zero mass. It is interesting to notice, however, that several authors "disproved" this conjecture recently. For example see S. Kamefuchi and H. Umezawa [Nuovo Cimento (10) **31** (1964), 429–446] and M. Baker, K. Johnson and B. W. Lee [see #3753 below]. This is a typical example which shows that a "formal proof" in quantum field theory sometimes goes wrong.

Y. Takahashi (Dublin)

Cheshkov, A. A. [Češkov, A. A.]; Shirokov, Yu. M. [Širokov, Ju. M.] 3713

Invariant parametrization of local operators.

Ž. Èksper. Teoret. Fiz. **44** (1963), 1982–1992 (*Russian. English summary*); *translated as Soviet Physics JETP* **17** (1963), 1333–1339.

This is a paper with the purpose of giving general formulae which express the matrix elements of various local operators in terms of invariant form factors.

J. M. Jauch (Geneva)

Rohrlich, F. 3714

Functional differential calculus of operators.

J. Mathematical Phys. **5** (1964), 324–331.

The functional derivative with respect to operators of operator functionals is defined for operators which satisfy commutation relations of interest in quantum field theory. From this definition, a functional differential calculus is developed for functionals of tensor, as well as spinor, fields.

Let $u_\alpha(x)$ be an operator field which transforms according to the irreducible Lorentz group representation and let $\bar{u}_\alpha(x)$ be its adjoint with the following commutation relations:

$$(1)\qquad [u_\alpha(x), u_\beta(y)] = -i\Delta_{\alpha\beta}(x-y).$$

Then the functional derivative of a functional F of u and $\bar{u}$ is given implicitly by

$$(2)\qquad [u_\alpha(x), F]_s = -i\int \Delta_{\alpha\lambda}(x-y)\,\frac{\delta F}{\delta u_\lambda(y)}\,dy,$$

where $s=-1$ [$+1$] when F transforms like a tensor [spinor] representation. When $\Delta_{\alpha\beta}$ contains no more than one derivative of the Δ function, (2) can be solved and an explicit expression can be obtained for the functional derivative. The analytic form of the functional derivative can be derived from this expression and the problem of the so-called "anticommuting c numbers" does not arise.

Y. Kato (Kobe)

Roman, P.; Marathay, A. S. 3715

Analyticity and phase retrieval. (Italian summary)

Nuovo Cimento (10) **30** (1963), 1452–1464.

Authors' summary: "The general problem of constructing a spectrum $g(\nu)$ from the knowledge of the magnitude of its Fourier transform $|\gamma(\tau_r)|$ is considered. The question reduces to locating the zeros of the analytic continuation $\gamma(\tau)$ in the upper half-plane. It is shown that if $g(\nu)$ is real, the complex zeros of $\gamma(\tau)$ in the upper half-plane either are on the imaginary axis or occur pairwise in symmetrical position. If, in addition, $g(\nu)\geqq 0$, the zeros on the imaginary axis disappear. The condition $g(\nu)\geqq 0$ also leads to the requirement that $\gamma(\tau)$ must be representable as a convolution of a function $h(\tau)$ with itself. The analytic properties of $h(\tau)$ and the equations to determine it are discussed. Possible ways to obtain the solution of the ensuing nonlinear eigenvalue problem are suggested."

J. W. Moffat (Baltimore, Md.)

Fronsdal, C.; Norton, R. E. 3716

Integral representations for vertex functions.

J. Mathematical Phys. **5** (1964), 100–108.

The triangle Feynman diagram is considered as a function of the external masses squared and arbitrary real values of the internal masses. It is well known that the diagram satisfies a dispersion relation in any one of the external variables when the remaining variables are held fixed in a certain range of real values. Starting from this situation, dispersion relations are derived for more general values of the fixed variables by analytic continuation. The diagram is then considered as a function of two external variables and double dispersions derived for fixed real values of the remaining variables. For certain values of the fixed variables (e.g., for negative values of the fixed external mass squared) a Mandelstam type representation is shown to hold, but in general, due to the presence of complex singularities on the physical sheet of the two dispersed variables, this is not so. In this situation a transformation to certain new variables is made in which a Mandelstam type representation is obtained. It is shown that alternatively one can obtain the same representation in the old variables by applying the Bergman-Oke-Weil formula. With a suitable choice of cuts the integration is over real regions only.

The presence of a non-Landau singularity on the physical sheet (but far from the physical region) of one of the variables for certain values of the fixed variables is pointed out. This result has been obtained independently by Bronzan and Kacser [Phys. Rev. (2) **132** (1963), 2703–2711].

J. N. Islam (College Park, Md.)

Mozrzymas, J. 3717

The solution of the Feynman-Gell-Mann equation in the spinor space. The finite dimensional representations of the Lorentz group.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 309–315.

This is a continuation of a previous note [same Bull. **11** (1963), 181–186; MR **27** #3332] and the author states that the discussion and physical interpretation of his results will be given in subsequent papers.

A. O. Barut (Boulder, Colo.)

Mozrzymas, J. 3718a

Rzewuski's iso-space as a curved complex space.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 593–598.

Mozrzymas, J. 3718b

The inversion of the angle of helicity as a conformal transformation in Minkowski's space-time.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 599–601.

It is known that the complex Lorentz group which leaves the complex norm z^2 invariant is isomorphic to the direct product of two 2-by-2 unimodular groups (12-parameter group). In the two papers under review the geometry of the four-dimensional complex space (which the author calls the Rzewuski space) is studied. It is shown that the same geometry can also be interpreted in a three-dimensional curved complex space. The complex transformations can be considered as rotations of a sphere in this complex space about its center. The argument of the radius vector of the sphere is interpreted as the helicity of the particle. *A. O. Barut* (Boulder, Colo.)

Bastai, A.; Bertocchi, L.; Fubini, S.; Furlan, G.; Tonin, M. 3719
On the treatment of singular Bethe-Salpeter equations. (Italian summary)
Nuovo Cimento (10) **30** (1963), 1512–1531.

Authors' summary: "The properties of the bound-state solutions of the Bethe-Salpeter equation in the ladder approximation are investigated in the case of potentials strongly singular at the origin. The analogy with potential scattering and the connexion to field theory are discussed. A method is given which allows the reduction of the singular problem to a regular one and explicit solutions for eigenvalue and eigenfunctions are presented in a particular case." *M. Schwartz* (Garden City, N.Y.)

Bastai, A.; Bertocchi, L.; Furlan, G.; Tonin, M. 3720
On the solutions of the Bethe-Salpeter equation. (Italian summary)
Nuovo Cimento (10) **30** (1963), 1532–1554.

Authors' summary: "We study the solutions of the Bethe-Salpeter equation for bound states, in the ladder approximation, in the case of potentials both regular and singular near the origin. Two methods are presented, one in momentum space, the other in coordinate space, which allow the complete determination of eigenvalues and eigenfunctions in the case where both the total energy E of the state and the mass of the exchanged particle are zero." *M. Schwartz* (Garden City, N.Y.)

Namysłowski, Józef; Wit, Romuald 3721
Asymptotic properties and zeros of the forward scattering amplitude.
Acta Phys. Polon. **23** (1963), 197–203.

Assuming that $T^{(1)}(\omega)$, the isotopic spin nonflip forward pion-nucleon scattering amplitude, behaves like a first-degree polynomial of ω as $|\omega|\to\infty$, the authors show that $\lim_{\omega\to\infty} [\mathrm{Re}\ T^{(1)}(\omega)/\mathrm{Im}\ T^{(1)}(\omega)]=0$, and find out that there are four or two zeros in the domain of analyticity of $T^{(1)}(Z)$ according to whether $T^{(1)}(\mu)$ is positive or negative. The latter statement is the same as those obtained by A. A. Ansel'm et al. [Ž. Èksper. Teoret. Fiz. **41** (1961), 619–628; MR **25** #3724], the reviewer [Progr. Theoret. Phys. **28** (1962), 479–488], and M. Sugawara and A. Tubis [Phys. Rev. (2) **130** (1963), 2127–2131; MR **27** #4523]. *S. Aramaki* (Urbana, Ill.)

Shapiro, I. S. [Šapiro, I. S.] 3722
Restrictions on Regge pole trajectories.
Nuclear Phys. **48** (1963), 58–64.

Restrictions on the vacuum Regge pole trajectory $\alpha(t)$ are derived. α is assumed to be univalent ($\alpha(t_1)\neq\alpha(t_2)$) and regular either in the left half-plane $\mathrm{Re}\,t<4\mu^2$ or in the entire t-plane with the cut $(4\mu^2, \infty)$. The condition of univalency is weaker than the requirement that α be a Herglotz function (i.e., that it map the upper half-plane into itself). It is argued that univalence is a reasonable requirement, since it is consistent with the known property of α that $\alpha'(t)\geqq 0$, $0\leqq t\leqq 4\mu^2$. The restrictions on α are derived directly from theorems on univalent functions which can be found in W. K. Hayman [*Multivalent functions*, Cambridge Univ. Press, Cambridge, England, 1958; MR **21** #7302]. The restrictions take the forms $R_1(a_1, t)\leqq\alpha(t)\leqq R_2(a_1, t)$, $S_1(t)\leqq\alpha'(t)/a_1\leqq S_2(t)$, $T_1(t)\leqq\alpha'(t)/[1-\alpha(t)]\leqq T_2(t)$, where $a_1=\alpha'(0)$. R_i, S_i, and T_i are given rational functions of t, which depend on whether one assumes univalence in the cut plane or only in the left half-plane. An attempt is made to compare the inequalities with data on p-p scattering at energies greater than 2 BeV. No conflict with the data is found. However, as the author points out, there is no sound reason to suppose that this scattering is dominated by the vacuum pole. The question of convexity of the vacuum trajectory is also considered. Here a function of z is said to be convex if it maps $|z|<1$ into a convex domain. The function of z in question is $\alpha(t(z))$, where $t(z)$ maps $|z|<1$ into the cut plane. It is asserted that if a resonance lies on the vacuum trajectory (i.e., if $\alpha(t)$ assumes a complex value close to a positive integer when $t>4\mu^2$), then α is not convex. Assuming that α is convex, the author finds another set of inequalities. They contradict the experimental data. *R. L. Warnock* (Chicago, Ill.)

Regge, T. 3723
Progressi recenti della teoria dei momenti angolari complessi.
Nuovo Cimento Suppl. (1) **1** (1963), 173–178.

Trueman, T. L.; Yao, T. 3724
High-energy scattering amplitude in perturbation theory.
Phys. Rev. (2) **132** (1963), 2741–2748.

The authors examine the high-energy behaviour of scattering amplitudes of scalar particles interacting through a scalar Yukawa interaction, retaining the ladder diagrams of the perturbative expansion in the coupling constant. If only the leading contribution $(\ln s)^n/s$ of each diagram as $s\to\infty$ is retained, the evaluation of the corresponding series yields a Regge behaviour, i.e., $s^{\alpha(t)}$. The authors show that, while the contribution to the series from the terms $(\ln s)^n/s^2$ is negligible, the terms of lower order in the logarithm in the nth graph, $(\ln s)^p/s$, $p<n$, give a contribution which strongly depends on the class of terms taken into account. If certain subclasses of those terms are computed, the result changes strongly, either modifying the coefficient of the asymptotic behaviour, or being even dominant over the sum of the leading terms in any graph. The addition of further terms restores the original behaviour, destroying a previous divergence at the threshold $t=4m^2$ in the leading Regge trajectory. The paper emphasizes therefore the point that the question of the high-energy behaviour of the complete sum of the ladder graphs is still unsettled. *V. de Alfaro* (Princeton, N.J.)

Zwanziger, Daniel 3725

Construction of amplitudes with massless particles and gauge invariance in S-matrix theory.

Phys. Rev. (2) **133** (1964), B1036–B1045.

The method of constructing Lorentz-invariant scattering amplitudes using the n-dimensional spinor calculus [A. O. Barut, I. Muzinich and D. N. Williams, same Rev. (2) **130** (1963), 442–457] is extended here to massless particles. Massless particles of spin j have, under proper Lorentz transformations, one direction of polarization instead of $(2j+1)$ for massive particles. This fact is incorporated by using the projection matrices $\mathscr{D}^{0j}(\frac{1}{2}(1\mp\sigma_3))$. The scattering amplitude transforms in the spin index of a massless particle as a null-spinor ξ_α satisfying $\mathscr{D}_{\alpha\beta}{}^{0j}((k\cdot\sigma)/2k^0)\xi_\beta{}^j=\xi_\alpha{}^j$, where k is the momentum of the massless particles. To avoid confusion with the representations of the rotation group, it should be remarked that the matrices $\mathscr{D}^j$ throughout this paper should be replaced by $\mathscr{D}^{0j}$, the representations of the homogeneous Lorentz group. The latter is identical with the former only if the argument is a unitary matrix. One should also watch for a number of misprints concerning dotted and undotted indices.

Combining the results of this paper with the paper quoted above, it is now possible to write all the formulae regarding the construction of invariant amplitudes in such a way that they are automatically valid for both massive and massless particles. *A. O. Barut* (Boulder, Colo.)

Mandelstam, S. 3726

The Regge formalism for relativistic particles with spin. (Italian summary)

Nuovo Cimento (10) **30** (1963), 1113–1126.

Author's summary: "The scattering amplitude for particles of spin σ_1 and σ_2 is examined in the angular-momentum plane, and the perturbation terms are found to have poles at the positive integers below $\sigma_1+\sigma_2$ or at the corresponding half-integers if $\sigma_1+\sigma_2$ is half-integral. In the complete amplitude, the poles begin to move away from these values as the coupling is turned on. However, the amplitude in the j-plane, obtained by analytically continuing the amplitude from values of j greater than $\sigma_1+\sigma_2-1$, will not be equal to the physical partial-wave amplitudes at the points in question. In the presence of a third double-spectral function, the states of the wrong signature will have essential singularities of the Gribov-Pomeranchuk type at these points. Our results are also valid in processes which can have an intermediate state with particles of spin σ_1 and σ_2. If the spinning particles are themselves Regge particles, all these statements may require modification." *F. Calogero* (Rome)

Mandelstam, S. 3727

Cuts in the angular-momentum plane. I, II. (Italian summary)

Nuovo Cimento (10) **30** (1963), 1127–1147; *ibid.* (10) **30** (1963), 1148–1162.

It is an important problem to find out if scattering amplitudes have other singularities in angular momentum variable and how these affect the cross-sections in the direct and crossed channels (low and high energies, respectively). The exchange of a Regge pole, when inserted in the elastic unitarity condition, implies immediately the existence of cuts in angular momentum, as was shown by Amati, Fubini and Stanghellini [Phys. Lett. 1 (1962), 29–32; MR **27** #1188]. These authors, however, considered only part of the dispersion diagram. It is shown in the first paper under review that these cuts cancel in the complete dispersion diagram associated with such an exchange of the Regge pole. In the second part it is shown that there are more complicated diagrams in which, however, cuts are present. Although the existence of cuts would complicate the situation, it may have also a good side in that under certain conditions the essential singularities found by Gribov and Pomeranchuk [ibid. **2** (1962), 239–241; MR **26** #2197] do not contribute to the asymptotic behavior of the amplitude in the physical region. *A. O. Barut* (Boulder, Colo.)

Wilkin, C. 3728

Cuts and poles in the angular-momentum plane.

Nuovo Cimento (10) **31** (1964), 377–396.

The problem of branch-point singularities in the angular momentum plane due to certain graphs [S. Mandelstam, #3727 above] is discussed again here, treating these diagrams as Feynman integrals without using the unitarity condition. The same conclusions are reached as by Mandelstam. *A. O. Barut* (Boulder, Colo.)

Aitchison, I. J. R. 3729

Logarithmic singularities in processes with two final-state interactions.

Phys. Rev. (2) **133** (1964), B1257–B1266.

This is an interesting paper in which the author discusses thoroughly the type of behavior which can be expected from triangle type singularities. He confines himself to the detailed mathematical formulation for the π^-p to $\pi^+\pi^-n$ process which once promised to be a rich source of experimental probe for such types of anomalous or logarithmic singularities. This hope has not been fully realized because of the large width associated with the (3, 3) isobar; however, the method can be usefully applied to processes involving strange particles where experimental discovery of the type of effect he discusses can better be analysed. *S. F. Tuan* (Lafayette, Ind.)

Feinberg, G.; Pais, A. 3730

A field theory of weak interactions. II.

Phys. Rev. (2) **133** (1964), B477–B486.

The mathematical methods introduced in an earlier paper [same Rev. (2) **131** (1963), 2724–2761; MR **27** #6537] to deal with higher-order effects in weak interactions are further developed and applied to leptonic processes. In particular, the treatment of the Bethe-Salpeter equation embodying the uncrossed ladder graphs is refined. Among other results, it appears that while the theory can deal sensibly with weak interactions mediated by vector bosons, it predicts zero amplitudes for four-Fermion contact interactions. *G. Barton* (Brighton)

Balachandran, A. P. 3731

Uniqueness of the partial-wave amplitudes.

Phys. Rev. (2) **132** (1963), 894–895.

The author proves the following result: Given all but a

finite number of partial-wave amplitudes in a two-body scattering process, the remaining amplitudes are uniquely determined up to an additive constant if either (a) the scattering amplitude has pure crossing symmetry, or (b) there is an energy region in one of the crossed channels where the scattering is purely elastic, and the amplitude satisfies a Mandelstam representation with a finite number of subtractions.

The relevance of this result to the hypothesis that all particles are Regge poles is discussed.

A. C. Hearn (Stanford, Calif.)

Pócsik, G. 3732

Fermion self-masses and Lehmann's spectral representation.

Nuclear Phys. **49** (1963), 286–288.

Lehmann's spectral representation of the propagator [H. Lehmann, Nuovo Cimento (9) **11** (1954), 342–357; MR **17**, 332] is extended to the case of the spinor field, and fermion self-masses are investigated especially for a spinor self-coupling. The relation of the present treatment to Nambu's self-consistent mass equation [Y. Nambu and G. Jona-Lasinio, Phys. Rev. (2) **122** (1961), 345–358; ibid. (2) **124** (1961), 246–254] is discussed.

Y. Kato (Kobe)

Zăgănescu, M. 3733

Periods of Mandelstam's boundary curves. (Italian summary)

Nuovo Cimento (10) **30** (1963), 784–790.

Author's summary: "Starting from the well-known equation of Mandelstam's boundary curve of a square diagram, we investigate the elliptic integral associated with this curve. Legendre's modulus k^2 depends exclusively on the ratio m/μ of the two masses, the mass of the 'nucleon' and of the 'pion'. For $\mu=m$ and $\mu=2m$, the genus of the curve is zero. A birational correspondence can be established in this case between the coordinates of the two curves so that the corresponding diagrams are, in a certain sense, equivalent. Finally, the values of the spectral density and scattering amplitude on the boundary curve are calculated. The scattering amplitude is given by an elliptic integral of the third kind."

G. F. Dell'Antonio (Naples)

Olive, D. I. 3734

On the analytic continuation of the scattering amplitude through a three-particle cut. (Italian summary)

Nuovo Cimento (10) **28** (1963), 1318–1336.

The scattering amplitude is continued analytically onto the sheet reached by encircling the lowest three-particle branch point in an anticlockwise direction by using generalized normal threshold discontinuity formulae derived from unitarity and extended unitarity. The desired continuation, denoted by $A_{22}{}^i$ (i denotes the sheet in question and A_{ij} represents the amplitude for $i\to j$), is obtained from integral equations which express $A_{22}{}^i$ in terms of $A_{22}{}^+$ and $A_{23}{}^i$ and the latter in terms of $A_{33}{}^+$ (including the disconnected parts associated with it) and itself (the + denotes the boundary value on the physical edge of the cut). The integral equations are not of Fredholm type due to the presence of delta functions arising from the connectedness structure. The inversion is carried out by using an algebraic trick and an extension of the Fredholm theory to singular equations.

From the solution is verified the existence on the sheet in question of a complex normal threshold branch point corresponding to the combination of a stable particle and a resonance. The presence of this branch point was indicated by Landshoff [Nuovo Cimento (10) **28** (1963), 123–131; MR **26** #7356] by using analyticity in the coupling constant. This result has also been obtained independently by Zwanziger [Phys. Rev. (2) **131** (1963), 888–898; MR **27** #4559]—some of whose techniques are used in the present paper—using integral equations derived from unitarity, in which the branch point appears as a pinch between the resonance pole of the scattering amplitude in the integrand and an end point of the phase space integration contour.

Results very similar to those derived in this paper have been obtained independently by J. Gunson ["Unitarity and on-mass shell analyticity as a basis for S-matrix theories", to be published].

J. N. Islam (College Park, Md.)

Nguyen Van Hieu 3735

Regge poles and the asymptotic behavior of the cross sections for some weak-interaction processes.

Ž. Èksper. Teoret. Fiz. **45** (1963), 544–547 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 374–376.

Author's summary: "The asymptotic behavior of the cross-sections for some weak-interaction inelastic processes is examined assuming the existence of Regge poles with $\alpha(0)=1$."

Halliday, I. G.; Polkinghorne, J. C. 3736

High-energy behavior in production processes.

Phys. Rev. (2) **132** (1963), 852–855.

Authors' summary: "Regge pole-like terms in the high-energy behavior of production amplitudes are evaluated by using the leading asymptotic behavior of sums of Feynman diagrams. The forms obtained depend on the way variables are allowed to tend to infinity."

M. Fowler (College Park, Md.)

Halliday, I. G. 3737

High-energy behaviour in perturbation theory. (Italian summary)

Nuovo Cimento (10) **30** (1963), 177–192.

Author's summary: "The high-energy behaviour due to contributions from the edge of the hypercontour of integration of a large class of Feynman graphs is determined. It is shown that the sum of the leading terms of graphs with a certain type of iterated structure has a high-energy behaviour consistent with Regge poles which tend to $l=-1,-2,-3,\cdots$ as the coupling constant tends to zero. Expressions are obtained for the trajectories and residues of these associated Regge poles."

M. Fowler (College Park, Md.)

Islam, J. N. 3738

Leading Landau curves of some Feynman diagrams. (Italian summary)

Nuovo Cimento (10) **30** (1963), 259–265.

From the author's summary: "It is shown that the leading Landau curves of some Feynman diagrams possess no parts associated with positive Feynman parameters when some of the external and internal masses satisfy certain inequalities. The assumed validity of the Mandelstam representation then implies that the corresponding leading curves are nonsingular on the physical sheet."
S. Aramaki (Urbana, Ill.)

Nedelkov, I. P. 3739
Darstellungen der Streuamplituden. (English summary)
Ann. Physik (7) **12** (1963), 1–13.
The author writes a scattering amplitude $h(z)$ as a sum of functions, each of which is an analytic function of z regular in a domain which is larger than the regularity domain of $h(z)$ itself. This is called a "component representation" of $h(z)$. The determination of each "component" which is regular in a rather large domain is discussed in two ways, viz., in terms of an algebraic system of equations and in terms of a Cauchy problem. In the first method, the author makes extensive use of some results obtained earlier by himself and co-workers and presented in an unpublished report from Dubna [P. H. Burnev, W. A. Mesterjakov, I. P. Nedelkov, "Über eine Randwertaufgabe der Dispersionstheorie" (in Russian), Dubna, 1962]. As this paper is not available to the reviewer, the corresponding part of the paper under review is essentially unintelligible to him.
G. Källén (College Park, Md.)

Van Hove, L. 3740
Exchange contributions to high energy scattering and imaginary character of the elastic amplitude.
Phys. Lett. **7** (1963), 76–77.
This letter deals with the properties of the elastic scattering amplitude at very high energies, and continues a previous investigation by the author on the same subject [same Lett. **5** (1963), 252–253; MR **27** #4571]. In the previous work it was shown that the existence at large s (s = (c.m. energy squared)) of an s-independent, nonvanishing differential cross-section $d\sigma_{el}/dt$ ($t = -$(c.m. momentum transfer squared)) implies that the scattering amplitude is purely imaginary for $s \to \infty$, when one of the scattering particles is self-conjugate or belongs to an isospin multiplet containing an element invariant under charge conjugation C. In the present paper this result is generalized by considering the exchange of quantum numbers $C = \pm 1$ in the crossed channel; if the two contributions for $C = \pm 1$ have different asymptotic dependence on s, the dominant one must be $C = +1$. Therefore the method described above [loc. cit.] can be applied in general to prove the imaginary character of the elastic amplitude at very high energies. *E. Ferrari* (Rome)

Eden, R. J. 3741
Legendre transforms and Khuri representations of scattering amplitudes.
Phys. Rev. (2) **132** (1963), 912–913.
A representation of a scattering amplitude is described based on a Legendre transform which differs from the partial wave amplitude by the choice of variable of integration. A detailed account of the use of this transform is given in a separate article [Nuovo Cimento (10) **31** (1964), 998–1012]. The representation obtained is another example of a class that has similar characteristics to the crossing symmetric Sommerfeld-Watson transformation developed by Khuri [Phys. Rev. (2) **132** (1963), 914–926; MR **28** #891a]. *S. Aramaki* (Urbana, Ill.)

Gorshkov, V. G. [Gorškov, V. G.]; Frolov, G. V. 3742
Asymptotic relations between cross sections for large angle scattering.
Ž. Èksper. Teoret. Fiz. **44** (1963), 1747–1749 (*Russian*); *translated as Soviet Physics JETP* **17** (1963), 1174.
The amplitudes for nucleon Compton effect, photoproduction of pions and for πN scattering are coupled via elastic unitarity conditions. Therefore, if one of the amplitudes is dominated by a pole in the angular momentum plane, then all amplitudes should have a pole with the same angular momentum. The unitarity condition then gives relations between the residues of the poles, which in turn imply relations between the cross-sections at large angles. Similar relations in the forward direction, therefore for total cross-sections, have been obtained before [M. Gell-Mann, Phys. Rev. Lett. **8** (1962), 263–264; MR **25** #3730; V. N. Gribov and I. Ya. Pomeranchuk, ibid. **8** (1962), 343–345; MR **25** #1862]. It should be remarked that only elastic unitarity has been used without the pole terms, and it is not clear how accurate these relations would be. *A. O. Barut* (Boulder, Colo.)

Frolov, G. V. 3743
Perturbation theory and fermion Regge poles in electrodynamics.
Ž. Èksper. Teoret. Fiz. **44** (1963), 1746–1747 (*Russian*); *translated as Soviet Physics JETP* **17** (1963), 1173.
This is a brief note pointing out that in the Compton effect on an electron the asymptotic behavior of the amplitude under the assumption of a dominant single pole of definite signature in angular momentum is in contradiction with what one obtains in perturbation theory. The imaginary part of the amplitude vanishes in second order in the latter case and not in the former case.
A. O. Barut (Boulder, Colo.)

Patashinskiĭ, A. Z. [Patašinskiĭ, A. Z.]; 3744
Pokrovskiĭ, V. L.; Khalatnikov, I. M. [Halatnikov, I. M.]
Regge poles in quasiclassical potential well problems.
Ž. Èksper. Teoret. Fiz. **44** (1963), 2062–2078 (*Russian. English summary*); *translated as Soviet Physics JETP* **17** (1963), 1387–1397.
Authors' summary: "The problem of poles of the scattering phase-shift (Regge poles) is investigated for the case of a rectangular and spherically symmetric potential well. In this case the scattering phase-shift has an explicit expression in terms of Bessel functions. In looking for poles of the scattering phase-shift, a previously developed method is used to trace the properties of the phase shift along the level lines. Two series of poles are found: 'physical' and 'unphysical'. The character of the motion of the poles as the energy varies is then clarified. Finally, some general relations are established between the number of levels and the number of resonances. The simplest form

of the potential well has been chosen in order not to complicate the calculations with inessential details. However, the results remain in essence of general validity for potentials which are singular outside the point $r=0$."
E. Predazzi (Turin)

Geshkenbeĭn, B. V. [Geškenbeĭn, B. V.]; Ioffe, B. L. 3745
Trajectories of Regge vacuum poles.
Ž. Èksper. Teoret. Fiz. **45** (1963), 346–348 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 240–241.
Certain restrictions on the behavior of the vacuum (or Pomeranchuk) pole trajectory are presented, based on assumed analyticity properties and on experimental data.
F. Calogero (Rome)

Squires, E. J. 3746
On the nature of the singularity of a partial-wave scattering amplitude at the negative integral value of J.
Phys. Lett. **7** (1963), 363–364.
Gribov and Pomeranchuk [same Lett. **2** (1962), 239–241; MR **26** #2197] have suggested that the presence of a third spectral function in the Mandelstam representation for a relativistic scattering amplitude gives rise to an essential singularity at $J=-1,-2,\cdots$ of the partial wave amplitude analytically continued in the angular momentum J. In this paper an alternative possibility is advanced, namely, the presence of singularities at $J=-1,-2,\cdots$ whose nature changes character with energy. This possibility is substantiated by a mathematical example based on the N/D decomposition, and its physical consequences are discussed.
F. Calogero (Rome)

Charap, John M.; Squires, Euan J. 3747
On complex angular momentum in many-channel potential-scattering problems. III.
Ann. Physics **25** (1963), 143–154.
This is a continuation of two previous papers [same Ann. **20** (1962), 145–156; MR **26** #2212a; ibid. **21** (1963), 8–28; MR **26** #2212b] on the introduction of complex angular momentum in many-channel potential scattering with arbitrary spin. A transformation of the Schrödinger equation is introduced so that the transformed S-matrix does not have any kinematical singularities in the angular momentum plane. It is interesting that it is this transformed S-matrix which enters in the partial wave amplitudes in the helicity representation. The results agree with that of Desai and Newton [Phys. Rev. (2) **129** (1963), 1437–1444; MR **26** #7352] in the case of spin $\frac{1}{2}$ particles. The form of unitarity is given in the new representation.
A. O. Barut (Boulder, Colo.)

Nishijima, K. 3748
Unsubtracted dispersion relations in weak interactions and the Goldberger-Treiman relation.
Phys. Rev. (2) **133** (1964), B1092–B1104.
This is an important paper proposing a very beautiful hypothesis about the inter-relation between weak and strong interactions.

In strong interactions, the traditional rule is to assume one subtraction in the dispersion relation for each matrix element to which corresponds a coupling written directly into the renormalizable strong Lagrangian. By contrast, no subtractions at all are supposed to be needed for more complicated processes, which would correspond to non-renormalizable couplings. This rule is based on the general assumption that non-renormalizable strong couplings do not really occur in nature, so that processes corresponding to them are in fact "induced" in their entirety by other, renormalizable, couplings.

The author proposes to generalize this rule to the weak interactions. Because the latter certainly correspond to non-renormalizable couplings, he assumes that all weak interaction matrix elements satisfy unsubtracted dispersion relations. Since the weak interactions violate the symmetries respected by the strong interactions, the absence of adjustable subtraction constants (i.e., weak coupling parameters) necessitates convergence conditions on the dispersion integrals, which constrain the strong coupling parameters. In general, these constraints are not self-consistent; to make them so, the strong coupling parameter must be subjected to eigenvalue-type equations. Constraints also emerge on the different weak coupling parameters; only their common overall normalization needs to be taken from experiment.

In the present paper this program is applied to strangeness-conserving leptonic decays; an earlier paper [McCliment and the author, same Rev. (2) **128** (1962), 1970–1982] dealt with nonleptonic hyperon decays. In particular, the connection between different previous derivations of the Goldberger-Treiman formula for $\pi\to\mu\nu$ is discussed.

In the detailed execution of his project, the author relies on the ladder approximation for the form factors due to Federbush, Goldberger and Treiman [ibid. (2) **112** (1958), 642–665, Appendix B; MR **23** #B295].

{The reviewer, though deeply impressed by the author's basic hypothesis, does not believe that in the light of more recent ideas it is tenable to rely on this particular approximation for the asymptotic behaviour of form factors, which, here, is precisely the crucial problem.}
G. Barton (Brighton)

Nath, Pran; Shaw, G. L.; Iddings, C. K. 3749
The uncoupled phase method for interactions with hard cores.
Phys. Rev. (2) **133** (1964), B1085–B1089.
The "uncoupled phase" method developed by Ross and Shaw [Ann. Physics **9** (1960), 391–415] is extended to the case of potentials with a hard core. A numerical investigation shows the region of validity of the method to be larger than expected.
J. L. Gammel (College Station, Tex.)

Grynberg, M.; Koba, Z. 3750
Four-pion wave functions.
Acta Phys. Polon. **23** (1963), 501–526.
Authors' summary: "Following the same procedure as in the case of three pions, we can work out a complete set of orthonormal wave functions for four free pions. These wave functions describe systems of pions, which satisfy the Bose statistics and possess definite values of the total

energy, the total angular momentum, and the parity in the center of mass system in the non-relativistic approximation, as well as a definite total isospin and a definite charge parity when the system is neutral. The procedure of construction of this set of wave functions is described in detail and their explicit expressions are given up to $\Lambda = 3$, and the number of these eigenstates are listed up to $\Lambda = 5$, where Λ is the 'effective angular momentum'."

Ohnuki, Yoshio; Maki, Ziro; Yamamoto, Hiroshi 3751

Quantum mechanical approach to the composite model of elementary particles.

Structure of elementary particles [*Progr. Theoret. Phys. Suppl. No.* 19 (1961)], *pp.* 89–124.

In this article the authors discuss the dynamical aspects of the symmetry properties and the mass relations among the composite particles in the framework of conventional quantum field theory. In order to examine this, the authors take the interaction Lagrangian of the four-fermion interaction type among the three basic fields. Instead of constructing the Bethe-Salpeter amplitude for the composite system the present authors obtain a two-body propagator by a perturbational method analogous to a one-body propagator so that the bound state can be examined in terms of the propagators. Corresponding to scalar, pseudoscalar, vector and axial vector mesons they obtain the eigenvalue equations for the determination of the masses of the particles in terms of the coupling constant. As a direct application of this treatment of composite system, the authors solve the eigenvalue equation based on the "chain-approximation" method using the four-fermion direct interaction with the introduction of a cut-off of the order of nucleon mass. A vector-type four-fermion interaction with a coupling strength nearly 0.414×10^{-43} erg cm^3 can reproduce the observed mass of the π and K-meson. *S. N. Biswas* (Bombay)

Roos, Matts 3752

Data on elementary particles and resonant states, November 1963.

Nuclear Phys. **52** (1964), 1–24.

Extensive tables of elementary particles and resonances, known as of November 1963, are given. The tables list isospin, spin, parity, G-parity, strangeness, mass in MeV and in pion mass unit, width, lifetime, production process and lab momentum, decay modes, branching ratio, and Q-value.

An explanatory introduction, detailed footnotes, and a long list of references completes the tables (see also Phys. Lett. **8** (1964), 1–4 [MR **28** #2844]). *P. Roman* (Boston, Mass.)

Baker, M.; Johnson, K.; Lee, B. W. 3753

Broken symmetries and zero-mass bosons.

Phys. Rev. (2) **133** (1964), B209–B213.

Nambu and Jona-Lasinio have shown that the homogeneous Bethe-Salpeter (B.S.) equation for the particle-antiparticle system has a zero-mass ($q^2 = 0$) solution which is consistent with the equation for the fermion mass. However, the inhomogeneous γ_5 vertex equation has no solution. For $q^2 \neq 0$, the converse is true. Since in this case from the nature and the existence of a zero-mass solution to the homogeneous B.S. equation, one can conclude that the pseudoscalar vertex has a pole at $q^2 = 0$, it follows that a zero-mass pseudoscalar particle is present. In this paper the authors have obtained a solution to the zero-momentum B.S. equation for any γ_5 invariant theory with symmetry-breaking solution. However, by taking the example of quantum electrodynamics with zero bare electron mass, it is found that the equation for the vertex $\Gamma_5(q^2)$ has no solution for any q and therefore it is not possible to conclude that $\Gamma_5(q^2)$ has a pole at $q^2 = 0$. Hence one does not know whether zero-mass bosons are present. *B. P. Nigam* (Buffalo, N.Y.)

Greenberger, Daniel M. 3754

The scale transformation in physics.

Ann. Physics **25** (1963), 290–308.

Author's summary: "Some criteria are established to test whether invariance under a continuous transformation will lead to a conservation law. The scale transformation is examined as a special case, and shown not only to lead to no general conservation law, but also in fact to be of a trivial nature. This is due to the rather artificial way in which scale invariance is usually introduced. A theory is then constructed by introducing an internal coordinate of dimension (length) in order to allow only the dimensionless ratio of lengths to enter, and by exploiting the gaugelike structure of the scale transformation. In this theory the scale transformation does lead to a new conserved current (as well as to an 'almost conserved' one), and the internal coordinate is shown to play the same role for the scale transformation as the internal coordinate spin plays for the case of rotations, and allows the theory to treat massive particles." *R. Haag* (Urbana, Ill.)

Peshkin, Murray 3755

Spin and parity analysis at all production angles.

Phys. Rev. (2) **133** (1964), B428–B430.

A method is proposed for the determination of the spin and parity of an unstable particle X ($\to D + E$), produced in a reaction $A + B \to C + X$. The particles A, B, C, D should be spinless, while E could be a photon or a spinless particle. The author makes use of a rule given by A. Bohr [Nuclear Phys. **10** (1959), 486–491], which relates the polarizations of the particles involved in a parity-conserving collision to their intrinsic parities, i.e., to $P_A P_B P_C P_X$ being 1 or -1. The angular distribution of the decay in the rest system of X (having spin j) is expressed in terms of Legendre polynomials as

$$T_\delta(\theta) = \tfrac{1}{2} \sum_{l=0}^{2j} a_l(\delta) P_l(\cos\theta),$$

where δ is the production angle and a_l is dependent upon the dynamics of the production process. Using Bohr's rule and certain properties of the vector coupling coefficients, it is shown that the highest coefficient $a_{2j}(\delta)$ cannot vanish and its sign determines the parity of X. Moreover, the numerical value of $a_{2j}(\delta)$ must lie between bounds which are independent of δ. These bounds are calculated for several simple cases ($j = 0$ to 4). The article concludes by stressing certain experimental advantages of the method. *P. Singer* (New York)

Amati, D. 3756

On the correspondence between resonances and S-matrix poles for broken symmetries.

Phys. Lett. **7** (1963), 290–292.

In the original paper of Oakes and Yang [Phys. Rev. Lett. **11** (1963), 174–178], they questioned the validity of unitary symmetry mass formulae. They assumed that the poles of the S-matrix must be only those corresponding to observed resonances and must, therefore, be on the sheet that can be reached from the physical one by crossing the real-energy axis at the position of the resonance. On this basis they arrived at the conclusion that when elementary particle mass differences are eliminated, either the resonances become bound states or they behave in such a different way under the symmetry-breaking interaction that there is not much hope that any consideration based on symmetry can retain some meaning in the actual physical situation. The author would like to show in a simple example (dynamical calculation in a two-channel case) that this situation is not necessarily the case, the initial assumption of Oakes and Yang being an unrealistic one.

{The reviewer would like to call attention to a recent talk by Yang on 'The mass formula of SU_3' [Proc. Argonne User's Group Meeting (December, 1963)] where he discussed the relevance of comments made by other authors on the Oakes-Yang paper [loc. cit.]. It is the opinion of the reviewer that the question under debate is not one of principles necessarily, but one of detailed dynamical consideration for the cases evaluated. However, should the recently discovered $Y_1^*(1660)$ be determined experimentally as $P_{3/2}$, as suggested by preliminary evidence [Taher-Zadeh et al., Phys. Rev. Lett. **11** (1963), 470–474], serious problems will arise for the assignment of baryon resonances of spin-parity (3/2+) as a decuplet with an equal-spacing mass formula.}

S. F. Tuan (Lafayette, Ind.)

Majewski, M. 3757

Formula for angular distribution in the statistical theory of multiple particle production.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 535–539.

The angular distribution for the pions produced in the statistical theory of multiple-particle production involves sums over terms of the form $|C(l\lambda L;\ mM-m)Y_l^m(\hat{\mathbf{r}})|^2$. The author reduces these sums by standard angular momentum techniques (Racah and Wigner coefficients) to explicit Legendre series. The general properties of the result (forward-backward symmetry, etc.) are very briefly discussed, and a few remarks concerning practical applications are added. *L. C. Biedenharn* (Durham, N.C.)

Ademollo, M.; Gatto, R. 3758

Complete spin tests for fermions.

Phys. Rev. (2) **133** (1964), B531–B541.

Authors' summary: "Complete spin tests for fermions of arbitrary spin, produced from a spin-zero boson on an unpolarized spin-½ fermion and decaying into a spin-zero boson and a spin-½ fermion, are derived. The tests constitute a set of necessary and sufficient conditions for a particular spin assignment, in the absence of more detailed dynamical information. Essential use is made of the R invariance of the parity conserving production process. More general tests, applicable to arbitrary production processes, are also discussed."

P. Singer (New York)

de Broglie, Louis 3759

★Introduction to the Vigier theory of elementary particles.

With a chapter by Jean-Pierre Vigier. Translated by Arthur J. Knodel.

Elsevier Publishing Co., Amsterdam-London-New York, 1963. vi+139 pp. $11.00.

In spite of the spectacular results obtained by using modern dispersion relation techniques and other methods directly devised for dynamical calculations, it is still true—perhaps even more than ever—that group-theoretic methods and, more generally, the search for basic symmetries are yet the most promising tools of the theoretician when he attempts to unravel the mysteries of elementary particles. Many higher symmetry schemes have been suggested in the last few years, the most successful amongst them appearing to be the octet model based on U_3. Whatever the future of this or other higher symmetry schemes, we shall have to face ultimately the question regarding the origin of the underlying symmetry group. Very little work has been done so far along these lines. It may appear to many physicists that perhaps the most promising approach concerning the origin of internal symmetries might be one which is based on a deeper study of ordinary space-time properties. One interesting approach in this spirit was undertaken by the author, Halbwachs, Hillion, Vigier, Takabayasi, and others. The purpose of the book under review is to summarize, elaborate and to extend somewhat the results reached in these researches.

In a sense, the basic ideas of this theory can be summarized thus: The framework of thought is connected to the theory of double solution introduced many years ago by the author. A particle is then conceived of as a clearly localized region of space wherein a vortex-like wave field develops comparable to a rotating liquid droplet. In order to be in accord with relativity, a very close study of the relativistic theory of a liquid drop in connection with the hydrodynamics of Dirac's electron theory had to be undertaken. It turned out that relativistic rotary motion is much more complex than had been suspected and makes it possible to give an explanation to the complex structure of particles. By quantizing the particle's internal motion, quantum numbers characterizing different types of particles can be established. These quantum numbers can be related to isospin, strangeness, and baryon number.

The exposition is clear and systematic, and is always based on well-known concepts. It would be out of place to discuss here the merits and shortcomings of the theory. The only comment which we yet want to make is that, unfortunately, during the period of two years which lapsed between the publishing of the original French [Gauthier-Villars, Paris, 1961] and the present English edition, many new developments, both experimental and theoretical, have taken place, so that a certain amount of this monograph is by now superseded.

P. Roman (Boston, Mass.)

Jouvet, B.; Adam, J. P. 3760
Particules et champs composites résultant d'interactions entre plusieurs courants. (English and Italian summaries)
Nuovo Cimento (10) **29** (1963), 1275–1289.
Authors' summary: "It is shown that when composite bosons result from Fermi couplings between two different pairs of unlike fermions, one can construct an equivalent Yukawa-type theory which postulates at least two boson fields whose renormalization constants are zero. The number of boson fields can be reduced when the Fermi coupling constants fulfil certain relations. To the Fierz group of rearrangement of the fermion pairs, there can be associated a group of Yukawa-type theories which are equivalent, although postulating boson fields of different types." *E. R. Caianiello* (Naples)

★**Lectures on strong and electromagnetic interactions.** 3761
Brandeis Summer Institute in Theoretical Physics, 1963, Vol. 1. Lectures by P. T. Matthews, D. R. Yennie, M. E. Mayer. Notes by A. Phillips, S. Brodsky, L. Heiko.
Brandeis University, Waltham, Mass., 1964. v + 343 *pp.* $3.00.
This book is comprised of informally edited lecture notes on strong interactions of elementary particles (Matthews), topics in quantum electrodynamics (Yennie), and unitary symmetry of strong interactions (Mayer). Individual reviews follow immediately below [#3762–#3764].
P. Roman (Boston, Mass.)

Matthews, Paul T. 3762
Strong interactions of elementary particles.
Notes by Anthony Phillips.
Lectures on Strong and Electromagnetic Interactions (Brandeis Summer Institute in Theoretical Physics, 1963, *Vol.* 1), *pp.* 1–163. *Brandeis Univ., Waltham, Mass.*, 1964.
This well-organized and clearly written article gives a most useful introduction to the dynamics of strong interactions. After a brief introduction on generalities, the basic notions concerning the S-matrix are developed. This is followed by the exposition of the fundamental concepts of the substitution law (crossing relations), unitarity, and analyticity. Chapter 6 gives for completeness a brief summary of invariance and conservation laws. Chapter 7 extends the covariant formalism to include spin one-half particles and discusses phase shifts in more detail. The second part of the review is concerned with applications, notably π-N scattering, $\bar{K}$-N scattering, resonances, and a brief discussion of high-energy scattering.
P. Roman (Boston, Mass.)

Yennie, Donald R. 3763
Topics in quantum electrodynamics.
Notes by S. J. Brodsky.
Lectures on Strong and Electromagnetic Interactions (Brandeis Summer Institute in Theoretical Physics, 1963, *Vol.* 1), *pp.* 165–258. *Brandeis Univ., Waltham, Mass.*, 1964.
The purpose of these lecture notes is to provide physical insight into the theory without the assumption of a sophisticated knowledge of quantum field theory and without the use of advanced mathematical techniques. It deals with two major topics: (1) operator techniques in the Lamb shift calculations; (2) the infrared divergence problem and radiative corrections to high-energy scattering. *P. Roman* (Boston, Mass.)

Mayer, Meinhard E. 3764
Unitary symmetry of strong interactions.
Notes by Lance Heiko.
Lectures on Strong and Electromagnetic Interactions (Brandeis Summer Institute in Theoretical Physics, 1963, *Vol.* 1), *pp.* 259–343. *Brandeis Univ., Waltham, Mass.*, 1964.
These informal lecture notes serve as a first and very useful introduction to the currently popular SU_3 symmetry of elementary-particle interactions. A brief introduction on generalities concerning elementary particles, the concept of internal symmetry, and the isospin formalism is followed by the introduction of SU_3 symmetry in different ways, and a simple group-theoretic treatment of its tensor algebra, infinitesimal operators, weight diagrams, product representations, Clebsch-Gordan coefficients, and the generalization of the Wigner-Eckart theorem. Chapter 3 reviews very briefly basic applications, such as the mass formula, electromagnetic effects, and relations between cross-sections.
An appendix by L. Heiko gives a new derivation of the electromagnetic mass difference formula for baryons.
P. Roman (Boston, Mass.)

Bonnevay, G. 3765
A model for final-state interactions. (Italian summary)
Nuovo Cimento (10) **30** (1963), 1325–1343.
Translation of the author's summary: "By a method of analytic continuation, a model is studied for the disintegration of one particle into three identical particles, where only two-body elastic interactions in a single dominant partial-wave are taken into account. An integral equation is obtained, with a regular kernel in the physical region. If a resonance occurs in the two-body interactions, the importance of the first two rescatterings is shown. The higher Riemann sheets contribute to rescattering of higher order." *E. Ferrari* (Rome)

Coleman, Sidney 3766
On partially conserved currents.
Ann. Physics **24** (1963), 37–45.
Author's summary: "It is observed that the hypothesis of partially conserved leptonic weak interaction currents, which has been proposed by several authors, implies that the fundamental meson-baryon interaction be invariant under a certain group constructed from these currents. The only candidate for this group that is consistent with experiment and that does not require the introduction of an exorbitant number of spinless mesons is the eight-dimensional rotation group. The associated meson-baryon coupling is one first proposed by Gürsey. This group does not seem to have any connection with the approximate symmetries of the strong interactions; thus we reject the notion of partially conserved currents."

{Reviewer's remarks: (a) The author's results are based upon the assumption that $|\Delta I| = \frac{3}{2}$ currents have been observed in leptonic decays. This is still somewhat uncertain. If they do not exist, a solution to the problem could be provided by SU_3 [see M. Gell-Mann, Phys. Lett. 8 (1964), 214–215]. (b) If they do exist, a connection with the strong interactions symmetry may be provided by the inclusion of SU_3 in SO_8, though this would still require additional assumptions [see the reviewer, ibid. 4 (1963), 81–83; ibid. 4 (1963), 312; also, M. Gourdin, Nuovo Cimento (10) 30 (1963), 587–602].}

Y. Ne'eman (Pasadena, Calif.)

Macfarlane, A. J.; Mukunda, N.; Sudarshan, E. C. G. 3767

Electromagnetic and decay properties of G_2 multiplets.

Phys. Rev. (2) **133** (1964), B475–B477.

Authors' summary: "In this paper we write down consequences of exact and approximate invariance under the group G_2, for the baryons, mesons, and resonances. We use the method of Weyl reflections and the Shmushkevich algorithm, previously applied to the group SU_3, to derive electromagnetic mass and magnetic moment relationships, and also relationships between the one- to two-particle decay vertices (coupling constants). The assignment of multiplets is the usual one, using the singlet, the 7-fold, and the 14-fold representations of G_2."

S. L. Glashow (Stanford, Calif.)

Diu, B. 3768

Elementary and composite particles in the Lee model. (French and Italian summaries)

Nuovo Cimento (10) **28** (1963), 834–842.

In this paper the author proposes to study the difficult question whether it is possible to make a distinction in principle between a composite particle and a fundamental particle by examining the Lee model. The result is inconclusive, in contradistinction to an earlier paper by Vaughn, Aaron, and Amado [Phys. Rev. (2) **124** (1961), 1258–1268] on the same subject. *J. M. Jauch* (Geneva)

Mahmoud, Hormoz; Cooper, Richard K. 3769

Application of the theory of the symmetric group to the several-nucleon problem.

Ann. Physics **26** (1964), 222–239.

Authors' summary: "The problem of separating the spin and isospin dependence from the equation of motion of a system consisting of a small number of nucleons is considered. Certain coefficients, analogous to those used in the theory of angular momentum, are introduced and it is demonstrated that with their use the equation of motion may be reduced to a system of coupled differential equations involving the position coordinates only. Some of the properties of these coefficients and their connection with the permutation group are discussed. Tables of coefficients for three-nucleon and four-nucleon problems are also included."

Greiner, Walter; Green, Anthony M. 3770

Nuclear bremsstrahlung.

Nuclear Phys. **49** (1963), 481–488.

From the authors' summary: "A formalism for the bremsstrahlung accompanying scattering of nucleons in which the nuclear potential is explicitly worked into the radiation matrix elements. The result is an expansion of the matrix element in a power series of v/c and $\hbar\omega/\mu c^2$. As an application of this formalism we calculate the low-energy neutron-proton bremsstrahlung for a square well and a square well plus hard-core nuclear potential."

I. Białynicki-Birula (Warsaw)

Nakai, Shinzo 3771

On the theory of collective motion.

Nuclear Phys. **48** (1963), 45–57.

Tomonaga's method [Progr. Theoret. Phys. **13** (1955), 467–481; ibid. **13** (1955), 482–496; MR **18**, 172] for describing collective motion is extended to the three-dimensional case. The approximate canonical momentum operators defined by Tomonaga do not commute when their definition is extended to three-dimensional form. Previous calculations overcame this difficulty by using the method of redundant variables, but this leads to complicated subsidiary conditions. In this paper the author derives the correct canonical momentum by using the theory of point transformations and makes a comparison between Tomonaga's method and the exact method of canonical transformations. The formulation is such that it allows exact calculation of the commutators for the canonical variables. Only after this stage are approximations necessary to simplify the calculations.

As an application of the method, nuclear surface oscillations are investigated. It is shown how much more directly Bohr's phenomenological theory [Bohr, Danske Vid. Selsk. Mat.-Fys. Medd. **26** (1952), no. 14; Bohr and Mottelson, ibid. **27** (1953), no. 16] (the molecular model of atomic nuclei) can be derived than in the method of redundant variables. *I. M. Barbour* (Rome)

Moszkowski, S. 3772

Two-body interactions and the Nilsson potential.

Phys. Lett. **6** (1963), 237–238.

In calculations with the Nilsson model, the total energy of the nucleus is not the sum of energies of single particles, for such a sum will count the two-particle interactions twice. The author gives an example of a two-body interaction for which the total energy is the sum of single-particle energies in the Nilsson potential.

K. K. Gupta (Bombay)

Meligy, A. S. 3773

Coulomb wave functions for low energies.

Proc. Cambridge Philos. Soc. **60** (1964), 209–215.

Author's summary: "The irregular radial Coulomb wave function is expanded in a convergent series of Bessel functions in which the coefficients are expressed in powers of the energy and the argument of the Bessel functions depends on the radius only and not on the energy. This formulation is suitable for low-energy particles. The corresponding expansion for the regular Coulomb function can be deduced from previous work by Tricomi."

Nielson, C. W.; Koster, George F. 3774

★Spectroscopic coefficients for the p^n, d^n, and f^n configurations.

The M.I.T. Press, Cambridge, Mass., 1963. xi+275 *pp.* $8.50.

In the Racah analysis of atoms in p^n, d^n and f^n configurations one calculates matrix elements of operators with known transformation properties. The Clebsch-Gordan coefficients (related $3-j$ symbols), Racah W coefficients (or related $6-j$ symbols), and the coefficients of fractional parentage are useful. The first two tables are compiled in M. Rotenberg, R. Bivins, N. Metropolis and J. K. Wooten, Jr., *The 3−j and 6−j Symbols* [Tech. Press of M.I.T., Cambridge, Mass., 1959]. The present book tabulates the coefficients of fractional parentage, and their algebraic combinations occurring in the "reduced matrix elements".

A. O. Barut (Boulder, Colo.)

Combet Farnoux, Françoise 3775

Sur la détermination de certaines intégrales intervenant dans le calcul d'une limite inférieure des niveaux d'énergie des atomes.

C. R. Acad. Sci. Paris **258** (1964), 3439–3442.

Author's summary: "Contribution des parties angulaires de fonctions atomiques s et p aux intégrales faisant intervenir l'opérateur $[(1/r_{ij}r_{ik})+(1/r_{ij}r_{jk})+(1/r_{ik}r_{jk})]$ dans la détermination de l'intégrale $I_2=\int \Phi^*H^2\Phi\,d\tau$ pour un système atomique donné."

Robinson, P. D. 3776

Electron correlation in the ground state of helium.

Proc. Phys. Soc. **82** (1963), 659–663.

It is shown that the expectation value of the electron-nuclear interaction for the ground state of helium is stationary when evaluated with uncorrelated wave functions. This result, taken together with the virial theorem, provides an explanation of a linear relationship which exists between the error in the energy and the expectation value of the electron-electron interaction.

A. Dalgarno (Belfast)

Neugebauer, Th. 3777

Berechnung der Lichtzerstreuung mit doppelter Frequenz aus der Schrödingergleichung. (Russian summary)

Acta Phys. Acad. Sci. Hungar. **16** (1963), 217–226.

The effect of applying a periodic electric field to an atomic or molecular system is calculated by solving the Schrödinger equation to second order in the magnitude of the electric field, and the scattering of light in which the frequency is doubled is investigated.

A. Dalgarno (Belfast)

Neugebauer, Th. 3778

Berechnung der Lichtzerstreuung mit doppelter Frequenz aus der auf das n-Teilchenproblem verallgemeinerten Diracgleichung. (Russian summary)

Acta Phys. Acad. Sci. Hungar. **16** (1963), 227–248.

This is similar to the previous paper [#3777] except the theory is developed using a many-body Dirac equation.

A. Dalgarno (Belfast)

Corinaldesi, E. 3779

Perturbation method for diatomic molecules. (Italian summary)

Nuovo Cimento (10) **30** (1963), 105–111.

Author's summary: "The homopolar binding of diatomic molecules is treated by a method employing a modified Schrödinger equation for a multi-component wave function."

A. C. Hurley (Melbourne)

Kummer, Hans 3780

Das Eindeutigkeitsproblem in der hochauflösenden Protonenresonanzspektroskopie.

Helv. Phys. Acta **36** (1963), 901–936.

In der Kernresonanzspektroskopie benutzt man einen sogenannten Modell-Hamiltonoperator, in dem solche Parameter eingehen, die man nicht unmittelbar quantenmechanisch berechnen, sondern höchstens deren Grösse abschätzen, kann. Es ist deshalb ein wichtiges Problem aus dem gemessenen Absorptionsspektrum die Werte dieser Parameter zu bestimmen. Bei Molekülen ist der stationäre Term in dem Hamiltonoperator, den man in der Protonenresonanzspektroskopie benutzt, ein reiner Spinoperator, welcher die Form

$$(1)\qquad H=\sum_{i=1}^{n}\Omega_i I_{i3}+\sum_{i<k}\mathcal{F}_{ik}\mathbf{I}_i\mathbf{I}_k$$

besitzt. ($\mathbf{I}_i$ ist der Spinvektor des i-ten Protons.) Das erste Glied auf der rechten Seite von (1) ist einfach ein Zeeman-Term, der von der Wechselwirkung der einzelnen Kernspins mit dem statischen Magnetfeld H_0 (das in der Richtung der 3-Achse angenommen wird) herrührt. Das auf diese Kernspins einwirkende Feld ist jedoch nicht H_0, sondern $H_i=H_0(1-\Delta_i)$, wo Δ_i davon verursacht wird, dass das äussere Feld magnetische Momente in den Elektronenwolken induziert. Für die Ω_i haben wir:

$$(2)\qquad \Omega_i=\frac{1}{2\pi}\gamma H_i=\frac{1}{2\pi}\gamma H_0(1-\Delta_i),$$

wo γ das gyromagnetische Verhältnis des Protons bedeutet. Das zweite Glied in (1) beschreibt die durch die Elektronenwolken vermittelten Wechselwirkungen zwischen den Kernspins der das fragliche Molekül aufbauenden Atome. Die Kopplungskonstanten $\mathcal{F}_{ik}$ und die Δ_i, welche zusammen der Verfasser phänomenologische Parameter nennt, sind die Grössen, welche aus dem Absorptionsspektrum bestimmt werden müssen.

Ist das linear polarisierte Wechselfeld in Richtung der 1-Achse orientiert, so muss man die 1-Komponente des totalen Spins $F_1=\sum_{i=1}^{n}I_{i1}$ berechnen. Die Frequenz einer Absorptionslinie ist dann gleich der Differenz zweier Eigenwerte von H. Die Menge aller Hamiltonoperatoren der Form (1) stellt einen Teilraum N des Raumes P, der zu dem Spinraum der Protonen gehörenden symmetrischen Operatoren, dar. Andererseits wird der Begriff der virtuell möglichen Absorptionsspektren eingeführt und die Menge von allen diesen Spektren wird mit M bezeichnet. Die Berechnungsvorschrift lässt sich jetzt auffassen als eine Abbildung B vom Vektorraum N in die Menge M. Ein Element aus $B(N)$ ist ein Absorptionsspektrum. Die Frage welche der Verfasser in der vorliegenden Arbeit bespricht, ist das Problem der eindeutigen Umkehrbarkeit der Berechnungsvorschrift B. Physikalisch bedeutet dieses Problem die Frage, ob einem gemessenen Absorptionsspektrum nur eine Serie von

phänomenologischen Parametern entspricht, oder ob das selbe Spektrum durch mehrere solche Serien deutbar ist. Es wird bewiesen, dass bis auf die Umnumerierung der Teilchen, sowie der selektiven Umkehrung der Vorzeichen der in (1) stehenden Glieder die erwähnte Abbildung tatsächlich eindeutig umkehrbar ist.

Um diese Behauptung beweisen zu können, bespricht der Verfasser den Zustandskomplex eines quantenmechanischen Systems mit endlich vielen Eigenzuständen, die Beziehung zwischen dem Frequenz/Intensitäts-Kettenpaar und dem experimentellen Spektrum, die Beziehungen zwischen den phänomenologischen Parametern und dem Energiespektrum und endlich die Forminvarianzgruppe des Hamiltonoperators.

T. Neugebauer (Budapest)

Schultz, T. D. 3781

★Quantum field theory and the many-body problem.
Gordon and Breach, Science Publishers, New York-London, 1964. viii + 150 *pp.* $6.95.

This book is based on two series of lectures given by the author in 1960. The notes have been amended and well edited by the author in early 1962. The book is intended to give a first introduction to one particular aspect of field-theoretic methods of the many-body problem which is the Green's function method in perturbation theory approach via diagrams.

The treatise starts with a very short introduction to second quantization and the Heisenberg and Schrödinger pictures. A rather thorough discussion of many-fermion systems at absolute zero is then followed by a survey of the electron-phonon system at absolute zero. The last chapter is concerned with many-fermion systems at finite temperatures. Unfortunately, the discussion of superconductivity and of liquid helium had to be omitted.

It appears to the reviewer that the book serves its professed purpose extremely well, and prepares the absolute beginner for more elaborate further studies in this field. The style is clear and emphasizes physical ideas to a great extent. Extensive references are added to each chapter. Of course, no attempt to completeness is made. *P. Roman* (Boston, Mass.)

Ando, Tsuyoshi 3782

Properties of fermion density matrices.
Rev. Modern Phys. **35** (1963), 690–702.

This excellent paper, which summarizes and extends the work of previous authors, contains nineteen theorems pertaining to (i) the eigenvalues of reduced fermion density matrices, (ii) the problem of approximating the wave-function ψ of an N-fermion system by a sum of the form $\sum_{i=1}^{k} \psi_i^p \psi_i^q$, where ψ_i^p, ψ_i^q are, respectively, functions of p and q particles with $p+q=N$, and (iii) the expansion of ψ as a sum of Slater determinants. About one-half of the results of this paper occur also in a paper by the reviewer [same Rev. **35** (1963), 668–689; MR **27** #5571]; however, for most of these, the author gives alternative proofs and in one case makes a minor correction.

A. J. Coleman (Kingston, Ont.)

Talman, James D. 3783

Excitation spectrum of a many-boson system.
Phys. Rev. (2) **132** (1963), 955–958.

Author's summary: "It is shown within the framework of certain approximations that the excitation spectrum of a many-boson system satisfies the dispersion equation $E_{\mathbf{q}} = |\mathbf{q}|v$, where $\mathbf{q}$ is the momentum of the excitation and v is the macroscopic sound velocity. A method of avoiding the difficulties which arise in discussing a system with attractive forces is suggested, so that it may be possible to perform a calculation of the properties of liquid helium."

S. Gartenhaus (Lafayette, Ind.)

Faddeev, L. D. 3784

On the separation of self-action and scattering effects in perturbation theory. (Russian)
Dokl. Akad. Nauk SSSR **152** (1963), 573–576.

In order to circumvent the well-known difficulties inherent in the perturbational treatment of self-energy effects, a new method is proposed which is based more on intuitive perturbation theory rather than being mathematically rigorous. It is shown that a canonical (unitary) transformation can be constructed such that the energy operator breaks up into two terms— the first including all self-action effects, and the second containing only scattering effects. The eigenfunctions of the first term can be considered as asymptotic states.

P. Roman (Boston, Mass.)

Pruski, S. 3785

On an approximate second-order Fermion density matrix.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 615–619.

If $\mu(1;1')$ is the 1-matrix (or first-order reduced density matrix) of a system of identical fermions, then, as was shown by the reviewer [Quantum Chemistry Group Uppsala Univ. Tech. Note No. 80 (1962); Rev. Modern Phys. **35** (1963), 668–689; MR **27** #5571] it is possible to define a 2-matrix $\tau(12;1'2')$ as an infinite sum of powers of μ which not only enjoys the obvious symmetry properties but also is such that its $(2,2')$ trace equals $\mu(1;1')$. The present paper simplifies and makes much more explicit the reviewer's treatment of τ, and also proves the convergence of the infinite series by which it was originally defined. Since τ coincides with the exact 2-matrix for a system described by a single Slater determinant, the author proposes to explore its value in going beyond a one-configuration approximation.

A. J. Coleman (Kingston, Ont.)

Matsubara, Takeo; Thompson, Colin J. 3786

Zumino's theorem in the quasi-chemical equilibrium theory.
J. Austral. Math. Soc. **3** (1963), 456–467.

A theorem on the canonical form of an antisymmetric matrix, due to Zumino, can be used to give a much simpler derivation of certain expectation values essential in the theory of superconductivity. This new derivation is not only simpler, but avoids difficulties of principle in the previous derivation by the reviewer, which have been pointed out by M. Girardeau. The final results are in agreement with the reviewer's results.

J. M. Blatt (Kensington)

Kitamura, Masanao 3787

Non-linear integral equations of the Hammerstein type.

Progr. Theoret. Phys. **30** (1963), 435–442.

Author's summary: "The integral equations which have been studied recently by Bardeen, Cooper and Schrieffer and many others in connection with the theory of superconductivity are studied in rather general form. They are the non-linear integral equations of the kind that were studied by Hammerstein. Some of his results can be applied in their original form to the study of the Bardeen-Cooper-Schrieffer equation (BCS equation). However, the BCS equation is so singular that the usual proof of the convergence of the successive approximation cannot be applied in this case. It is shown here that with a slight but important modification the convergence of the successive approximation can be assured provided the BCS equation satisfies certain conditions."

L. N. Cooper (Providence, R.I.)

Privorotskiĭ, I. A. [Privorockiĭ, I. A.] 3788

The problem of pairings with nonzero angular momentum in a Fermi system.

Ž. Èksper. Teoret. Fiz. **44** (1963), 1401–1408 (*Russian. English summary*); *translated as Soviet Physics JETP* **17** (1963), 942–947.

Author's summary: "The properties of a Fermi system with an interaction in the case of pairs of nonzero angular momentum are investigated. A decoupling of the three-particle Green's function is proposed which allows one to obtain solutions both with an isotropic and anisotropic gap. It is shown that the solution with an anisotropic gap is asymptotically exact. A discussion is given of the difficulties arising in obtaining a basis for solutions with an isotropic gap."

L. N. Cooper (Providence, R.I.)

Grossmann, Siegfried 3789

Occupation number representation with localized one particle functions. (Macroscopic description of quantum gases. I.)

Physica **29** (1963), 1373–1392.

Author's summary: "The following considerations shall contribute to our understanding of the macroscopic-stochastic and macroscopic-deterministic properties of quantum gases. We shall begin with a physical formulation of the problem, which resembles Boltzmann's point of view in classical mechanics. Just the difference to Gibb's method, to which in quantum mechanics corresponds the discussion of the Wigner distribution function, is important to understand the properties of the many body system as seen by a macroscopic observer. We use the following method: Taking localized one particle functions (localized in configuration space as well as in momentum space) we define particle number operators, whose proper values or expectation values are the Boltzmann distribution function of the considered quantum gas, if measured in suitable 'macroscopical ensembles'. In order to discuss its time development the Hamiltonian of the gas is translated into the occupation number representation with these localized wave functions. The result is: There is not only a change of the distribution function due to the macroscopically well-known effects of streaming and collisions, but there would also be typical microscopical quantum effects, if the one particle functions are not chosen in a certain form together with a restriction in the desired observation. Thus the possible macroscopic observations in quantum gases are marked out. Our considerations also give a concrete example for the general concept of macroscopic observables."

R. Balescu (Brussels)

STATISTICAL PHYSICS, STRUCTURE OF MATTER

See also 3626.

Balescu, R. 3790

★Statistical mechanics of charged particles.

Monographs in Statistical Physics and Thermodynamics, Vol. 4.

Interscience Publishers [John Wiley & Sons, Ltd.], London-New York-Sydney, 1963. xii + 477 *pp*. $15.00.

In recent years, stimulated partly by applications in astrophysics and fusion research, interest has arisen in the statistical mechanics and kinetic theory of systems of electrically charged particles. From a fundamental point of view this subject offers many interesting aspects typical to it, such as shielding, collective oscillations, instabilities, etc. For this reason the appearance of a book completely devoted to it is to be welcomed. This book has a special flavor. It presents the subject from a unified point of view developed by the author in collaboration with I. Prigogine. In essence this is a sophisticated version of the perturbation theory of the Liouville equation. The fundamental quantities are the spatial Fourier components of the many-particle density function. The complicated relations governing their time development are represented symbolically by certain geometric structures (the Prigogine-Balescu diagrams), and selective summation is carried out for specific problems.

The book begins by the explanation of the general method. Chapters 2–5 deal with various aspects of collective oscillations, phenomena that generally occur on the shortest time scale. Chapters 6–11 deal with phenomena on a longer (the "kinetic") time scale. Chapter 12 is a brief discussion of thermal equilibrium and Chapter 13 deals with transport problems. The second part of the book (Chapters 14–18) contains a selection of topics from the quantum theory of Coulomb many-body systems. Singular integral equations play an important role in the mathematics and a summary of the pertinent information is given in a series of appendices.

The most attractive feature of the book is its unified point of view. At the same time this has certain disadvantages, because some of the derivations may appear unduly elaborate and motivated more by the requirements of the mathematics than by physical criteria. Also little attention is given to alternate derivations or points of view. Nevertheless, the book is bound to become an important part of the growing literature on the subject and is recommended for every serious student of kinetic theory.

A. Lenard (Princeton, N.J.)

Frieman, Edward A. 3791

On a new method in the theory of irreversible processes.

J. Mathematical Phys. **4** (1963), 410–418.

The starting point of this theory is the "BGBKY"

hierarchy of equations for the s-particle reduced distribution functions. It is well-known since Bogoljubov's classic work [*Problems of a dynamical theory in statistical physics* (Russian), GITTL, Moscow, 1946; MR **13**, 196; English transl. in Studies in Statistical Mechanics, Vol. I, pp. 1–118; North-Holland, Amsterdam, 1962; MR **24** #B2419] that a straightforward perturbation expansion of the solution fails for long times (of the order of the relaxation time) because of the occurrence of secular time variation. The author suggests an asymptotic expansion based on the existence of time scales of widely different lengths and illustrates it in the case of a weakly coupled gas. Assuming an expansion of the form

$$f_1 = f_1^{(0)}(t, \varepsilon\tau, \varepsilon^2\theta, \cdots) + \varepsilon f_1^{(1)}(t, \varepsilon\tau, \varepsilon^2\theta, \cdots) + \cdots,$$

where ε measures the strength of the interactions, and τ and θ are solutions of the equations $d\tau/dt = 1$, $d\theta/dt = 1, \cdots$, he expands the solution in powers of ε. At each order the secular terms are required to vanish. The Fokker-Planck equation is obtained rather simply from this condition. It is also shown that the effect of the initial correlations is lost by a mechanism of phase mixing in a short time.

The theory is very simple; it corresponds exactly to Bogoljubov's ideas, though being mathematically simpler. Its main conclusions (in particular, the phase mixing mechanism) also agree with the latest developments of Prigogine's theory [I. Prigogine, *Non-equilibrium statistical mechanics*, Interscience, New York, 1962].

The main drawback of the theory, in the reviewer's opinion, seems to be its rather essential limitation to the long-time study of systems in which there exists a small parameter. It cannot apparently be used for the study of general systems in which the time scales are not sharply separated (the latter obey non-Markoffian equations which cannot be obtained by asymptotic methods). Also, the application to non-uniform systems introduces some rather arbitrary classifications of orders of magnitude.

R. Balescu (Brussels)

Penrose, Oliver 3792

The remainder in Mayer's fugacity series.

J. Mathematical Phys. **4** (1963), 1488–1494.

Lieb [same J. **4** (1963), 671–678; MR **27** #6566] has proposed a method for obtaining upper and lower bounds on the distribution and thermodynamic functions for a classical system of particles with two-body interactions. Lieb's method is applied to the fugacity expansion. For a particular class of two-body interaction potentials, it is shown that successive partial sums in the expansions of the pressure, density, and distribution functions in powers of the fugacity constitute alternate upper and lower bounds for these functions. The interaction potential must either be non-negative or else have a hard core and decrease faster than r^{-3} at large distances. The results hold for positive values of the fugacity, and apply to lattice gases also. *R. J. Rubin* (Washington, D.C.)

Marshall, T. W. 3793

Random electrodynamics.

Proc. Roy. Soc. Ser. A **276** (1963), 475–491.

A study is made of a statistical ensemble of classical harmonic oscillators in interaction with an electromagnetic field. *D. ter Haar* (Oxford)

Byckling, Eero 3794

Occupation number representation in classical statistical mechanics.

Ann. Acad. Sci. Fenn. Ser. A VI No. **134** (1963), 34 pp.

An approximation scheme is developed for the Ising model, and is used to calculate the equation of state of a simple cubic lattice gas with an attractive interaction between first, second, and third nearest neighbors. The behavior of small convex clusters of lattice sites (in the application cubes of eight) is treated in detail, and one goes from these to the whole lattice in a way that is reminiscent of molecular field theory with the clusters as molecules. This reviewer would have found illuminating a discussion of the asserted exactness to order $1/N$ of certain steepest descent evaluations, as well as a comparison of the results of the method with those of some of the older techniques such as the Bethe-Peierls approximation.

N. D. Mermin (La Jolla, Calif.)

Kelbg, G. 3795

Theorie des Quanten-Plasmas.

Ann. Physik (7) **12** (1963), 219–224.

A discussion is given of the quantum-statistical partition function of a system with long-range interactions.

D. ter Haar (Oxford)

Taylor, P. L. 3796

The Boltzmann equation for conduction electrons.

Proc. Roy. Soc. Ser. A **275** (1963), 200–208.

Author's summary: "The paper presents the general procedure of solving the energy-dependent Boltzmann equation in plane geometry. The particular solutions are found and then it is proved that the general solutions can be formed by superposition of particular solutions. As an illustration, the fully degenerate kernel is considered in detail and a solution in a closed form is obtained."

E. J. Verboven (Nijmegen)

Bednarz, Roman J.; Mika, Janusz R. 3797

Energy-dependent Boltzmann equation in plane geometry.

J. Mathematical Phys. **4** (1963), 1285–1292.

Authors' summary: "Methods of solution of the Boltzmann equation for the distribution of electrons in an anisotropic metal are discussed. Some general properties of integral equations are used to develop a method suitable for use in numerical calculations." *E. J. Verboven* (Nijmegen)

Handscomb, D. C. 3798

A Monte Carlo method applied to the Heisenberg ferromagnet.

Proc. Cambridge Philos. Soc. **60** (1964), 115–122.

This paper is a continuation of an earlier paper by the author [same Proc. **58** (1962), 594–598; MR **26** #3453]. The author estimates, using the Monte-Carlo method, order parameters in a simple Heisenberg ferromagnet and compares his results with those obtained by other methods for the case of a simple cubic lattice.

D. ter Haar (Oxford)

Fujita, S. 3799

On the generalized Boltzmann equation of a quantum gas. II.

Physica **29** (1963), 1087–1104.

The generalized Boltzmann equation for a homogeneous imperfect gas derived by the author in an earlier paper [Physica **27** (1961), 940–956; MR **23** #B2906] "is found to be inadequate". A new equation is derived for the case of Fermi-Dirac and Maxwell-Boltzmann statistics. For technical reasons the case of Bose-Einstein statistics is not discussed. *R. J. Rubin* (Washington, D.C.)

Fajn, V. M. [Faĭn, V. M.] 3800

Das Prinzip der Entropiezunahme und die Quantentheorie der Relaxation.

Fortschr. Physik **11** (1963), 525–582.

In this review article, the author discusses various attempts to derive master equations, the question of increase in entropy, and the various methods for obtaining expressions for the relaxation of systems, such as ferro- and antiferro-magnetics. *D. ter Haar* (Oxford)

Faĭn, V. M. 3801

Principle of entropy increase and quantum theory of relaxation.

Uspehi Fiz. Nauk **79** (1963), 641–690 (*Russian*); *translated as Soviet Physics Uspekhi* **6** (1963), 294–323.

An English translation from the Russian; the German translation is reviewed above [#3800].

Verboven, E. 3802

The master equation with special transition probabilities.

J. Mathematical Phys. **4** (1963), 266–270.

The paper starts with a review of the work of Van Hove [Physica **23** (1957), 441–480; MR **19**, 696] and of Janner [Helv. Phys. Acta **35** (1962), 47–68; MR **25** #1891] on the master equation, wherein it is proved that certain quantum-mechanical many-body systems do reach microcanonical equilibrium. Following that, the case of an electron in a system of random elastic scatterers, previously treated by an approximation [Van Hove and the author, Physica **27** (1961), 418–432; MR **22** #11573], is treated numerically for a simple form of the transition probabilities. It is shown that while the previous approximate solution approached equilibrium by damped oscillations, the numerical solution shows only very slight oscillations. *G. Gioumousis* (Palo Alto, Calif.)

Peterson, Robert L.; Quay, Paul M. 3803

Generalized master equations for the density matrix.

J. Mathematical Phys. **5** (1964), 85–89.

Van Hove [Physica **23** (1957), 441–480; MR **19**, 696] and Janner [Helv. Phys. Acta **35** (1962), 47–68; MR **25** #1891] have derived generalized master equations for systems with an infinite number of degrees of freedom using the diagonal singularity properties. Swenson [J. Mathematical Phys. **3** (1962), 1017–1022; MR **26** #4739] has shown that the equations formally identical to those of Van Hove and Janner can be derived for finite systems and without any special assumptions about the properties of the perturbations. The present authors develop techniques of rewriting the von Neumann equation for the density matrix describing a system of an arbitrary size. They demonstrate that a number of generalized master equations can be derived according to chosen principles. In the reviewer's opinion it is desirable to formulate a theory which is independent of any special representation in which the density operator or its equivalent is expressed. *S. Fujita* (Brussels)

Lebowitz, J. L. 3804

Exact solution of generalized Percus-Yevick equation for a mixture of hard spheres.

Phys. Rev. (2) **133** (1964), A895–A899.

Author's summary: "The Percus-Yevick approximate equation for the radial distribution function of a fluid is generalized to an m-component mixture. This approximation which can be formulated by the method of functional Taylor expansion, consists in setting $\exp[-\beta\varphi_{ij}(r)]C_{ij}(r)$ equal to $g_{ij}(r)[e^{-\beta\varphi_{ij}(r)}-1]$, where C_{ij}, g_{ij}, and φ_{ij} are the direct correlation function, the radial distribution function and the binary potential between a molecule of species i and a molecule of species j. The resulting equation for C_{ij} and g_{ij} is solved exactly for a mixture of hard spheres of diameters R_i. The equation of state obtained from $C_{ij}(r)$ via a generalized Ornstein-Zernike compressibility relation has the form

$$p/kT = \{[\sum \rho_i][1+\xi+\xi^2]-18/\pi \sum_{i<j} \eta_i\eta_j(R_i-R_j)^2 \times[R_i+R_j+R_iR_j(\sum \eta_l R_l^2)]\}(1-\xi)^{-3},$$

where $\eta_i=\pi/6$ times the density of the ith component and $\xi=\sum \eta_i R_i^3$. This equation yields correctly the virial expansion of the pressure up to and including the third power in the densities and is in very good agreement with the available machine computations for a binary mixture. For a one-component system our solution for $C(r)$ and $g(r)$ reduces to that found previously by Wertheim and Thiele, and the equation of state becomes identical with that found on the basis of different approximations by Reiss, Frisch, and Lebowitz." *H. L. Frisch* (Murray Hill, N.J.)

Hemmer, P. C.; Hauge, E. Hiis 3805

Yang-Lee distribution of zeros for a van der Waals gas.

Phys. Rev. (2) **133** (1964), A1010–A1015.

Authors' summary: "The distribution of zeros of the grand partition function for a gas obeying van der Waals' equation of state (together with Maxwell's rule) is studied. For infinite temperature the zero distribution is located on part of the negative real axis, but with decreasing temperatures the distribution branches off the real axis circumventing the origin on both sides. Below the critical temperature the distribution forms a closed curve around the origin with a diameter decreasing exponentially to zero as $T\to 0$. An additional tail of the distribution remains on the negative real axis at all temperatures, but with a density of zeros going linearly to zero with $T\to 0$."

Mühlschlegel, B.; Zittartz, H. 3806

Gaussian average method in the statistical theory of the Ising model. (German summary)

Z. Physik **175** (1963), 553–573.

The partition function of the Ising model whose Hamiltonian for the case of spin $\frac{1}{2}$ is given by $H = -\varepsilon \sum_i^N \sigma_i - \frac{1}{2} \sum_{i,j}^N V_{ij}\sigma_i\sigma_j$ is expressed as the Gaussian average, where the summations run over the N lattice sites of the system and σ_i is the third Pauli spin matrix. The partition function is expressed in the form

$$Z = \int dz^N e^{-G},$$

$$G = \pi \sum_i z_i^2 - \sum_i \log 2 \cosh (\beta\varepsilon + 2\pi \sum_j A_{ij}z_j),$$

where the A_{ij} are given by

$$\sum_k A_{ik}A_{kj} = \frac{\beta}{2\pi} V_{ij}.$$

The variation method is applied to obtain the approximate solution, by making the free energy itself stationary, in contrast with the usual steepest descent method. The variation parameters employed are (1) the magnetization, (2) the N eigenvalues of a matrix which is connected to the spin-spin correlation, and (3) a parameter which keeps the total number of spins constant. The results are compared with those derived by the approximation based on the linked cluster expansion, which has been developed in modern many-body theory. It is found that in the case of high temperature expansion for the nearest-neighbor interaction in the simple cubic lattice the difference appears only from the sixth order on. *S. Ono* (Tokyo)

Reiss, H. 3807

Role of thermodynamic activity in rate processes.

J. Chem. Phys. **40** (1964), 1783–1791.

Author's summary: "This paper contains an investigation of the relation between rates of diffusion and evaporation and thermodynamic activity. The method of proceeding involves the use of the one-dimensional Ising model for which an exact statistical treatment can be given. The actual system treated may be thought of as an edge dislocation with adsorbed impurity atoms. As a byproduct, some interesting results concerning the formation of cluster and pinning points along a dislocation are derived. However, the main purpose of this work remains the investigation of the connection between rate and activity. For the one-dimensional model (for any degree of coupling between molecules) it can be shown exactly that the rate of evaporation is proportional to the thermodynamic activity with constant coefficient and that the rate of diffusion is similarly proportional to the gradient of the thermodynamic activity with constant coefficient. These results, which have been suggested before, therefore receive theoretical justification in terms of this simple model. Certain not-too-restrictive conditions must be applied, however. The most important of these is the requirement that every subsystem of the system be at equilibrium with respect to all variations save the one which corresponds to the rate process."

Bowe, Joseph C. 3808

Electron velocity distributions in gases.

Amer. J. Phys. **31** (1963), 905–912.

Starting with the well-known Maxwell-Boltzmann transport equation for a one species gas, subject to the action of an external electric field, an analytical approximation for the velocity distribution is obtained. The author assumes that the distribution function is independent of space and time and that the applied external electric field is a d.c. field. The collision term is regarded as the sum of two terms, one due to elastic collisions and the other to inelastic collisions. A reference energy u_1 is introduced in terms of which different approximations are obtained for the high- and low-energy regions. Also the distribution function is assumed to be given as the sum of two terms one of which is isotropic and a smaller anisotropic term due to the presence of the electric field. From the assumption that the cross-section for inelastic collisions is proportional to $(u-u_1)^2$, an approximate analytical solution for the high-energy range is given in terms of Hankel functions of the first kind. Physical interpretations and the effects of inelastic collisions are emphasized.

J. K. Thurber (New York)

Mandelbrot, Benoit 3809

On the derivation of statistical thermodynamics from purely phenomenological principles.

J. Mathematical Phys. **5** (1964), 164–171.

The author derives the laws of statistical thermodynamics from purely phenomenological principles. He makes the following assumptions. The zeroth principle: A system known to have an energy U contained between u and $u+du$ can be found with equal probabilities in either of $dG(u)$ states, where $G(u)$ is a non-decreasing function of u. The first principle: Energy is the unique invariant of certain physical transformations, those resulting from thermal interaction. If the thermal interactions are weak, the energy is an additive expression.

It is shown that Gibbs' canonical distribution can be derived from these principles. The concepts of entropy and heat can also be derived provided one further assumes the second principle which states that $G(u)$ is an adiabatic invariant, that is, no energy level can either be created or annihilated by a change of volume of the enclosure. The author develops this approach to prove some properties of entropy and other thermodynamic functions.

F. C. Auluck (Delhi)

Nettleton, R. E. 3810

Antireciprocity and memory in the statistical approach to irreversible thermodynamics.

J. Chem. Phys. **40** (1964), 112–116.

Author's summary: "The macroscopic state of a large, closed system is specified by the ensemble averages α_j of a set of even dynamical variables A_j, together with the averages v_j of the quantities iLA_j, obtained by operating on the A_j with the self-adjoint Liouville operator L. Phenomenological equations for the time rates of change, $\dot{\alpha}_j$ and $\dot{v}_j$, are derived by a technique due to Zwanzig, in which one operates on Liouville's equation with a projection operator which projects out the part which is 'relevant' to the phenomenological description employed. These phenomenological equations are shown to exhibit the Onsager-Casimir reciprocity relations, including antisymmetry relations whose derivation is found to require a slight modification of Zwanzig's mathematical assumptions. Since $v_j = \dot{\alpha}_j$, these equations also show that irreversible thermodynamics can be extended to the case

where second-order time derivatives appear representing memory effects, as well as nonlinear terms in the α_j, provided the equations are still required to be linear in the v_j. Furthermore, Onsager's equations are obtained while allowing the phenomenological matrices to be functions of the variables α_j, and not merely of constants of the motion. This serves to generalize and extend Zwanzig's earlier treatment."

Sone, Yoshio 3811

Kinetic theory analysis of the linearized Rayleigh problem.

Phys. Fluids 7 (1964), 470–471.

The paper discusses the linearized Rayleigh problem utilizing the B-G-K statistical model. In particular, the author evaluates the slip velocity and the stress on the plate in the following two cases: (i) when the time is very much smaller than the inverse of the mean collision frequency, and (ii) when the time is very much greater than the inverse of the mean collision frequency. It appears that the B-G-K model predicts the desired characteristics of the problem.

P. L. Bhatnagar (Bangalore)

Gião, António 3812

Sur la loi de distribution de Maxwell-Boltzmann.

Arquivo Inst. Gulbenkian Ci. Sec. A Estud. Mat. Fís.-Mat. 1 (1963), 1–30.

On obtient la loi de distribution d'équilibre statistique d'un système relativiste de N particules à partir d'un schéma hamiltonien. L'évolution du système vers l'équilibre statistique est ensuite étudiée et le théorème H déduit des considérations précédentes.

J. Naze (Bergen)

Andersen, Knud; Shuler, Kurt E. 3813

On the relaxation of the hard-sphere Rayleigh and Lorentz gas.

J. Chem. Phys. 40 (1964), 633–650.

Authors' summary: "As part of a study of the relaxation of nonequilibrium systems, the (translational) relaxation of a hard-sphere Rayleigh and Lorentz gas is investigated. From a detailed analysis of the collision dynamics an exact expression is derived for the kernel $A(x|x')$ of the collision integral which gives the probability per unit time for a change of the reduced kinetic energy from x' to x during a binary collision between a subsystem and a heat bath particle. A master equation, i.e., a linearized Boltzmann equation, incorporating this kernel is then formulated to represent the time variation of the distribution function of the subsystem particles. Making use of the special property of this kernel that it is a strongly peaked function around $x-x'=0$ for both the Rayleigh and Lorentz gas, a technique is developed for transforming this integral master equation into differential Fokker-Planck equations consistent in the order of the expansion parameter λ, the ratio of the mass of the heat bath particles to the subsystem particles. The Fokker-Planck equation for the Rayleigh gas is solved analytically and explicit solutions are presented for the relaxation of initial Maxwell and initial δ-function distributions of the energy. It is shown that an initial Maxwell distribution of the energy (or speed) of the subsystem particles relaxes to the final equilibrium Maxwell distribution via a continuous sequence of Maxwell distributions. The mean energy of the subsystem particles is shown to relax exponentially to its equilibrium value independent of the form of the initial distribution. For the hard-sphere Lorentz gas the Fokker-Planck equation is not susceptible of an analytical solution. Machine solutions are presented for various initial distributions which show that the Maxwell distribution is not preserved in the relaxation of the hard-sphere Lorentz gas. Finally, a brief discussion is given of the relation between the hard-sphere model ($r^{-\infty}$) considered here and the more general model of the Rayleigh and Lorentz gas with an r^{-s} repulsive central force law."

Cercignani, Carlo; Daneri, Adelia 3814

Flow of a rarefied gas between two parallel plates.

J. Appl. Phys. 34 (1963), 3509–3513.

Authors' summary: "The Poiseuille flow of a rarefied gas between two parallel plates is analyzed numerically for an inverse Knudsen number ranging from 0 to 10.5. The Bhatnagar, Gross, and Krook model is used and the transport integro-differential equation is reduced to a purely integral one, which is solved numerically by the discrete ordinate method. The plot of the volume flow rate vs. pressure is shown to have the expected minimum; besides, it fits well with experimental results and previous approximate calculations. In particular, the results given by Takao, properly corrected, are in good agreement with ours."

P. L. Bhatnagar (Bangalore)

Darrozès, Jean 3815

Étude de la structure d'un choc faible avec le modèle cinétique de Krook-Boltzmann. (English summary)

Recherche Aérospat. No. 95 (1963), 17–22.

The one-dimensional steady shock-layer problem is formulated in terms of the Bhatnagar-Gross-Krook kinetic model. This equation is transformed into a system of integral equations in the density and velocity of temperature of the gas. Weak shock-wave theory is considered in detail. It is shown that the leading asymptotic (in shock strength) term is the same as that obtained from the asymptotic solutions of the Navier-Stokes equation for the identical gas.

L. Sirovich (Providence, R.I.)

Guiraud, J. P. 3816

Théorie cinétique des phénomènes de dissipation dans les mélanges de gaz; application au calcul des coefficients de dissipation par une méthode simplifiée.

Rarefied Gas Dynamics (*Proc. 3rd Internat. Sympos., Palais de l'UNESCO, Paris*, 1962), *Vol. I, pp.* 226–251. *Academic Press, New York*, 1963.

Author's summary: "Une tentative pour échapper à la méthode de Chapman-Cowling dans le calcul des coefficients de viscosité, conductibilité thermique, et diffusion des mélanges de gaz est présentée, dans le but de simplifier radicalement la théorie. La technique utilisée est très analogue à celle mise en oeuvre par Prigogine pour montrer que la solution de Chapman-Cowling est en accord avec l'équation de l'entropie dérivée du concept de Gibbs et de la mécanique des milieux continus, mais ici l'ordre des opérations est inversé car on se sert de l'équation de l'entropie pour calculer les coefficients de transport en

supputant la structure microscopique compatible aves les valeurs des flux, grâce à un principe d'écart minimum de la fonctionnelle H relativement à sa valeur d'équilibre. La méthode est sans doute critiquable en raison des approximations a priori qu'elle comporte mais c'est le prix qu'il faut payer pour la simplicité et la souplesse: des flux surabondants peuvent ainsi être introduits, pour serrer de plus près la réalité, qui sont ensuite déterminés par minimisation de la dissipation."

Shkarofsky, I. P. 3817
Cartesian tensor expansion of the Fokker-Planck equation.
Canad. J. Phys. **41** (1963), 1753–1775.

Author's summary: "The Fokker-Planck equation is expanded via Cartesian tensors. The results, valid for arbitrary mass ratio, reduce to known expressions for the f_0 (isotropic) and $\mathbf{f}_1$ (directional) parts of the distribution. The same analysis also gives the relation for the $\mathbf{f}_2$ (tensor) part of the distribution. Nonlinear and recoil terms are also investigated. The f_0 relation yields relaxation times and energy equipartition times for arbitrary velocity distributions. The ion recoil term is important for low-frequency waves in the electron $\mathbf{f}_1$ equation but is negligible in the electron $\mathbf{f}_2$ equation. The electron-electron contributions to the $\mathbf{f}_1$ and $\mathbf{f}_2$ collisional terms are of the same order as the electron-ion contributions. Except for momentum transfer calculations, the ion-ion collisional terms dominate over the ion-electron terms in the ion collision equations referring to the directional ($\mathbf{F}_1$) and tensor ($\mathbf{F}_2$) parts of the ion distribution."
R. Balescu (Brussels)

Shkarofsky, I. P. 3818
Inclusion of flow terms in the Cartesian tensor expansion of the Boltzmann equation.
Canad. J. Phys. **41** (1963), 1776–1786.

Author's summary: "The Cartesian tensor expansion of Boltzmann's equation as given by Johnston [Phys. Rev. (2) **120** (1960), 1103–1111; MR **22** #8741] is extended to include terms denoting gradients in flow velocity. The expansion is performed in intrinsic velocity space. The gradient velocity terms yield a linear contribution to the tensor ($\mathbf{f}_2$) part of the angle-integrated distribution function from which the zero-trace pressure tensor is calculable. It is shown that the standard moment equations are obtained by further integration over the magnitude of velocity. For the case of a completely ionized gas, collisional terms are inserted appropriately."
R. Balescu (Brussels)

Shkarofsky, I. P. 3819
Calculation of the pressure tensor in a fully ionized plasma.
Canad. J. Phys. **41** (1963), 1787–1800.

Author's summary: "The tensor part of the pressure arising from gradients in flow velocity is calculated from the Fokker-Planck equation by using the Cartesian tensor expansion in direction cosines of the intrinsic velocity vectors and the Laguerre expansion for the magnitude of the intrinsic velocities. Wavelike perturbations and magnetic fields are included and both the electron and ion pressure tensors are investigated. The results are compared with those derived by other approaches."
R. Balescu (Brussels)

Dupree, Thomas H. 3820
Kinetic theory of plasma and the electromagnetic field.
Phys. Fluids **6** (1963), 1714–1729.

An exact kinetic equation for plasma and the electromagnetic field is derived. This equation describes the fluctuations of the fields and particle distributions. The solution is obtained by expanding in a parameter which characterizes the amplitude of these fluctuations. A systematic procedure is given for generating the solution to arbitrary order in the expansion. Some typical applications of the theory are presented. These include calculations of a collision integral, incoherent scattering, and bremsstrahlung emission and absorption.
I. M. Cohen (Providence, R.I.)

Frieman, E. A.; Book, D. L. 3821
Convergent classical kinetic equation for a plasma.
Phys. Fluids **6** (1963), 1700–1706.

A kinetic equation is derived for the description of the approach to equilibrium of a plasma. This equation combines the collective features of the Coulomb collisions with the hard short-range behavior. As a result, the equation is free from divergences, as well at long distances as at short ones. The method used is an application of the first author's recently developed theory of kinetic equations [see #3791 above].
R. Balescu (Brussels)

Gurevich, A. V. [Gurevič, A. V.] 3822
Smearing out of inhomogeneities in a weakly ionized plasma in a magnetic field (ambipolar diffusion).
Ž. Èksper. Teoret. Fiz. **44** (1963), 1302–1306 (*Russian. English summary*); *translated as Soviet Physics JETP* **17** (1963), 878–881.

The author considers an infinite plasma so weakly ionized that collisions of charged particles with neutrals dominate all other types of collisions and inertial effects of charged particles. The plasma is uniform except for small initial perturbations in the charged particle densities and velocities. Utilizing the macroscopic, linearized, continuity and momentum equations for electrons and ions in the isothermal approximation, the author describes the development in time due to diffusion of given initial perturbations. The problem is solved by Fourier transforming in space and Laplace transforming in time.

Assuming the initial perturbation to be many Debye spheres in extent, it is found that there are two characteristic times associated with two characteristic types of diffusion. The first characteristic time, which is very short compared to the second, is the time scale for the initial net charge density to vanish. The second characteristic time is much longer, and is associated with ordinary ambipolar diffusion.

Supposing the plasma is immersed in a uniform magnetic field which is much stronger than any magnetic perturbations caused by the plasma itself, the author shows that the resulting ambipolar diffusion coefficient D

cannot in general be separated into $D_{\parallel}$ and $D_{\perp}$ characterizing diffusion along and across the magnetic field, respectively. However, if the initial inhomogeneity extends much farther across than along the field, one has approximately,

$$D_{\parallel} = \kappa(T_i + T_e)/M\nu_i,$$

$$D_{\perp} = \kappa(T_i + T_e)/M\nu_i[1 + (\Omega/\nu_i)^2],$$

where M is ion mass, ν_i is ion neutral collision frequency, $T_{i,e}$ is the (ion, electron) temperature, and Ω is the ion cyclotron frequency. On the other hand, if the initial inhomogeneity extends much farther along than across the field, one has $D_{\parallel}$ as above, but

$$D_{\perp} = \kappa(T_i + T_e)/M\nu_i\left[1 + \frac{m}{M}\frac{\omega^2}{\nu_i\nu_e}\right],$$

where m is electron mass, ω is electron cyclotron frequency, and ν_e is electron neutral collision frequency.

R. A. Gerwin (Seattle, Wash.)

Berk, Herbert L. 3823

Frequency- and wavelength-dependent electrical transport equation for a plasma model.

Phys. Fluids **7** (1964), 257–262.

Author's summary: "The plasma model used by Dawson and Oberman is extended to frequencies embracing the collision frequency and to finite wavelengths. To the so-called dominant order, the collisional term is a Fokker-Planck equation which is frequency dependent. At low frequencies, the electrical conductivity is found to agree with conventional Lorentz-gas theories, while at high frequencies, the electrical conductivity is that predicted by Oberman and Dawson. The finite wavelength corrections to the electrical conductivity are obtained."

Czerwonko, Jerzy 3824

Quantum statistical mechanics of antiferromagnetics. I.

Acta Phys. Polon. **22** (1962), 445–475.

Author's summary: "The thermodynamical perturbation method for isotropic antiferromagnetics is studied. Average values are computed by the summation of an infinite number of diagrams in the representation of Matsubara. The theory may be applied to lattices divisible into two sublattices such that the nearest neighbours of atoms of the first sublattice lie in the second one."

S. Fujita (Brussels)

Kaščeev, V. 3825

On the theory of the Mossbauer effect. (Russian)

Latvijas PSR Zinātņu Akad. Vēstis **1963**, *no.* 8 (193), 43–51.

The anharmonic terms are taken into account in order to calculate the influence of the thermal vibrations on the Mossbauer frequency. The average vibrational energy for the third and fourth order anharmonic terms has been calculated and it is found that they lead to a shift of the maximum of the Mossbauer line.

J. Zak (Cambridge, Mass.)

Schlup, W. 3826

Die Regularisierung der interpolierten Clusterentwicklung für die Eigenfrequenzverteilung einer ungeordneten binären Kette.

Helv. Phys. Acta **36** (1963), 886–900.

Verfasser fasst seine Ergebnisse wie folgt zusammen: Die lokal negative Verteilungsdichte $f(\omega^2)$ der Eigenfrequenzen von Gitterschwingungen einer isotopen linearen Kette wird in symmetrischer Weise in erster Ordnung in den beiden Konzentrationen interpoliert und in Form der komplexen Clusterentwicklung algebraisch regularisiert. Das Resultat wird bis auf die Willkür bei der Interpolation eindeutig aus der Wahrscheinlichkeitsdichte $f(\omega^2) \geqq 0$ und dem Rayleigh-Theorem festgelegt. Dabei hat man unter den verschiedenen Methoden zur analytischen Berechnung der Eigenfrequenzverteilung eines ungeordneten Gitters vier Gruppen zu unterscheiden: (I) Exakte Lösungen, die nur für sehr spezielle Modelle bekannt sind, (II) Mit Hilfe der Methode der Entkopplung von temperaturabhängigen Greenfunktionen gefundene Näherungslösungen, (III) Verbesserte Methoden einer analytischen Störungstheorie, (IV) Lösungen der einfachen analytischen Störungstheorie, die, physikalisch interpretiert, als Clusterentwicklung bezeichnet wird.

Im Hinblick auf die komplexe Form der Clusterentwicklung wird die Regularisierung mittels einer komplexen Hilfsfunktion durchgeführt (§ 2). § 3 untersucht die Invarianzeigenschaften der Frequenzverteilung und die Symmetriesierung der Clusterentwicklung. Die Frage, ob die Wahrscheinlichkeitsbedingung und das Rayleigh-Theorem alle algebraischen Regularisierungen ausschließen, bleibt offen.

M. Pinl (Moscow, Idaho)

Brown, E. 3827

Bloch electrons in a uniform magnetic field.

Phys. Rev. (2) **133** (1964), A1038–A1044.

Author's summary: "The physical periodicity of a space lattice is not destroyed by the presence of a uniform magnetic field. It is shown that a ray group of unitary operators, isomorphic to pure translations, commutes with the Hamiltonian in this case. Such a group has the characteristic property that $AB = \exp[i\phi(A, B)]C$, where A, B, and C are elements of the group and ϕ is a numerical factor. Representation theory applied to this group yields the characteristic degeneracies of levels in magnetic fields, as well as the transformation properties of eigenfunctions. By means of these it is possible to construct an effective Hamiltonian appropriate to finite magnetic fields in crystals."

Bhagavantam, S.; Pantulu, P. V. 3828

Magnetic symmetry and physical properties of crystals.

Proc. Indian Acad. Sci. Sect. A **59** (1964), 1–13.

Authors' summary: "Application of group-theoretical methods to a study of the effect of symmetry on the physical properties of the 32 crystal classes as given earlier by Bhagavantam has been extended in this paper to a similar study of the magnetic properties of the 90 magnetic crystal classes. Results obtained in various cases agree with those derived earlier by more elaborate methods."

Marinchuk, A. E. [Marinčuk, A. E.]; Moskalenko, V. A. 3829
Thermodynamics of a crystal lattice.
Fiz. Tverd. Tela **5** (1963), 575–580 (*Russian*); *translated as Soviet Physics Solid State* **5** (1963), 418–421.
Authors' summary: "On the basis of a diagrammatic technique, an expression is derived for the thermodynamic potential Ψ of a homogeneous lattice considering cubic and quartic anharmonicities. The Dyson equation for the polarization operator is obtained, and a variational theorem for Ψ is established."

Fletcher, N. H. 3830
Crystal interfaces.
J. Appl. Phys. **35** (1964), 234–240.
Author's summary: "An expression for the energy of an interface of general form between two crystals of arbitrary structures and relative orientations is derived in a form suitable for a variational calculation. This variational approach is applied to a one-dimensional interface between two two-dimensional crystals of differing lattice constants. In general there is a minimum in the interfacial energy when the lattice distances in the two crystals are in the ratio of small integers. Particular cases in which the surface potential is either sinusoidal or parabolic are discussed and detailed curves of interfacial energy as a function of lattice misfit are calculated."

Enderlein, R. 3831
A new method in the quantum mechanical transport theory and the dissipation-fluctuation theorem for thermal disturbances.
Phys. Lett. **7** (1963), 326–329.
A non-uniform local equilibrium is described in terms of "superlattice states". In this representation, the author derives the Kubo formula of electron transport for a model of electrons interacting with impurities.
R. Kubo (Philadelphia, Pa.)

Nakano, H. 3832
A variation principle in the quantum theory of irreversible processes.
Proc. Phys. Soc. **82** (1963), 757–777.
The transport coefficients or generalized admittance can be rigorously expressed in terms of certain correlation functions of relevant physical quantities. Such expressions may be regarded as extreme values of certain functionals properly defined for the respective problem. The variational principle is then equivalent to a rigorous equation of motion of a density matrix. Examples are discussed for the cases of electric and magnetic susceptibilities. Generally the extremum is only stationary, but it becomes a minimum principle for dissipation parts. Perturbational approximations of the variation problem are also discussed.
R. Kubo (Philadelphia, Pa.)

Salpeter, E. E.; Treiman, S. B. 3833
Multiple scattering in the diffusion approximation.
J. Mathematical Phys. **5** (1964), 659–668.
Authors' summary: "The passage of classical particles through a grainy scattering medium can be described by a linearized Boltzmann equation. A discussion is given of the physical conditions which justify the use of the Fokker-Planck diffusion approximation to this equation. Some limiting properties of the solutions of the diffusion equation are first discussed for the initial value problem in an infinite medium characterized by a diffusion length D. For a total path length $l \ll D$ convenient formulas are given for the distribution of scattering angles θ and, for given θ, the first few moments of the final position vector are computed. These results are taken as a basis for approximate treatment of steady-state boundary value problems. The case of a particle beam incident on a thin plane parallel slab of thickness $d \ll D$ is considered. Approximate formulas are given for the angular distribution of the transmitted beam and for the (very small) fraction of the beam which emerges from the entrance face. Errors are assessed, and the behavior for grazing angles of incidence or exit is discussed in a conjectural way."

McLennan, James A., Jr.; Swenson, Robert J. 3834
Theory of transport coefficients in low-density gases.
J. Mathematical Phys. **4** (1963), 1527–1536.
The evaluation of transport coefficients, based on correlation-function expressions, is discussed. The calculation is restricted to the low-density case, although the methods employed permit a wider range of application. The analysis is essentially based on a generalized master equation [see, e.g., R. J. Swenson, same J. **3** (1962), 1017–1022; MR **26** #4739] and the factorization theorem of M. Kac [Proc. Third Berkeley Sympos. Math. Statist. and Probability, 1954/55, Vol. III, pp. 171–197, Univ. California Press, Berkeley, Calif., 1956; MR **18**, 960]. The results obtained are in agreement with those usually found from the Boltzmann equation.
E. J. Verboven (Nijmegen)

Swenson, Robert J. 3835
A note on the formal theory of transport coefficients.
Physica **29** (1963), 1174–1180.
The present reviewer has proven a theorem [Physica **27** (1961), 693–706; MR **24** #B335] stating that the transport coefficient characterizing the steady non-equilibrium state is determined by the Markoffian long-time kinetic equation alone, instead of the full non-Markoffian equation which characterizes the approach of the system to equilibrium. In the present paper an alternative proof of this theorem is given for the quantum-mechanical case, and the theorem is shown to be valid also for low-frequency transport coefficients. The proof is within the Van Hove-Janner formalism [see, e.g., Janner, Helv. Phys. Acta **35** (1962), 47–68; MR **25** #1891]. The mathematics is simple and elegant.
R. Balescu (Brussels)

Fujita, S.; Mayné, F. 3836
Theory of transport coefficients. IV. Electrical conductivity in the presence of a strong magnetic field.
Physica **29** (1963), 1201–1213.
Part III (by Fujita) appeared in J. Mathematical Phys. **3** (1962), 1246–1250 [MR **26** #7384]. Authors' summary: "The transverse electrical conductivity of an electron-phonon system under a strong static magnetic field is calculated up to the second order in the electron-phonon interaction starting from the Nakano-Kubo formula and

using the Feynman diagram techniques. The result is in agreement with that obtained by Titeica from intuitive grounds. It is found that the initial electron correlations contained in the canonical distribution $\exp[-\beta(H_0+H_I)]$ contribute to the conductivity in contradistinction to the case of a magnetic field-free system, where the (static) conductivity in general can be expressed in terms of asymptotic cross-sections defined between states free from correlations. This fact necessarily implies the impossibility of properly describing the magneto-resistance on the basis of Boltzmann-like equations." *R. Balescu* (Brussels)

Čujanov, V. A. 3837
The inverse problem in the theory of a nuclear reactor. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 35–51.
The flow of neutrons $\Phi(r, v)$ of velocity v at a point r in a nuclear reactor of a given form, of given dimensions and with certain operating conditions, is described by a singular integral equation for $\Phi(r, v)$. The direct problem of the theory of a nuclear reactor consists of determining the characteristic value λ for which the mentioned integral equation has a positive solution. In practical calculations of a nuclear reactor it is usually required to determine the critical concentration ρ_l of the nuclei. The development of atomic power stations has given rise to new problems related to the mentioned integral equation. Examples of such problems are given by G. Goertzel and W. A. Loeb [Nucleonics **12** (1954), no. 9, 42–45]. An important problem of this type involves the space distribution of the quantity

$$Q(r) = \rho_1(r)\int_0^\omega \sigma_f(v)\Phi(r, v)\,dv,$$

which is proportional to the heat release of the reactor. Here $\rho_1(r)$ is the concentration of nuclei of the fission element and σ_f is the cross-section of the fission. In the simpler cases of a given temperature field, one can assign the desired form to the function $Q(r)$. This form can be obtained by selecting the appropriate distribution of the concentration of the nuclei ρ_l in the reactor. The function can be given in explicit form and depends on the concentration ρ_l. The determination of $\Phi(r, v)$ and $\rho_l(r)$ by means of the original integral equation and certain auxiliary conditions is called by the author the inverse problem in the theory of nuclear reactors. Its solution yields the required constitution of the reactor in terms of the properties of a desired neutron flow, in contrast to the direct problem of determining the flow for a given reactor compound. The problem leads to a nonlinear integral equation. It is solved by the method of successive substitutions. The solution is shown to exist for certain characteristic values. An illustrative numerical example is given with tabulated results.
H. P. Thielman (Alexandria, Va.)

RELATIVITY

See also 2955, 3568, 3693, 3862, 3870, 3871.

Infeld, L. 3838
"Uniformly accelerated" motion and relativity.
Acta Phys. Polon. **23** (1963), 69–75.
The quadratic line element of a de Sitter universe is shown to be unchanged by a coordinate transformation which represents a transition from rest to a uniformly accelerated motion. In the two descriptions of the universe, different expressions for the red-shift are obtained.

Although the author does not draw attention to the fact, it should be noted that the universe with "uniformly accelerated" motion is kinematically equivalent to the system of relatively accelerated observers proposed by L. Page [Phys. Rev. (2) **49** (1936), 254–268], which was shown by H. P. Robertson [ibid. (2) **49** (1936), 755–760] to have a formal connection with the transformation theory of a de Sitter universe.
C. Gilbert (Newcastle upon Tyne)

Nožička, František 3839
Elementareigenschaften der Weltlinien und gleichförmig beschleunigte Bewegung in der Minkowskischen Mechanik.
Comment. Math. Univ. Carolinae **3** (1962), *no.* 2, 3–31.
The author considers some kinematical properties of the motion of a particle in Minkowski space. In particular, he discusses in geometrical terms the following Lorentz-invariant characterization of "motion with constant acceleration": $\dot u_j \dot u^j = \text{const}$, $W_j W^j = 0$, $W_j \ddot u^j = 0$, with $W_j = e_{jklm} u^k \dot u^l \ddot u^m$, where e_{jklm} is the usual skew-symmetric numerical tensor, u^j is the four-velocity, and dots denote differentiation with respect to proper time.
H. A. Buchdahl (Canberra)

Som, M. M. 3840
Cylindrically symmetric radial electrostatic fields in general relativity.
Proc. Phys. Soc. **83** (1964), 328–330.
The author shows that a singularity-free cylindrically symmetric radial electrostatic field has a negative definite value for its total Møller energy [cf. a similar result of W. B. Bonnor, Proc. Phys. Soc. Sect. A **66** (1953), 145–152; MR **14**, 1133]. *A. Raychaudhuri* (Calcutta)

Collinson, C. D. 3841
Symmetry properties of Harrison space-times.
Proc. Cambridge Philos. Soc. **60** (1964), 259–263.
Solutions of the field equations of general relativity in a space-time admitting a coordinate system in which the metric tensor is of the form $g_{ij} = \delta_{ij} A_i^2(x^0, x^1) B_i^2(x^0, x^3)$ were investigated by B. K. Harrison [Phys. Rev. (2) **116** (1959), 1285–1296; MR **22** #2420]. The present paper is devoted to a study of the group structure of such metrics. A solution is called degenerate if, after suitable transformations, it depends only on two coordinates. All possible symmetries of non-degenerate Harrison space-times are found: such metrics admit only a one-parameter group of motions. Any Harrison metric admitting an r-parameter group of motions, with $r > 1$, must admit a 2-parameter Abelian subgroup. *H. Rund* (Pretoria)

Kaĭgorodov, V. R. 3842
The classification of gravitational fields of general type in terms of the motion groups. V. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1963**, *no.* 5 (36), 51–55.

Part IV (by Petrov, the author, and Abdullin) appeared in same Izv. **1962**, no. 1 (26), 130–142 [MR **25** #3784]. In this paper the author considers a Riemannian space V_4 admitting a transitive group of motions G_6. If LC_{AB} is the Lie derivative of the complex representation C_{AB} of Weyl's tensor with respect to the Killing's vector ξ^k and $\xi_{k,l}$, he shows that the condition for V_4 to admit G_6 is that $r \leqq 2$, where r is the rank of the equations $LC_{AB}=0$.

A. H. Klotz (Liverpool)

Capella, Alphonse 3843

Théorie minkowskienne de la gravitation.

C. R. Acad. Sci. Paris **258** (1964), 87–89.

A flat-space theory of gravitation is proposed in which a suitable Lagrangian results in the field equations $-\frac{1}{2}\partial^\rho\partial_\rho(h_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}h^{\sigma\tau}g_{\sigma\tau})=T_{\alpha\beta}$, where $h_{\alpha\beta}$ describes the field and $T_{\alpha\beta}$ the matter. The equations of motion are deduced from the conservation of the sum of an "energy-momentum tensor of the metric field" and the matter tensor. Formally, they agree with those postulated by Birkhoff in the static case, and so lead to the three "crucial" effects, without, however, the need for adjusting any free parameters. *W. Rindler* (Dallas, Tex.)

Kichenassamy, S. 3844

Sur une tentative d'interprétation physique de la Relativité générale. Application au décalage vers le rouge des raies spectrales.

C. R. Acad. Sci. Paris **258** (1964), 470–473.

The author discusses a form of weak equivalence principle, which involves approximating curved space-time locally not by its flat tangent space, but by a conformally flat tangent space. The formalism is used to examine the red-shift and to distinguish its gravitational and Doppler components. *W. Rindler* (Dallas, Tex.)

Stanjukovič, K. P. 3845

A generalized variational formalism in general relativity. (Russian)

Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1964**, *no.* 1, 62–70.

The author discusses the field equations deriving from the Lagrangian $L=-\frac{1}{2}R_{mn}B^{mn}$, where B^{mn} is an arbitrary tensor function of g_{ik} and its first and second derivatives, of R and of various "external" parameters λ^α.

A. Peres (Haifa)

Pachner, Jaroslav 3846

Cosmological considerations on the relativity of inertia.

Acta Phys. Polon. **23** (1963), 133–148.

The author extends his treatment of isotropic cosmological models [Ann. Physik (7) 8 (1961), 60–75; MR **24** #B1993], and compares the resulting models with observation.

{The treatment of mass and inertia is not clear; for example, no distinction is made between gravitational and inertial mass.}

G. F. R. Ellis (Cambridge, England)

Popovici, A.; Demayo, A. 3847

Les tenseurs de courbure de VI-e ordre dans V_n. I. Identités algébriques.

Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (*N.S.*) **4** (**52**) (1961), *no.* 1–2, 91–105 [1963].

On démontre plusieurs identités algébriques pour des tenseurs de courbure d'ordre $p>4$ dans un V_n riemannien. Ces identités vont servir à l'étude des équations covariantes de la fusion, pour la particule de spin maximum 2 dans V_6. *A. Haimovici* (Iași)

Dweck, E. E. 3848

A note on the principle of Doppler in a first order pool.

Matrix Tensor Quart. **11** (1960/61), 69–73.

This paper fulfills a long delayed duty. It states in the language of tensor systems that the frequency or phase changes related to the relative velocity of source and observer have to be accepted as a fundamental entity, which fact arises from the basic notions of a wave, a frequency, a wavelength and a phase. Nowhere in the general approach have any restrictions been imposed on the intrinsic mobility of the source. Thus the Doppler principle is strongly related to the fundamental postulation of the constancy of light velocity. The paper is hard to read for one not familiar with the geometry of non-holonomic spaces and transformations of diakoptical processes. The interested reader is therefore recommended to the cited literature, especially to a paper of K. Kondo [same Quart. **11** (1960/61), 1–10]. However, the paper deserves great acknowledgment for its basic considerations. *P. R. Arendt* (Eatontown, N.J.)

Raab, Werner 3849

Über Riemannsche Spinräume.

Acta Phys. Austriaca **16** (1963), 348–349.

The author investigates the relation between the group of motions of a certain space and that of the associated spin space. If the metrical ground forms G and H of the two spaces are connected by the relation $G=U'(H\otimes H)U$, and if $B'HB=H$, then it is shown that $A'GA=G$, where $A=U^{-1}(B\otimes B)U$. The forms of G, U, and H in the case of Minkowski space are given, and the possibility of extending the results to Riemannian spaces is discussed.

S. Datta Majumdar (Calcutta)

Nevzglyadov, V. G. [Nevzgljadov, V. G.] 3850

Relativistic theory of motion in the external field for a system with internal degrees of freedom. (Italian summary)

Nuovo Cimento (10) **29** (1963), 118–147.

"The Lorentz covariant theory of motion in an external field is developed for a mechanical system with 12 degrees of freedom: 3 translational, 3 rotational and 6 deformation degrees. The system is called by the author a relativistically homogeneous deformation body." It is a generalization of the author's previous theory [Dokl. Akad. Nauk SSSR **141** (1961), 1328–1331; MR **24** #B1793] of the 'homogeneous deformation body' and the 'absolute solid body'.

In the present paper the problem is set up in a more general way which includes the relativistic top as a special case. The equations are discussed at length and the first

integrals of motion are obtained. The system is quantized in the usual manner by treating the dynamical variables as linear operators and the generalized forms of Schrödinger's and Dirac's relativistic equations in the author's theory are obtained. As an illustration the magnetic moment of the proton is calculated on the basis of this theory and is found to be in tolerably good agreement with the experimental value.
S. Datta Majumdar (Calcutta)

Kirija, V. S. 3851
An approximate solution of the two-body problem in the general theory of relativity. (Russian. Georgian summary)
Soobšč. Akad. Nauk Gruzin. SSR **32** (1963), 307–310.
There have been some speculations [N. L. Balazs, Z. Physik **154** (1959), 264–266; L. I. Schiff, Amer. J. Phys. **28** (1960), 340–343; MR **22** #10729] according to which the Schwarzschild line element can be obtained directly from the equivalence principle, without the explicit use of the Einstein field equations. However, it was shown by Sexl [Z. Physik **167** (1962), 265–272; MR **25** #1926] that only the lowest-order term in g_{00} can be obtained in such a way. The author, apparently unaware of Sexl's argument, tries to derive higher-order terms in the laws of planetary motion, and obtains a perihelion precession smaller than the correct result. *A. Peres* (Haifa)

Deser, Stanley 3852
Méthode canonique en relativité générale et problèmes de quantification.
Cahiers de Phys. **17** (1962/63), 357–373.
An account in French of work by Arnowitt, Misner and the author on the canonical formalism in general relativity. {For an account in English see their chapter in *Gravitation: An introduction to current research*, pp. 227–265 [Wiley, New York, 1962; MR **26** #1182].}
P. W. Higgs (Edinburgh)

Deser, Stanley 3853
Propriétés asymptotiques du champ gravitationnel.
Cahiers de Phys. **17** (1962/63), 374–381.
An account in French of the application of the canonical formalism [#3852 above] to the study of asymptotic properties of the gravitational field. {For an account in English see Arnowitt, the author and Misner [Phys. Rev. (2) **121** (1961), 1556–1566; MR **22** #10728; ibid. (2) **122** (1961), 997–1006; MR **23** #B991].}
P. W. Higgs (Edinburgh)

Wellner, Marcel; Sandri, Guido 3854
Scalar gravitation.
Amer. J. Phys. **32** (1964), 36–39.
Authors' summary: "Nordström's scalar theory of gravitation is discussed from a modern point of view, and compared with some aspects of general relativity. We discuss the equations of motion for test masses, the field equations in the presence of matter, and the extent to which the principle of equivalence, Mach's principle, and the expansion of the universe are contained in this model. The theory implies what amounts to a Riemannian metric, and features a positive energy density for the gravitational field." *H. A. Buchdahl* (Canberra)

Melvin, M. A. 3855
Pure magnetic and electric geons.
Phys. Lett. **8** (1964), 65–68.
An exact static cylindrically-symmetric solution of the Einstein-Maxwell equations is derived. The solution may be described as representing "a parallel bundle of magnetic, or electric, flux held together by its own gravitational pull". *C. Gilbert* (Newcastle upon Tyne)

Scherrer, Willy 3856
Bericht über eine einheitliche Feldtheorie.
Z. Physik **174** (1963), 351–352.
A brief report on some consequences of the author's unified field theory [Z. Physik **152** (1958), 319–327; MR **20** #6297], and earlier references quoted there.
W. Israel (Edmonton, Alta.)

Nariai, Hidekazu; Kimura, Toshiei 3857
Electrodynamics in the expanding universe.
Progr. Theoret. Phys. **29** (1963), 915–932.
The authors obtain the Green's function of the electromagnetic field in the expanding universe subject to the generalized Lorentz condition. The authors find that in contrast to their result on gravitons [the authors, Progr. Theoret. Phys. **29** (1963), 296–320] there is no breakdown of macroscopic causality for photons and, further, the tail term in the ponderomotive equation [cf. DeWitt and Brehme, Ann. Physics **9** (1960), 220–259; MR **23** #B1528] does not appear. *A. Raychaudhuri* (Calcutta)

ASTRONOMY

See also 3556, 3562, 3838, 3846, 3851.

Aksenov, E. P.; Grebenikov, E. A.; Demin, V. G. 3858
The generalized problem of motion about two fixed centers and its application to the theory of artificial earth satellites.
Astronom. Ž. **40** (1963), 363–372 (*Russian*); *translated as Soviet Astronom. AJ* **7** (1963), 276–282.
It is well known that the potential of the oblate Earth can be approximated by the potential of two fixed centers of attraction placed along an axis of symmetry, and separated by an imaginary distance. Therefore an approximation to the theory of the motion of a satellite in the Earth's field can be obtained from the solution of the problem of "two fixed centers". It was established by Euler that this latter problem can be reduced to quadratures. The present authors solve the problem to the extent of obtaining explicit integrals (but not going so far as to express them in standard elliptic forms) and a set of six constants.

The authors claim that their solution is obtained more simply than that of Vinti in his well-known theory, which essentially makes use of the same potential. This is true

so far as it goes. However, the solution as presented here is of little more than academic interest; it is not in a suitable form for calculations, and the constants are not interpreted physically. By way of contrast, Vinti developed his solution for practical use, and he has, in addition, extended his solution to cover some deviations of the potential of the Earth from the potential of two fixed centers. *J. M. A. Danby* (New Haven, Conn.)

Kyner, W. T. 3859

Qualitative properties of orbits about an oblate planet.

Comm. Pure Appl. Math. **17** (1964), 227–236.

The author also considers [cf. #3860] a disk mapping for the satellite problem, but in different coordinates. He goes on to specify points in the disk using a system of polar coordinates with origin given by the fixed point corresponding to the nearly circular periodic solution given by Conley. If the oblateness of the planet is small (he considers $J_n = O(J_2^n)$, $n > 2$) the mapping is shown to be "close" to a rotation about the fixed point, where the angle of rotation varies with the radius. Then, from a theorem of Moser, it is shown that there exist invariant curves of this mapping. This family of curves has finite measure, but is not uniformly distributed over the disk. (Its existence is not sufficient to settle the question of whether the problem possesses a third uniform integral.)

Physically, a consequence of the existence of these curves is that oscillations of the inclination and eccentricity of any orbit are sharply restricted. The results are shown to be valid whether or not the planet has symmetry about its equator.

J. M. A. Danby (New Haven, Conn.)

Conley, C. C. 3860

A disk mapping associated with the satellite problem.

Comm. Pure Appl. Math. **17** (1964), 237–243.

The author considers properties of the motion of a satellite of a planet having symmetry about a plane and an axis normal to that plane. The existence of the integrals of energy and the component of angular momentum along the axis of symmetry permits a reduction of the system of differential equations of motion to one of the fourth order; the fact that the longitude is an ignorable coordinate permits a further reduction to the third order. Orbits having negative energy are considered in the corresponding three-dimensional phase space.

The set of points lying in this space and in the equatorial plane of the planet is a sphere. An orbit cutting the sphere at P will next cut it at its image point P' (orbits with zero inclination excepted), and this defines the mapping of the sphere onto itself. Further, by identifying points that differ only in the coordinate of velocity along the axis of symmetry, there are identified a disk, and a corresponding disk mapping that is area-preserving.

For potentials that are "close" to that of a point mass, and for definite energy and angular momentum, no boundary point of the mapping can be fixed; therefore the mapping has a fixed point interior to the disk, corresponding to an inclined orbit. This is periodic (although only if the longitude is ignored). In the case of the potential of a point mass, such an orbit is circular, having the appropriate energy and angular momentum.

J. M. A. Danby (New Haven, Conn.)

Chebotarev, G. A. [Čebotarev, G. A.] 3861

Motion of an artificial earth satellite in an orbit of small eccentricity.

AIAA J. **2** (1964), 203–208.

A solution is given for the motion of a satellite in the field of the Earth, subject to the following restrictions: The Earth has axial symmetry, and its oblateness is considered only through the second harmonic in its potential. The eccentricity of the orbit is small, such that its square is neglected. The solution is developed to the first order only in secular and short period terms. [The possibility of the appearance of long period terms in the solution is not considered.]

The solution is found from an integration of the planetary equations. The elements considered are (in the usual notation) a, i, Ω, $h = e \sin(\tilde{\omega} - \Omega)$, $l = e \cos(\tilde{\omega} - \Omega)$, $\lambda_0 = \varepsilon - \Omega$.

Practical problems of determining and improving the constants and calculating the coordinates of the satellite are considered in some detail.

J. M. A. Danby (New Haven, Conn.)

Weisfeld, Morris 3862

The convergence of an approximation used in the relativistic mass-center problem.

J. Math. Anal. Appl. **8** (1964), 282–286.

The equation

$$\frac{d^2U}{d\varphi^2} + U = 1 + \lambda U^2,$$

where λ is very small, arises in considering the motion of a test particle about a mass center in general relativity. Because of the smallness of λ, perturbation methods are commonly used for obtaining a solution in the form of a power series in λ. The convergence of this series is studied in this paper by direct estimation of its terms, and sufficient conditions for convergence are obtained. These estimates are uniform in φ and hence are valid for the entire motion. *D. Brouwer* (New Haven, Conn.)

Newton, Robert R. 3863

Damping of a gravitationally stabilized satellite.

AIAA J. **2** (1964), 20–25.

Author's summary: 'The librations of a prolate, axially symmetric satellite can be coupled to the longitudinal oscillations of a spring-mass system connected to the satellite. The oscillations of the spring can be heavily damped; thus, the librations can be damped. The coupling for librations in the plane of the orbit is linear in the libration amplitude, and hence is effective for all amplitudes. Coupling for librations normal to the orbital plane is quadratic in the amplitude, and has low effectiveness for small amplitudes. The theory of the damping is developed, and optimum values of the system parameters are found." *J. M. A. Danby* (New Haven, Conn.)

Yu, E. Y. 3864

Long-term coupling effects between librational and orbital motions of a satellite.

AIAA J. **2** (1964), 553–555.

If a non-spherical satellite moves in a non-uniform gravitational field, there is coupling between the motion

of the center of mass and the motion around the center of mass. The author considers the long-term effects of this coupling on the orbit traced out by the center of mass for a special model. This consists of two masses connected by a hinge joint that can damp the relative motion, in such a way that the centers of mass of the two bodies always coincide. The satellite is assumed to move in the field of a point mass in an orbit of small eccentricity (its square is neglected). One principal axis of inertia in each body is assumed to remain perpendicular to the plane of the orbit, and small departures of another are considered from the local vertical.

It is shown that the orbit suffers a decrease in the semimajor axis and an advance in perigee. However, in a numerical example, these effects are shown to be negligible.
J. M. A. Danby (New Haven, Conn.)

Aoki, Shinko 3865
Note on variability of the time standard due to the relativistic effect.
Astronom. J. **69** (1964), 221–223.
Author's summary: "By using the Schwarzschild fundamental line element, the difference between the coordinate time and the proper time is derived for the planetary motions. The amount is shown for the earth:

$$dt/ds = 1 + 1.48 \times 10^{-8} + 3.3 \times 10^{-10} \cos w,$$

where t is the coordinate time, s is the proper time, and w is the true anomaly. The identification of the coordinate time as Ephemeris Time and that of the proper time as Atomic Time are discussed, although these are open for experiments." *G. Clemence* (New Haven, Conn.)

Hadjidemetriou, John D. 3866
Two-body problem with variable mass: A new approach.
Icarus **2** (1963), 440–451.
Author's summary: "We have proved that the isotropic loss of mass from a binary system is equivalent to a force proportional to the rate of loss of mass and to the velocity vector, in the direction of motion. Using this force as a perturbation, we have derived the equations of the variation of the elements of the orbit; and from these equations approximate relations for the secular variation of the semi-major axis and the eccentricity have been found. The eccentricity is proved to remain secularly constant, while the semi-major axis increases secularly. On the other hand, the radiation pressure does not alter the results obtained for the isotropic loss of mass, if the total mass of the system is augmented by an additional term depending on the luminosities of the two components."

Obala, Jacques 3867
Sur des relations exprimant les moments des vitesses pour un amas globulaire.
C. R. Acad. Sci. Paris **258** (1964), 2021–2022.
Author's summary: "Nous établissons les formules exprimant les moments de la distribution des vitesses en fonction de la distribution des positions des étoiles et du potentiel gravitationnel pour un amas d'étoiles stationnaire possédant la symétrie sphérique."
J. M. A. Danby (New Haven, Conn.)

Lanzano, Paolo 3868
A class of periodic solutions for the restricted three-body problem.
Icarus **2** (1963), 364–375.
Author's summary: "Periodic orbits have been established in the planar case of the restricted three-body problem of Celestial Mechanics, which can be generated from circular Keplerian orbits and are valid for any mass ratio of the finite bodies. Extending a procedure by Siegel, already applied by this author to Hill's lunar theory, a periodic series solution has been obtained for the equation of motion, whose numerical coefficients can be calculated from a consistent set of recurrent relations. Fourier series representations have been given for the coordinates of the third body with respect to the synodic reference, where the coefficients of the trigonometric functions appear as power series of a parameter related to the period of the orbit. Approximate expansions for the radius vector, angular velocity, magnitude of velocity vector, and the Jacobi constant of motion have been furnished which exhibit how the motion occurring along a periodic orbit of this class deviates from a uniform, circular one. Using the majorant method of series and adapting a theorem on implicit functions, the series solution has been proved to be convergent for small values of the period but for an arbitrary mass ratio of the two primary bodies. The results include as a limiting case Hill's periodic solutions. The procedure is fit for extensive numerical work."

Moore, Derek W.; Spiegel, Edward A. 3869
The generation and propagation of waves in a compressible atmosphere.
Astrophys. J. **139** (1964), 48–71.
Authors' summary: "The equations governing the aerodynamic generation and the propagation of waves in a compressible atmosphere are exhibited. The fluctuating terms which are the turbulent sources for aerodynamic noise are approximated by an externally applied, time-harmonic, point force. Lighthill's results for the asymptotic radiation field in an anisotropic medium are then applied to an isothermal atmosphere. In this way, the surfaces of constant phase, group velocity, and intensity of the far field are computed. For finite frequencies above the critical frequency for vertical propagation, a monopole component is produced in the field by gravitational effects. The propagation problem is also studied for arbitrary temperature profiles, and it is found that in certain regions in the solar atmosphere there exist finite bands of non-propagating frequencies. It is suggested that the oscillations in the solar atmosphere result from forced excitations of these non-propagating frequencies and that these are excited by turbulence arising from shear instability and penetrative convection."

Pachner, Jaroslav 3870
Nonconservation of energy during cosmic evolution.
Phys. Rev. Lett. **12** (1964), 117–118.
The motion of a test body in an expanding universe, in the neighborhood of the local field of an attracting body, is considered in relation to previous work by the author [Phys. Rev. (2) **132** (1963), 1837–1842; MR **28** #1948]. The total energy of the body increases with time, and in

the case of a galaxy taking part in the general cosmic recession, which comes under the influence of a local field, there will be a relatively high increase of its total energy during the process of capture.

C. Gilbert (Newcastle upon Tyne)

Chambers, Ll. G. 3871

The Hund gravitational equations and the expanding universe.

Canad. J. Phys. **42** (1964), 84–89.

The modification of Maxwell's equations, devised by Hund [Z. Physik **124** (1948), 742–756; MR **11**, 410] to illustrate the non-linearity of Einstein's gravitational equations in the first approximation, is considered with a view to a cosmological solution.

A spherically symmetric solution is obtained in an approximate form valid for points of space not too distant from the origin. The apparent density of matter and its exhibited red shift are calculated as functions of position and epoch in the model and compared with the observational data. It is concluded that the solution agrees with the data within the observational error.

W. Davidson (London)

GEOPHYSICS

See also 3520, 3593.

Budden, K. G.; Jull, G. W. 3872

Reciprocity and nonreciprocity with magnetoionic rays.

Canad. J. Phys. **42** (1964), 113–130.

Reciprocity and antireciprocity are considered between the currents i_1 and i_2 induced in two short-circuited antennas A and B when identical voltages V are applied at the terminals of B and A, respectively. The medium between the antennas consists partly of free space and partly of a magneto-ionic medium such as the ionosphere; the object of the paper is to consider reciprocity only when ray theory is applicable, the results of the paper not being necessarily true for a full wave treatment. The ray theory approximation is discussed in terms of the local Appleton refractive index and in terms of the local ray refractive index. The medium is not necessarily horizontally stratified as in the more familiar case when the Booker q-function is used. Generalisations of the W.K.B. approximation are derived, based on the function $\exp(-ik\int \mathscr{M}\,ds)$, $\mathscr{M}$ being the local ray refractive index. To connect the two antennas, a certain solenoidal vector $\mathbf{W}$ is introduced, defined by

$$\mathbf{W} = \tfrac{1}{4}(\mathbf{E}\times\bar{\mathbf{H}}+\bar{\mathbf{E}}\times\mathbf{H})\exp\left(2ik\int \mathscr{M}\,ds\right),$$

where $\bar{\mathbf{E}}$ and $\bar{\mathbf{H}}$ denote a certain fictitious field system. $\mathbf{W}$ is parallel to the ray in the medium. The question of limiting polarisation is discussed, and finally the two currents i_1 and i_2 are derived. The equations $i_1 = \pm i_2$ are examined generally and in special cases, conditions for reciprocity and antireciprocity being derived. The paper closes with an examination of polarisation fading.

J. Heading (Southampton)

Golizdra, G. Ya. [Golizdra, G. Ja.] 3873

The distribution of the singular points of a gravitational field for one class of two-dimensional bodies.

Izv. Akad. Nauk SSSR Ser. Geofiz. **1963**, 1701–1708 (*Russian*); *translated as Bull.* (*Izv.*) *Acad. Sci. USSR Geophys. Ser.* **1963**, 1030–1034.

An analytic continuation downward of a mapped gravimetric anomaly in the direction of the disturbing masses, cause of the anomaly, may be useful in the solution of the inverse problem of gravimetric prospecting. In this paper the author studies the type of singularities of a gravitational potential due to horizontal cylindrical bodies (two-dimensional case) located in most cases at angular points of the normal section's boundary curve.

E. Kogbetliantz (New York)

Parkyn, D. G. 3874

Forced oscillations in a cylindrical atmosphere.

J. Atmospheric Sci. **21** (1964), 61–67.

Author's summary: "The vertical propagation of small oscillations in a viscous, heat-conducting cylindrical atmosphere is discussed as being analogous to the more complex problem of a spherical atmosphere. It is shown that in the case of either viscosity or heat conduction acting alone, the differential equation for the amplitude factor of forced temperature oscillations as a function of height may be solved explicitly. Contrary to classical discussions of tides in an undamped atmosphere it is shown that the effects of the damping terms predominate at even the lowest levels."

ECONOMICS, OPERATIONS RESEARCH, GAMES

See also 3554, 3555, 3887.

Cooper, Leon 3875

Heuristic methods for location-allocation problems.

SIAM Rev. **6** (1964), 37–53.

Author's summary: "A number of heuristic algorithms have been devised for solving large location-allocation problems. The algorithms have been extensively tested and several are in use in at least one industrial corporation. It has been found that one of the algorithms, the random destination algorithm, appears to provide satisfactorily close approximations to the optimal solution to location-allocation problems in reasonable amounts of computation time."

Arrow, Kenneth J. 3876

Optimal capital adjustment.

Studies in applied probability and management science, pp. 1–17. *Stanford Univ. Press, Stanford, Calif.*, 1962.

A firm has an initial stock of capital goods K_0 at time $t=0$, and wishes to choose a program of capital investment, that is, a function $K(t)$ for $t\geq 0$ such that $K(0)=K_0$. For such a program, the rate of expenditure on capital goods is defined by $I(t)=K'(t)+\delta K(t)$, where δ is the depreciation rate. The present value of $K(t)$ is $V=\int_0^\infty e^{-\alpha t}(\pi(K(t))-I(t))\,dt$, where α is the external interest rate and $\pi(K)$ is the rate of profit return from a stock K of capital goods. It is desired to find a program $K(t)$

that will maximize V, subject to the condition $I(t) \geqq 0$ for all t (the sale of capital goods is impossible). It is assumed that the function $P(K) = \pi(K) - (\alpha + \delta)K$ (the net profit rate) is decreasing for large K, has a finite number of local maxima, and takes distinct values for distinct local maxima. An optimal solution is found, which has the following property: There is a level $\bar{K}$ of capital stock that should be "striven for", in the sense that if $\bar{K} \geqq K_0$, then $\bar{K} - K_0$ should be acquired at once, and if $\bar{K} < K_0$, then the current stock should be allowed to depreciate until its value is $\bar{K}$; in either case, the level $\bar{K}$ should be maintained once achieved. A different application of this model, to advertising rather than capital adjustment, is also indicated.

In order to render the problem meaningful and justify the operations used in the proof (for example, integration by parts), it is presumably necessary to place some restrictions—in the nature of piecewise continuity or differentiability or the like—on the functions $K(t)$ and $\pi(K)$. These restrictions are not made explicit in the paper.

R. J. Aumann (Jerusalem)

Fortet, R. 3877

La circulation des voitures sur une route ou dans les artères d'une ville, à partir des lois de l'hydraulique. (English summary)

Houille Blanche 18 (1963), 909–916.

This is a brief review of the literature on fluid models of traffic flow prior to 1960. *G. Newell* (Providence, R.I.)

Abadie, Jean M.; Williams, A. C. 3878

Dual and parametric methods in decomposition.

Recent advances in mathematical programming, pp. 149–158. *McGraw-Hill, New York*, 1963.

The decomposition method of Dantzig and Wolfe [Econometrica **29** (1961), 767–778; MR **25** #1953] is a "primal" method for dealing with large special structured linear programs. The method may be described as follows. The large problem is reformulated through use of the observation that feasible solutions to the whole must be convex combinations of extreme feasible solutions to one or more independent subsets of constraints. The transformed problem is a linear program with relatively few constraints or rows and many variables or columns, each column representing an extreme solution to one of the independent subsets of the original constraints. A primal (revised) simplex method is used on the transformed problem. This means that a feasible solution is at hand and a step seeks a new column with negative "reduced cost" to enter the basis or on which to "pivot". This is done by solving a linear program with constraints corresponding to one of the subsets. The row of the pivot is determined by the usual ratio rule.

In this paper a "dual" method for solving the same problem is given. The same reformulation is made. However, a dual (revised) simplex method is used on the transformed problem. This means that a dual feasible solution is at hand. The row on which to pivot is chosen as usual. A step, then, seeks a new column on which to pivot which will preserve the dual feasibility. This entails solving a program with linear constraints and rational objective (i.e., a ratio of linear functions). The authors provide an algorithm for solving this problem through the solution of a sequence of linear programs. Finally, it is shown that this same step permits certain types of parametric solutions of the large problem which, as is pointed out, allows certain post-optimal solution studies.

M. L. Balinski (Princeton, N.J.)

Gilmore, P. C.; Gomory, R. E. 3879

A solvable case of the traveling salesman problem.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 178–181.

Let D be an $n \times n$ matrix of non-negative numbers d_{ij}, let τ be a cyclic permutation, and let (1) $C(\tau) = \sum_{i=1}^{n} d_{i\tau_i}$ and (2) $MC(\tau) = \operatorname{Max}_{1 \leq i \leq n} \{d_{i\tau_i}\}$. The traveling salesman and bottleneck traveling salesman problems are, respectively, the problems of finding a τ that minimizes (1) and (2). This paper outlines the solution to these problems for the case in which D is obtained from n pairs of non-negative numbers R_i and S_i by defining

$$d_{ij} = \max\{0, S_j - R_i\}.$$

J. J. Stone (White Plains, N.Y.)

Graves, Robert L. 3880

Parametric linear programming.

Recent advances in mathematical programming, pp. 201–210. *McGraw-Hill, New York*, 1963.

L'auteur applique la méthode du simplexe pour étudier les programmes linéaires dans lesquels les seconds membres et les coefficients de la fonction économique sont des polynômes dépendants d'un paramètre. Il montre ainsi que l'intervalle de variation du paramètre se subdivise en un nombre fini d'intervalles, et que, sur chacun de ces intervalles, ou bien la solution n'existe pas, ou bien les coordonnées de la solution sont des polynômes dépendants du paramètre. Sa méthode peut s'étendre à toute famille de fonction dépendant de paramètres comprenant les fonctions constantes, stable pour l'addition et la multiplication, et n'ayant qu'un nombre fini de zéros dans l'intervalle de variation du paramètre.

A. Ghouila-Houri (Paris)

Ventzel, E. S. [Ventcel', E. S.] 3881

★Lectures on game theory.

Authorized English edition translated from the Russian. International Monographs on Advanced Mathematics and Physics.

Hindustan Publishing Corp., Delhi; Gordon and Breach Publishers, Inc., New York, 1961. ii + 78 pp. $4.50.

This book is a popular treatment of the theory of games based on the elements of probability theory and a little analysis. The ground covered is: Fundamental concepts, lower and upper values, the minimax principle, pure and mixed strategies, examples of solutions of 2×2 and $2 \times n$ games, linear programming, approximate methods of solution, and solution of a few infinite games. The translation is adequate, the printing is clear and legible, and the price is high. *N. D. Kazarinoff* (Ann Arbor, Mich.)

BIOLOGY AND BEHAVIORAL SCIENCES

See 3556.

INFORMATION, COMMUNICATION, CONTROL

See also 2976, 3207, 3217, 3271, 3352, 3353, 3541, 3648.

Parzen, Emanuel 3882

A new approach to the synthesis of optimal smoothing and prediction systems.

Mathematical optimization techniques, pp. 75–108. *Univ. of California Press, Berkeley, Calif.*, 1963.

This paper describes a new approach in which a wide class of smoothing and predicting problems is incorporated. The approach is based on the author's idea that reproducing-kernel Hilbert spaces provide a unified framework for such problems, and the theoretical elaborations of this idea were first developed by the author in "Statistical inference of time series by Hilbert space methods, I" [Dept. of Statist., Stanford Univ. Tech. Rep. No. 23 (1959)]. The present paper is a further development of examples and applications, and problems of prediction, smoothing, smoothing and prediction, parameter estimation, and signal extraction and detection are treated. It is shown that each type of problem has a characteristic statistical structure, which calls for a coordinate system in which there is a natural way of expressing quantities such as inner products and data-handling procedures. One of the important innovations is the treatment of minimum-variance linear unbiased prediction.

E. A. Robinson (Uppsala)

Astapov, Yu. M. [Astapov, Ju. M.]; Medvedev, V. S. 3883

The statistical characteristics of noise for level quantization.

Avtomat. i Telemeh. **24** (1963), 164–171 (*Russian. English summary*); *translated as Automat. Remote Control* **24** (1963), 158–164.

Although directed toward the study of a nonlinear quantizing device, this study in essence finds the power spectrum of the zero-mean time series formed by subtracting unity from a sequence of positive Dirac-delta or impulse functions, where the content of each impulse equals the mean spacing between impulses. Assuming that the intervals between successive impulses are random, statistically independent, and identically distributed, a closed form is obtained for the power spectrum in terms of the characteristic function corresponding to the interval distribution. The derivation employs a theorem relating the power spectrum of a stationary time series to a limit taken on the mean of the squared magnitude of its Laplace-transformed truncation.

R. Price (Lexington, Mass.)

Stratonovič, R. L.; Sosulin, Ju. G. 3884

On the calculation of the detection characteristics of fluctuating signals. (Russian)

Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1964**, *no.* 1, 43–49.

Based on the likelihood ratio signal detection procedure, exact values for the miss probability (β) and the false alarm probability (α) are derived for an optimal receiver in the case of correlated non-degenerate Gaussian signals in the presence of correlated non-degenerate Gaussian noise. The method of derivation is based on the standard procedure of diagonalization of correlation matrices, and the error probabilities are expressed in terms of the appropriate eigenvalues. The corresponding values of β and α for a quadratic signal detection scheme (with the threshold function $\sum(x_i^2+y_i^2)$, where x_i and y_i are the components of the input signal and the noise, respectively), which is not an optimal procedure for the correlated case, are also presented. Some particular cases are briefly discussed.

The authors propose to use their formulae for tabulation of the detection characteristics (without indicating, however, a method of classification).

The results of this paper are linked to previous investigations by one of the authors [Stratonovič, Radiotehn. i Èlektron. **7** (1962), 187–194; Avtomat. i Telemeh. **22** (1961), 1163–1174].

S. Kotz (Toronto, Ont.)

Gross, M. 3885

Linguistique mathématique et langages de programmation.

Rev. Française Traitement Information [*Chiffres*] **6** (1963), 231–253.

C'est un excellent exposé de l'aspect syntaxique des langages de programmation, à la lumière des grammaires génératives introduites par Chomsky. Après certaines considérations préliminaires, on discute la notion de grammaire formelle et sa liaison avec la notion de système formel. On décrit ensuite la machine de Turing. On signale des analogies entre l'étude des langues naturelles et l'étude des langages de programmation. Les langages réguliers de Kleene sont présentés sous divers hypostases: grammaires, automates et structures algébriques. On montre que ALGOL n'est pas un langage régulier, mais COMIT est régulier. D'une façon plus détaillée sont présentés les langages "context-free" et le type correspondant d'automate: l'automate à mémoire PUSHDOWN. On constate que la plupart des langages de programmation (ALGOL, FORTRAN, LISP, etc.) sont tels qu'on peut en donner une description syntaxique "context-free". On signale des langages et structures non "context-free" et on donne une présentation rapide des grammaires séquentielles.

S. Marcus (Bucharest)

Greibach, Sheila A. 3886

The undecidability of the ambiguity problem for minimal linear grammars.

Information and Control **6** (1963), 119–125.

Author's summary: "The ambiguity problem for general context-free phrase structure grammars has been shown undecidable. This paper proves the undecidability of a new form of Post's correspondence problem. Using this result, the undecidability of the ambiguity problem for minimal linear grammars is obtained."

E. Shamir (Berkeley, Calif.)

Gercbah, I. 3887

Formulation of certain optimal problems in the theory of reliability. (Russian)

Latvijas PSR Zinātņu Akad. Vēstis **1963**, *no.* 8 (193), 25–31.

The author considers several problems of the following sort. One has a system, with several components perhaps, which can be in any one of several states: e.g., full

operating order, replacement or regeneration of some or all of the components, partial or total failure. He denotes these states by E_i. (In fact, $i=1, 2$.) The system starts out, say, in E_1; after the nth transition it will be in some state I_n, the time elapsed between the $(n-1)$st and nth transition being called X_n. One then assumes appropriate distribution functions relating these variables, and "loss" variables Y_i connected with the systems going into states of less than full operating order. The task is to calculate a limiting loss per unit time, $\lim_{t\to\infty} U_{(p)}(t)/t$, which will be a functional of the assumed distributions, and then to choose these distributions so as to minimize this loss.

Suppose

$$Q_k(t) = \Pr(I_{n+1} = E_k, X_{n+1} \le t | I_0, I_1, X_1, \cdots, I_n, X_n)$$
$$= \Pr(I_{n+1} = E_k, X_{n+1} \le t | I_n = E_r),$$

and let $p_{rk} = Q_{rk}(\infty)$, $\mu_{rk} = \int_0^\infty t\, dQ_{rk}(t)$. Then, using the usual kind of recurrence equations, the author finds that the limiting loss per unit time is

$$\frac{p_{21}E(Y_1)+p_{12}E(Y_2)}{p_{21}(\mu_{11}+\mu_{12})+p_{12}(\mu_{21}+\mu_{22})}.$$

He uses this relation in the several applications.

J. Chover (Stanford, Calif.)

Krasovskii, N. N. [**Krasovskiĭ, N. N.**] 3888

On the stabilization of unstable motions by additional forces when the feedback loop is incomplete.

Prikl. Mat. Meh. **27** (1963), 641–663 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 971–1004.

The problem being considered is the following: the controlled system is described by the vector differential equation $\dot{x}=p(t, x, u)$, where x is an n-vector and u is an r-vector. The state $(x, u)=(0, 0)$ is an unstable equilibrium state $(p(t, 0, 0)=0)$. It is not possible to measure x but only $y=q(t, x)$. One wishes to derive a control law which stabilizes the system and which also may be required in some sense to be optimal. In particular, he considers a control law of the form $\dot{u}(t)=U(t, y_t, u_t)$, where U is a functional and y_t and u_t are functions defined by $y_t(\theta)=y(t+\theta)$, $u_t(\theta)=u(t+\theta)$, $-\tau\le\theta\le 0$. The control law depends on a portion of the past history of the system. An example is given where stabilization is not possible if the control law is generated by an ordinary differential equation $\dot{u}(t)=S(t, y(t), u(t))$.

The problem is shown to have a solution in the linear case ($\dot{x}=P(t)x+B(t)u$, $y=Q(t)x$ with a linear functional differential equation generating the control law) when certain conditions of "controllability" and "observability" are satisfied. The existence of an optimal control law is also established when the criterion of optimality is the minimization of a quadratic functional. Determining the control law requires the calculation of the fundamental matrix solution of $\dot{x}=P(t)x$, the solution of a system of linear algebraic equations, and the solution of a Cauchy problem for a system of ordinary differential equations. An example is presented with computations carried out on a "URAL 1". *J. P. LaSalle* (Baltimore, Md.)

Blaquière, A. 3889

La méthode du diagramme de Nyquist généralisé dans le domaine non-linéaire. (**Russian summary**)

Qualitative methods in the theory of non-linear vibrations (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. II*, 1961), *pp.* 55–83. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

From the author's summary: "La méthode étudiée dans ce mémoire a été exposée antérieurement dans différentes publications. Nous essaierons de la présenter ici de façon plus systématique, et nous étendrons son champ d'applications en montrant comment on peut l'adapter à de nouveaux problèmes." For references to the author's earlier work see the review of his earlier paper [Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics, pp. 413–420, Academic Press, New York, 1963; MR **26** #6522]. *W. M. Wonham* (Baltimore, Md.)

Kirszenblat, Albert 3890

Sur une classe de systèmes linéaires paramétriques.

C. R. Acad. Sci. Paris **258** (1964), 444–447.

The author considers, as a generalization of a final value control system where the error is made to vanish at a final time, a class of systems described by nth order ordinary differential equations such that system errors vanish periodically, and he obtains conditions on the coefficients of the differential equations.

M. Aoki (Los Angeles, Calif.)

Lewis, D. C.; Mendelson, P. 3891

Contributions to the theory of optimal control. A general procedure for the computation of switching manifolds.

Trans. Amer. Math. Soc. **110** (1964), 232–244.

The system of differential equations representing the control system is $\dot{x}=h(x)+a\varepsilon$, where $\dot{x}=dx/dt$, x is an n-vector, h is an n-vector function, a is a constant nonzero n-vector. The scalar ε (the control variable) is bounded $|\varepsilon|\le 1$, and the object is to select ε to bring the system to the origin in minimum time. In many cases time optimality is achieved by taking $|\varepsilon|=1$. The problem is to determine where in the state space ε should switch from one extreme value to the other. The method of construction of the switching manifolds is, in general, local and may not in the large even for controllable linear systems yield time optimal control. The case $d^4x/dt^4=\varepsilon$ is analyzed in detail. *J. P. LaSalle* (Baltimore, Md.)

Moseenkov, B. I. 3892

An investigation of non-stationary single-frequency regimes of oscillations in systems with distributed parameters. (**Russian. English summary**)

Applications of the methods of non-linear vibrations to the problems of physics and technology (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. III*, 1961), *pp.* 286–304. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

Author's summary: "The paper deals with the method of asymptotic integration and with the energy interpretation of the first approximation equations which have been elaborated by Yu. A. Mitropolsky to investigate non-stationary one-frequency regimes of oscillations in systems with N degrees of freedom. This method is formally extended to oscillatory systems with an infinite number of degrees of freedom and thus to systems with distributed

parameters on investigating their nonstationary one-frequency regimes of oscillations. A number of practically important problems investigated by Yu. A. Mitropolsky and by the author following the standard scheme based on the proposed method are presented in the paper. Some of them are described in detail and illustrate the efficiency of this method."

Aronovič, G. V.; Beljustina, L. N.; Kartvelišvili, N. A.; Ljubimcev, Ja. K. 3893

Application of the methods of the theory of oscillations to stability problems of steady-state operation of hydroelectric stations and power systems. (Russian. English summary)

Applications of the methods of non-linear vibrations to the problems of physics and technology (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. III*, 1961), *pp.* 9–33. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

From the authors' summary: "The regimes of synchronous machines in a system of infinite power are found by the methods of the qualitative theory of differential equations. The conditions for synchronous and asynchronous rotation of the rotor are established by analyzing the possible structurally stable topological structures and bifurcations on the phase cylinder."

Popov, E. P. 3894

Approximate methods of investigating non-linear oscillations in automatic systems. (Russian. English summary)

Applications of the methods of non-linear vibrations to the problems of physics and technology (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. III*, 1961), *pp.* 376–397. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

Author's summary: "Closed-loop automatic systems are considered, whose dynamics are described by means of an ordinary differential equation of arbitrary order containing a nonlinearity of the type $y=F(x, px)$, where $p=d/dt$. Unlike the usual substitution (for a small-parameter method)

$$y = F(x, px) = k_1x+k_2px+\mu f(x, px),$$

where μ is a small parameter, we use the substitution

$$y = F(x, px) = F(x_1, px_1)+\varepsilon\Phi(t),$$

where $x_1=A_1 \sin \Omega_1 t$ is the first harmonic of the unknown periodic solution $x=x_1+\varepsilon z(t)$; ε is a small parameter. A complete solution comprising all the harmonics is considered. This leads to approximate equations to find separately the first harmonic (not a small one) and each of the higher harmonics which are small in respect to the variable x and not small in respect to the variable y. This is a generalization of the harmonic linearization method or the harmonic balance method. The solution turns out to be true provided a certain mathematical condition is satisfied, coinciding with the known physical postulate about the filter property of the linear part of the system. Apart from such approximate forms of the first and the higher harmonics of the periodic solution, the method extends to the investigation of slow processes in oscillatory systems, which is of particular importance for practice. A special method of splitting the system equation into two equations is proposed, the nonlinear interrelation between their solutions being retained. System oscillations superimposed on a slowly-changing component may be either self-oscillating or forced ones. A similar approach may also be used when analyzing random processes in automatic systems."

Bridgland, T. F., Jr. 3895

Stability of linear signal transmission systems.

SIAM Rev. **5** (1963), 7–32.

A survey of various known results concerning mainly the exponential asymptotic stability of linear systems of differential equations and difference equations.

H. A. Antosiewicz (Los Angeles, Calif.)

Babunašvili, T. G. 3896

Synthesis of linear optimal systems. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 295–298.

Consider a control system described by the equation (1) $\dot{x}=A(t)x+B(t)u+f(t)$, where x is an n-dimensional phase vector, u is an r-dimensional control vector, and A, B, and f are integrable matrix functions. The author considers the following problem (the time-optimal regulator problem): Given a point x_0 in n-space, find a control function $u(t)$, whose range belongs to a given convex, compact polyhedron U in r-space containing the origin, such that the solution $x(t)$ of (1) with $x(0)=x_0$ attains the origin in minimum time. According to the Pontrjagin maximum principle [L. S. Pontrjagin et al., *The mathematical theory of optimal processes* (Russian), Fizmatgiz, Moscow, 1961; English transl., Interscience, New York, 1962], if $u(t)$ is a time-optimal control, then for almost all t, (2) $\psi(t)B(t)u(t)=\|\psi(t)B(t)\|$, where $\|\psi B\|=\max_{u\in U}\psi Bu$, and the row vector $\psi(t)$ is a nontrivial solution of the equation (3) $\dot{\psi}=-\psi A$. Assuming that the system is nondegenerate (i.e., that $u(t)$ is uniquely determined by (2) for any nontrivial solution of (3) and almost all t), the time-optimal control is determined by (2) once the initial value $\psi(0)$ of (3) is known. The author shows that if ψ_0 is a suitable initial value, and T is the minimum transfer time, then

$$(4)\quad \psi_0 z(T) = \int_0^T \|\psi_0 K(t)\|\,dt = \min_{\chi\in\Lambda}\int_0^T \|\chi K(t)\|\,dt,$$

where $z(t)$ is a function that depends only on A, f and x_0, K depends only on A and B, and Λ is the plane defined by $\chi z(T)=\psi_0 z(T)$. Conversely, if T is the smallest positive number for which (4) has a solution ψ_0, then T is the minimum transfer time, and any solution ψ_0 of (4) is a suitable initial value for (3). Thus it is only necessary to find a solution of (4) with smallest value $T>0$. An iterative procedure is suggested for solving (4). In this procedure, estimates for T and ψ_0 are alternatively made. The $(i+1)$st estimate for ψ_0 is the minimum of the function $g_i(\chi)=\int_0^{t_i}\|\chi K(t)\|\,dt$ subject to the constraint $\chi z(t_i)=\chi_i z(t_i)$, where χ_i and t_i are the ith estimates for ψ_0 and T, respectively. This minimum is to be found by the method of steepest descent, the gradient of $g_i(\chi)$ being readily obtainable. The times t_i form an increasing sequence which converges to T, and the vectors $\chi_i/\|\chi_i\|$ converge to a compact set any member of which is a candidate for ψ_0.

No results of any computational experience are presented. Related computational methods have been derived by the reviewer [J. Math. Anal. Appl. **1** (1960), 484–493; MR **23** #B1612; J. Soc. Indust. Appl. Math. Ser. A Control **1** (1962), 16–31; MR **26** #2707] and by Eaton [J. Math. Anal. Appl. **5** (1962), 329–344; MR **25** #4214].
L. W. Neustadt (Ann Arbor, Mich.)

Dubovickiĭ, A. Ja.; Miljutin, A. A. 3897
Some optimal problems for linear systems. (Russian. English summary)
Avtomat. i Telemeh. **24** (1963), 1616–1625.

Consider the vector differential equation (1) $\dot{x}=Ax+bu$, where A and b are constant matrices, and the forcing (control) function u is bounded and measurable. Let $g(x)$ and $\phi(u)$ be convex functions where g is continuously differentiable, ϕ has a unique minimum at $u=0$ and the levels of ϕ are compact. Given a point P, let K denote the class of controls $u(t)$ for which the corresponding trajectory $x(t;u)$ of (1), with $x(0;u)=0$, satisfies the condition $x(T_0;u)=P$ for some $T_0>0$. The authors consider the following optimization problems: (A) Given real numbers g_0 and ϕ_0, where $g(P)<g_0$ and $g(0)<g_0$, find a $u\in K$ subject to the constraints (2) $\phi(u(t))\leqq\phi_0$ and (3) $g(x(t;u))\leqq g_0$ for every $t\in[0,T_0]$ that minimizes T_0. (B) Given T_0 and g_0, find a $u\in K$ subject to the constraint (3) that minimizes vrai max $\phi(u(t))$. (C) Same as (B) but T_0 is free. (D) Given T_0 and ϕ_0, find a $u\in K$ subject to the constraint (2) that minimizes max $g(x(t;u))$. (E) Same as (D), but T_0 is free. The authors sketch a derivation of an equation (referred to as the Euler equation) that must be satisfied by every variation of an optimal trajectory and control. On the basis of this equation, the authors derive six theorems. Some of these are similar to the Pontrjagin maximum principle [L. S. Pontrjagin et al., *The mathematical theory of optimal processes* (Russian), Chapter 6, Fizmatgiz, Moscow, 1961; English transl., Interscience, New York, 1962]. Others provide sufficiency conditions. The most significant of these is the fact that every $u\in K$ that satisfies (2), (3) and the Euler equation is a solution of problem (A). The results are applied to three second-order examples.

{The material is unfortunately presented in an extremely obscure fashion. The difficulty lies partly in the notation, which is sometimes undefined, often sloppy, and occasionally ambiguous and/or inconsistent. In addition, there is an inexcusably large number of typographical errors.} *L. W. Neustadt* (Ann Arbor, Mich.)

Zadeh, L. A.; Eaton, J. H. 3898
An alternation principle for optimal control.
Avtomat. i Telemeh. **24** (1963), 328–330 (*Russian. English summary*); *translated as Automat. Remote Control* **24** (1963), 305–306.

Let a system be described by the relationship $x(t)=A[x(t_0),u(t_0,t)]$, where $x(t)$ is the state at time t, $x(t_0)$ is the initial state and $u(t_0,t)$ is the input function over the interval $[t_0,t]$. A policy is defined to be a function $\pi(t,x)$ that determines $u(t_0,t)$ at time t by means of the relationship $u(t_0,t)=\pi(t,x(t))$. The authors formulate a principle for obtaining a policy that is better than any one of N given policies by using portions of the given policies. The presentation is heuristic and is intended to stimulate discussion. *L. D. Berkovitz* (Lafayette, Ind.)

Egorov, A. I. 3899
On optimal control of processes in distributed objects.
Prikl. Mat. Meh. **27** (1963), 688–696 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1045–1058.

The process is described by a system of quasi-linear partial differential equations

$$\frac{\partial^2 u_i}{\partial x\partial t}+b_i(x,t)\frac{\partial u_i}{\partial x}+c_i(x,t)\frac{\partial u_i}{\partial t}=f_i(x,t,u,v),\quad i=1,\cdots,n,$$

where v is a control function assuming values in a convex (open or closed) set in r-dimensional Euclidean space. The criterion of optimality is the minimization of a function $J=\int_0^l\int_0^\pi f_0(x,t,u,v)\,dt$. A necessary condition for optimality is given (a maximum principle) and it is shown to be for linear equations sufficient for local optimality.
J. P. LaSalle (Baltimore, Md.)

Lehman, Alfred 3900
Wye-delta transformation in probabilistic networks.
J. Soc. Indust. Appl. Math. **11** (1963), 773–805.

This paper is concerned with the problem of reducing a probabilistic network to a single branch, with respect to a given pair of terminals. Each branch of the network has a certain connection probability which may be a function of time. The terminal connection probability is represented as a Boolean function of the branch variables. It is desired that the connection probability in the reduced graph be the same as in the original graph, i.e., that the reduction procedure is exact. Exact procedures are known, such as the enumeration of minimal paths. However, even moderate complexity of the network requires a large amount of labor in the execution of these procedures.

By assigning values to the delta and wye branch variables in the network Boolean functions, the author derives relationships which lead to delta-to-wye, wye-to-delta, and combined wye-delta transformations. In each case a maximum error bound is derived, all less than .01, but for certain relationships between the branch variables the transformation is exact. An example is presented where the derived transformation is used in conjunction with well-known series-parallel techniques to reduce a network. It is shown how a table-hook-up procedure facilitates the reduction process. The author is careful to limit the applicability of his procedure. It is not convenient to use it for functions which have no associated graph, and then not all graphs are wye-delta reducible. Also, since in most cases an error is introduced, the size of the graphs which can effectively be reduced is limited.
W. H. Kim (New York)

Muroga, S. 3901
The principle of majority decision logical elements and the complexity of their circuits. (French, German, Russian and Spanish summaries)
Information processing, pp. 400–407. *UNESCO, Paris; R. Oldenbourg, Munich; Butterworths, London*, 1960.

The author considers the synthesis of combinatorial switching circuits using majority decision (MD) logical

elements. He defines an MD element as a device which receives inputs $x_1, \cdots, x_p$ which are either 0 or 1 and produces an output $f(x_1, \cdots, x_p)$ which is 1 if more than half of the inputs are 1, and is 0 otherwise. He then introduces unequal relative amplitudes of the inputs which he calls "coupling numbers". Thus, to each x_i he assigns an integral coupling number w_i. There may also be a constant input which is either 0 or 1 with coupling number w_c. The "threshold" of the MD element is $\frac{1}{2}(w_1 + \cdots + w_p + 1 + w_c)$ if the constant input is 1 and $\frac{1}{2}(w_1 + \cdots + w_p + 1 - w_c)$ if the constant input is 0. The output is then 1 if the sum of the coupling numbers of the variables which have input value 1 is greater than or equal to the threshold. Otherwise, the output is 0. For example, any symmetric polynomial in p variables can be realized by an MD element in which $w_i = 1$, $1 \leq i \leq p$, and w_c is chosen suitably.

The paper goes on to analyze the number of MD elements required to synthesize an arbitrary Boolean function of n variables. It is shown that if there is no bound on the threshold, then a circuit can be synthesized with at most $2^{n-1}+1$ MD elements in two stages. If three stages are used, an upper bound is $2^{(n-1)/2} + 2^{(n+1)/2} + \frac{1}{2}(n+1)$ for odd n and $2^{n/2+1} + \frac{1}{2}n + 1$ for n even. For MD elements with a maximum allowable sum of coupling numbers equal to 5, a Boolean function of n variables can be synthesized by a circuit with less than $(2^n/n)(1+\varepsilon)$ MD elements, where $\varepsilon \to 0$ as $n \to \infty$.

E. K. Blum (Middletown, Conn.)

Flegg, H. Graham 3902

★Boolean algebra and its application, including Boolean matrix algebra.

John Wiley & Sons, Inc., New York, 1964. xv + 261 *pp.* \$9.00.

Notwithstanding the preface, this book is written for engineers. Without giving a formal definition of a Boolean algebra, the author discusses various applications of it (not the mathematics of it). He considers ever so briefly the algebra of classes and the propositional calculus. The major part of the book (80%) is devoted to combinational switching theory. [There is practically no sequential switching.] This includes several standard simplification methods, representation on the n-cube, symmetric functions, and Boolean matrices. This is the first book the reviewer has seen which covers the topic of applying Boolean matrices to switching circuits reasonably well. The value of the book is enhanced with an extensive bibliography.

In summary, the book is suitable as a text for combinational switching. *S. Ginsburg* (Van Nuys, Calif.)

Talancev, A. D. 3903

The logic of filters. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 227–238.

Author's summary: "The theory of a certain class of automata transforming symbolic information is considered. Particular logical-mathematical objects called nature objects are axiomatically determined, as well as so-called stationary functions of such objects. Operators realizing the functions are called filters. The multitude of nature objects and a set of selected filters are used to form an algebra. The algebra laws are ascertained. A problem of synthesizing arbitrary filters with n inputs is solved."

Gluškov, V. M. [Глушков, В. М.] 3904

★The design of numerical automata [Синтез цифровых автоматов].

Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1962. 476 *pp.* 1.41 r.

The best source of information on automata theory the reviewer knows of is R. McNaughton's paper [Advances in Computers, Vol. 2, pp. 379–421, Academic Press, New York, 1961; MR **24** #B2520], which was published about a year before the book under review, but still it seems to be years ahead of it. The various types of automata (deterministic and probabilistic, finite and growing, discrete and continuous, etc.) are not even mentioned in the book (after an introductory chapter (Chapter 1) on such elementary and intuitive notions as information, algorithm, transformation of information, automata). The author chooses the set-theoretical point of view and defines an automaton (Chapter 2) as a 6-tuple $\langle X, B, A, \delta, \lambda, a_0 \rangle$, where X, B are the sets of input and output signals, respectively, A is the set of states, δ is the transition function, λ is the output function and a_0 $(\in A)$ is the initial state.

Following Kleene, the notion of event is introduced, then Kleene's fundamental result is proved (i.e., that an event is representable in a finite automaton if and only if it is regular) in a slightly more general setting.

The "only if" part is so simple in Kleene's original paper that it is hard to understand why the author found it necessary to introduce more than a dozen new notions and notations. The idea of the "if" part comes from a paper of McNaughton and Yamada [IRE Trans. Electronic Computers **EC-9** (1960), 39–47] (to which, at this place, no reference is given). There is only one mistake in the adaptation, namely, that the concept "the place α is connected with the place β" is only intuitively defined. The rules on pp. 97–98 should be considered as definitions, rather than obvious statements. The reader who cannot comprehend too many definitions should consult the original papers.

The minimization problem of automata is discussed to some extent, giving the (now) classical result of Auenkampt and Hopf. Unfortunately, the more delicate problem of minimizing a partial automaton is not discussed (the interested readers should consult Ginsburg's book on this subject).

Chapters 3–5 take their topics from switching theory. In Chapters 3 and 4 Boolean functions are discussed; Post's result on functional completeness and some minimization processes (e.g., that of Quine) are the main features. In Chapter 5 the universality of a special switching circuit is proved and then the connection of switching circuits and Boolean matrices is explored. The reliability of an automaton is the topic of Chapter 6, where a few examples are used to illustrate some ideas of Moore, Shannon, and von Neumann. An example of an error detecting and correcting code (due to Hamming) is also given.

Chapter 7 contains some general ideas on flow diagrams of programs for, and the general structure of, digital computers.

The best feature of the book is that it contains several

illustrative examples which are worked out in every detail. The most serious disadvantage is a lack of references in the text; as a matter of fact, the long list of references (which is far less extensive than that of McNaughton [loc. cit.]) at the end of the book is rather deceptive, since these papers are very seldom referred to. Since in the Introduction it is claimed that "This (second) chapter is based almost entirely on the author's own results, . . ., and also on certain new results, presented for the first time", the reader might get the false impression that the author invented state graphs, transition matrices, and so on. *G. Grätzer* (University Park, Pa.)

Stearns, R. E.; Hartmanis, J. 3905
Regularity preserving modifications of regular expressions.
Information and Control **6** (1963), 55–69.

A set of elements of the free monoid over a finite alphabet A is regular if it is defined by a (possibly non-deterministic) finite automaton. The authors develop a technique for constructing finite automata defining various modifications of the original regular set. The modifications considered are "derivations", some proportional deletions and certain errors or noiselike changes.

E. Shamir (Berkeley, Calif.)

AUTHOR INDEX

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★Proceedings of Symposia in Applied Mathematics. Vol. XVI: Stochastic processes in mathematical physics and engineering. 3906
American Mathematical Society, Providence, R.I., 1964. viii + 318 *pp.* $7.60.

This volume contains the proceedings of a symposium held in New York City from 30 April to 2 May 1963. Most of the papers will be reviewed individually.

★American Mathematical Society Translations. Series 2, Vol. 37: 22 papers on algebra, number theory and differential geometry. 3907
American Mathematical Society, Providence, R.I., 1964. v + 429 *pp.* $5.50.

This volume contains translations of three papers of V. A. Kurbatov, two each by Ju. V. Linnik, Ku Chao-Hao, three by V. A. Toponogov, and one each by I. M. Gel'fand, M. S. Gel'fand, L. A. Ljusternik, Ju. I. Manin, G. V. Dorofeev, I. R. Šafarevič, V. I. Šneĭdmjuller, M. S. Calenko, Wang Yuan, L. D. Kudrjavcev, G. F. Laptev, and I. M. Gel'fand and M. I. Graev.

★American Mathematical Society Translations. Series 2, Vol. 38: 16 papers on topology and 1 on game theory. 3908
American Mathematical Society, Providence, R.I., 1964. iv + 340 *pp.* $4.70.

This volume of translations contains fifteen papers on topology, one on game theory, and one on the theory of functions of a complex variable. There are four papers by Ju. M. Smirnov, three by V. I. Ponomarev, four by Chang Su-cheng, three by Wu Wen-tsun, and one each by Wu Che-jen, Chow Hsu-kwan and N. N. Vorob'ev.

★American Mathematical Society Translations. Series 2, Vol. 39: 15 papers on topology and logic. 3909
American Mathematical Society, Providence, R.I., 1964. iv + 298 *pp.* $4.40.

This volume contains translations of three papers by I. M. Mel'nik, two by V. A. Efremovič, and one each by V. A. Rohlin, L. M. Abramov, E. G. Skljarenko, Ja. G. Sinaĭ, A. L. Garkavi, V. I. Ponomarev, E. S. Tihomirova, B. A. Trahtenbrot, A. I. Mal'cev, and V. S. Černjavskiĭ.

★Transactions of the Third Prague Conference on Information Theory, Statistical Decision Functions, Random Processes. 3910
Held at Liblice near Prague, from June 5 to 13, 1962.
Publishing House of the Czechoslovak Academy of Sciences, Prague, 1964. 846 *pp.* *Kčs* 66.00.

The papers resulting from the conference described in the heading will be reviewed individually. The conference was dedicated to the memory of the late Professor Antonin Špaček.

★Séminaire P. Dubreil, M.-L. Dubreil-Jacotin et C. Pisot, 14ième année: 1960/61. Algèbre et théorie des nombres. Fasc. 1, 2. 3911
Faculté des Sciences de Paris.
Secrétariat mathématique, Paris, 1963. *Fasc.* 1 (*Exp.* 1–15), iii + 184 *pp.*; *Fasc.* 2 (*Exp.* 16–26), iii + 187 *pp.*

Fascicule 1: (1) L. Lesieur, Cœur d'un module, I; (2) J.-P. Serre, Sur les modules projectifs; (3) P. Eymard, Suites équiréparties dans un groupe compact; (4) J. Lévy-Bruhl, Produit symétrisé; (5) J. Riguet, Caractérisation de la notion de structures et de structures locales chez N. Bourbaki et C. Ehresmann, I; (6) K. E. Aubert, Sur la théorie des x-idéaux; (7) E. R. Kolchin, Le théorème de la base finie pour les polynômes différentiels; (8) M. Coz, Immersion d'un demi-groupe, vérifiant la règle de simplification, dans un groupe, d'après Ptak; (9) M. Curzio, Treillis des sous-groupes de composition; (10) P. Lefebvre, Sur certaines équivalences simplifiables d'un demi-groupe; (11) F. Châtelet, L'arithmétique des corps quadratiques; (12) J. Guérindon, Sur une classe de modules gradués; (13) J. Petresco, Problème des mots appartenant au sous-groupe d'un groupe libre; (14) M.-P. Brameret, Sur les Baer*-demi-groupes; (15) M. Grandet, Dérivés d'un ensemble d'entiers algébriques.

Fascicule 2: (16) P. Dubreil, Sous-groupes d'un demi-groupe. Demi-groupe des endomorphismes d'un groupe; (17) R. Croisot, Cœur d'un module, II; (18) J. G. van der Corput, Calcul des neutrices; (19) G. Birkhoff, Lattice-ordered demigroups; (20) G. Birkhoff, Positivité et criticalité; (21) G. Poitou, Points rationnels sur les courbes; (22) O. Borůvka, Transformations des équations différentielles linéaires du deuxième ordre; (22) bis. O. Borůvka, Décompositions dans les ensembles et théories des groupoïdes; (23) Š. Schwarz, Sur les caractères des demi-groupes compacts; (23) bis. Š. Schwarz, Les mesures dans les demi-groupes; (24) A. Almeida Costa, Sur la théorie générale des demi-anneaux, I; (25) A. Almeida Costa, Sur la théorie générale des demi-anneaux, II; (26) R. Deheuvels, Théorie de l'homologie.

Les exposés (8), (18) et (20) n'ont pas été rédigés, et ne seront pas multigraphiés.

★Séminaire P. Dubreil, M.-L. Dubreil-Jacotin et C. Pisot, 15ième année: 1961/62. Algèbre et théorie des nombres. Fasc. 1, 2. 3912
Faculté des Sciences de Paris.
Secrétariat mathématique, Paris, 1963. *Fasc.* 1 (*Exp.* 1–11), iii + 149 *pp.*; *Fasc.* 2 (*Exp.* 12–23), iii + 142 *pp.*

Fascicule 1 : (1) M. Curzio, Les transporteurs d'un groupe fini dans les sous-groupes de Pitting et Frattini; (2) P. Grillet, Equivalences compatibles. Equivalences prépermises; (3) J. Querré, Equivalence de fermeture dans un demi-groupe résidutif; (4) J. Fort, Radical tertiaire d'un sous-module, sous-modules tertiaires, dans un module sur un anneau non nécessairement commutatif; (5) P. Lefebvre, Demi-groupes admettant des complexes minimaux pour un résidu à droite ou bilatère donné; (6) M. Krasner, Nombre d'extensions d'un degré donné d'un corps de nombres p-adiques; (7) G. Preston, Equivalences régulières dans des demi-groupes complètement simples; (8) G. Preston, Caractérisation des cardinaux réguliers dans des demi-groupes associés; (9) G. Vors, Neutrices; application à la formule sommatoire d'Euler et aux équations aux différences finies; (10) M.-P. Brameret, Groupes finis d'ordre impair; (11) R. Desq, Etude de quelques relations d'équivalence définies dans un demi-groupe D.

Fascicule 2: (12) G. Mattenet, Sur les quasi-groupes [MR 28 #2166]; (13) C. Joulain, Sur les anneaux non commutatifs. I. Radical; (14) C. Joulain, Sur les anneaux non commutatifs. II. Anneaux noethériens et artiniens à gauche; (15) G. Renault, Sur les anneaux non commutatifs. III. Enveloppe injective d'un module; (16) G. Renault, Sur les anneaux non commutatifs. IV. Modules isotypiques; (17) G. B. Preston, Les congruences dans les demi-groupes abéliens et libres; (18) J.-J. Payan, Construction des corps abéliens de degré 5; (19) M. Ego, Structure des demi-groupes dont le treillis des sous-demi-groupes est semi-modulaire; (20) A. H. Clifford, La décomposition d'un demi-groupe commutatif en ses composantes archimédiennes; (21) A. H. Clifford, Caractères d'un demi-groupe commutatif; (22) A. G. Kuroš, Les radicaux en théorie des groupes; (23) A. G. Kuroš, Groupes avec multi-opérateurs.

Les exposés (6), (7) et (8) n'ont pas été rédigés, et ne seront pas multigraphiés.

HISTORY AND BIOGRAPHY

Burkill, J. C. 3913
Charles-Joseph de la Vallée Poussin.
J. London Math. Soc. **39** (1964), 165–175.

An account of de la Vallée Poussin's life, together with a selected bibliography.

Rankin, R. A. 3914
Thomas Murray MacRobert.
J. London Math. Soc. **39** (1964), 176–182.

An account of MacRobert's life, together with an extensive bibliography. Another biography has been given by R. P. Gillespie and A. Erdélyi [Proc. Glasgow Math. Assoc. **6** (1963), 57–64; MR **27** #5673].

Selberg, Sigmund 3915
Thoralf Albert Skolem. (Norwegian)
Norske Vid. Selsk. Forh. (Trondheim) **36** (1963), 165–168.

A brief account of the career of Skolem. There is no bibliography.

LOGIC AND FOUNDATIONS

See also 3943, 3944, 4413, 4542, 4692, 4957, 4979.

Barker, Stephen F. 3916
★Philosophy of mathematics.
Foundations of Philosophy Series.
Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964. xiv + 111 *pp.* $1.50.

This is a brief and eminently readable introduction to the philosophy of mathematics. Table of Contents: Introduction, Euclidean Geometry, Non-Euclidean Geometry, Numbers and Literalistic Philosophies of Number, Transition to a Non-literalistic View of Number.

P. J. Davis (Providence, R.I.)

Newsom, Carroll V. 3917
★Mathematical discourses: The heart of mathematical science.
Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964. ix + 125 *pp.* $5.00.

The first half of this booklet is rather historical. In the second half, examples of mathematical deduction from axiomatic systems (groups, plane projective geometry, Boolean algebra, completely ordered fields) are exhibited. It is written for the non-mathematician, yet it does not belong to the usual body of material written by superficial authors for superficial readers. Genuine erudition, a scientific spirit and teaching responsibility pervade the book. Nevertheless, there is a certain feeling that the author could have given more than he did. The author's historical knowledge, apparently stemming from secondary sources, is not sufficiently integrated. This lack of integration is also felt in the systematic part. Painstaking rigor in deriving conclusions from axioms does not make much sense if the axioms themselves are formulated in a comparatively slovenly manner. Despite this criticism, the booklet may be recommended, perhaps rather to first-year college students, than to plain non-mathematicians.

H. Freudenthal (Utrecht)

★Essays on the foundations of mathematics. 3918
Dedicated to A. A. Fraenkel on his seventieth anniversary.
Edited by Y. Bar-Hillel, E. I. J. Poznanski, M. O. Rabin, and A. Robinson for The Hebrew University of Jerusalem.
The Magnes Press, The Hebrew University, Jerusalem, 1961. x + 351 *pp.* (1 *plate*) $10.00.

The papers in this volume, which contains a bibliography of the published work of Fraenkel, will be reviewed individually.

Łoś, Jerzy 3919

Remarks on foundations of probability. Semantical interpretation of the probability of formulas.

Proc. Internat. Congr. Mathematicians (*Stockholm*, 1962), *pp.* 225–229. *Inst. Mittag-Leffler, Djursholm*, 1963.

Suppose that $\mathcal{M}$ is a model and A is its set of elements. Denote by $\mathcal{A}$ the set of sequences of elements from A. A probability function ν defined for an algebra of subsets of $\mathcal{A}$, called a probability in $\mathcal{M}$, is defined for all sets $\mathcal{M}(\alpha)$ with α in S, the set of all formulas. Suppose that $\{\mathcal{M}_t\}_{t\in T}$ is a family of models and that for every $t \in T$, ν_t is a probability in $\mathcal{M}_t$ and that there is a σ-additive probability function μ on a σ-algebra of subsets of T such that all functions $f_\alpha(t)=\nu_t(\mathcal{M}_t(\alpha))$, $\alpha \in S$, are μ-measurable. Then (Theorem 1) $p(\alpha) = \int_T f_\alpha(t)\, d\mu(t)$ is a probability on S. Theorem 2: Every continuous probability on the algebra of sentences S may be represented as an integral $p(\alpha)= \int_T \nu_t(\mathcal{M}_t(\alpha))\, d\mu(t)$, where $\mathcal{M}_t$ are suitable models, ν_t are probabilities in these models, and μ is a σ-additive probability function in the set of models.

Beth, Evert W. 3920

★Formal methods. An introduction to symbolic logic and to the study of effective operations in arithmetic and logic.

Synthese Library.

D. Reidel Publishing Co., Dordrecht, 1962. xiv + 170 *pp.* *Dfl.* 23.50.

The author has written a textbook on the lines of the more formal parts of his book [*The foundations of mathematics*, North-Holland, Amsterdam, 1959; MR **22** #9445] to which it forms an excellent introduction. Philosophical issues have been relegated to a forthcoming companion volume [*Mathematical thought*, Reidel, Dordrecht, 1964]. Thus there is no discussion of the paradoxes or of ontology. Elementary logic is dealt with in the first four chapters. The deduction-theoretic, semantic, and axiomatic methods are all brought in in the first chapter as co-equals. The aim is to avoid giving the beginner the impression that any one method has some intrinsic priority. The method of semantic tableaux is given a prominent place.

Effective operations are covered in the fifth, sixth and seventh chapters. Syntactic and semantic incompleteness are given separate treatment, and the celebrated results on the incompleteness of arithmetic are expounded. In the chapter on the theory of definition he gives an outline of the methodology of the deductive sciences and an account of Padoa's method of demonstrating independence. Incompleteness in the theory of definition is demonstrated.

In the final chapter he covers some of the recent work on logic machines and, in particular, attempts to adapt computers to such methods as the method of semantic tableaux. The outcome is that a great deal more work needs to be done both on the machines and on the logic.

Contents: Chapter 1, Purely implicational logic; Chapter 2, Full sentential logic; Chapter 3, Theory of quantification, equality, and functionality; Chapter 4, Completeness of elementary logic; Chapter 5, The formalisation of arithmetic and its limitations; Chapter 6, The theory of definition; Chapter 7, On machines which prove theorems; Appendix: Supplementary explanations; List of schemata and axioms; Bibliography; Index.

J. Tucker (Bangor)

Cuesta Dutari, N. 3921

Implication structures. (Spanish)

Acta Salmant. Ci. (*N.S.*) **6**, *no.* 1, 22 *pp.* (1962).

An implication structure is a set P of propositions partitioned into two cells, the set W of true propositions and the set $P-W$ of false propositions. A binary relation $\rightarrow$ of implication that is reflexive and transitive is defined on $P\times P$, and W is closed under transitivity of this relation. $w\rightarrow f$ is false whenever w belongs to W and f belongs to $P-W$. A relation such as material implication is maximal under this definition since all implications except those prohibited by the last statement are true. The author defines a postulate in a set A of implications as an element α such that $A-\alpha$ is not a deductive basis of A. Among non-postulates an implication is an axiom if it is necessarily an element of some deductive basis of A. The author treats examples of implication structures, pointing out the distinctions among postulates, axioms and non-axioms. The paper closes with discussion of the relation between implication and negation, and contraposition.

E. J. Cogan (Bronxville, N.Y.)

Bull, R. A. 3922

A note on the modal calculi S 4.2 and S 4.3.

Z. Math. Logik Grundlagen Math. **10** (1964), 53–55.

This note gives a proof that the modal calculi S 4.2 and S 4.3, introduced by Dummett and the reviewer, possess the finite modal property in the sense of Harrop. Algebraic methods are used. The result was claimed by Dummett and the reviewer, but on insufficient grounds [same Z. **5** (1959), 250–264; MR **28** #27], so that the present note fills a gap in the proofs of that paper.

E. J. Lemmon (Claremont, Calif.)

Nakamura, Akira 3923

Truth-value stipulations for the von Wright system M' and the Heyting system.

Z. Math. Logik Grundlagen Math. **10** (1964), 173–183.

In an earlier paper [same Z. **8** (1962), 147–164; MR **26** #3593] the author axiomatized infinitely many-valued threshold logics with infinite ordered sequences as "truth-values", and showed that von Wright's system of modal logic M'' is equivalent to a special case of such threshold logic. In the present paper he modifies these earlier results in such a way that von Wright's system of modal logic M' is shown to be equivalent to a "truth-value stipulation" of infinitely many-valued logic with enumerably infinite sequences serving as truth-values.

Since M' is known to be equivalent to the system of strict implication $S4$, Nakamura's truth-value stipulation for M' is also equivalent to $S4$. By utilizing this fact, together with the method of Tarski and McKinsey for translating Heyting's system into $S4$, the author is able to give an infinite truth-value stipulation for the Heyting intuitionistic system. In establishing these results, convincing evidence is given that the major difficulty is not in proving plausibility, but rather in obtaining a proof of deductive completeness. This is probably true in general, but one would certainly expect it to be true of axiomatic systems like $S4$ for which there is no finite characteristic matrix. *A. Turquette* (Champaign, Ill.)

Pogorzelski, H. A. 3924

Primitive words in an infinite abstract alphabet.

Z. Math. Logik Grundlagen Math. **10** (1964), 193–198.

A unique factorisation theorem is stated for words in the alphabet $a_1, a_2, \cdots$ with respect to a multiplication $X \bigcirc^k Y$ defined by $X \oplus \Lambda = X$, $X \oplus a_\mu Y = a_\mu(X \oplus Y)$, $X \bigcirc^k \Lambda = \Lambda$, $X \bigcirc^k a_\mu Y = \sigma_\mu{}^k(X) \oplus (X \bigcirc^k Y)$, where Λ is the empty word and the subscript function $\sigma_\mu{}^k$ is defined by $\sigma_\mu{}^k(\Lambda) = \Lambda$, $\sigma_\mu{}^k(a_\nu X) = \sigma_\mu{}^k(X) \oplus a_{\mu[k]\nu}$, $y[k]x$ being a certain class of primitive recursive functions.

R. L. Goodstein (Leicester)

Grzegorczyk, A. 3925

A note on the theory of propositional types.

Fund. Math. **54** (1964), 27–29.

Henkin's formulation of the theory of propositional types [Fund. Math. **52** (1963), 323–334; MR **27** #3497] uses (besides variables of all types, the abstractor λ and the application-function $(A_{\alpha\beta} B_\beta)$) a denumerable set of constants $Q_{(o\alpha)\alpha}$. The author shows: (1) From the original set of constants one may retain only $Q_{(oo)o}$ and add as new primitives the (formerly defined) terms of conjunction $\wedge$ and falsity F; (2) Alternatively, one may retain $Q_{(o(oo))(oo)}$ and $Q_{(o(o(oo)))(o(oo))}$ and add as a new primitive the term of equivalence. In either case each $Q_{(o\alpha)\alpha}$ can be defined in the new terms.

E. Engeler (Zürich)

Grzegorczyk, A. 3926

Recursive objects in all finite types.

Fund. Math. **54** (1964), 73–93.

In this paper the author shows that several concepts of recursive functionals of finite type can be represented in a system $\mathscr{R}$ which is practically identical with a form of the basic theory of functionality in combinatory logic; see the reviewer and R. Feys [*Combinatory logic*, Vol. 1, Chap. 9, North-Holland, Amsterdam, 1958; MR **20** #817] (hereafter cited as CLg). The author appears to be unaware of this relationship; he cites CLg only for certain terminology, and he states as an "open problem" a question to which an affirmative answer was given, in principle, by the reviewer [Trans. Amer. Math. Soc. **50** (1941), 454–516, pp. 477–478; MR **3**, 129]. The author's formulation of $\mathscr{R}$ differs from that current in combinatory logic in two principal respects: first, he generates the numbers from atoms representing zero and the successor function, whereas in combinatory logic (until very recently) they were defined, following Church [Ann. of Math. (2) **34** (1933), 839–864], as certain "iterators" Z_n (see CLg pp. 174 ff.); second, he assigns type indices to the combinators, and regards instances of a combinator with different type indices as distinct entities. The former of these changes requires only slight revisions in previous work, provided one postulates a Z (with suitable properties) mapping the numbers on the Z_n; the latter is, in view of Theorem 9B1 and § 9C of CLg, an unnecessary complication. Thus §§ 1, 2 of the paper, which deal with these matters, are interesting chiefly for showing that one can derive certain properties of such a system independently. In § 3 he makes a real innovation, viz., a definition of ordered pair such that the function and its inverses have functional characters for certain cases where the two constituents are not of the same type. In the later sections he shows that all primitive recursive functionals in the sense of Kleene [Trans. Amer. Math. Soc. **91** (1959), 1–52; MR **21** #1273] are included in $\mathscr{R}$ but do not exhaust $\mathscr{R}$; in fact, a doubly recursive function which is not primitive recursive is shown to be in $\mathscr{R}$. He ends by saying that the class $\mathscr{R}$ may be considered identical to the class of recursive functionals considered by Gödel [Dialectica **12** (1958), 280–287; MR **21** #1275] and Kreisel [Constructivity in Mathematics (Proc. Colloq., Amsterdam, 1957), pp. 101–128, North-Holland, Amsterdam, 1959; MR **21** #5568], and by presenting a refined proof of Gödel's interpretation theorem. There is a rather large number of misprints and other inaccuracies apparently caused by inadvertence.

H. B. Curry (University Park, Pa.)

Dekker, J. C. E. 3927

The minimum of two regressive isols.

Math. Z. **83** (1964), 345–366.

Let $E = \{0, 1, 2, \cdots\}$, and let $P(E)$ be the class of all subsets of E. Then $\alpha, \beta \in P(E)$ are called recursively equivalent if there exists a 1-1 partial recursive function p such that α is contained in the domain of p and $p(\alpha) = \beta$. Then $\langle\alpha\rangle$ denotes the equivalence class of α, and the set Λ of isols is the set of all $\langle\alpha\rangle$ such that α has no infinite recursively enumerable subset. An α is called retraceable if there is a uniform effective method which, when applied to an element of α, yields the next smaller element of α (if any). The set Λ_R of regressive isols is the set of all $x \in \Lambda$ such that $x = \langle\alpha\rangle$ for some retraceable α. The author extends the function $f(x, y) = \min(x, y)$, $f: E \times E \to E$, to a function $F(x, y)$, $F: \Lambda_R \times \Lambda_R \to \Lambda_R$, as follows. If $\alpha, \beta \in P(E)$, let $\phi(\alpha, \beta)$ consist of all numbers of the form $2^x 3^y$ such that $x \in \alpha$, $y \in \beta$, and the two sets $[u \in \alpha | u < x]$, $[u \in \beta | u < y]$ have the same number of elements. If α, β are retraceable, put $F(\langle\alpha\rangle, \langle\beta\rangle) = \langle\phi(\alpha, \beta)\rangle$. The reviewer [Ann. of Math. (2) **73** (1961), 362–403; MR **24** #A1215] gave a procedure extending each relation R among natural numbers to a relation R_Λ among isols. Dekker's student Barback has recently shown that F is f_Λ restricted to $\Lambda_R \times \Lambda_R$. It is noteworthy that since f is not almost combinatorial, f_Λ is not defined on all of $\Lambda \times \Lambda$. The paper under review is the first extensive investigation of a partial function of isols which is essentially non-total.

A. Nerode (Ithaca, N.Y.)

Hořejš, Jiří 3928

Note on definition of recursiveness.

Z. Math. Logik Grundlagen Math. **10** (1964), 119–120.

Let D be a non-recursive set of integers, and let $S_i = \{0, 1\}$ if $i \in D$ and $S_i = \{0\}$ if $i \notin D$. Then S_i, $i = 0, 1, 2, \cdots$, is a sequence of finite sets such that the relation $1 \in S_i$ is non-recursive; and if c_i is the cardinal of S_i, both the relations $c_i = 1$, $c_i = 2$ are non-recursive.

R. L. Goodstein (Leicester)

Lacombe, Daniel 3929

Deux généralisations de la notion de récursivité.

C. R. Acad. Sci. Paris **258** (1964), 3141–3143.

If N is the set of natural numbers, E is a denumerable set, E^n the set of all ordered n-tuplets of elements of E, P_i a subset of E^{p_i} and ϕ_j a mapping of E^{q_j} into E, then an ordered set $(P_1, \cdots, P_m, \phi_1, \cdots, \phi_n)$ is said to be existentially recursive if there is a bijection θ from N onto E such that each set $\theta^{-1}(P_i)$ and each function $\theta^{-1}(\phi_j)$ is recursive. A subset P of E^p is said to be universally recursive if $\theta(P)$ is recursive for every bijection of E onto N, and P is said to be Boolean if there is a set $a_1, a_2, \cdots, a_n$ of elements of E and a sentence Φ, a

conjunction of equations of the form $x_i = x_j$ or $x_i = a_k$, or their negations, such that $(x_1, \cdots, x_p) \in P \leftrightarrow \Phi(x_1, \cdots, x_p)$. It is stated that P is universally recursive if and only if it is Boolean, and various results about existential recursion are stated without proof. *R. L. Goodstein* (Leicester)

Lacombe, Daniel 3930

Deux généralisations de la notion de récursivité relative.

C. R. Acad. Sci. Paris **258** (1964), 3410–3413.

In this paper the concepts of the previous paper [#3929] are relativised. For example, P is said to be universally recursive in $(Q_1, \cdots, Q_n)$ if $\theta^{-1}(P)$ is recursive in $(\theta^{-1}(Q_1), \cdots, \theta^{-1}(Q_n))$ for all bijections θ from N onto E. To generalise the notion of a Boolean set the author uses an enumeration $\mathscr{E}$ of sets

$$(F, R_1, \cdots, R_n, x_1, \cdots, x_p, a_1, \cdots, a_h)$$

for given integers $q_1, \cdots, q_n, p, h$, where F is a finite set, R_i is a subset of F^{q_i} and each x_j, a_k belongs to F; let $U \subset \mathscr{E}$, then, for each subset Q_i of E^{q_i}, $\Phi_{U, a_1, \cdots, a_h}(Q_1, \cdots, Q_n)$ denotes the set of all $(x_1, \cdots, x_p)$ such that there is a finite subset F of E containing $x_1, \cdots, x_p, a_1, \cdots, a_h$ and

$$(F, F_1, \cdots, F_n, x_1, \cdots, x_p, a_1, \cdots, a_h),$$

where F_i is the restriction of Q_i to F, is isomorphic to a member of U. Then P is said to be recursively Boolean in $Q_1, \cdots, Q_n$ if there exist $a_1, \cdots, a_h$ and recursively enumerable subsets U, V of $\mathscr{E}$ such that $P = \Phi_{U, a_1, \cdots, a_h}(Q_1, \cdots, Q_n)$ and $\mathscr{C}P = \Phi_{V, a_1, \cdots, a_h}(Q_1, \cdots, Q_n)$, where $\mathscr{C}P$ is the complement of P in E^p. It is stated that for denumerable sets E, P is universally recursive in $(Q_1, \cdots, Q_n)$ if and only if it is recursively Boolean in $(Q_1, \cdots, Q_n)$. *R. L. Goodstein* (Leicester)

Vopenka, Petr [Vopěnka, Petr] 3931

Submodels of models of set theory. (Russian. German summary)

Z. Math. Logik Grundlagen Math. **10** (1964), 163–172.

Let $\bar{E}$ be a relation corresponding to the membership predicate of Gödel set theory [*The consistency of the continuum hypothesis*, Princeton Univ. Press, Princeton, N.J., 1940; MR **2**, 66] and $\bar{\psi}$ a property corresponding to the predicate calculus. For $(\bar{E}, \bar{\psi})$ a model of set theory and for a nonempty subset $\bar{E}_1$ of $\bar{E}$ the notions of being internal, closed relative to a list of set-theoretic operations, and almost universal in $\bar{E}$ are defined. It is shown that these conditions on $\bar{E}_1$ are sufficient that $(\bar{E}_1, \bar{\psi}_4)$ be a model, where $\bar{\psi}_4$ is a confinement of $\bar{\psi}$ to $\bar{E}_1$. These methods are applied to ultraproducts in a second section, where two submodels are investigated. Finally, it is shown that every ordinal number confinal with θ (the first number for which nontrivial mass exists) is also confinal with θ in a model Δ. These notions and notation are related to those of the author [Z. Math. Logik Grundlagen Math. **8** (1962), 281–292; MR **26** #3610; ibid. **9** (1963), 161–167; MR **27** #39]. *E. J. Cogan* (Bronxville, N.Y.)

Lorenzen, Paul 3932

★Metamathematik.

B·I-Hochschultaschenbücher, Bd. 25.

Bibliographisches Institut, Mannheim, 1962. 173 *pp.* 3.80 *DM*.

This small book provides a clear and direct introduction to a number of the most important ideas and results in metamathematics. Although it presupposes no special knowledge, the reader who studies this book with understanding cannot fail to acquire considerable subtlety of mathematical thought. As a general introduction to metamathematics this book ranks high, but the reviewer feels its special distinction lies in its careful development of the constructivist approach.

The author expresses the reasonable opinion that, in all metamathematics, the metalogic should be constructivist. This judgment appears to be based on the nature of the enterprise, and not on any philosophical bias. The object theory may, of course, be constructivist or classical at will; herein the author subsumes the semantic method (theory of models) of Tarski, where the object theory is taken to include an axiomatic theory of sets adequate for the semantic discussion. In accordance with this position, the discussion in this book proceeds on constructivist grounds, without prejudice or obscurantism; classical theories and the semantic method receive throughout adequate attention in proper perspective.

Chapter I develops the formalism of both classical and effective (constructive) predicate logic. Effective logic is introduced by consideration of 'dialogues' or 'games' in which a proponent defends a thesis against the attack of an opponent; the reviewer has the impression that this approach, which recurs throughout the book, is more than a heuristic device, and is perhaps to be taken as philosophically prior to the technical formalism that grows out of it. It is perhaps thankless to question the author's scrupulous avoidance of polemic, but, in the absence of supporting argument and further detail, the reviewer does not find this approach either naively suggestive or philosophically compelling. In any case, one arrives quickly at an exact formulation, in the manner of Gentzen, of the constructivist and classical predicate logics. These are extended to include equality.

Chapter II, Formalisierung der Arithmetik, introduces the usual notation of arithmetic and specifies the convention whereby quantifiers are to be instantiated only by numerals. The distinction between existence of a proof and of truth, the existence of a 'winning strategy', becomes especially clear in this context. A somewhat parallel discussion of the scope of the induction axiom reconciles the claim to categoricity of Peano arithmetic with Skolem's critique. The greater part of Chapter II is devoted to a proof and discussion of Gentzen's theorem asserting the consistency of classical Peano arithmetic.

Chapter III, Arithmetisierung der Formalismen, begins with a very careful development of the syntax of a formal theory. This serves both as a basis for a definition of recursive enumerability, and for the proof of Gödel's incompleteness and undecidability results; in particular, the essential undecidability of R. M. Robinson's fragment of arithmetic is established.

Chapter IV deals with decidability and completeness. A constructivist interpretation of Gödel's completeness theorem is given. The central examples are the undecidability of the theory of rings; the completeness of the theory of algebraically closed fields of characteristic 0, with the Lefschetz principle; and the decidability of that of real closed fields, with the analogous 'Tarski principle'. In this discussion, as elsewhere, classical and semantic methods are not neglected.

R. C. Lyndon (Ann Arbor, Mich.)

Gladkiĭ, A. V. 3933

On the recognition of replaceability in recursive languages. (Russian)

Algebra i Logika Sem. **2** (1963), *no.* 3, 5–22.

Finite sequences of elements of a finite set $\mathfrak{B}$ are called chains, and are denoted by Latin letters. A language $\mathfrak{A}$ is a set of chains. X is replaceable by Y in $\mathfrak{A}$ if $Z_1XZ_2 \in \mathfrak{A}$ implies $Z_1YZ_2 \in \mathfrak{A}$. The set of all chains by which X is replaceable in $\mathfrak{A}$ is denoted by $\phi_X(\mathfrak{A})$. The set of all chains Y for which $Y \in \phi_X(\mathfrak{A})$ and $X \in \phi_Y(\mathfrak{A})$ is denoted by $\psi_X(\mathfrak{A})$. $\phi(\mathfrak{A})$ $[\psi(\mathfrak{A})]$ denotes the set of ordered pairs (X, Y) for which $Y \in \phi_X(\mathfrak{A})$ $[Y \in \psi_X(\mathfrak{A})]$.

With $\mathfrak{B} = \{\alpha, \beta\}$, languages $\mathfrak{A}_i$ are constructed such that: $\mathfrak{A}_1$ is recursive, but $\phi_{\beta\beta}(\mathfrak{A}_1)$ and $\psi_{\beta\beta}(\mathfrak{A}_1)$ are not; $\phi(\mathfrak{A}_2)$ is recursive, but $\mathfrak{A}_2$ is not; all $\phi_X(\mathfrak{A}_3)$ and $\psi_X(\mathfrak{A}_3)$ are recursive, but $\mathfrak{A}_3$ and $\psi(\mathfrak{A}_3)$ are not; $\mathfrak{A}_4$, all $\phi_X(\mathfrak{A}_4)$ and all $\psi_X(\mathfrak{A}_4)$ are recursive, but $\psi(\mathfrak{A}_4)$ is not; $\psi(\mathfrak{A}_5)$ is recursive, but $\mathfrak{A}_5$ and $\phi_\beta(\mathfrak{A}_5)$ are not; $\mathfrak{A}_6$ and $\psi(\mathfrak{A}_6)$ are recursive, but $\phi_{\beta\beta}(\mathfrak{A}_6)$ is not; and $\mathfrak{A}_7$ is the recursive type I language generated by a type I grammar [N. Chomsky, Information and Control **2** (1959), 137–167; MR **21** #4107], but $\psi_{\beta\beta}(\mathfrak{A}_7)$ is not recursive.

The author proves that if $\mathfrak{B} = \{\alpha\}$, then all $\phi(\mathfrak{A})$ are recursive, but their triviality cannot be effectively determined for the recursively enumerable $\mathfrak{A}$.

M. Greendlinger (Ivanovo)

Lavrov, I. A. 3934

Undecidability of elementary theories of certain rings. (Russian)

Algebra i Logika Sem. **1** (1962), *no.* 3, 39–45.

Let K_{nm} be the free ring (non-associative and non-commutative) on n generators of nilpotency class m (i.e., products of m elements equal to 0). Theorem: If $n \geq 2$ and $m \geq 3$, the elementary theory of K_{nm} is undecidable, i.e., there is no algorithm for determining which first-order predicate calculus formulas containing no non-logical constants other than $+$ and $\cdot$ are valid for K_{nm}.

This result is also proven for the associate and/or commutative cases. Allowing formulas to contain n fixed free generators as individual constants, the author obtains analogous theorems for Lie rings (with or without nilpotency) and anti-commutative rings (associative or non-associative). He generalizes these results to algebras in which coefficients of the generators and their products belong to any ring with an undecidable elementary theory.

M. Greendlinger (Ivanovo)

Kuznecov, A. V. 3935

Undecidability of the general problems of completeness, solvability and equivalence for propositional calculi. (Russian)

Algebra i Logika Sem. **2** (1963), *no.* 4, 47–66.

The author considers the class $\mathfrak{P}$ of all "usual propositional calculi" (UPC). The common frame is determined by the alphabet &, ∨, ⊃, ¬, (,), p, q, r, s, t, 0, 1, ⋯, 9, where p, q, r, s, t (with possible indices 0, 1, ⋯, 9) are variables and where formulas are defined to be variables and connections of formulas by means of &, ∨, ⊃ and ¬. The rules of deduction accepted are substitution (formulas for letters) and modus ponens. Particular UPC are obtained by addition of more specific axioms. Intuitionistic calculus is denoted by K.

A partial order $\leq$ is introduced in $\mathfrak{P}$ by $J_1 \leq J_2$ if all formulas which are deducible in J_1 are deducible in J_2. If, moreover, $J_1 \neq J_2$, the notation $J_1 < J_2$ is used. $J_1 \sim J_2$ means $J_1 \leq J_2$ and $J_2 \leq J_1$. Main theorem: If $K \leq J_0$, the sets $\{J | J \in \mathfrak{P}$ and $J \sim J_0\}$ and $\{J | J < J_0$ and $J < K\}$ are recursively inseparable.

The recursive unsolvability of the general problem of equivalence in $\mathfrak{P}$ (i.e., is $J_1 \sim J_2$ for any $J_1, J_2 \in \mathfrak{P}$?) and of the overconstructivity in $\mathfrak{P}$ (i.e., is $K \leq J$ for any $J \in \mathfrak{P}$?) follow as an easy consequence. Also, the problem of deducibility for $J \in \mathfrak{P}$ is not recursively solvable.

The author states the hypothesis that the subclass $\mathfrak{P}'$ of all overconstructive calculi (i.e., of all $J \in \mathfrak{P}$ such that $K \leq J$) satisfies the property of finite approximability (a calculus J is finite approximable if there exists a model M of J such that all axioms of J are satisfied in M, and such that for any formula $A \in J$ or A is provable in J or A is not satisfied in M). If this hypothesis is true, the problems of deducibility, completeness and equivalence in $\mathfrak{P}'$ are recursively solvable.

The author points out that the abstract of Linial and Post [Bull. Amer. Math. Soc. **55** (1949), 50] contains less general results, as connectives ⊃ and & are not in UPC of that abstract.

V. Vučković (Notre Dame, Ind.)

Kratko, M. I. 3936

Algorithmic unsolvability of the problem of completeness recognition for finite automata. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 35–37.

The concepts of "logical net over a basis that consists of delay elements and logical elements" and of "realization of an operator in a logical net" [see N. E. Kobrinskiĭ and B. A. Trahtenbrot, *Introduction to the theory of finite automata* (Russian), esp. p. 122, Fizmatgiz, Moscow, 1962; MR **26** #4866] are extended as follows: In lieu of delay elements or logical elements, arbitrary finite automata are usable as elements; then there is the requirement that any arbitrary oriented cycle of a logical net should lead through an element whose exit channel is not subordinate to an entrance channel belonging to the same cycle. Also, every entrance or exit channel of an element may be in any of a finite number of states indexed by members of a set of letters M (i.e., not necessarily $M = \{0, 1\}$). Operators realized in logical nets then transform sequences $x(1), x(2), \cdots$ into sequences $z(1), z(2), \cdots$, where $x(t)$ and $z(t)$ consist of letters from M. Further, either of the following two conventions for bases may be adopted: (i) the basis elements are initial automata, and the initial state of the logical net over the basis is the state in which all its elements are in their initial states ("initial basis"); (ii) the basis elements are not initial automata, and arbitrary internal states may occur as initial states ("non-initial basis"). If there exists such a logical net over a (non-)initial basis B in which an operator T is realized, then it is said that T is realizable in (non-)initial basis B. The (non-)initial basis is said to be complete if in it arbitrary o-d operators in the sense of Kobrinskiĭ and Trahtenbrot [loc. cit.] are realizable. The problem of finding the conditions for such basis completeness then acquires special importance and has so far been solved for some particular cases only: E. L. Post [Amer. J. Math. **43** (1921), 163–185] and S. V. Jablonskiĭ [Trudy Mat. Inst. Steklov. **51** (1958), 5–142; MR **21** #3331] solved the problem positively for functions of the algebra of

logic conceived as automata without memory; V. B. Kudrjavcev [Problemy Kibernet. 8 (1962), 91–115] established completeness criteria for automata without feedback; A. A. Letičevskiĭ [Seminar po teorii avtomatov, Kiev, 1963] provided completeness conditions for bases containing Moore automata or memoryless automata.

Theorem 1: For an arbitrary o-d operator T the problem of recognizing whether it is realized in (non-)initial bases is algorithmically unsolvable. Theorem 2: The problem of recognizing the completeness of (non-)initial bases is algorithmically unsolvable. Theorem 3: Let M_A and M_B be the sets of all o-d operators realizable in given bases A and B, respectively. Then with respect to arbitrary (non-)initial bases A and B the following problems are algorithmically unsolvable: (i) $M_A = M_B$ or not? (ii) $M_A \subset M_B$ or not? (iii) $M_A \cap M_B = \emptyset$ or not?

A not too concise proof is sketched for Theorem 1. The author points out that all three theorems have been established by him for bases with elements whose number of entrance channels does not exceed three.

E. M. Fels (Munich)

Marcus, Solomon 3937

Sur un modèle de H. B. Curry pour le langage mathématique.

C. R. Acad. Sci. Paris **258** (1964), 1954–1956.

The author makes three remarks as follows: (1) If K_1, K_2, K_3 are, respectively, the languages consisting of all words of the forms ab^n, ab^mcab^n, ab^ncab^n (introduced by the reviewer [Proc. Sympos. Appl. Math., Vol. XII, pp. 56–68, esp. p. 58, Amer. Math. Soc., Providence, R.I., 1961]), then K_1 and K_2 are finite-state languages but K_3 is a context-free language which is not a finite-state one. (2) No language in which the length of every word is a perfect square can be a finite-state language. (3) Certain corrections are made to the author's previous article [C. R. Acad. Sci. Paris **256** (1963), 3571–3574; MR **26** #6020]. (For terminology see N. Chomsky [Information and Control **2** (1959), 137–167; MR **21** #4107; ibid. **2** (1959), 393–395; MR **21** #7815]; proofs, where needed, are to be found in the author's forthcoming book [*Grammars and finite automata* (Romanian), Editura Acad. R. P. Romîne, Bucharest, 1964].)

H. B. Curry (University Park, Pa.)

Ljubič, Ju. I. 3938

Estimates for the number of states arising from the determination of an indeterminate autonomous automaton. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 41–43.

Let φ be a finite oriented graph interpreted as an additive mapping of all subsets of a finite set S into subsets of S [see the reviewer, Czechoslovak Math. J. **8 (83)** (1958), 487–495; MR **22** #1520a]. Suppose that S has n elements and denote by Σ the family of all subsets of S. Take a one-point set and denote by Σ_0 the smallest subfamily of Σ which contains this set and is invariant with respect to φ. Denote by $d(\varphi)$ the cardinality of Σ_0. The author calls the passage from (S, φ) to (Σ_0, φ) the determinization of the indeterminate automaton (S, φ). The number $d(\varphi)$ will be the number of states of the determinate automaton (Σ_0, φ). The paper states (without proof) a number of estimates for $d(\varphi)$. If φ is primitive, $d(\varphi) \leqq n^2 - 2n + 3$, and if the index of imprimitivity is h, then $d(\varphi) \leq h^{-1}(n^2 - 2nh + 4h^2)$. In the general case when φ has r irreducible components with cardinalities $n_1, \cdots, n_r$ and indices of imprimitivity $h_1, \cdots, h_r$,

$$d(\varphi) \leqq m(h_1, \cdots, h_r) + \sum_{k=2}^{r} m(h_1, \cdots, h_k) + \sum_{k=1}^{r} h_k^{-1}(n_k^2 - 2n_k h_k + 4h_k^2),$$

where m is the greatest common divisor. Asymptotic estimates of the growth of $d(\varphi)$ are also given.

V. Pták (Prague)

SET THEORY

See also 4068.

Davies, Roy O. 3939

Covering the plane with denumerably many curves.

J. London Math. Soc. **38** (1963), 433–438.

Let us call a plane set a curve if each line in some fixed direction intersects it in exactly one point. Mazurkiewicz proved that the plane is not the union of a finite number of curves. Sierpiński proved that the continuum hypothesis implies that the plane is the union of denumerably many congruent curves. It follows from this that the continuum hypothesis ($2^{\aleph_0} = \aleph_1$) implies the proposition (R): The plane is the union of denumerably many curves. Sierpiński [Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10** (1951), 406–411; MR **13**, 828; Fund. Math. **38** (1951), 1–13; MR **14**, 26; *Cardinal and ordinal numbers*, Państwowe Wydawnictwo Naukowe, Warsaw, 1958; MR **20** #2288] has raised the question whether conversely (R) implies the continuum hypothesis. The author shows in this paper that (R) can be established without assuming any hypothesis other than the axiom of choice. The author proves (with the aid of the axiom of choice) the following result, which is stronger than (R): Let θ_i $(i = 1, 2, \cdots)$ be any given infinite sequence of directions in the plane, no two of which are parallel. Then the plane can be decomposed into $\aleph_0$ sets S_i such that each line in the direction θ_i intersects S_i $(i = 1, 2, \cdots)$ in at most a single point.

G. Fodor (Szeged)

Mayer-Kalkschmidt, J.; Steiner, E. 3940

Some theorems in set theory and applications in the ideal theory of partially ordered sets.

Duke Math. J. **31** (1964), 287–289.

The authors show that if a family of sets is closed under chain union (property of Zorn), it is closed under directed union. They use a lemma of Iwamura which states that every infinite directed set D is the union of a chain of directed subsets, each of cardinality less than that of D.

This provides an improvement of a theorem of Birkhoff and the reviewer [Trans. Amer. Math. Soc. **64** (1948), 299–316; MR **10**, 279] as follows: A family of sets is the family of all subalgebras of an abstract algebra with finitary operations if and only if it has a largest element and is closed under arbitrary intersection and chain union. The original version had directed union instead of chain union. A family with these properties may be called an algebraic family.

They next show that if a family F of sets is closed under arbitrary intersection, then the family of all directed unions of members of F is an algebraic family. It follows that the smallest algebraic family containing a family A is the family of all directed unions of arbitrary intersections of members of A. Hence the family of all ideals of a partially ordered set is the smallest algebraic family containing the principal ideals.

They also show that a complete lattice in which every element is the join of a chain of join-inaccessible elements is isomorphic to the lattice of all ideals of a partially ordered set. *O. Frink* (University Park, Pa.)

Piccard, Sophie 3941

Quelques propriétés des systèmes déterminants d'ensembles et de leurs noyaux.

C. R. Acad. Sci. Paris **258** (1964), 1369–1372.

For every finite sequence $(n_1 \cdots n_k)$ of natural numbers let $E_{n_1 \cdots n_k}$ be a subset of a given set E. Define for every countable ordinal α sets $E^{\alpha}_{n_1 \cdots n_k}$ by transfinite induction as follows:

$$E^0_{n_1 \cdots n_k} = E_{n_1 \cdots n_k},$$

$$E^{\alpha}_{n_1 \cdots n_k} = E^{\alpha^*}_{n_1 \cdots n_k} \cap \bigcup_{n_{k+1}=1}^{\infty} E_{n_1 \cdots n_k n_{k+1}}$$

if $\alpha = \alpha^* + 1$ and $E^{\alpha}_{n_1 \cdots n_k} = \bigcap_{0 \le \xi < \alpha} E^{\xi}_{n_1 \cdots n_k}$ if α is a limit number. Furthermore, define

$$T^{\alpha} = \bigcup_{k, n_1, \cdots, n_k} (E^{\alpha}_{n_1 \cdots n_k} - E^{\alpha+1}_{n_1 \cdots n_k}).$$

The author shows that if $0 \le \alpha < \beta < \Omega$, then $T^{\alpha} \supseteq T^{\beta}$.

G. Bruns (Hamilton, Ont.)

Piccard, Sophie 3942

Quelques propriétés des systèmes déterminants d'ensembles et de leurs noyaux. Les systèmes déterminants réguliers.

C. R. Acad. Sci. Paris **258** (1964), 1663–1665.

For terminology see #3941 above. A system

$$\{E_{n_1 \cdots n_k} | k, n_1, \cdots, n_k = 1, 2, \cdots\}$$

is called regular if it satisfies $E_{n_1 \cdots n_k} \supseteq E_{n_1 \cdots n_k n_{k+1}}$. For such a system define $S^{\alpha} = \bigcup_{n_1=1}^{\infty} E_{n_1}$, $U^{\alpha} = S^{\alpha} - T^{\alpha}$, $N = \bigcup_{n_1 n_2 \cdots} \bigcap_{k=1}^{\infty} E_{n_1 \cdots n_k}$, $\mathfrak{N} = E - N$, $N_0 = U^0$, $\mathfrak{N}_0 = E - S^0$, $N_{\alpha} = U^{\alpha} - \bigcup_{0 \le \xi < \alpha} U^{\xi}$, $\mathfrak{N}_{\alpha} = \bigcap_{0 \le \xi < \alpha} S^{\xi} - S^{\alpha}$. The author proves several relations between these sets.

G. Bruns (Hamilton, Ont.)

Hanf, W. 3943

Incompactness in languages with infinitely long expressions.

Fund. Math. **53** (1963/64), 309–324.

It is known that the first-order predicate calculus is compact; that is, if every finite subset of a set of its sentences has a model, then the set itself has a model. The author here defines for each cardinal α of a comprehensive class a logic L_{α} that includes, besides equality and negation, predicates of fewer than α arguments, conjunctions of fewer than α terms, and quantification over fewer than α variables, and demonstrates that such logics L_{α} are incompact. The classes of cardinals for which results are obtained are approached as follows. A cardinal α is said to be singular if for some $\beta < \alpha$ there is a β-termed sequence γ such that $\gamma_{\eta} < \alpha$ for $\eta < \beta$ and $\alpha = \bigcup_{\eta < \beta} \gamma_{\eta}$, accessible if α is singular or $\alpha \le 2^{\beta}$ for some cardinal $\beta < \alpha$, and strongly accessible if α is singular or else the successor of a smaller cardinal. A cardinal α is said to be β-incompact if there is a set Γ of sentences of L_{α} that has power β, that has no model, and such that every subset of Γ of power $< \alpha$ has a model. The cardinal α is incompact if it is β-incompact for some $\beta \geqq \alpha$, and strongly incompact if it is α-incompact.

The author first shows that the members of a class of accessible cardinals including all strongly accessible cardinals are strongly incompact. To obtain comprehensive classes of nondenumerable inaccessible cardinals for which the incompactness result can be proved, the author uses methods of generating these classes from the class of accessible cardinals similar to those introduced by P. Mahlo [Ber. Verh. Sächs. Ges. Wiss. Leipzig Math.-Phys. Kl. **63** (1911), 187–225].

E. J. Cogan (Bronxville, N.Y.)

Hanf, W. 3944

On a problem of Erdős and Tarski.

Fund. Math. **53** (1963/64), 325–334.

The author considers properties P_1–P_4 and Q of an infinite cardinal α first investigated by P. Erdős and A. Tarski [*Essays on the foundations of mathematics*, pp. 50–82, Magnes Press, Hebrew Univ., Jerusalem, 1961] and a property S, equivalent to Q: There is an α-complete set algebra which is α-generated by α elements and in which every α-complete prime ideal is principal. It is shown that S implies P_1, so that P_1, P_2, and S are equivalent. *E. J. Cogan* (Bronxville, N.Y.)

Hajnal, A. 3945

Remarks on the theorem of W. P. Hanf.

Fund. Math. **54** (1964), 109–113.

A very special case of one of the theorems of the author states that if m_0 is the first inaccessible cardinal, then $m_0 \not\to (m_0, 4)^3$; in other words, if G is a set of power m_0, there is a splitting of the unordered triples of the distinct elements of G into two classes so that every subset G_1 of G of power m_0 contains a triple of the second class and every quadruple of G contains a triple of the first class. For the details the reader must be referred to the article.

P. Erdős (Budapest)

COMBINATORIAL ANALYSIS

See also 4259, 4641.

Tauber, S. 3946

On multinomial coefficients.

Amer. Math. Monthly **70** (1963), 1058–1063.

The main results of this paper are extensions to multinomial coefficients of two binomial coefficient identities; first, the Vandermonde convolution

$$\sum_{j=0}^{k} \binom{p}{j}\binom{q}{k-j} = \binom{p+q}{k}$$

and next the "orthogonal" identity

$$\sum_{s=m}^{n} (-1)^{m+s}\binom{n}{s}\binom{s}{m} = \delta_{nm},$$

with δ_{nm} the Kronecker delta. If $[N; k_1, \cdots, k_n]$ is the multinomial coefficient $N!/k_1!\cdots k_n!$, $k_1+\cdots+k_n=N$, the Vandermonde multinomial convolution is

$$\sum [p; j_1, \cdots, j_n][q; k_1-j_1, \cdots, k_n-j_n] = [p+q; k_1, \cdots, k_n].$$

The sum is over the first $n-1$ of the j's with $j_i = 0, 1, \cdots, k_i$. As noted by the author, this has also been proved by L. Carlitz [Elem. Math. **18** (1963), 37–39; MR **27** #56]. The orthogonal multinomial identity is

$$\sum (-1)^{m+s}[s_n; s_0, s_1-m_1, m_1-s_0, \cdots, s_n-m_n, m_n-s_{n-1}] = \prod_{k=1}^{n} \delta_{s_k, s_{k-1}},$$

with $m=m_1+\cdots+m_n$, $s=s_0+\cdots+s_{n-1}$, and summation over all the m_i variables with $m_i = s_{i-1}, s_{i-1}+1, \cdots, s_i$. {Reviewer's note: The multivariable form of the orthogonal identity of immediate combinatorial interest is that implied by the inverse relations

$$F(n_1, \cdots, n_s) = \sum (-1)^k \binom{n_1}{k_1}\cdots\binom{n_s}{k_s} G(k_1, \cdots, k_s),$$

$$G(n_1, \cdots, n_s) = \sum (-1)^k \binom{n_1}{k_1}\cdots\binom{n_s}{k_s} F(k_1, \cdots, k_s),$$

with $k=k_1+\cdots+k_s$. This has been given by L. C. Hsu [Math. Student **22** (1954), 175–178; MR **16**, 893] and by M. T. L. Bizley [J. Inst. Actuar. Students' Soc. **16** (1960), 147–151].} *J. Riordan* (Murray Hill, N.J.)

ORDER, LATTICES

See also 3940, 4346, 4502, 4507, 4508.

Dokas, Lambros 3947

Certains complétés des ensembles ordonnés munis d'opérations: Complété de Krasner.

C. R. Acad. Sci. Paris **256** (1963), 3937–3939.

The author claims to construct an extension of a partially ordered set. The reviewer is, however, unable to understand the construction, even taking into consideration the correction given in the paper reviewed below [#3948]. *G. Bruns* (Hamilton, Ont.)

Dokas, Lambros 3948

Sur certains complétés des ensembles ordonnés munis d'opérations: Rectifications à mes Notes précédentes.

C. R. Acad. Sci. Paris **258** (1964), 30–33.

Corrections of two previous papers [same C. R. **256** (1963), 2504–2506; MR **26** #4941; #3947 above]. The first extension of a partially ordered set E constructed by the author turns out to be the well-known MacNeille completion [Trans. Amer. Math. Soc. **42** (1937), 416–460]. The second extension can be described as follows. Define a cut in E as a pair (A, B) of disjoint subsets of E such that, for all $x \in E$, $x \in A$ if and only if $x<y$ for all $y \in B$ and $x \in B$ if and only if $x>y$ for all $y \in A$. Let $C_i(E)$ and $C_s(E)$ be the system of all lower classes and upper classes of cuts, respectively. Define $E' = E \cup C_i(E) \cup C_s(E)$ and identify in E' an element $e \in E$ with the lower class $(\leftarrow, e]$ and the upper class $[e, \rightarrow)$ if these belong to E'. Partially order the resulting system in a natural way. In the last part the author claims to correct the second of the above-mentioned papers. *G. Bruns* (Hamilton, Ont.)

Vuilleumier, Monique 3949

Théorèmes du type de Toeplitz-Schur dans l'ensemble ordonné des suites.

C. R. Acad. Sci. Paris **258** (1964), 1974–1975.

For any pre-order, $\leqq$, on a set E: an upper [lower] bound a of a subset X is called strict if there is no element b of X such that $a \leqq b$ and $b \leqq a$. The envelope of X is the set of lower bounds of the set of all upper bounds of X. X is called closed if $x \in X$ and $t \leqq x$ implies $t \in X$.

Consider the pre-orders between sequences of complex numbers: $x_n = O(y_n)$, with $(x_n) \asymp (y_n)$ as the corresponding equivalence relation; $x_n = o(y_n)$ or $(x_n) \sim (y_n)$, which is defined by $(x_n) \sim (y_n) \Leftrightarrow : \exists \lambda \neq 0 : \forall \varepsilon > 0\, \exists n_\varepsilon : n \geqq n_\varepsilon \Rightarrow |x_n - \lambda y_n| \leqq \varepsilon |y_n|$. Theorem: Let (s_n) be a sequence of complex numbers, Y a set of sequences with a common upper bound and (a_{nk}) an infinite complex matrix. (a) In order that $(\sum_{k=1}^{\infty} a_{nk} x_k) \in Y$ for all $(x_n) \sim (s_n)$ it is necessary that $(\sum_{k=1}^{\infty} |a_{nk} s_k|)$ belong to the O-envelope of Y. (b) In order that $(\sum_{k=1}^{\infty} a_{nk} x_k) \in Y$ for all $(x_n) \asymp (s_n)$ it is necessary that $(\sum_{k=1}^{\infty} |a_{nk} s_k|)$ be a strict lower bound of the set of all upper bounds of Y. (c) If Y is closed $(\sum_{k=1}^{\infty} |a_{nk} s_k|) \in Y$ implies $(\sum_{k=1}^{\infty} a_{nk} x_k) \in Y$ for all $(x_n) \asymp (s_n)$. *O. Pretzel* (Berlin)

GENERAL MATHEMATICAL SYSTEMS

Nöbauer, Wilfried; Philipp, Walter 3950

Die Einfachheit der mehrdimensionalen Funktionenalgebren.

Arch. Math. **15** (1964), 1–5.

On considère, sur un ensemble quelconque R, les opérations σ d'indice $n+1$, définies, avec les notations d'un précédent travail [Monatsh. Math. **66** (1962), 441–452; MR **26** #3633] par la relation

$$\forall \mathfrak{Q} \in R^n,\ \sigma(f, g_1, g_2, \cdots, g_n)\mathfrak{Q} = f(g_1\mathfrak{Q}, g_2\mathfrak{Q}, \cdots, g_n\mathfrak{Q}).$$

Alors, si $n>1$, l'ensemble de toutes les fonctions f à n dimensions sur R, définies par les σ est une algèbre simple. *A. Sade* (Marseille)

Wrona, Włodzimierz 3951

★Matematyka. Podstawowy wykład politechniczny. Część I [Mathematics. Fundamental polytechnical lectures. Part I].

Państwowe Wydawnictwo Naukowe, Warsaw, 1964. 567 pp. zl 60.00.

The book begins with a discussion of real numbers, inequalities, and induction. Chapter 2 is on analytic geometry and Chapter 3 is on elementary functions. Chapter 4 discusses limits, Chapter 5 is on differentiation, and Chapter 6 treats integration. The standard items of elementary calculus are covered. Finally, Chapter 7 is on probability theory. *R. Carroll* (New Brunswick, N.J.)

THEORY OF NUMBERS

See also 3994, 4110, 4558.

Shanks, Daniel 3952

★Solved and unsolved problems in number theory. Vol. I.

Spartan Books, Washington, D.C., 1962. ix + 229 *pp.* $7.50.

The title of Chapter I of this stimulating book is "From perfect numbers to the quadratic reciprocity law". Amongst the conjectures about prime numbers: Goldbach, twin primes (including the Hardy-Littlewood conjecture), Mersenne's primes.

Chapter II ("The underlying structure") begins with interesting information on Gauss and the *Disquisitiones arithmeticae* [J. Springer, Berlin, 1889]. On p. 82 occurs Conjecture 14 (Artin) which states, for example, that the number of primes $p \leq N$ for which 2 is a primitive root (mod p) is asymptotic to $\prod_p \{1-1/p(p-1)\}\pi(N)$. Here $\pi(N)$ is the number of primes less than N. The value of the constant $.37395\cdots$ is given to 40 decimal places (computed by J. W. Wrench, Jr.).

The last Chapter (III) is titled "Pythagoreanism and its many consequences". Amongst much interesting information in this chapter is Chebyshev's theorem (p. 145): $\int x^U(A+Bx^V)^W\,dx$ (where U, V, W are rational numbers) is integrable in terms of elementary functions if and only if $(U+1)/V$ or W or $(U+1)/V+W$ is an integer.

Euler's conjecture that an nth power is never equal to a sum of fewer than n nth powers is mentioned (p. 158). But a reference to his further conjecture that $A^n = B_1{}^n + \cdots + B_n{}^n$ always has a non-trivial solution in positive integers (proved only for $n \leq 5$) is missing. There is an account of Waring's problem (see, in particular, p. 211) but no reference to Mahler's theorem that the formula $g(k) = 2^k + [(\frac{3}{2})^k] - 2$ holds for $k > k_0$.

S. Chowla (University Park, Pa.)

Linnik, Ju. V. 3953

Five lectures on some topics in number theory and probability theory. (Russian. Hungarian and English summaries)

Magyar Tud. Akad. Mat. Kutató Int. Közl. **4** (1959), 225–258.

The lectures are on the following topics. (I) The variance method for solving certain binary additive problems; (II) Applications of the variance method, the divisor problem, the Hardy-Littlewood problem, and other problems; (III) Integer points on a sphere and Markov chains, and an analogue of the ergodic theorems for integral matrices; (IV) Some properties of infinitely divisible laws; (V) Contributions to the theory of big deviations for sums of independent random variables, and a problem on Fourier transforms.

Ayoub, Raymond 3954

★An introduction to the analytic theory of numbers.

Mathematical Surveys, No. 10.

American Mathematical Society, Providence, R.I., 1963. xiv + 379 *pp.* $10.20.

The scope of the book may be judged from the chapter headings: (I) Dirichlet's theorem on primes in an arithmetic progression; (II) Distribution of primes; (III) The theory of partitions; (IV) Waring's problem; (V) Dirichlet L-functions and the class number of quadratic fields; Appendix A (gamma function); Appendix B (functional equations).

Any book of modest size on such a vast subject as the analytic theory of numbers must be highly selective, and each reader will no doubt have his own view of the ideal selection within a given compass. No comment is offered here on the general plan, which must be largely arbitrary; but some questions may be raised on the organization of material within the chosen fields. The book is an "Introduction" and does not aim at reaching the frontiers of knowledge, but there is a certain unevenness in the balance between classical and more modern results in the various fields. Thus, the chapter on Waring's problem includes some of Vinogradov's work, but the chapter on distribution of primes does not include an elementary proof of the prime number theorem. This omission is naturally linked in the sophisticated reader's mind with the inclusion of several pages on "equivalences" between various theorems of prime number theory. Such equivalences were important and fascinating under the old classification of "elementary" and "transcendental", and may still serve as an unofficial guide to strategy in the selection and arrangement of proofs of the main theorems. But they lost most of their significance when the prime number theorem itself became "elementary"; and the reader for whom this book is indeed an introduction may wonder what logical meaning he is to attach to a formal statement that two theorems (involving no variable element) are equivalent, beyond the fact that both are true.

In order to cater for different classes of reader the author has adopted the policy of leading up to his final results by way of simpler ones. It may be thought that this has been carried too far in some places; but this is a matter of taste, and many readers will no doubt welcome the leisurely approach to the more comprehensive results, with the opportunity of breaking off before the end is reached.

At this point it must be stated, with regret, that the presentation is marred by some serious defects. There is a certain lack of economy, in that details are needlessly repeated in special cases when a general result might be quoted. Thus, on p. 19 and on p. 20, most of the calculation could be avoided by appeals to (I) Lemma 3.1. Again (pp. 102, 105) individual instances of Euler's identity are worked out without reference to the general result. This entails some bold expansions that really call for comment if they are written down at all; but they need not be written down if (I) Theorem 1.5 is quoted. But the main ground for criticism is the inordinately large number of inaccuracies of detail. These occur with depressing regularity throughout the book, and a comprehensive list is out of the question. But we note some typical instances. (Numbers refer to pages.)

4. Conditions for Euler's identity (Theorem 1.5) incorrectly stated. Counterexample: $f(n)$ multiplicative; $f(p) = -1$, $f(p^2) = 1 + p^{-2}$, $f(p^\nu) = 0$ $(\nu \geq 3)$. (Product $\prod(1+p^{-2})$ absolutely convergent, series $\sum f(n)$ not convergent.) Correct statement: identity holds if either side is convergent when every $f(.)$ in it is replaced by its modulus.

21. "there exists a region enclosing the real axis for $s > \sigma_0$ in which the series (23) converges uniformly." False unless the series converges at $s = \sigma_0$.

37. "n is a prime if and only if $n!+1$ is divisible by n."

62–64, Theorems 4.6, 4.7, 4.8. Inequalities $\log T \geqq |t| > 1$ should presumably read $T \geqq |t| > 1$. (See application on pp. 68–9.)

92. "The requisite properties (and more) have already been established in Corollary 2 of Theorem 4.2." Since this theorem precedes any consideration of zeros, this might suggest that the Hardy-Littlewood proof of the prime number theorem is free from such considerations. Admittedly, this impression is corrected by the preceding remark about regularity of $-\zeta'(s)/\zeta(s)$ for $\sigma \geqq 1$, $s \neq 1$, but the sentence quoted is in itself misleading.

98. The possibility $L=\infty$ is ignored.

106, Theorem 7.6. Definitions and identities should be for all $x \geqq 1$. Equivalence does not extend to all real x unless $f(x)$ and $g(x)$ are defined as 0 for $x<1$. (But the author is not the only delinquent here.)

129. Numerical values of $\pi(x)$ are taken from Gauss's works, and not from more recent (and presumably more accurate) sources.

153. "$H=\ldots=O(\sigma^{1/4})$." Apparently $O(..)$ must be (mis)read as meaning 'exactly of order ..'; for the application requires an inequality $H>\ldots$.

218, 222, 237–8, etc. Back-handed technique for convergence proofs. It is scarcely logical to conduct such proofs in terms of symbols that are not known to be meaningful until after the required convergence has been established. Nor is absolute convergence correctly expressed by saying that the infinite series or integral in question is "$O(1)$". Such techniques are acceptable when summand or integrand is non-negative (so that there is always a 'value', finite or infinite), but this is not the case here.

237. Fallacy with (first) mean-value theorem when the factor retained in the integrand is not of fixed sign.

238. Argument of doubtful validity as it stands, since "absolute convergence" (lines 8–9) refers, not to the multiple integral, but to a special arrangement as a repeated integral.

241. Meaningless group of symbols of the form $\lim_{n\to\infty} f(n)=g(n)$.

256, Theorem 8.1. False for $p=2$, $r \geqq 3$ (and the reference on p. 257 only covers the case $r=1$).

296. $L(1,\chi)$ written as product over p with no mention of convergence.

364. "the interchange of integration and summation being clearly justified." No mention of the condition $\sigma>1$; indeed it is (wrongly) stated later that the integral is "well behaved for $\sigma>0$".

365, (12). Variable y retained after integration over $-1 \leqq y \leqq 1$.

There are also some systematic peculiarities of style, such as: (1) incomplete statement of conditions in enunciations (as in Waring's problem, where the text may have to be searched for the current restriction on s), or incorrect statement (as in (I) Lemma 3.2, where continuity is not enough to make sense of the enunciation); (2) omission of modulus signs in several places; (3) a tendency to ignore questions of regularity (as on pp. 81–2, where several statements may be false or meaningless under the restriction $\gamma \geqq 0$, or even $\gamma>0$). The bibliography of 30 items, and the references scattered through the book, contain several mistakes in the spelling of authors' names and in the titles of books or papers.

In spite of its imperfections, however, the book has some positive merits. Complicated arguments are usually prefaced by clear and accurate explanations of the basic ideas, before ideas tend to become obscured by details. The subject matter includes topics not usually found in books in English, such as Hua's contribution to Waring's problem (which was overshadowed by Vinogradov's work but not rendered entirely obsolete), Rademacher's identity for the partition function in a self-contained treatment including proofs of the transformation formulae from first principles, and a connected account of various matters relating to the class number $h(d)$. The author does not always choose what might seem to be the simplest methods available, but it is clear from the problems and notes at the ends of the chapters that he is aware of the alternatives and has therefore made his choice deliberately. The generous collections of problems give the reader an opportunity of extending his knowledge and of picking up some interesting historical points.

{On p. 71, and again on p. 131, currency is given to the belief that the prime number theorem has been proved with error $O(xe^{-c\lambda(x)})$, where $\lambda(x)=(\log x)^{3/5}$. This result was claimed by I. M. Vinogradov and by N. M. Korobov in 1958, and has been widely quoted. So far as the reviewer is aware, however, no proof has been published, and various workers who have tried to reconstruct the details have had to be content with the less elegant result involving

$$\lambda_1(x) = (\log x)^{3/5}(\log\log x)^{-1/5}$$

in place of $\lambda(x)$. See the remarks on pp. 226–7 of A. Walfisz, *Weylsche Exponentialsummen in der neueren Zahlentheorie* [VEB Deutscher Verlag der Wiss., Berlin, 1963; author's reference 29, corrected]. It is highly desirable that the claim to the stronger and neater result should be substantiated or withdrawn without further delay.}

A. E. Ingham (Cambridge, England)

Delone, B. N.; Faddeev, D. K. 3955

★**The theory of irrationalities of the third degree.**
Translations of Mathematical Monographs, Vol. 10.
American Mathematical Society, Providence, R.I., 1964. xvi+509 *pp.* $11.10.

Most of this book (pp. 1–452) consists of a translation of a Russian monograph [Trudy Mat. Inst. Steklov. **11** (1940); MR **2**, 349]. As an unexpected bonus, the volume contains two translations not mentioned on the title-page or cover of the volume, namely, Delone's supplement to the Russian translation of Dirichlet's *Vorlesungen über Zahlentheorie* [Vieweg, Braunschweig, 1863; cf. pp. 370–403 of the Russian translation, Moscow, 1936] and ten pages of a paper of Delone and Faddeev on the geometry of Galois theory [Mat. Sb. (N.S.) **15 (57)** (1944), 243–284; MR **6**, 200].

Kubilius, J. 3956

★**Probabilistic methods in the theory of numbers.**
Translations of Mathematical Monographs, Vol. 11.
American Mathematical Society, Providence, R.I., 1964. xviii+182 *pp.* $8.60.

The original Russian was reviewed earlier [second, enlarged edition, Gosudarstv. Izdat. Politič. i Naučn. Lit. Litovsk. SSR, Vilna, 1962; MR **26** #3691].

Jacobson, Bernard 3957

Sums of distinct divisors and sums of distinct units.

Proc. Amer. Math. Soc. **15** (1964), 179–183.

For a given positive integer M, let $\alpha(M)$ be the number of positive integers n which can be written as the sum of a set of positive divisors of M, and let $\beta(M)$ denote the same thing if we also admit negative divisors. Obviously $\alpha(M) \leq \sigma(M)$, $\beta(M) \leq \sigma(M)$, where $\sigma(M)$ is the sum of all positive divisors of M. A number of questions concerning $\alpha(M)$ were considered by B. M. Stewart [Amer. J. Math. **76** (1954), 779–785; MR **16**, 336]; the present paper deals with similar questions about $\beta(M)$. The numbers M satisfying $\beta(M) = \sigma(M)$ are completely characterized as

$$M = 2^b 3^c \prod_{i=1}^{k} p_i^{t_i},$$

b and c not both zero, $k \geq 0$, and if $k > 0$, also $3 < p_i < p_j$ for $i < j$, $p_1 \leq 2\sigma(2^b 3^c) + 1$ and $p_{j+1} \leq 2\sigma(2^b 3^c \prod_{i=1}^{j} p_i^{t_i}) + 1$ for $1 \leq j \leq k-1$. Furthermore, it is shown that the numbers $\beta(M)/\sigma(M)$ lie everywhere dense in the interval $[0, 1]$. Finally the author proves that in the quadratic field $\mathrm{Ra}(\sqrt{2})$ every integer is the sum of a finite set of units. *N. G. de Bruijn* (Eindhoven)

Sierpiński, W. 3958

Sur une propriété des nombres naturels.

Elem. Math. **19** (1964), 27–29.

L'auteur donne une démonstration directe de la proposition suivante; tout nombre naturel est d'une infinité de manières une différence de deux nombres naturels dépourvus de diviseurs premiers carrés. L'auteur remarque que cette proposition est contenue dans le théorème de T. Nagell [Abh. Math. Sem. Hamburg Univ. **1** (1922), 179–194, p. 188]. *A. Schinzel* (Columbus, Ohio)

Stoneham, R. G. 3959

The reciprocals of integral powers of primes and normal numbers.

Proc. Amer. Math. Soc. **15** (1964), 200–208.

Let p be a prime, and let g be a primitive root mod p^2; then g is a primitive root mod p^n for each $n \geq 1$. The author considers the expansion of p^{-n} to the base g. The recurring period consists of $(p-1)p^{n-1}$ digits. Let j be one of the integers $1, \cdots, [n \log_g p]$, and let B_j be any sequence of j digits (a "digit" being one of the numbers $0, 1, \cdots, g-1$). The author investigates how often this sequence B_j occurs as a set of j consecutive digits in the recurring period. If j is given, this number hardly depends on the sequence: it equals $[p^n/g^j] - [p^{n-1}/g^j] + m$, where $|m|$ is at most 2. There are a number of corollaries which relate this fact to the concept of normal numbers. *N. G. de Bruijn* (Eindhoven)

Carlitz, L. 3960

A note on the Eulerian numbers.

Arch. Math. **14** (1963), 383–390.

The function $H_n(\lambda)$, defined by the identity

$$\frac{1-\lambda}{e^x - \lambda} = \sum_{n=0}^{\infty} H_n(\lambda) \frac{x^n}{n!},$$

is of the form

$$H_n(\lambda) = (\lambda - 1)^{-n} \sum_{s=1}^{n} A_{ns} \lambda^{s-1}.$$

The present note is devoted to the study of arithmetical properties of the integers A_{ns}. Denote, for example, by $\nu(n)$ the number of odd numbers among A_{ns} $(1 \leq s \leq n)$. It is shown that, if $n = 2^{e_1} + \cdots + 2^{e_k}$, where $0 \leq e_1 < \cdots < e_k$, then $\nu(2n+2) = 2^{k+1}$, $\nu(2n+1) = 2^k$. However, the greater part of the paper is taken up with the discussion of the behaviour of the A_{ns} (mod 3). The entire question is closely linked with the study of arithmetical properties of binomial coefficients. *L. Mirsky* (Sheffield)

Carlitz, L. 3961

Recurrences for the Bernoulli and Euler numbers.

J. Reine Angew. Math. **214/215** (1964), 184–191.

The Bernoulli numbers in the even suffix notation may be defined by the recurrence $B_0 = 1$, $\sum_{r=0}^{n-1} \binom{n}{r} B_r = 0$ $(n > 1)$. The author uses the Staudt-Clausen theorem to show the impossibility of a recurrence of the type $\sum_{r=0}^{k} A_r(n) B_{n-r} = A(n)$, where k is independent of n and $A_r(n)$, $A(n)$ are polynomials in n with integral coefficients. Similar results are proved for Bernoulli numbers of order t and for Euler numbers. *T. M. Apostol* (Pasadena, Calif.)

Carlitz, L. 3962

A note on the generalized Wilson's theorem.

Amer. Math. Monthly **71** (1964), 291–293.

Assume p is a prime > 5, $p^r | m$, $r \geq 2$, and let P_m denote the product of the integers $\leq m$ and prime to p. The author proves that

$$P_m \equiv ((p-1)!)^{m/p}(1 + mp^2 B_{p-3}/18) \pmod{p^{r+3}},$$

where B_{p-3} is a Bernoulli number in the even suffix notation. *T. M. Apostol* (Pasadena, Calif.)

Sokolov, N. P. 3963

On some multidimensional determinants with integral elements. (Russian)

Ukrain. Mat. Ž. **16** (1964), 126–132.

It is first shown that the sum of the 2^{p-1} determinants associated with a p-dimensional matrix with integral coefficients is divisible by 2^{p-1}. If n is the order of the matrix and each coefficient is the g.c.d. of its indices, that is, $A = [(i_1, \cdots, i_p)]$, it is shown that those determinants with some indices signant equal $\phi(1)\phi(2)\cdots\phi(n)$. This result extends Smith's identity $(p=2)$. When the coefficients are $(i_1, \cdots, i_p)^{r+1}$, $r = 0, 1, \cdots$, these determinants equal $\prod \phi_r(k)$, where $\phi_r(k) = k^{r+1} \prod (1 - p_i^{-r-1})$, $p_i | k$. This extends a result of Gyires [Publ. Math. Debrecen **5** (1957), 162–171; MR **19**, 731]. Finally, if $A = [a + d \sum (i_\alpha - 1)]$, where a and d are in any field and some indices are signant, then each of the determinants equals zero except for $n = 2$ with two indices signant, in which case one gets $-2^{p-2} d^2$. This extends Hankel's identity $(p=2)$. *B. Vinograde* (Ames, Iowa)

Cohen, Eckford 3964

Arithmetical notes. XII. A sequel to Note VI.

Norske Vid. Selsk. Forh. (Trondheim) **36** (1963), 10–15.

Part VI appeared in Michigan Math. J. **9** (1962), 277–282 [MR **25** #3897]. For positive integers m, n let $Q(m, n)$ denote the number of sets of ordered pairs $[x_1, x_2]$, $[y_1, y_2]$ of positive integers x_1, x_2, y_1, y_2 such that

$$(1)\quad m = x_1+y_1,\ n = x_2+y_2,\ (x_1, x_2) = (y_1, y_2) = 1.$$

The author proves the theorem: If $n \geqq m$, then

$$Q(m, n) = \alpha((m, n))mn + O(n \log^2 m) \text{ if } n \geqq m^2,$$
$$= \alpha((m, n))mn + O(mn^{1/2} \log^2 m) \text{ if } n < m^2.$$

Here

$$\alpha(t) = \prod_{p|t}\left(1-\frac{1}{p^2}\right)\prod_{p\nmid t}\left(1-\frac{2}{p^2}\right)$$

and (x, y) denotes, as usual, the greatest common divisor of x and y. *S. Chowla* (University Park, Pa.)

Cohen, Eckford 3965

A generalization of Axer's theorem and some of its applications.

Math. Nachr. **27** (1964), 163–177.

Let Z denote the multiplicative semi-group of the positive integers, and for a fixed $r \in Z$, let Z_r denote the subsemigroup of Z consisting of the integers relatively prime to $r(Z = Z_1)$. Let T_r denote the set of all functions from Z_r into the complex field and place $T = T_1$. A function f of T induces a function $f^{(r)}$ of T_r by placing $f^{(r)}(n) = f(n)$ for each $n \in T_r$. Let $\varphi(x, r)$ denote the number of n in Z_r not exceeding x. The relative mean value M_r of a function f of T_r is defined to be the limit (if it exists)

$$(1)\qquad M_r(f) = \lim_{x\to\infty}\frac{1}{\varphi(x, r)}\sum_{\substack{n\leq x\\ n\in Z_r}} f(n).$$

The relative norm N_r of f in T_r is defined by the equation (1) when we change M to N, replace lim by lim sup, and replace $f(n)$ by $|f(n)|$.

The author proves the theorem: Let f and g be functions of T_r related by $f(n) = \sum_{d|n} g(d)$, $n \in Z_r$, and suppose that g has relative mean value α_r and is of finite relative norm. Then

$$\lim_{x\to\infty}\left\{\frac{r}{x\phi(r)}\sum_{\substack{n\leq x\\ n\in Z_r}} f(n) - \sum_{\substack{n\leq x\\ n\in Z_r}}\frac{g(n)}{n}\right\} = -\frac{\alpha_r}{r}\{\Phi(r)+\beta\phi(r)\},$$

where $\beta = 1-\gamma = \Gamma'(2)$ and $\Phi(r)$ is defined for each r in Z by

$$\Phi(r) = \sum_{d\delta = r}\mu(d)\delta\log d.$$

The author cites the following example. Let $C(n)$ denote the number of solutions of the congruence

$$x_1^2+x_2^2+x_3^2-x_4^2-x_5^2-x_6^2 \equiv 1 \pmod n;$$

then the mean value of $C(n)/n^5$ over the odd integers exists and is equal to $96/\pi^4$.

S. Chowla (University Park, Pa.)

Spence, E. 3966

Formulae for sums involving a reduced set of residues modulo n.

Proc. Edinburgh Math. Soc. (2) **13** (1962/63), 347–349.

Let n be a positive integer >1 and m the square-free part of n. Let $1 = a_1 < a_2 < \cdots < a_{\phi(n)} = n-1$ be the $\phi(n)$ positive integers $\leq n$ which are relatively prime to n. The author proves that

$$\sum_{j=1}^{\phi(n)} ja_j = \frac{\phi(n)}{24}(8n\phi(n)+6n+2\phi(m)(-1)^{\omega(m)}-2^{\omega(m)}),$$

where $\omega(m)$ is the number of prime factors of m. Similar formulae for $\sum_{j=1}^{\phi(n)} j^2a_j$ and $\sum_{j=1}^{\phi(n)} ja_j^2$ are also obtained. The proof depends on evaluating the double sum $\sum_{j=1}^{\phi(n)}\sum_{i=1}^{\phi(n)}(a_{i+j}-a_j)^2$ in two different ways.

S. L. Segal (Rochester, N.Y.)

Bernstein, Leon 3967

Periodische Jacobische Algorithmen für eine unendliche Klasse algebraischer Irrationalzahlen vom Grade n und einige unendliche Klassen kubischer Irrationalzahlen.

J. Reine Angew. Math. **214/215** (1964), 76–83.

This paper is a German version of the author's English article entitled "Periodical continued fractions for irrationals of degree n by Jacobi's algorithm" [same J. **213** (1963), 31–38; MR **27** #5727].

E. Frank (Chicago, Ill.)

Graham, R. L. 3968

On finite sums of unit fractions.

Proc. London Math. Soc. (3) **14** (1964), 193–207.

This paper is concerned with the general question of when a given rational number can be written as the sum of reciprocals of a finite number of distinct integers taken from some given infinite sequence. The main theorem asserts that if $S = (s_1, s_2, s_3, \cdots)$ is a sequence of positive integers such that (a) every sufficiently large integer is the sum of a finite number of distinct products of the s_i and (b) s_{n+1}/s_n is bounded, then a rational p/q ($(p, q) = 1$) is the sum of a finite number of reciprocals of distinct products of s_i if and only if (c) p/q can be approximated arbitrarily closely by such sums and (d) q divides some finite product of the s_i.

Among the corollaries are the facts that p/q is a finite sum of reciprocals of distinct squares of integers if and only if it lies in $[0, \frac{1}{6}\pi^2-1) \cup [1, \frac{1}{6}\pi^2)$, and of distinct square-free integers if and only if q is square-free.

H. S. Wilf (Philadelphia, Pa.)

Aigner, Alexander 3969

Brüche als Summe von Stammbrüchen.

J. Reine Angew. Math. **214/215** (1964), 174–179.

The paper is concerned with the problem of representing a given fraction as the sum of r unit fractions, in other words, with the Diophantine equation

$$\frac{m}{n} = \frac{1}{x_1}+\cdots+\frac{1}{x_r}\qquad (x_i \geq 1),$$

where $m \geq 1$, $n \geq 1$, $(m, n) = 1$. The case $r = 3$ is of particular interest. L. Bernstein [same J. **211** (1962), 1–10; MR **26** #77] discussed the case $m = 4$. The present paper contains some criteria for the representation of a fraction of the type m/p, p prime, as a sum of three unit fractions; in particular, it is shown that the criteria are the same for primes in reciprocal residue classes (mod m), that is, $kp+1 \equiv 0$, $p+k \equiv 0 \pmod m$. The cases $m = 5, 6, 7$ are discussed at some length. It is shown that $5/p$ is not a

sum of three unit fractions only when the numbers $(p+1)/2$, $(p+2)/3$, $(2p+1)/3$ are products of primes of the form $10x+1$; in particular, $5/p$ is such a sum for $p<30000$. Finally, $6/p$ and $7/p$ are certainly sums of three unit fractions for $p<2000$ ($p>2$ when $m=7$).

L. Carlitz (Durham, N.C.)

Chowla, S. 3970

On a conjecture of Artin. I, II.

Norske Vid. Selsk. Forh. (*Trondheim*) **36** (1963), 135–141.

The author proves that there is an absolute constant c_1 such that if k is an odd prime, $s>c_1k\log k$, and $a_1, \cdots, a_s$ are rational integers, then the form $\sum_1^s a_ix_i^k=0$ has nontrivial solutions in each of the p-adic fields. The proof is based on ideas in an earlier paper [J. Indian Math. Soc. (N.S.) **25** (1961), 47–48; MR **25** #3893]. The author also states (but does not prove) that there is an absolute constant c_2 such that if k is an odd prime, $s>c_2k\log k$, $a_1, \cdots, a_s$ are rational integers, then $\sum a_ix_i^k=0$ has a nontrivial rational solution. Davenport and the reviewer have shown this result to hold for all integers $k\geqq 18$ when $s\geqq k^2+1$ [Proc. Roy. Soc. Ser. A **274** (1963), 443–460; MR **27** #3617].

D. J. Lewis (Ann Arbor, Mich.)

Chowla, S.; Shimura, G. 3971

On the representation of zero by a linear combination of k-th powers.

Norske Vid. Selsk. Forh. (*Trondheim*) **36** (1963), 169–176.

Let $\Gamma^*(k)$ be the minimum integer s such that for every sequence $a_1, \cdots, a_s$ of rational integers the equation $\sum_{i=1}^s a_ix_i^k=0$ has nontrivial solutions in each of the p-adic fields. Davenport and the reviewer [Proc. Roy. Soc. Ser. A **274** (1963), 443–460; MR **27** #3617] have shown that $\Gamma^*(k)\leqq k^2+1$, and it is well known that there are infinitely many k for which $\Gamma^*(k)=k^2+1$. The actual value of $\Gamma^*(k)$ depends in a complicated way on the arithmetic structure of k.

The authors prove that if k is an odd integer, then $\Gamma^*(k)<(2/\log 2+\varepsilon)k\log k$, provided $k>k_0(\varepsilon)$. Also, there are infinitely many odd integers k such that $\Gamma^*(k)>k\log k/\log 2$. The proofs rely on the box principle together with the usual Hensel-Newton method for obtaining solutions in p-adic fields.

D. J. Lewis (Ann Arbor, Mich.)

Chowla, S.; Walum, H. 3972

On the divisor problem.

Norske Vid. Selsk. Forh. (*Trondheim*) **36** (1963), 127–134.

Let $[x]$ be the greatest integer in x, and let $\psi(x)=x-[x]-\frac{1}{2}$. The well-known conjecture that the error term in Dirichlet's divisor problem is $O(x^{1/4+\varepsilon})$ for every $\varepsilon>0$ is, as is well known, equivalent to $\sum_{n\leq\sqrt{x}}\psi(x/n)=O(x^{1/4+\varepsilon})$ for every $\varepsilon>0$. The authors prove that

$$\sum_{n\leq\sqrt{x}} n\left(\psi^2\left(\frac{x}{n}\right)-\frac{1}{12}\right) = O(x^{3/4}).$$

This leads them to the following conjecture. For $0<x\leqq 1$, let $g_1(x)=x-\frac{1}{2}$, $g_2(x)=x^2-x+\frac{1}{6}$, $\cdots$, where $g_r(x)$ is determined recursively by the conditions $g_r'(x)=rg_{r-1}(x)$ and $\int_0^1 g_r(x)\,dx=0$, and define $g_r(x+1)=g_r(x)$. Let $G_{a,r}(x)=\sum_{n\leq\sqrt{x}}n^ag_r(f(x/n))$ where a is a non-negative integer and $f(x)$ is the fractional part of x. (This is clearly the intended definition of $G_{a,r}(x)$; in the text ψ is written for f.) The authors then conjecture that $G_{a,r}(x)=O(x^{a/2+1/4+\varepsilon})$ for every $\varepsilon>0$.

It will be observed that the result of this paper is somewhat stronger in the case $a=1$, $r=2$, while the divisor problem conjecture is the case $a=0$, $r=1$. The work of Hardy and Ingham on the divisor problem shows that the ε cannot be omitted in this latter case of the conjecture.

There are several minor misprints in addition to the one noted above.

S. L. Segal (Rochester, N.Y.)

Burgess, D. A. 3973

A note on L-functions.

J. London Math. Soc. **39** (1964), 103–108.

The author [Proc. London Math. Soc. (3) **12** (1962), 193–206; MR **24** #A2570] has shown that if $L_\chi(s)$ is the Dirichlet L-function belonging to the non-principal character χ (mod p), t is a fixed real number and ε is an arbitrary fixed positive real number, then $L_\chi(\frac{1}{2}+it)\ll k^{3/16+\varepsilon}$. In this paper the author shows that when k is a prime, then k^ε may be replaced by $\log k$. The proof relies on further refinements of the author's estimates on the size of exponential sums.

D. J. Lewis (Ann Arbor, Mich.)

Lang, Serge 3974

★Algebraic numbers.

Addison-Wesley Publishing Co., Inc., Reading, Mass.-Palo Alto-London, 1964. ix+163 *pp*. $7.00.

We list the titles of the ten chapters: Algebraic Integers, Completions, The Different and Discriminant, Cyclotomic Fields, Parallelotopes, Ideles and Adeles, Functional Equation, Density of Primes and Tauberian Theorem, The Brauer-Siegel Theorem, Explicit Formulas.

To illustrate the highly refreshing style of the author we quote the following from the section on Minkowski's constant (pp. 78–81): "I copied the following table of values for the Minkowski constant in a course of Artin 12 years ago. . . . We conclude by an example of which Artin was very fond. Consider the equation $f(X)=X^5-X+1\ldots.$ Let α be a root of $f(X)$ and $k=Q(\alpha)$ . . . and (oh, miracle!) every ideal is principal." Chapter 8, "Density of Primes", contains general theorems containing as special cases the theorems of Hecke on equidistribution of ideals and primes in sectors of circles. The statement on page 131 that the numbers $\{\log p\}$, p all primes, and $\log n$, n all natural numbers, are "equidistributed" is false if this means "uniform distribution" in the sense of H. Weyl.

Chapter 9 contains a proof of the Brauer-Siegel theorem: If k ranges over a sequence of number fields Galois over Q of degree N and absolute value of the discriminant d, such that $N/\log d$ tends to zero, then we have (h is the class-number of k over Q)

$$\log(hR)\sim\log d^{1/2}.$$

{A theorem of Ankeny, Brauer and the reviewer [Amer. J. Math. **78** (1956), 51–61; MR **18**, 565] says essentially that this theorem cannot be improved. Also, sharper estimates of h when k is the cyclotomic field are available

[Ankeny and Chowla, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 529–532; MR **11**, 230].}

S. Chowla (University Park, Pa.)

Hmyrova, N. A. 3975

On polynomials with small prime divisors. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 1268–1271.

Let $f(y)$ denote a polynomial of the form

$$y^n + a_1 y^{n-1} + \cdots + a_n$$

with integral coefficients. Denote by $F_f(x, z)$ the number of $m \leqq x$ such that $p|f(m)$ implies $p \leqq z \leqq x$. Here $\alpha = \log z/\log x$. If

$$\frac{\log\log x}{\log x} \leqq \alpha \leqq 1,$$

the author obtains the estimate

$$F_f(x, z) \leqq c(K)x \exp\left(-\frac{1}{4\alpha}\log\frac{1}{\alpha}\right),$$

where $c(K)$ is a constant depending only on the normal field K generated by the roots of f.

The literature cited includes references to papers by I. M. Vinogradov, Ju. V. Linnik, A. A. Buchstab, N. G. de Bruijn, the reviewer and W. E. Briggs, A. I. Vinogradov. One might add also the reviewer and T. Vijayaraghavan, V. Ramaswami. *S. Chowla* (University Park, Pa.)

Chandrasekharan, K.; Narasimhan, Raghavan 3976

On the mean value of the error term for a class of arithmetical functions.

Acta Math. **112** (1964), 41–67.

The authors continue their study of functional equations which formally resemble the functional equation of the Riemann ζ-function. Earlier papers dealt with Ω- and O-results [Acta Arith. **6** (1960/61), 487–503; MR **23** #A3719; Ann. of Math. (2) **76** (1962), 93–136; MR **25** #3911] and approximate functional equations [Math. Ann. **152** (1963), 30–64; MR **27** #3605]. In this paper corresponding problems about the mean value of error terms are considered.

Let $s = \sigma + it$. Suppose $\sum_{n=1}^\infty a_n \lambda_n^{-s}$ $(0 < \lambda_1 < \cdots < \lambda_n \to \infty$, not all a_n are 0) is a Dirichlet series absolutely convergent in some half-plane where it represents a meromorphic function $\phi(s)$. Let

$$E(y) = \sum_{\lambda_n \leqq x}' a_n - \frac{1}{2\pi i}\int_C \frac{x^s}{s}\phi(s)\,ds,$$

where C is a simple closed path containing all the singularities of the integrand in its interior and $\sum'$ means that the last term is to be halved if $\lambda_n = x$. The authors obtain estimates for $\int_1^x (E(y))^2\,dy$ when $\phi(s)$ satisfies the same general conditions as in the last two papers cited above when these satisfy further numerical restrictions. The general conditions are: Suppose $\sum_{n=1}^\infty b_n \mu_n^{-s}$ $(0 < \mu_1 < \cdots < \mu_n \to \infty$, not all b_n are 0) is also absolutely convergent in some half-plane where it represents a function $\psi(s)$. Let δ be real. Let $\Delta(s) = \prod_{\nu=1}^N \Gamma(\alpha_\nu s + \beta_\nu)$ where $N \geqq 1$, β_ν is complex and $\alpha_\nu > 0$. The functional equation $\Delta(s)\phi(s) = \Delta(\delta - s)\psi(\delta - s)$ is said to hold if there exists a domain D which is the exterior of some closed bounded set S in which there exists a holomorphic function χ such that $\lim_{|t|\to\infty} \chi(\sigma + it) = 0$ uniformly in every interval $-\infty < \sigma_1 \leqq \sigma \leqq \sigma_2 < \infty$ and $\chi(s) = \Delta(s)\phi(s)$ for $\sigma > c_1$, $\chi(s) = \Delta(\delta - s)\psi(\delta - s)$ for $\sigma < c_2$, where c_1 and c_2 are constants.

The estimates for $\int_1^x (E(y))^2\,dy$ are obtained subject to the additional conditions: $\delta > 0$, $\sum_{\nu=1}^N \alpha_\nu = A \geqq 1$, $\lambda_n = c''n$, $\mu_n = c'n$, where c'' and c' are constants, and also certain order conditions are imposed on $\sum_{\mu_n \leqq x} b_n^2$ and sometimes also on $\sum_{\lambda_n \leqq x} a_n^2$.

When specialized to particular $\phi(s)$ the estimates lead sometimes to new results, sometimes to the best known old ones, and sometimes to slightly poorer than the best known results. In particular, the results for $\zeta_K(s, \mathscr{L})$, the Dedekind ζ-functions for the ideal class $\mathscr{L}$ in the algebraic number field K of degree n over the rationals, are new.

There are several minor misprints.

S. L. Segal (Rochester, N.Y.)

Bulota, K. 3977

An approximate functional equation for the Hecke ζ-function for an imaginary quadratic field. (Russian. Lithuanian and English summaries)

Litovsk. Mat. Sb. **2** (1962), *no.* 2, 39–82.

Let K be a quadratic imaginary field, $\mathfrak{K}$ a class of ideals of the field K, $\mathfrak{m} \neq 0$ an integer ideal, $\Xi(\mathfrak{a})$ Grössencharakter of Hecke for ideals mod $\mathfrak{m}$ with exponent m [Math. Z. **6** (1920), 11–51], $N(\mathfrak{a})$ the norm of the ideal $\mathfrak{a}$. The author considers the zeta-function of Hecke, defined for Re $s > 1$ by the Dirichlet series

$$Z(s, \Xi, \mathfrak{K}) = \sum_{\mathfrak{a}\in\mathfrak{K}} \Xi(\mathfrak{a})N(\mathfrak{a})^{-s},$$

where $\mathfrak{a}$ runs over all integer ideals of $\mathfrak{K}$. Hecke has proved the analytic continuation of $Z(s, \Xi, \mathfrak{K})$ over the whole plane of s and the functional equation

$$Z(s, \Xi, \mathfrak{K}) = \psi(s, \Xi)Z(1-s, \bar{\Xi}, \mathfrak{K}^*),$$

where $\bar{\Xi}$ denotes the complex conjugate character, $\mathfrak{K}^*\mathfrak{K}$ is the class of ideals which contains $\mathfrak{m}\mathfrak{o}$, $\mathfrak{o}$ is the different of K, $\psi(s, \Xi)$ the gamma factor. The author proves the approximate functional equation for $Z(s, \Xi, \mathfrak{K})$: if x and y are sufficiently large positive numbers, H a fixed positive number, $xy = (dN(\mathfrak{m}))^2|\mathrm{Im}\, s|/2\pi$, d the discriminant of K, then for $|\mathrm{Re}\, s| < H$

$$Z(s, \Xi, \mathfrak{K}) = \sum_{\substack{\mathfrak{a}\in\mathfrak{K}\\ N(\mathfrak{a})<x}} \Xi(\mathfrak{a})N(\mathfrak{a})^{-s} + \psi(s, \Xi)\sum_{\substack{\mathfrak{b}\in\mathfrak{K}^*\\ N(\mathfrak{b})<y}} \bar{\Xi}(\mathfrak{b})N(\mathfrak{b})^{s-1} + O(R).$$

He gives some expressions for R, e.g., if $x = y$, $m = O(|\mathrm{Im}\, s|)$, then $R = x^{1/2-\mathrm{Re}\, s}(\ln x)^{1/2}$.

J. Kubilius (Vilnius)

Rankin, R. A. 3978

The difference between consecutive prime numbers. V.

Proc. Edinburgh Math. Soc. (2) **13** (1962/63), 331–332.

Part IV appeared in Proc. Amer. Math. Soc. **1** (1950), 143–150 [MR **11**, 644]. Let p_n denote the nth prime. In a previous paper [J. London Math. Soc. **13** (1938), 242–247] the author showed that

$$p_{n+1} - p_n > A\log p_n(\log_2 p_n)(\log_4 p_n)(\log_3 p_n)^{-2}$$

infinitely often, where $\log_2 = \log\log$, $\log_3 = \log\log_2$, $\log_4 = \log\log_3$, and A is any positive constant $< \frac{1}{3}$. Recently

A. Schönhage proved [Arch. Math. **14** (1963), 29–30; MR **26** #3680] that A may be any constant $< \frac{1}{2}e^{\gamma}$ (where γ is Euler's constant). The author shows now that $\frac{1}{2}e^{\gamma}$ can be replaced by e^{γ}. He uses estimates of the reviewer [Nederl. Akad. Wetensch. Proc. Ser. A **54** (1951), 50–60; MR **13**, 724] for the number of integers $\leq x$ which are composed of primes $\leq y$ only.

N. G. de Bruijn (Eindhoven)

Kogan, L. A. 3979

On the number of representations of an integer by the quadratic forms $x^2+y^2+z^2+pt^2$, $x^2+p(y^2+z^2+t^2)$. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. **1960**, *no.* 6, 24–33.

J. Liouville gave without proof explicit formulae for the number of representations of an integer n by the quadratic forms $x^2+y^2+z^2+5t^2$, $x^2+5(y^2+z^2+t^2)$, $x^2+y^2+2z^2+2zt+3t^2$, $5(x^2+y^2)+2z^2+2zt+3t^2$. Later writers supplied proofs involving the theory of elliptic functions. The present author gives a simpler proof.

{Reviewer's remark: In formula (14) x^2 in the exponent should be replaced by $2x^2$.}

S. Knapowski (Poznań)

Hahn, Hwa S. 3980

On the relative growth of differences of partition functions.

Pacific J. Math. **14** (1964), 93–106.

Let A be an arbitrary set of positive integers; let $p(n)$ denote the number of partitions of n into parts taken from A, repetitions being allowed. If $k>0$, then $p^{(k)}(n)$ denotes the kth difference, and if $k \leq 0$ it is the summatory function of order $-k$. Thus, for all integers k,

$$f_k(X) = \sum_{n=0}^{\infty} p^{(k)}(n)X^n = (1-X)^k \prod_{a\in A} (1-X^a)^{-1}.$$

It was conjectured by Bateman and Erdős [Mathematika **3** (1956), 1–14; MR **18**, 195] that $p^{(k+1)}(n)/p^{(k)}(n) = O(n^{-1/2})$ (if k is fixed, and if the elements have g.c.d. unity, even if any arbitrary set of k elements is removed from A). The author assumes stronger conditions (as introduced by Roth and Szekeres [Quart. J. Math. Oxford Ser. (2) **5** (1954), 241–259; MR **16**, 797]), namely, that $n(u)=u^{\alpha+o(1)}$ ($n(u)$ is the number of $a\in A$ with $a\leq u$; α is a constant, $0<\alpha\leq 1$), and

$$(\log m)^{-1} \inf \sum_{i=1}^{m} \|\beta a_i\|^2 \to \infty \qquad (m\to\infty).$$

Here $\|x\|$ is the distance of x to the nearest integer, and the infimum is taken over all β with $(2a_m)^{-1}<\beta\leq\frac{1}{2}$.

Under these conditions it is proved that the Bateman-Erdős conjecture holds, and even that $p^{(k+1)}(n)/p^{k(n)} \sim \sigma_n$, where σ_n is the solution of $n=\sum_{a\in A} a(e^{\sigma a}-1)^{-1}$.

Under the further assumption that $n(u)\sim u^{\alpha}L(u)$, where $0<\alpha\leq 1$ and L is slowly oscillating in the sense of Karamata, it is proved that

$$p^{(k+1)}(n)/p^{(k)}(n) \sim n^{-1/(1+\alpha)}L_1(n),$$

where L_1 is again slowly oscillating.

N. G. de Bruijn (Eindhoven)

Cheema, M. S.; Gordon, Basil 3981

Some remarks on two- and three-line partitions.

Duke Math. J. **31** (1964), 267–273.

A k-line partition of the positive integer n is a representation of n in the form $n=\sum_{i=1}^{k}\sum_{j=1}^{\infty} a_{i,j}$, where $a_{i,j}$ are non-negative integers satisfying $a_{i,j}\geq a_{i,j+1}$ and $a_{i,j}\geq a_{i+1,j}$; the number of such partitions of n is denoted by $t_k(n)$. It is known that $\sum_{n=0}^{\infty} t_k(n)x^n = F_k(x)G_k(x)$, where $F_k(x)=\prod_{m=1}^{\infty}(1-x^m)^{-k}$ and $G_k(x)=\prod_{m=1}^{k-1}(1-x^m)^{k-m}$ [P. A. MacMahon. *Combinatory analysis*, Vol. 2, pp. 171–245, Cambridge Univ. Press, London, 1916]. The authors give a purely combinatorial proof of this identity for $k=2$ and use it to derive various identities involving $t_k(n)$ and the coefficients of $F_k(x)$, as well as deriving the congruences $t_2(n)\equiv 0 \pmod 5$ if $n\equiv 3$ or $4 \pmod 5$ and $t_3(n)\equiv 0 \pmod 3$ if $n\equiv 2 \pmod 3$. A table of values of $t_2(n)$ and $t_3(n)$ is given for $n\leq 50$.

H. W. Brinkmann (Swarthmore, Pa.)

Jacobowitz, Ronald 3982

Multiplicativity of the local Hilbert symbol.

Pacific J. Math. **14** (1964), 187–190.

The author gives a proof of the second inequality of local class field theory in the case of quadratic extensions by employing essentially the same method as H. Hasse [J. Fac. Sci. Univ. Tokyo Sect. I **2** (1934), 477–498].

T. Tamagawa (New Haven, Conn.)

Kuroda, Shigenobu 3983

On a theorem of Minkowski. (Japanese)

Sûgaku **14** (1962/63), 171–172.

Using a theorem of Minkowski, the author proves the following theorem: Let k be a number field of degree m over the rational field such that $|D_k| < m^{2m}\pi^m/(2m)!$ for the discriminant D_k. Then there exists no non-trivial unramified extension over k. He also shows that if the class number of k is 1 (namely, if k has no non-trivial unramified abelian extension), then the condition on D_k can be replaced by $|D_k| < (15m)^{2m}\pi^m/((30m)!)^{1/15}$. Applying the result, the author finds 22 algebraic number fields of degree $m\leq 5$ which have no non-trivial unramified extensions.

K. Iwasawa (Cambridge, Mass.)

Kondo, Takeshi 3984

On abelian extensions of the Gaussian field. (Japanese)

Sûgaku **15** (1963), 110.

The author proves the following result, correcting an argument in a paper by T. Takagi [J. College Sci. Univ. Tôkyô **19** (1903), article 5]: Let q be an odd prime. Let K be a number field such that (i) K is a non-abelian Galois extension of the rational field Q, (ii) K is an abelian extension of type (q,q) over the Gaussian field $Q(i)$, $i^2=-1$, and (iii) K has a subfield which is an abelian extension of degree $2q$ over Q. Then K also contains a subfield which is a non-Galois extension of degree $2q$ over Q. The proof is purely group-theoretical so that it applies to any Galois extension of the same type as K/Q.

K. Iwasawa (Cambridge, Mass.)

Koch, Helmut 3985

Über Darstellungsräume und die Struktur der multiplikativen Gruppe eines p-adischen Zahlkörpers.

Math. Nachr. **26** (1963), 67–100.

Let G be a (finite) group. Let M be a representation module of G over a field k, and let χ be a linear character of G with values in k. The author considers a non-degenerate, anti-symmetric, k-linear pairing $[a, b]$ of $M \times M$ into k such that $[\sigma a, \sigma b] = \chi(\sigma)[a, b]$, $\sigma \in G$. A representation module associated with such a pairing is called a representation space of G (relative to the character χ).

The author first considers the representation spaces of finite abelian groups. He then studies the case where k is the prime field of characteristic $p \neq 2$, and G is a finite group of order ef $(e, f \geq 1)$, generated by two elements σ and τ satisfying $\sigma^f = \tau^e = 1$, $\sigma^{-1}\tau\sigma = \tau^q$, q being a power of p with $q^f \equiv 1 \pmod e$. He proves in particular the following result: Let M be a representation space of G over k relative to a character χ such that M is, as a representation module, the direct sum of m copies $(m \geq 1)$ of the group ring of G over k, and such that M is also the direct sum of two isotropic (i.e., self-orthogonal) submodules. Then the structure of such a representation space M is uniquely determined by χ and m: $M = M(\chi, m)$. Furthermore, the structure of $M(\chi, m)$ can be simply described if m is even.

Now, let F be an extension of degree m over the p-adic number field $(p \neq 2)$, and let K be a finite, tamely ramified, Galois extension of F, containing a pth root of unity $\zeta \neq 1$. The Galois group G of K/F acts on $M = K^*/K^{*p}$ in the obvious manner so that M becomes a representation module of G over the prime field k of characteristic p. Let χ be the character of G in k defined by $\zeta^\sigma = \zeta^{\chi(\sigma)}$, $\sigma \in G$. Then Hilbert's norm residue symbol in K for the power p defines a pairing on M and makes M a representation space relative to χ. In the main theorem of the paper, the author proves that the representation space M is the orthogonal sum of $M(\chi, m)$ and a two-dimensional subspace with a basis a, b such that $[a, b] = 1$, $\sigma a = a$, $\sigma b = \chi(\sigma)b$ for $\sigma \in G$. *K. Iwasawa* (Cambridge, Mass.)

FIELDS AND POLYNOMIALS

See also 3985, 4042, 4672.

Ax, James 3986

Zeroes of polynomials over finite fields.

Amer. J. Math. **86** (1964), 255–261.

Using some ideas of Dwork, the author proves the following theorem: Let $F = F(X_1, \cdots, X_n)$ be a polynomial of degree d over a finite field k of characteristic p with q elements, and let $N(F)$ denote the number of solutions of $F(X_1, \cdots, X_n) = 0$ in k. Let a be the largest integer strictly less than n/d. Then $N(F)$ is divisible by q^a. The theorem includes the well-known result of Chevalley and Warning (the divisibility of $N(F)$ by p when $n > d$), to which the author also gives a new simple proof independently. Examples of $F(X_1, \cdots, X_n)$ are given such that q^a is the exact power of p dividing $N(F)$.

The author also notes that the above theorem is equivalent to the following: Let $Z(t)$ denote the zeta-function of the hypersurface defined over k by $F(X_1, \cdots, X_n) = 0$. Then each zero and each pole of $Z(t)$ has p-adic valuation at least q^a. Here the zeros and the poles are considered as elements of an algebraic extension of the p-adic number field, and the valuation is normalized so that $|p| = p^{-1}$.

K. Iwasawa (Cambridge, Mass.)

Barbilian, D.; Radu, N. 3987

Résolution abstractée par des radicaux. (Romanian. Russian and French summaries)

Acad. R. P. Romîne Stud. Cerc. Mat. **13** (1962), 377–418.

This paper presents a generalization of the classical theory of solvability of algebraic equations by radicals. First, a general notion of group derivation is defined: This is an operation Δ which, to every group G of a certain class of groups, assigns a subgroup ΔG of G belonging to the same class and which is submitted to certain conditions (for instance, if Ω is any homomorphism of G, $\Omega\Delta G \subset \Delta\Omega G$). A special case of such a Δ is the usual derivation (ΔG is the commutator group of G); other examples are given. One defines in the obvious way Δ-abelian and Δ-solvable (in the paper: "Δ-metabelian") groups. Second, a general notion of radical in fields is introduced, the precise definition of which is not quite clear to the reviewer (from the authors' own summary: "La notion de radical est introduite . . . comme une fonction dont le domaine de définition est constitué par un élément unique, qui est le sous-ensemble d'un corps de la classe donnée ayant la valeur respective dans une extension du corps qui se trouve dans la classe considérée . . ."); anyway, a certain given group derivation Δ plays a role in that definition. Finally, a theorem is proved to the effect that, under additional conditions on Δ, if a set X of quantities is expressible in terms of (generalized) radicals over a field K, the Galois group of $K(X)/K$ is Δ-solvable. Part of the content of this paper has been given in a less detailed form in three earlier notes of the first author [Acad. R. P. Române Bul. Şti. Secţ. Şti. Mat. Fiz. **2** (1950), 769–776; ibid. **3** (1951), 75–86; ibid. **3** (1951), 97–106]. *J. L. Tits* (Bonn)

Fröhlich, A. 3988

The module structure of Kummer extensions over Dedekind domains.

J. Reine Angew. Math. **209** (1962), 39–53.

Le présent travail est une description élégante (grâce, en particulier, à l'emploi du langage cohomologique) et l'étude approfondie des algèbres kummériennes (c'est-à-dire algèbres galoisiennes à groupe abélien) sur le corps quotient d'un anneau dédekindien. Dans ce but, l'auteur introduit des invariants de nature arithmétique, dont certains étaient plus ou moins connus dans le cas classique, où l'algèbre considérée est un surcorps d'un corps (de degré fini) de nombres algébriques (bien que, même dans ce cas, le point de vue de l'auteur clarifie leur rôle) et d'autres semblent tout-à-fait nouveaux. Le résultat le plus remarquable de ce travail est la solution, en termes des invariants mentionnés, du problème de l'existence d'une base normale de l'anneau des entiers de l'algèbre considérée (c'est-à-dire de ses éléments entiers par rapport à l'anneau dédekindien mentionné).

Une algèbre galoisienne sur un corps K est une K-algèbre unitaire Λ de rang fini, possédant un groupe G de K-automorphismes, et isomorphe, en tant que $K(G)$-module (où $K(G)$ est l'anneau du groupe G sur K), à $K(G)$. Elle est dite kummérienne si: (a) son groupe G est abélien; (b) le corps de base K contient les racines n-ièmes de l'unité, où n est quelque multiple de l'exposant de G; (c) l'algèbre est commutative et semi-simple (ce qui entraîne que l'ordre de G est premier à la caractéristique de K).

Soit $X = X(\Lambda/K)$ le groupe des caractères de G qu'on dénotera aussi $G(\Lambda/K)$. Si $\chi \in X$, soit Λ^χ l'ensemble des $\alpha \in \Lambda$ tels que, pour tout $\sigma \in G$, on ait $\alpha^\sigma = \alpha\chi(\sigma)$. Alors, Λ est la somme directe $\bigoplus_\chi \Lambda^\chi$ des Λ^χ et Λ^χ est un K-module cyclique $\Lambda_\chi = K\alpha_\chi$. On a $\alpha_\chi\alpha_\psi = \alpha_{\chi\psi}a_{\chi,\psi}$, où le système des $a_{\chi,\psi}$ est, visiblement, un cocycle de dimension 2 de X dans K^* (où K^* est le groupe multiplicatif de K) qui définit un élément $a_{\Lambda/K}$ du 2-ième groupe de cohomologie $H^2(X, K^*)$ de X dans K^*, indépendant du choix des α_χ et dit invariant de Hasse de Λ/K. Cet invariant algébrique définit Λ/K à l'isomorphie près.

o étant un anneau dédekindien de K, dont K est le corps quotient, soit O l'ordre maximal de Λ correspondant à o, et soit, pour $\chi \in X$, $O^\chi = O \cap \Lambda^\chi$. $\tilde{O} = \bigoplus_\chi O^\chi$ est dit l'ordre kummérien de Λ/K. Si $\mathfrak{p}$ est un idéal premier de o et si $o_\mathfrak{p}$ est l'anneau de valuation du corps local correspondant $K_\mathfrak{p}$, posons, pour tout o-module M, $M_\mathfrak{p} = o_\mathfrak{p} \otimes_o M$ (en particulier, si M est un K-module, on a aussi $M_\mathfrak{p} = K_\mathfrak{p} \otimes_K M$). On voit que $\Lambda_\mathfrak{p}$ s'identifie canoniquement avec $K_\mathfrak{p}(G)$, ce qui identifie canoniquement chaque $\chi \in X$ avec un caractère $\chi_\mathfrak{p} \in X(\Lambda_\mathfrak{p}/K_\mathfrak{p})$. On a $O(\Lambda_\mathfrak{p}) = O_\mathfrak{p}$ et $(O^\chi)_\mathfrak{p} = (O_\mathfrak{p})^{\chi_\mathfrak{p}}$, d'où résulte $\tilde{O}(\Lambda_\mathfrak{p}) = \tilde{O}_\mathfrak{p}$.

On a $O_\mathfrak{p}^\chi = o_\mathfrak{p}\beta_{\chi,\mathfrak{p}}$, où $\beta_{\chi,\mathfrak{p}}$ est défini modulo le groupe multiplicatif $U_\mathfrak{p}$ des unités de $K_\mathfrak{p}$. On a, donc, $\beta_{\chi,\mathfrak{p}}\beta_{\psi,\mathfrak{p}} = \beta_{\chi\psi,\mathfrak{p}}b_{\chi,\psi,\mathfrak{p}}$, où $b_{\chi,\psi,\mathfrak{p}} \in K_\mathfrak{p}^* \cap o_\mathfrak{p}$. Ainsi, $\{b_{\chi,\psi,\mathfrak{p}}\}$ est (pour un $\mathfrak{p}$ fixé) un cocycle $\in C^{(2)}(X, K_\mathfrak{p})$, qui définit, indépendamment du choix des $\beta_{\chi,\mathfrak{p}}$, un élément $b_\mathfrak{p}$ de

$$C^2(X, K_\mathfrak{p}^*)/B^2(X, U_\mathfrak{p}),$$

où $B^2(\Gamma, A)$ est le groupe des cobords de dimension 2 du groupe Γ dans un Γ-module A. Puisque les α_χ sont les éléments d'une $f(\chi) = (f(\chi)_\mathfrak{p})$ définit, quand on le considère comme fonction de χ, un élément de $\mathfrak{M}(X, J_K)/\mathfrak{M}(X, K^*)\mathfrak{M}_0(X, U_K)$, qui sera dénoté $\tilde{N}_{\Lambda/K}$ et qui est un invariant arithmétique plus fin que $N_{\Lambda/K} = \{N_\chi; \chi \in X\}$, car ce dernier invariant est l'élément de $\mathfrak{M}(X, J_K)/\mathfrak{M}(X, K^*)\mathfrak{M}(X, U_K)$ défini par le même idèle $f(\chi)$. L'auteur prouve que $O = o(G)$ (autrement dit que l' "anneau des entiers" O de Λ possède une base normale sur celui o de K) si et seulement si Λ/K est non-surramifiée (autrement dit, $\mathfrak{b}_{\Lambda/K} = (1)$) et $\tilde{N}_{\Lambda/K} = 1$ (l'auteur donne encore une autre formulation légèrement différente de ce critère).

Remarque du référent: Les résultats du § 6 se localisent facilement, c'est-à-dire se déduisent immédiatement des résultats analogues pour les $\Lambda_\mathfrak{p}/K_\mathfrak{p}$. Autrement dit, on peut se borner, en fait, au cas où Λ est un corps et K est un corps discrètement valué et complet. Or, dans ce cas particulier, une partie des résultats considérés par l'auteur est étroitement liée avec certains résultats de la théorie de la ramification qui datent de la période 1920–1940 et sont dûs à A. Speiser, O. Ore, J. Herbrand et au référent. On rappelle quelques définitions. Soient $|\cdot\cdot|$ et $\omega(\cdot\cdot) = -\mathrm{Log}\,|\cdot\cdot|$ la valuation et l'ordre valuatif, qui correspondent à l'idéal premier $\mathfrak{p}$ de K, normalisés de manière que l'idéal premier $\mathfrak{P}$ de Λ ait l'ordre $\omega(\mathfrak{P}) = 1$; $V_{\Lambda/K}$ étant le groupe de ramification de Λ/K, c'est-à-dire l'ensemble des $\gamma \in G$ tels que, pour tout $\alpha \in \Lambda$ non-nul, on ait $|\alpha^\gamma - \alpha| < |\alpha|$, soient $v_0, v_1, \cdots, v_{s-1}, v_s = +\infty$ les nombres de ramification (écrits dans l'ordre croissant) de Λ/K, c'est-à-dire les valeurs que prend le nombre caractéristique $v(\gamma) = \mathrm{Min}_{|\alpha| \leq 1}\,\omega(\alpha^\gamma - \alpha)$ quand γ parcourt $V_{\Lambda/K}$, et soit n_q $(0 \leq q \leq s)$ l'ordre du q-ième groupe de ramification $V_q = \{\gamma;\ v(\gamma) \geq v_q\}$. Posons $E = \omega(p)$, A. Speiser [même J. **149** (1919), 174–188] a prouvé, quand K est $\mathfrak{p}$-adique, que $v_{s-1} \leq 1 + E/(p-1)$ et O. Ore [Math. Ann. **100** (1928), 650–673] en a déduit l'inégalité plus générale $(v_q - 1)n_q \leq E(p/(p-1))$. D'autre part, il résulte du § 9 du travail du référent [Mathematica (Cluj) **13** (1937), 72–191] que si Λ/K est défini par une équation d'Eisenstein binôme $x^n - \pi = 0$, les inégalités précédentes sont des égalités. Par contre, en vertu d'un résultat de Speiser que, dans le cas galoisien, tous les v_q sont congrus (mod p), il résulte que, dans le cas contraire, ces inégalités sont strictes pour tous les q. Λ_χ étant le sous-corps de Λ appartenant au sous-groupe $G_\chi = \{\gamma;\ \chi(\gamma) = 1\}$ de G, on s'aperçoit, en appliquant à Λ_χ/K les formules de Hasse, qui expriment l'ordre de conducteur en fonction des v_q, n_q et de l'ordre $n-1$ du groupe d'inertie, extension de K, que les composantes $(\alpha_\chi^n)_\mathfrak{p}$ de $\alpha^n \in K$, considéré comme un idèle principal de K, sont des unités pour presque tous les $\mathfrak{p}$; et, pour un tel $\mathfrak{p}$, on peut prendre $\beta_{\chi,\mathfrak{p}} = \alpha_{\chi,\mathfrak{p}} = \sqrt[n]{(\alpha_\chi^n)_\mathfrak{p}}$, ce qui donne $b_\mathfrak{p} \in B^{(2)}(X, U_\mathfrak{p})$. Ainsi, si J_K et U_K désignent les groupes des idèles et des idèles unitaires de K, le vecteur $b_{\chi,\psi} = (b_{\chi,\psi,\mathfrak{p}})$ est un idèle de K et le système $\{b_{\chi,\psi}\}$ est $\in C^{(2)}(X, J_K)$ et détermine, indépendamment du choix des $\beta_{\chi,\mathfrak{p}}$, un élément $b_{\Lambda/K}$ de $C^{(2)}(X, J_K)/B^{(2)}(X, U_K)$.

Visiblement, $\mathfrak{n}_{\chi,\mathfrak{p}} = \beta_{\chi,\mathfrak{p}}\alpha_{\chi,\mathfrak{p}}^{-1}$ est une unité pour presque tous les $\mathfrak{p}$ quelque soit le choix des α_χ et des $\beta_{\chi,\mathfrak{p}}$ possibles, et le vecteur $(\mathfrak{n}_{\chi,\mathfrak{p}})$ est un K-idèle, dont la classe N_χ (mod U_KK^*), qui s'identifie canoniquement avec une classe des idéaux de K, est dite la classe invariante de Λ/K.

D'autres invariants arithmétiques secondaires peuvent se construire à partir de $b_{\Lambda/K}$, à savoir: (a) la classe b_χ^* de $\prod_{i=0}^{n-1} b_{\chi,\chi^i}$ (mod U_K^n), qui ne depend pas du choix de cocycle $b_{\chi,\psi}$ de $b_{\Lambda/K}$; (b) la classe B_χ de $b_\chi^* \in J_K/U_K^n$ (mod U_K/U_K^n) (c'est donc un idéal de K); (c) le cocycle $\{R_{\chi,\psi}\}$ de X dans le groupe J_K/U_K des idéaux de K, qui est l'image de $b_{\Lambda/K}$ par l'application canonique

$$C^{(2)}(X, J_K)/B^{(2)}(X, U_K) \to C^{(2)}((X, J_K/U_K); d)R_\chi = R_{\chi,\chi^{-1}}.$$

Soit $a \in K$ tel que $a = \alpha^n$, où $\alpha \in \Lambda^\chi$. Soit $(a) = \mathfrak{a}_1\mathfrak{a}_2^{-n}$ la décomposition de l'idéal (a) telle que $\mathfrak{a}_1$ soit un idéal entier de K libre des facteurs puissances n-ièmes d'idéaux. L'auteur prouve que $B_\chi = \mathfrak{a}_1$, b_χ^* est la classe des idèles b tels que $(b) = \mathfrak{a}_1$ et que $ab^{-1} \in J_K^n$, N_χ est la classe de $\mathfrak{a}_2$, et R_χ est le produit des facteurs premiers de $\mathfrak{a}_1$.

D'autre part on a, localement, que $R_{\chi\,\mathfrak{p}} = (1)$ ou $= \mathfrak{p}$ selon que $O_\mathfrak{p}^\chi$ est engendré ou non par une unité de K, et B_χ peut se caractériser comme le p.g.c.d. des idéaux (b) tels que $b \in o$ et $= b^{1/n} \in O^\chi$.

Par considérations générales sur les $o(G)$-modules, où o est un anneau dédekindien et G un groupe abélien (§§ 4, 5), qui sont trop compliquées pour être résumées ici, l'auteur démontre (§ 6) les résultats suivants.

Soit m un entier rationnel. Définissons $\mathfrak{t}(m)$ comme produit (dont seul un nombre fini de facteurs sont distincts de (1)) $\prod_\mathfrak{p} \mathfrak{t}(m)_\mathfrak{p}$, étendu à tous les idéaux premiers de K, où $\mathfrak{t}(m)_\mathfrak{p} = (1)$ quand m est une unité de $o_\mathfrak{p}$, et quand m n'est pas une unité de $o_\mathfrak{p}$ et p est la caractéristique de $o/\mathfrak{p} = o_\mathfrak{p}/\mathfrak{p}o_\mathfrak{p}$ et ζ_p est une racine p-ième primitive de l'unité (qui est $\in K$ si m divise n)

$$\mathfrak{t}(m)_\mathfrak{p} = (p_\mathfrak{p})^{\omega_p(m) + 1/(p-1)} = (p^{\omega_p(m)}(\zeta_p - 1))_\mathfrak{p},$$

où $\omega_p(m)$ est l'ordre de m en p.

Une fonction $F(\chi)$, définie (pour toute algèbre kummérienne Λ/K) sur $X(\Lambda/K)$ et à valeurs dans le groupe des idéaux de K, sera dite un invariant entier si: (a) $F(\chi)$ est toujours un idéal entier; (b) $F(\chi_\mathfrak{p}) = F(\chi)_\mathfrak{p}$; (c) Si

$\Lambda/K \to \Lambda'/K'$ est un isomorphisme des algèbres kummériennes et $\chi \to \chi'$ l'isomorphisme des caractères de leurs groupes qu'il induit, on a $F(\chi)=F(\chi')$; (d) χ^0 étant le caractère principal de G, on a $F(\chi^0)=(1)$; (e) $F(\chi^i)=F(\chi)$ si i est premier à l'ordre $n(\chi)$ de χ. $\mathfrak{t}(n_\chi)$, R_χ et le conducteur $\mathfrak{f}_\chi$ de χ au sens d'Artin-Hasse sont de tels invariants entiers. L'auteur démontre, par les considérations mentionnées appliquées aux $o(G(\Lambda/K))$-modules $O(\Lambda)$ et $\widetilde{O(\Lambda)}$, que: $\mathfrak{b}_\chi=\mathfrak{f}_\chi R_\chi^{-1}$ en est encore un; $\mathfrak{b}_\chi$ divise $\mathfrak{t}(n_\chi)$; $\mathfrak{b}_{\Lambda/K}=\prod_\chi \mathfrak{b}_\chi$ est (1) si et seulement si Λ/K est non-surramifiée et, en particulier, $(\mathfrak{b}_{\Lambda/K})_\mathfrak{p}=(1)$ si et seulement si Λ/K n'est pas surramifiée en $\mathfrak{p}$; la valeur maximale de $\mathfrak{b}_{\Lambda/K}$ est $(\Lambda:K)^{(\Lambda:K)}$ et $\mathfrak{b}_{\Lambda/K}$ la prend si et seulement si $O(\Lambda)=\widetilde{O(\Lambda)}$; si, en plus, aucun facteur premier de n n'est une unité dans o, cette "ramification maximale" de Λ/K implique que Λ/K est cyclique.

E étant un ensemble et D étant un demi-groupe, soit $\mathfrak{M}(E,D)$ l'ensemble des applications de E dans D organisé en demi-groupe par la composition des valeurs $f_1f_2(e)=f_1(e)f_2(e)$ $(f_1, f_2 \in \mathfrak{M}(E,D),\ e \in E)$. Soit V un K-module et soit $f: X \to V$ une application de X dans V. Appelons f^* l'application $G \to V$ définie par $f(\gamma)=g^{-1}\sum_{\chi\in X} f(\chi)\chi(\gamma)$ où g est l'ordre de G (et de X). De même, si $h: G \to V$ est une application de G dans V, on définit $h_*: X \to V$ par $h_*(\chi)=\sum_{\gamma\in G} h(\gamma)\chi^{-1}(\gamma)$. Si V est une K-algèbre, on considère les $f: X \to V$ comme éléments de $\mathfrak{M}(X, V)$. En particulier, si N_χ est l'invariant défini plus haut, on peut considérer que le système $\{N_\chi;\ \chi \in X\}$ est un élément de $\mathfrak{M}(X, J_K)/\mathfrak{M}(X, K^*)\mathfrak{M}(X, U_K)$. Supposons que, pour tout $\mathfrak{p}$, Λ/K ne soit pas surramifiée, ce qui équivaut à $\mathfrak{b}_{\Lambda/K}=(1)$. Alors, en vertu d'un théorème d'Emmy Noether (redémontré dans le § 6 de ce travail), il existe un $\beta_\mathfrak{p} \in O_\mathfrak{p}$ qui engendre une base normale de $O_\mathfrak{p}/o_\mathfrak{p}$. Si l'on écrit $h: \gamma \to \beta_\mathfrak{p}^\gamma$, on posera $(\chi|\beta_\mathfrak{p})=h_*(\chi)$. L'auteur montre que $O_\mathfrak{p}^\chi=o_\mathfrak{p}((\chi|\beta_\mathfrak{p}))$ et que si l'on remplace $\beta_\mathfrak{p}$ par un autre élément, engendrant une base normale de $O_\mathfrak{p}/o_\mathfrak{p}$, $(\chi|\beta_\mathfrak{p})$ se multiplie par un facteur $\in \mathfrak{M}_0(X, U_K)$, où $\mathfrak{M}_0(X, U_K)$ est le sous-groupe de $\mathfrak{M}(X, U_K)$ formé par les éléments $f: X \to U_K$ de ce groupe tels que $f: X(\Lambda/K)_\mathfrak{p} \to U_\mathfrak{p}$ étant l'application locale induite par f, on ait, pour tout $\mathfrak{p}$ et pour tout χ, $f^*(\chi)_\mathfrak{p} \in o_\mathfrak{p}$. Par suite, si $\alpha_{\chi,\mathfrak{p}} f(\chi)_\mathfrak{p}=(\chi|\beta_\mathfrak{p})$, l'idèle que $\mathfrak{b}_\chi|\mathfrak{t}(n_\chi)$ équivaut aux inégalités indiquées, et que $\mathfrak{b}_\chi=\mathfrak{t}(n_\chi)$ a lieu si ou bien Λ/K est complètement ramifiée et, pour tout q, $n_q(v_q-1)=\mathrm{E}\dfrac{p}{p-1}$ (ce qui a lieu si et seulement si $R_\chi=\mathfrak{p}$, et entraîne $n_q=p^{s-q}$), ou bien $R_\chi=(1)$, $n=p^s$, $n_q=p^{s-q}$ et $n_qv_q=\mathrm{E}\dfrac{p}{p-1}$. Ceci donne un résultat non-aperçu par l'auteur: $R_\chi=\mathfrak{p}$ implique $\mathfrak{b}_\chi=\mathfrak{t}(n_\chi)$. Le fait que $\mathfrak{f}_\chi$ est un idéal de K (et pas seulement de Λ_χ) résulte des congruences $n_q(v_q-v_{q-1})\equiv 0 \pmod{n_{-1}}$ que Hasse a tiré de la théorie locale des corps des classes. Par contre, l'intégrité de cet idéal, résulte du fait élémentaire que $v(\gamma)$ est $\geqq 1$ pour tout élément γ du groupe d'inertie.

Quand K n'est pas un corps $\mathfrak{p}$-adique, mais un corps discrètement valué complet quelconque, les inégalités indiquées de Speiser et d'Ore, les congruences de Speiser et le résultat indiqué du référent restent encore valables si l'extension résiduelle de Λ/K est séparable (et cela a été indiqué dans la note du référent [C. R. Acad. Sci. Paris **220** (1945), 28–30; MR **7**, 364]), et, également, quand la surramification vient uniquement de l'extension non-séparable du corps résiduel de K et non de celle de son groupe de valuation. Les résultats de l'auteur montrent qu'il en est de même dans le cas "mixte" quand la surramification provient à la fois de l'extension inséparable du corps résiduel de K et de l'extension d'indice puissance de p de son groupe de valuation, ce qu'on ne peut pas prouver actuellement par des méthodes directes. D'autre part, le fait, prouvé par l'auteur, que $\mathfrak{f}_\chi$ est un idéal de K montre que, malgré ce qu'on ne connait pas, actuellement, d'analogue de la théorie locale abélienne des corps des classes, quand K discrètement valué et complet n'est pas localement compact, certaines conséquences de cette théorie restent vraies dans ce cas plus général.

M. Krasner (Paris)

Kuyk, W. 3989

On a theorem of E. Noether.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 32–39.

Let $k(X)=k(X_1, X_2, \cdots, X_n)$ be a purely transcendental extension of a field k, and let G and k_G be arbitrary transitive permutation groups on X and the field of all invariants under G in $k(X)$, respectively. Suppose that k_G is purely transcendental over k: $\exists U=\{U_1, \cdots, U_n\}\subset k_G$; $k_G=k(U)$. Then, as is well known, U gives a parametric representation of a set of polynomials in $k[t]$ with Galois group isomorphic into G. The author makes the following two remarks: (1) This set has a non-void intersection with every class of equivalent polynomials with Galois group isomorphic onto G, unless k is a finite field. (2) If $k(U)$ is any purely transcendental extension of k, and if $P(t)\in k(U)[t]$ has Galois group G, then all finite specializations $P^*(t)\in k[t]$ of $P(t)$ over a k-specialization of U into k that have zeros separable over k have Galois groups isomorphic into G. *K. Masuda* (Okayama)

Fenstad, Jens Erik 3990

An observation on the Gelfand-Mazur theorem.

Norske Vid. Selsk. Forh. (Trondheim) **36** (1963), 41–45.

Démonstration de la réciproque suivante au théorème de Gel'fand et Mazur: Si un corps valué K n'admet d'autre extension valuée que lui même, il est isomorphe au corps des nombres complexes. Ce résultat est conséquence immédiate de résultats classiques. *P. Samuel* (Paris)

Brehmer, S. 3991

Eine elementare Konstruktion von Winkelzahlen in der komplexen Zahlenebene.

Acta Math. Acad. Sci. Hungar. **15** (1964), 53–55.

Let K be an ordered commutative field and K^* its complex extension. A ray in K^* is any set of the form $\{rz|0\leqq r\in K\}$, for $0\neq z\in K^*$. The following theorem is proved: If the multiplicative subgroup Z of K satisfies $|Z\cap S|=1$ for every ray S of K^*, then Z is a homomorphic image of a group W which is similar to the additive group of K. This result permits an elementary definition of the trigonometric functions. *P. Dembowski* (Frankfurt a.M.)

Alling, Norman L. 3992

The valuation theory of meromorphic function fields over open Riemann surfaces.

Acta Math. **110** (1963), 79–96.

Let X be an open Riemann surface, A the algebra of all analytic functions on X, and M a free maximal ideal in

A, and K the residue class field associated with M. Then K has a natural valuation whose residue class field is the complex numbers and the value group of K is a divisible η_1-group.

Let A_M be the quotient ring of A with respect to M in F, the field of all meromorphic functions on X. It is shown that A_M is a valuation ring of F and the value group of A_M is shown to be a non-divisible near η_1-group. The structure of the prime ideals containing M is analyzed. The composite of the place of F (corresponding to A_M) and the place of K is shown to be a place of F over C onto C whose valuation is 1-maximal, and whose value group is a non-divisible η_1-group.

Let S be the space of all places of F over C. Under the weak topology S is compact. Let T be the closure of X in S and let S_A be the places that arise from maximal ideals in A. Then $X < S_A \subset T < S$.

H. L. Royden (Stanford, Calif.)

ABSTRACT ALGEBRAIC GEOMETRY

See also 4145, 4547.

Moišezon, B. G. 3993

A projectivity criterion of complete algebraic abstract varieties. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 179–224.

The object of this paper is to give a condition ensuring the projective embeddability of a complete abstract variety over an algebraically closed field of any characteristic. The results were previously stated [Dokl. Akad. Nauk SSSR **145** (1962), 996–999; MR **26** #117].

The condition is expressed in terms of a character denoted by $g_V[D]$, where $[D]$ is a line bundle over the irreducible variety V. $g_V[D]$ is defined inductively as follows. For a 0-dimensional variety V, $g_V[D]$ is taken as 1. Next, assume that $g_V[D]$ is defined for any line bundle over any irreducible variety of dimension $< s$. Finally, if $\dim V = s$, set $g_V[D] = \sum n_C[D]g_C[D]_C$, where C runs over all irreducible subvarieties of V of dimension $< s$, $[D]_C$ is the bundle over C induced by $[D]$, and the integers $n_C[D]$ are defined in terms of the transition functions of $[D]$ corresponding to an open affine covering of V. In fact, if a transition function is expressed as a fraction f/g with f and g in the local ring of C, then $n_C[D]$ is the difference of the multiplicities of the ideals in this ring generated by f and g. It is shown that $n_C[D]$ depends only on C and D and that it is non-zero for only finitely many C.

Next it is shown that for any variety V of dimension s $\chi(V, O_V[nD]) = (n^s/s!) \sum g_{V_i}[D]_{V_i}$ plus a polynomial in n of degree $< s$, where $O_V[nD]$ is the sheaf of sections of the line bundle $[nD]$ and the V_i are the irreducible components of V. Also it is proved that if V is a subvariety of X and $[D]$ is a line bundle over X, then, with a suitable definition of the intersection number $D^s \cdot V$ ($s = \dim V$), $g_V[D]_V = D^s \cdot V$.

Coming now to the main result of the paper, a line bundle $[D]$ over V is called non-degenerate if a sufficiently high multiple $[nD]$ corresponds to a biregular embedding in projective space in such a way that $[nD]$ is induced by the bundle over projective space corresponding to a hyperplane. The main theorem says that if V is a complete abstract variety and $[D]$ a line bundle over V such that $g_W[D]_W > 0$ for all irreducible subvarieties W, then $[D]$ is non-degenerate. This theorem is proved by first showing that there is a multiple $[E]$ of $[D]$ such that the elements of $H^0(V, O_V[nE])$ have no common zero for large n. The elements of a basis of this group are then used to define a map of V into a projective space, and this map turns out to have the required properties.

Using the above-mentioned connection between $g_W[D]_W$ and $D^k \cdot W$ (where $\dim W = k$), an immediate consequence of the main theorem is that if V is non-singular and complete and $\dim V = s$, and D is a divisor such that $D^k \cdot W_k > 0$ for all subvarieties W_k of dimension k $(1 \leq k \leq s)$, then the map into projective space corresponding to $[nD]$ for sufficiently large n is biregular.

Some applications of the main result are now given in the case of varieties over the complex numbers (to allow the use of homology theory, which is needed in the absence of an intersection theory for classes of algebraic equivalence). It is shown that if V is a non-singular complete variety over the complex numbers and if a connected algebraic group G acts on V in such a way that there is an open quasi-projective variety U for which the gU, $g \in G$, cover V, then V is projective. This is proved by constructing a divisor D such that all the $D^k \cdot W_k > 0$ for all subvarieties W_k of dimension k, $1 \leq k \leq \dim V$.

In particular, it is shown that if $\dim V = 3$ and if there is a linear system $|D|$ with no fixed component, then the condition $D \cdot W_1 > 0$ for all curves W_1 is sufficient to ensure that a multiple of $|D|$ will correspond to a biregular projective embedding of V. For in this case the other conditions on the $D^k \cdot W_k$ become vacuous.

A curve C on a variety is called absolutely fixed if no multiple hC $(h > 0)$ is in an infinite algebraic system. A variety of dimension 3 which is not projective has such a curve. For otherwise a D can be constructed satisfying the condition $D \cdot W_1 > 0$ for all curves W_1, contradicting the non-projectivity of V. In terms of this concept a problem of Chevalley is solved as follows. Suppose V (of dimension 3) has a finite number of absolutely fixed curves and suppose that every finite set on V is contained in an open affine subvariety. Then V is projective.

A further application of the same order of ideas is the finding of a condition that for a given linear system $|E|$ on a nonsingular variety V (any dimension) $|nE|$ shall not have fixed points for large n. The condition is that $E^k \cdot W_k > 0$ for all k-dimensional subvarieties W_k of the carrier of some divisor of $|E|$ $(1 \leq k \leq \dim V - 1)$. If $\dim V = 2$, this gives the theorem of Zariski that, on a normal variety, if $|E|$ is infinite with no fixed components, then for large n, $|nE|$ has fixed points only in special points. In the general case an example is given to show that the conditions $E^k \cdot W_k > 0$ cannot be dropped (thus contradicting a conjecture of Baldassari in his book [*Algebraic varieties*, Springer, Berlin, 1956; MR **18**, 508]).

There seems to be a close connection between the condition for non-degeneracy of a line bundle given in this paper and the condition given by Nakai [Amer. J. Math. **85** (1963), 14–26; MR **27** #1446] for a divisor on a projective variety to be ample, the result here being much stronger in that the variety is not taken to be given as projective.

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Serre, Jean-Pierre 3994

Sur les groupes de congruence des variétés abéliennes. (Russian summary)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 3–20.

From the author's Russian summary: "Let A be an abelian variety defined over a field k of finite degree over the rationals and let $A(k)$ be the group of points on A defined over k. A subgroup Γ of $A(k)$ is called a congruence subgroup if there exists a finite set I of valuations $\mathfrak{p}$ of k and open subgroups $U_\mathfrak{p}$ of $A(k_\mathfrak{p})$ for $\mathfrak{p} \in I$ such that $\Gamma = \bigcap_{\mathfrak{p} \in I} U_\mathfrak{p}$. (Open means here open in the $\mathfrak{p}$-adic topology.)

"In this paper the author investigates the nature of the closure $\widetilde{A(k)}$ of $A(k)$ in the injection $A(k) \to \prod_\mathfrak{p} A(k_\mathfrak{p})$ (cf. the addresses of the reviewer and Tate to the Stockholm Congress [Cassels, Proc. Internat. Congr. Mathematicians (Stockholm, 1962), pp. 234–246, Inst. Mittag-Leffler, Djursholm, 1963; Tate, ibid., pp. 288–295]). In particular, is it true that $\widetilde{A(k)} = \text{proj lim}_n\, A(k)/nA(k)$ (with the limit topology)? This is equivalent to the statement that every subgroup of $A(k)$ of finite index contains a congruence subgroup. The author considers the following more general property C_S: For every finite subset S of prime divisors on k and for every subgroup Γ of finite index, Γ contains a congruence subgroup defined by a set I of divisors which is disjoint from S.

"The principal result of the paper (Theorems 4 and 5) is that C_S is true if A is either (a) of dimension 1 or (b) an abelian variety with a sufficient number of complex multiplications.

"The proof depends on some other results of independent interest. Clearly it is enough to prove C_S for subgroups of index p^n, where p is any prime: we call the corresponding assertion $C_S(p)$. By considering the exact cohomology sequence associated with multiplication by p^n on A the author shows that $C_S(p)$ is true if

$$(*) \qquad \text{proj}\lim_n H_S{}^1(k, A_{p^n}) = 0,$$

where $H_S{}^1(k, A_{p^n})$ is the subgroup of elements of $H^1(G(\bar{k}/k), A_{p^n})$ which vanish when restricted to the splitting groups of the prime divisors outside S. Here $G(\bar{k}/k)$ is the Galois group of $\bar{k}/k$, where $\bar{k}$ is the algebraic closure of k and A_{p^n} is the group of points on A of order dividing p^n.

"Let G be a profinite group (projective limit of finite groups) and M any topological G-module. Following Tate, the author denotes by $H_*{}^1(G, M)$ the subgroup of $H^1(G, M)$ consisting of all the elements which vanish when restricted to any one-parameter subgroup of G (i.e., the closed subgroups with one topological generator). Let $T_p(A) = \text{proj lim}_n\, A_{p^n}$ be the Tate module of the abelian variety A and let $V_p(A)$ be the extension of this module to a module over the p-adic numbers. Then it is shown that (*) is true if

$$(**) \qquad H^1(G(\bar{k}/k), T_p(A)) = 0.$$

"It is well known that $T_p(A)$ is a free module over the ring $\mathbf{Z}_p$ of p-adic integers of dimension $2d$, where d is the dimension of A. Its group of automorphisms is just the linear group $GL(2d, \mathbf{Z}_p)$. Consequently, the action of $G(\bar{k}/k)$ on $T_p(A)$ gives a group homomorphism $\pi: G(\bar{k}/k) \to GL(2d, \mathbf{Z}_p)$. The group $GL(2d, \mathbf{Z}_p)$ is a p-adic Lie group in an appropriate sense and $\pi G(\bar{k}/k)$ is a closed subgroup. Hence it can be given the structure of a p-adic Lie group with a Lie algebra $\mathfrak{g}_p$. The representation π gives naturally a representation of $\mathfrak{g}_p$ in $V_p(A)$. The algebra $\mathfrak{g}_p$ and this representation are unchanged if k is replaced by any finite algebraic extension. The author shows that (**) is true if

$$(***) \qquad H_*{}^1(\mathfrak{g}_p, V_p(A)) = 0,$$

where $H_*{}^1$ is defined for Lie algebras as for groups. Finally he verifies that (***) is true in the two cases (a) and (b), and indeed shows that $H^1(\mathfrak{g}_p, V_p(A)) = 0$.

"Amongst others the author mentions the following unsolved problems: (1) Is there a relation between the $\mathfrak{g}_p$ for different p (e.g., do they have the same dimension)? (2) In the numerical examples of elliptic curves considered by the author $\mathfrak{g}_p$ is either abelian of dimension 2 (if A has complex multiplication) or the full matrix group (if not). Is this always true? (3) Is the main result of the paper true for all abelian varieties? (4) To what extent is the main result true for other algebraic groups? Examples are known of other groups for which it is true [Chevalley, J. Math. Soc. Japan **3** (1951), 36–44; MR **13**, 440], but Klein showed that it is false for the special linear group of dimension 2." *J. W. S. Cassels* (Cambridge, England)

Baily, Walter L., Jr. 3995

On the moduli of Abelian varieties with multiplications. *J. Math. Soc. Japan* **15** (1963), 367–386.

Let $\mathfrak{k}$ denote either a totally real number field or a purely imaginary quadratic extension of such a field, and let $\mathfrak{o}$ be the ring of integers of $\mathfrak{k}$. It is known [see Shimura, Ann. of Math. (2) **70** (1959), 101–144; MR **21** #6370; ibid. (2) **78** (1963), 149–192; MR **27** #5934] that the isomorphism classes of polarized abelian varieties of a given dimension of type σ (i.e., roughly, whose endomorphism ring contains a faithful image of σ) are parametrized by the points of the quotient space H/Γ of a bounded symmetric domain H by an arithmetically defined group Γ of automorphisms of H. It is shown here that H/Γ may be identified with a Zariski $\mathbf{Q}$-open subset of a projective variety V^* defined over the field $\mathbf{Q}$ of rational numbers, in such a way that the coordinates of a point $x \in H/\Gamma$ generate over $\mathbf{Q}$ the field of moduli of the polarized abelian variety of type σ represented by x. This generalizes earlier results of Shimura [Séminaire H. Cartan, 1957/58, Exp. 18–20, Secrétariat mathématique, Paris, 1958; MR **21** #2750] and of the author [Ann. of Math. (2) **75** (1962), 342–381; MR **29** #103] pertaining respectively to the Hilbert and Hilbert-Siegel groups. The method is a suitable generalization of that used before by the author, and in several points of the proof, the author just refers to the paper quoted above, or indicates how the corresponding step there has to be modified. He assumes the existence of a compactification V^* of H/Γ with properties similar to those known in the case of the Hilbert-Siegel group, also with respect to the operator Φ on automorphic forms, to be discussed elsewhere. The proof is based on the use of Θ-functions, whose discussion, including transformation formulae, addition theorem, etc., takes up the greater part of the paper. The embedding is defined first over the maximal abelian extension A of $\mathbf{Q}$ by means of Θ-functions (for suitable subgroups of finite index of Γ) with Fourier coefficients in A. It is then shown that by letting the Galois group of $A/\mathbf{Q}$ operate on the coefficients of these functions, one gets other mappings whose images satisfy Weil's conditions for the field descent from A to $\mathbf{Q}$.

A. Borel (Princeton, N.J.)

LINEAR ALGEBRA

See also 3963, 4065, 4135, 4259.

Janekoski, V. 3996

On the mixed product of three vectors. (Macedonian. French summary)

Bull. Soc. Math. Phys. Macédoine **13** (1962), 45–47.

Dionísio, J. J. 3997

On the characteristic roots of a matrix.

Univ. Lisboa Revista Fac. Ci. A (2) **8** (1960/61), 291–297.

Let A be an $n\times n$ complex matrix with characteristic roots $\lambda_1, \cdots, \lambda_n$. Let

$$s_{\Re}(A) = \max_{i,j} |\Re\lambda_i - \Re\lambda_j|, \quad s_{\Im}(A) = \max_{i,j} |\Im\lambda_i - \Im\lambda_j|,$$
$$(\bar{r}A) = \max_k |\lambda_k|, \quad \underline{r}(A) = \min_k |\lambda_k|,$$

where $\Re\lambda_i$, $\Im\lambda_i$ denote, respectively, the real and imaginary parts of λ_i. This paper deals with inequalities for the above scalars associated with A, these being obtained, in the main, by using the Hermitian decomposition of A and applying results due to Mirsky, Brauer and Mewborn, and Parker. *A. Geddes* (Glasgow)

Kochendörffer, R. 3998

★Determinanten und Matrizen.

Dritte Auflage. Mathematisch-Naturwissenschaftliche Bibliothek, 12.

B. G. Teubner Verlagsgesellschaft, Leipzig, 1963. v + 144 *pp.* *DM* 6.60.

A reprinting of the earlier 1957 edition [MR **19**, 1033].

Neiss, Fritz 3999

★Determinanten und Matrizen.

Sechste Auflage.

Springer-Verlag, Berlin-Göttingen-Heidelberg, 1962. vii + 111 *pp.* *DM* 8.60.

An earlier edition [Fünfte Aufl., 1959; MR **21** #2657] has already been reviewed.

Voevodin, V. V. 4000

The convergence of the orthogonal power method. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **2** (1962), 529–536.

For an arbitrary nonsingular matrix A and an arbitrary orthogonal starting matrix Q_0, the orthogonal matrices Q_k are determined so that $AQ_k = Q_{k+1}\Delta_{k+1}$, where Δ_{k+1} is upper triangular. Then $Q_k'AQ_k$ is (nearly) upper triangular for large k. This was proved in an earlier paper [the author, same Ž. **2** (1962), 15–24; MR **27** #2097] under restrictions that are now removed. *J. L. Brenner* (Palo Alto, Calif.)

Vorob'ev, N. N. 4001

An extremal matrix algebra. (Russian)

Dokl. Akad. Nauk SSSR **152** (1963), 24–27.

This paper summarizes properties of the products $C = A \circ B$, $D = A \circ B$, where $c_{ij} = \max_k (a_{ik}b_{kj})$; $d_{ij} = \min_k (a_{ik}b_{kj})$, and of the operators $F = A \max B$, $G = A \min B$, where $f_{ij} = \max(a_{ij}, b_{ij})$; $g_{ij} = \min(a_{ij}, b_{ij})$ and the Schur inverse S, $s_{ij} = a_{ij}^{-1}$ (defined in certain cases). These abstract operations occur in analysis, e.g., functionals in L^∞. The author cites also Bellman and Karush [Bull. Amer. Math. Soc. **67** (1961), 501–503; MR **24** #A1533]. *J. L. Brenner* (Palo Alto, Calif.)

Mesner, Dale M. 4002

Traces of a class of (0, 1)-matrices.

Canad. J. Math. **16** (1964), 82–93.

Let A be an $m\times n$ matrix of 0's and 1's, where without loss of generality we take $m \leqq n$. Trace t is said to be possible for A provided there exist permutation matrices P and Q such that the trace of PAQ is t. The author discusses the set of possible trace values for A, namely, the set of distinct traces of the class of matrices $\{PAQ\}$, where P and Q range over all permutation matrices of the appropriate orders. The largest and smallest possible trace values are denoted by $t_{\max}$ and $t_{\min}$, respectively. It is clear that $t_{\max}(A)$ equals the term rank of A and $t_{\min}(A) = m - t_{\max}(J-A)$, where J is the matrix of 1's. The following theorem summarizes the author's main results.

"Let A be an $m\times n$ matrix of 0's and 1's, $m\leqq n$, with maximum trace $t_{\max}$ and minimum trace $t_{\min}$. Then (1) if A is a rearranged direct sum of three or more square matrices of 1's, trace $m-1$ is impossible; $t_{\min} < m-1 < t_{\max} = m$; (2) if the complement of A is a rearranged direct sum of three or more square matrices of 1's, trace 1 is impossible; $0 = t_{\min} < 1 < t_{\max}$; (3) if A is square and is a rearranged direct sum of two matrices of 1's, trace t is impossible if $t_{\max} - t \equiv 1 \pmod 2$; $0 < t_{\max} - t_{\min} \equiv 0 \pmod 2$; (4) in every other case, if $t_{\min} \leqq t \leqq t_{\max}$, trace t is possible." *H. J. Ryser* (Syracuse, N.Y.)

Ehlich, Hartmut 4003

Determinantenabschätzungen für binäre Matrizen.

Math. Z. **83** (1964), 123–132.

The author investigates the maximal absolute value of the determinant of a matrix $A = [a_{ij}]$ of order n with entries $a_{ij} = \pm 1$. Let α_n denote this maximum over all $(+1, -1)$-matrices of order n and let $h_n = n^{n/2}$. Clearly $\alpha_n \leqq h_n$. Let $d \geqq 0$ and let $C = [c_{ij}]$ be a positive definite matrix with $c_{11} \geqq c_{22} \geqq \cdots \geqq c_{nn} \geqq d$ and $|c_{ij}| \geqq d$ for $i \neq j$. Then it is shown that

$$\det C \leqq (c_{nn} + (n-1)d) \prod_{i=1}^{n-1} (c_{ii} - d).$$

This result is then used to establish the following. If n is odd, then

$$\alpha_n^2 \leqq (n-1)^{n-1}(2n-1).$$

If n is even, then

$$\alpha_n^2 \leqq n^n = h_n^2 \text{ for } n \equiv 0 \pmod 4,$$
$$\leqq 4(n-2)^{n-2}(n-1)^2 \text{ for } n \equiv 2 \pmod 4.$$

The paper concludes with a discussion of equality and an analysis of the existence of "maximal" $(+1, -1)$-matrices A such that $AA^T = F_1$ or F_2, where

$$F_1 = \begin{bmatrix} n & & 1 \\ & \ddots & \\ 1 & & n \end{bmatrix} \text{ and } F_2 = \begin{bmatrix} F_3 & 0 \\ 0 & F_3 \end{bmatrix} \text{ with } F_3 = \begin{bmatrix} n & & 2 \\ & \ddots & \\ 2 & & n \end{bmatrix}.$$

H. J. Ryser (Syracuse, N.Y.)

Fiedler, Miroslav 4004
On inverting partitioned matrices. (Russian summary)
Czechoslovak Math. J. **13** (**88**) (1963), 574–586.
Let X be a vector space which is partitioned into a direct sum of spaces $X(u)$ where u is in an index set U. If A is a linear transformation on X, we get induced transformations $a(u, v)$ from $X(u)$ to $X(v)$ as follows: if $x = \sum x(u)$, $xA = y = \sum y(v)$; $y(v) = \sum x(u)a(u, v)$. Associate with A a directed graph whose vertices are elements of U and the (directed) edge from u to v exists if $a(u, v) \neq 0$. Finally, call the graph c-simple if there exists at most one path from u to v for any pair u, v of elements of U.

The author investigates the task of inverting transformations A possessing c-simple graphs. (A brief discussion deals with the importance of such transformations in the applications.) The inverse of a representative matrix can be evaluated recurrently and the problem is equivalent to that of solving a comparatively smaller system of (non-linear) matrix equations.

{In Theorem 3.4, part 1° of the conclusion, the first line should be preceded by 'either' and the second by 'or'.}
J. D. Swift (Los Angeles, Calif.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 3934, 3987, 4035, 4044, 4339.

Berger, Robert 4005
Differenten regulärer Ringe.
J. Reine Angew. Math. **214/215** (1964), 441–442.
Soient P un anneau local régulier, B sa fermeture intégrale dans une extension séparable de degré fini du corps des fractions de P, $\mathfrak{M}$ un idéal maximal de B, et $R = B_{\mathfrak{M}}$ (de sorte que R domine P). Si R est régulier, la différente de Kähler de R sur P est un idéal principal de R; comme elle est contenue dans les différentes de Noether et de Dedekind de R sur P, et qu'elle coïncide avec elles en codimension 1 [cf. l'auteur, Math. Z. **78** (1962), 97–115; MR **25** #2088], elle leur est égale. Extension à des anneaux réguliers non locaux. *P. Samuel* (Paris)

Nagahara, Takasi; Kishimoto, Kazuo; Tominaga, Hisao 4006
Supplementary remarks to the previous papers.
Math. J. Okayama Univ. **11** (1962/63), 159–163.
The authors begin by proving the following. Let R be a simple finite-dimensional algebra over a field F whose centre C is a separable extension field of F. If a is an element of R not in C, there exists a unit r in R such that $R = F[a, r]$. This is a slight generalization of earlier theorems in the literature, cf., e.g., F. Kasch [Norsk Mat. Tidsskr. **34** (1952), 97–99; MR **14**, 614] and A. Inatomi [Kōdai Math. Sem. Rep. **14** (1962), 149–159; MR **26** #2467]. This result is then used to give slight improvements of the Cartan-Brauer-Hua theorem and various results on the generation of subrings of Galois extensions of simple rings with minimum condition.
A. Rosenberg (London)

Nagahara, Takasi; Tominaga, Hisao 4007a
On Galois theory of simple rings.
Math. J. Okayama Univ. **11** (1962/63), 79–117.

Nagahara, Takasi; Tominaga, Hisao 4007b
On quasi-Galois extensions of division rings.
J. Fac. Sci. Hokkaido Univ. Ser. I **17** (1963), 73–78.

Continuing their study [see #4006 above] of the Galois theory of simple rings with minimum condition, the authors here try their hand at an infinite-dimensional theory for simple rings. Their definitions and results are too complicated to be reproduced here but they state that their theory specializes to the infinite Galois theory of division rings of J. H. Walter [Proc. Amer. Math. Soc. **10** (1959), 898–907; MR **22** #4750], as well as includes the older theory of N. Jacobson [*Structure of rings*, Amer. Math. Soc., Providence, R.I., 1956; MR **18**, 373] and N. Nobusawa [Osaka Math. J. **7** (1955), 1–6; MR **16**, 1084].

In the second paper under review the rings are specialized to be division rings and various extensions and generalizations of some of the results of the first paper to this more special case are given. *A. Rosenberg* (London)

Martindale, Wallace S., 3rd 4008
Lie isomorphisms of primitive rings.
Proc. Amer. Math. Soc. **14** (1963), 909–916.
If R and R' are associative rings, a Lie isomorphism f of R onto R' is a 1-1 mapping of R onto R' which satisfies (1) $\phi(x+y) = \phi(x) + \phi(y)$; (2) $\phi(xy - yx) = \phi(x)\phi(y) - \phi(y)\phi(x)$ for all $x, y \in R$. The following theorem is established: Let ϕ be a Lie isomorphism of the primitive ring R onto the primitive ring R', where the characteristic of R is different from 2 or 3 and R contains three non-zero orthogonal idempotents whose sum is the identity. Then $\phi = \sigma + \tau$, where σ is either an isomorphism or the negative of an anti-isomorphism of R into a primitive ring L', $L' \supseteq R'$, and τ is an additive mapping of R into the center of L' which maps commutators to zero.

The methods of proof, which are of interest in themselves, make use of modifications of arguments of Hua on matrix rings [J. Chinese Math. Soc. (N.S.) **1** (1951), 110–163; MR **17**, 123]. *E. H. Batho* (Durham, N.H.)

Brameret, Marie-Paule 4009
Anneaux et modules de largeur finie.
C. R. Acad. Sci. Paris **258** (1964), 3605–3608.
The width of an A-module M is the smallest integer m such that, given any $m+1$ elements of M, one of them is in the submodule generated by the others. If A is Artinian, the width of A is finite. Conversely, if A has a nilpotent radical, then every module of finite width has finite length. If $M \supset P_1 \supset P_2 \supset \cdots$, if the width of each M/P_i is not greater than n, and if every finitely generated submodule P has $\bigcap (P + P_i) = P$, then the width of M is also not greater than n. This has the expected corollaries on completions and Henselizations of local rings. The author also gives sufficient (but not necessary) conditions for a commutative ring to be Noetherian: The width of A is finite and, for every ideal I, there is a power N^s of the radical of A such that $I = J + (I \cap N^s)$ implies $I = J$. The note includes proofs. *D. Zelinsky* (Evanston, Ill.)

Sehgal, S. K. 4010
Ringoids with minimum condition.
Math. Z. **83** (1964), 395–408.

The main object of this paper is to prove the Wedderburn theorem for simple ringoids and to establish a structure theory for semi-simple ringoids. Here a ringoid differs from (and generalizes) a ring insofar as the operations of addition and multiplication are not universally defined. Actually, a ringoid in the author's sense is apparently more general than that of the morphisms of a small additive category, since he does not suppose abc defined whenever ab and bc are defined (although all his examples have this property). On the other hand, he does suppose that the disjoint additive abelian groups whose union constitutes a ringoid are none of them trivial, which is a strange condition to impose on the basic definition. Actually, many arguments would be simplified (indeed, trivialized) if the author supposed the condition mentioned above on products abc.

The final section studies the double coset ring $\Gamma(U, V)$ of two subgroups U, V of a finite group G over a field F. This ring is semi-simple and so, too, is the ringoid $\Gamma(G) = \bigcup_{U,V} \Gamma(U, V)$, where only products

$$\Gamma(U, V) \cdot \Gamma(V, W) \subseteq \Gamma(U, W)$$

are admitted, provided the characteristic of F does not divide the order of G. If it does, then $\Gamma(G)$ is not semi-simple. *P. J. Hilton* (Ithaca, N.Y.)

Steinfeld, O. 4011

Über die Operatorendomorphismen gewisser Operatorhalbgruppen.

Acta Math. Acad. Sci. Hungar. **15** (1964), 123–131.

The author generalizes some results on modules over rings to semimodules over semirings. Some of his theorems are: The semiring $E(M)$ of operator endomorphisms of a semimodule M over a semiring S, where M is a direct sum of semimodules $M_1, \cdots, M_k$ over S, is isomorphic to the semiring of all square matrices with k rows whose elements θ_{ij} are S-endomorphisms of M_i into M_j. If $M_1, \cdots, M_k$ are all isomorphic to M_1, then $E(M)$ is isomorphic to the semiring of all square matrices with k rows and elements in $E(M_1)$. If the semimodule M is simple, the semiring $E(M)$ is left-regular.

A semiring with unit which is a direct sum of k isomorphic left ideals L, is isomorphic to a complete matrix ring of order k with elements in $E(L)$. If the semiring S with zero and unit elements is a strong direct product of a finite number of minimal left ideals, then the semiring of endomorphisms of S, considered as a left module over itself, is isomorphic to S.

O. Frink (University Park, Pa.)

Weinert, H. J. 4012

Über Halbringe und Halbkörper. III.

Acta Math. Acad. Sci. Hungar. **15** (1964), 177–194.

L'auteur continue ses recherches sur les semi-anneaux (addition et multiplication associatives et distributivité de la multiplication) et les semi-corps (semi-anneaux dont les éléments non nuls, quand le 0 existe, forment un groupe multiplicatif) et donne une douzaine de théorèmes sur les semi-anneaux ordonnés et les extensions de semi-corps [cf. mêmes Acta **13** (1962), 365–378; MR **26** #3634; ibid. **14** (1963), 209–227; MR **26** #6219]. Un semi-anneau est ordonné s'il existe entre ses éléments une relation $a<b$ irréflexive, transitive et connexe, avec addition et multiplication monotones:

$$a < b \Rightarrow a+c < b+c \wedge c+a < c+b,$$

$$a < b \Rightarrow (ac < bc \wedge ca < cb) \vee (ac > bc \wedge ca > cb), \quad c \neq 0.$$

Un semi-anneau, $\mathfrak{R}$, avec addition abélienne satisfait à la propriété (*) si l'une au moins des équations en x, $a+x=b$, $b+x=a$ est résoluble sur $\mathfrak{R}$. Si $\mathfrak{R}$ est un semi-anneau ordonné et $\mathfrak{S}$ un semi-anneau quotient à droite, l'ordre sur $\mathfrak{R}$ induit de manière univoque un ordre sur $\mathfrak{S}$; théorème analogue pour les semi-anneaux différence. Pour qu'un semi-anneau $\mathfrak{R}$ avec addition régulière et commutative puisse être ordonné il faut, et si (*) est remplie, il suffit, ou bien que $\mathfrak{R}$ n'ait pas de 0, ou bien que celui-ci ne soit pas contenu dans le noyau de $\mathfrak{R}$. Puis viennent 8 théorèmes sur les semi-corps: extensions, anneaux de polynômes, éléments transcendants, idéaux, application au corps Δ des réels, à celui $\mathbf{Z}$ des nombres complexes. Par exemple, soit H un semi-corps propre, qui puisse être plongé dans un corps commutatif et dont les éléments algébriques sont sur $\mathbf{P}$. Alors on peut immerger H dans le semi-corps Θ des réels positifs toutes les fois qu'il existe un semi-corps vrai contenant H et possédant la propriété (*). {À cause de la prolifération actuelle des symboles dans la littérature—$\mathbf{P}$, Γ, $\mathfrak{K}_0$, K, R_0, Q, $\cdots$ rien que pour désigner le corps des rationnels!—un index eût facilité la lecture de ce travail.} *A. Sade* (Marseille)

NON-ASSOCIATIVE ALGEBRA

See also 4040, 4374.

Block, Richard E. 4013

On Lie algebras of rank one.

Trans. Amer. Math. Soc. **112** (1964), 19–31.

Kaplansky [same Trans. **89** (1958), 149–183; MR **20** #5799] proved that if L is a Lie algebra of dimension >3 over an algebraically closed field F of characteristic >3 and if L has a one-dimensional Cartan subalgebra such that, for every nonzero root α, multiplication between L_α and $L_{-\alpha}$ is nondegenerate, then all root spaces are one-dimensional and the roots form a group under addition. In the paper under review, the author proves that such an algebra L is necessarily an Albert-Zassenhaus algebra, i.e., an algebra with a basis $\{u_\alpha | \alpha \in G\}$, where G is a finite additive subgroup of F, and with multiplication

$$u_\alpha u_\beta = \{\alpha h(\beta) - \beta h(\alpha) + \alpha - \beta\} u_{\alpha+\beta},$$

where h is any additive mapping of G into F.

The author also states a conjecture that there are only finitely many non-isomorphic Albert-Zassenhaus algebras over F for each dimension p^n.

Rimhak Ree (Vancouver, B.C.)

Bernat, Pierre 4014

Sur le corps enveloppant d'une algèbre de Lie résoluble.

C. R. Acad. Sci. Paris **258** (1964), 2713–2715.

Let $\mathfrak{g}$ be a Lie algebra over a field k and let K be the left quotient ring of the universal enveloping algebra of $\mathfrak{g}$. This paper contains a sketch of a proof of the following

theorem: If k is the real field and if $\mathfrak{g}$ is solvable, then the center of K is a pure transcendental extension of k.

This result is known if $\mathfrak{g}$ has a composition series with one-dimensional factors [the author, same C. R. **254** (1962), 1712–1714; MR **24** #A2642], and this is applied here to the complexification of $\mathfrak{g}$.

G. Leger (Cambridge, Mass.)

Witthoft, William G. 4015

A class of nilstable algebras.

Trans. Amer. Math. Soc. **111** (1964), 413–422.

Let A be a finite-dimensional algebra over a field F of characteristic p not equal to 2 or 3 and satisfying an identity of the form (1) $\alpha_1(xz)y+\alpha_2(yz)x+\alpha_3(zx)y+\alpha_4(zy)x+\alpha_5x(zy)+\alpha_6y(zx)+\alpha_7x(yz)+\alpha_8y(xz)=0$. Kosier [same Trans. **102** (1962), 299–318; MR **24** #A3187] showed that the class $\mathfrak{A}$ of such algebras is defined by six residual identities, and each algebra in $\mathfrak{A}$ either satisfies one of these identities or is quasi-equivalent to an algebra satisfying one of them. One of these identities is (2) $x(xa)+(ax)x=2(xa)x$. In the same paper it was proved that if A is a semisimple, strictly power-associative algebra, then it has an identity and is the direct sum of simple algebras. Also, if in addition A is simple and of degree $t>2$ (t is the maximum number of pairwise orthogonal idempotents that occur in any scalar extension of A), A is flexible. If $t=2$ and A is a simple algebra over an algebraically closed field F, then A has an identity $1=e+f$, where e and f are primitive pairwise orthogonal idempotents. Then A can be decomposed into a vector space direct sum $A=A(1)+A(\frac{1}{2})+A(0)$, where $e\circ x=\frac{1}{2}(ex+xe)=\lambda x$ for all x in $A(\lambda)$, $\lambda=1,\frac{1}{2},0$. In addition, $A(1)=eF+N_1$, $A(0)=fF+N_0$, where N_i is the set of nilpotent elements of $A(i)$. It can be shown that $A(\lambda)\circ A(\frac{1}{2})\subseteq A(\frac{1}{2})+A(1-\lambda)$ for $\lambda=1,0$. If $A(\lambda)\subseteq A(\frac{1}{2})+N_{1-\lambda}$, then A is said to be nilstable. In this paper, A is assumed to be a finite-dimensional, power-associative of degree 2, nilstable algebra satisfying (2) over an algebraically closed field F. If A^+ is the algebra that is the same vector space as A but has a multiplication $x\circ y=\frac{1}{2}(xy+yx)$ where xy is the product in A, then A^+ is shown to be nilstable and Jordan. A trace argument is given to show that A^+ is simple and A is flexible. The multiplication table for such an algebra has been described by Kokoris [Proc. Amer. Math. Soc. **13** (1962), 335–340; MR **25** #1190]. If the characteristic is 0, then A is nilstable [Kokoris, Trans. Amer. Math. Soc. **77** (1954), 363–373; MR **16**, 442]. Hence a description is given of all algebras of degree two and characteristic 0 of this type.

R. H. Oehmke (Princeton, N.J.)

HOMOLOGICAL ALGEBRA

See also 3994, 4032, 4035, 4329a–d.

Bénabou, Jean 4016

Algèbre élémentaire dans les catégories avec multiplication.

C. R. Acad. Sci. Paris **258** (1964), 771–774.

The discussion of a category with multiplication (c.m.) initiated by the author [same C. R. **256** (1963), 1887–1890; MR **26** #6225] is augmented by considering commutative c.m.'s and monoid objects in a c.m. A simple construction is given of a semi-simplicial object associated with a co-monoid object with a co-unit. Many examples are mentioned. {It seems likely that a great deal of mathematics can be formulated in these terms, chiefly by use of the multiplicative structure in the endomorphism category of a given category and its ring objects determined by pairs of adjoint functors.}

J. W. Gray (Urbana, Ill.)

Gerstenhaber, Murray 4017

A uniform cohomology theory for algebras.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 626–629.

A cohomology theory is determined for categories of equationally defined algebras and homomorphisms by a "Yoneda" type process of considering appropriate extensions. In specific cases the groups agree with those, respectively, of Hochschild, Chevalley-Eilenberg, MacLane-Shupla, Dixmier, and MacLane-Eilenberg, but they disagree in general with those of Harrison. If every algebra is a quotient of a free algebra, and if projective resolutions exist, then the cohomology can be computed from a projective resolution of a generic singular extension. Finally, a dimension-shifting device is introduced by means of a canonical filtration of the cohomology groups.

J. W. Gray (Urbana, Ill.)

GROUP THEORY AND GENERALIZATIONS

See also 3985, 3994, 4011, 4012, 4418, 4422, 4424, 4425, 4850, 4857.

Lipschutz, Seymour 4018

An extension of Greendlinger's results on the word problem.

Proc. Amer. Math. Soc. **15** (1964), 37–43.

The main result of this paper is the following theorem: Let C_1 consist of groups G which admit solutions to their word problems and to their extended word problems with respect to the infinite cyclic group generated by any element W in G of infinite order. Let C_k, $k>1$, consist of groups G which are the generalized free products of groups in C_{k-1} with an infinite cyclic group amalgamated. Then any group G in C_k admits a solution to its word problem, and the extended word problem with respect to any infinite cyclic subgroup is solvable.

Solving the extended word problem of a group G with respect to a subgroup H means effectively determining whether or not elements of G belong to H. The proof makes use of B. H. Neumann's work on free products of groups with amalgamated subgroups. The author applies this theorem to a class of groups whose word problems were solved by the reviewer. He conjectures that this result, as well as his others, can be applied to classes of groups whose word problems were solved by Britton, Schiek and Tartakovskiĭ.

M. Greendlinger (Ivanovo)

Rapaport, Elvira Strasser 4019

Proof of a conjecture of Papakyriakopoulos.

Ann. of Math. (2) **79** (1964), 506–513.

Consider the group P^* generated by $a, b, x_1, x_2, \cdots, x_{2g-2}$, subject to the relation

$$aba^{-1}b^{-1}x_1x_2x_1^{-1}x_2^{-1}\cdots x_{2g-3}x_{2g-2}x_{2g-3}^{-1}x_{2g-2}^{-1} = 1,$$

and let N^* be the normal subgroup of P^* generated by the element $w^*a(w^*)^{-1}a^{-1}$, where w^* is any element of P^*. The author shows that P^*/N^* is torsion-free. This confirms a conjecture of Papakyriakopoulos [same Ann. (2) **77** (1963), 250–305, p. 251; MR **26** #3027]. The question arose from the work of Papakyriakopoulos on the Poincaré conjecture. The author proceeds as follows. Let H^* be the kernel of the obvious projection of P^*/N^* onto the free cyclic group generated by a. Then it need only be shown that H^* is torsion-free. Careful and intricate changes in the presentations of the various groups involved show that H^* is a generalised free product of certain groups H_r with certain subgroups amalgamated. The groups H_r have specific presentations and an application of a theorem of Karrass, Magnus and Solitar [Comm. Pure Appl. Math. **13** (1960), 57–66; MR **23** #A1696] shows that each H_r is torsion-free. The proof is then completed by appealing to the well-known theorem that a free product with amalgamation is torsion-free if its factors are torsion-free.

D. B. A. Epstein (Cambridge, England)

Head, T. J. 4020

Note on the occurrence of direct factors in groups.

Proc. Amer. Math. Soc. **15** (1964), 193–195.

Theorem 1: Each proper normal subgroup of a group G is contained in a proper direct factor of G if and only if G is a restricted direct product of simple groups. This generalises a theorem of the reviewer [J. London Math. Soc. **35** (1960), 310–320; MR **24** #A164]. Theorem 2: A group G is a subdirect product of simple groups if and only if for each non-unit normal subgroup N of G there is a proper normal subgroup N' such that $NN' = G$. Theorem 3: Every group can be embedded as a semidirect factor in a group G such that each non-unit normal subgroup of G contains a non-unit direct factor of G. Proofs of the first two theorems are elementary, relying only on Zorn's lemma; Theorem 3 is based on the result that every group can be embedded in a non-cyclic simple group.

J. Wiegold (Cardiff)

Ito, Noboru 4021

A note on transitive permutation groups of degree $p = 2q+1$, p and q being prime numbers.

J. Math. Kyoto Univ. **3** (1963), 111–113.

As a continuation of the author's investigations [Osaka Math. J. **14** (1962), 213–218; MR **26** #185; Bull. Amer. Math. Soc. **69** (1963), 165–192; MR **26** #5050] on transitive permutation groups of prime degree, the following theorem is proved in this paper. Let Γ be a nonsolvable transitive permutation group of degree $p = 2q+1$, where p, q are primes. If Γ has only one irreducible character of degree divisible by p, then $p = 5$, 7 or 11 and Γ is isomorphic to the linear fractional group $LF_2(p)$.

B. Chang (Vancouver, B.C.)

Curtis, Charles W. 4022

Groups with a Bruhat decomposition.

Bull. Amer. Math. Soc. **70** (1964), 357–360.

The groups of the title are characterized by axioms less restrictive than those of the reviewer [Canad. J. Math. **9** (1957), 347–351; MR **19**, 387] but more restrictive than those of J. Tits [C. R. Acad. Sci. Paris **254** (1962), 2910–2912; MR **25** #2149]. The author proves that the simple finite groups which satisfy his axioms relative to a Weyl group of type A_n, D_n or E_n are just the known ones that come from the Lie theory. By an inductive procedure he reduces the situation to the case of type A_2 for which the result is substantially due to Higman and McLaughlin [Illinois J. Math. **5** (1961), 382–397; MR **24** #A1069]. {Remark: At the colloquium on finite geometries held at Rome in 1963 Tits has announced, in terms of his own axioms, the corresponding result for arbitrary Weyl groups of rank at least three.}

R. Steinberg (Los Angeles, Calif.)

Dade, E. C. 4023

Lifting group characters.

Ann. of Math. (2) **79** (1964), 590–596.

Let G be a finite group and let π be a set of prime numbers. Every element x in G has a unique decomposition $x = x_\pi x_{\pi'} = x_{\pi'} x_\pi$, where x_π, $x_{\pi'}$ are π- and π'-elements, respectively. If S, T are subsets of G, define $S^T = \{s^t = t^{-1}st \mid s \in S, t \in T\}$ and $\varphi(S) = \{x \in G \mid x_\pi \in S^G\}$. Let H be a subgroup of G and let K be an invariant subset of H consisting of π-elements, let $K_1, \cdots, K_\nu$ be the conjugate classes of H contained in K. Let $\mathrm{Cl}(K)$ be the space of all complex-valued clan functions on H which vanish on $H-K$. If $f \in \mathrm{Cl}(K)$, define the function f^σ on G by $f^\sigma(x) = f(k_i)$ if $x \in \varphi(K_i)$, $k_i \in K_i$ for some $i = 1, \cdots, \nu$ and $f^\sigma(x) = 0$ otherwise. The following assumptions are relevant: (1.5) $\varphi(K_i) \cap \varphi(K_j)$ is empty for $i \neq j$, $i, j = 1, \cdots, \nu$, or equivalently, $K_i = K \cap K_i^G$ for $i = 1, \cdots, \nu$; (2.2) For each $k \in K$, the group $C_G(k)$ possesses a normal Hall π'-subgroup having $C_H(k)$ as a complementary subgroup. The author proves the following elegant results. Proposition 2: Suppose that (1.5) and (2.2) hold. Then σ is an isometry on $\mathrm{Cl}(K)$. Proposition 3: Suppose that (1.5) and (2.2) hold and $f \in \mathrm{Cl}(K)$; then

$$f^\sigma = -\sum_S \frac{(-1)^{|S|} f_S^*}{|H : N_H(S)|},$$

summed over all non-empty subsets S of K. (* denotes the induction map.) The proof of Proposition 2 is straightforward. However, Proposition 3 requires an ingenious argument. It is needed to show that σ maps generalized characters onto generalized characters, a fact which follows directly from it.

In showing that finite groups of odd order are solvable [the reviewer and J. G. Thompson, Pacific J. Math. **13** (1963), 775–1029] it was necessary to introduce the concept of a tamely imbedded subset of a group G. One of the essential properties of such a subset was that it enabled one to construct a linear isometry τ mapping certain generalized characters of subgroups onto generalized characters of G and having additional properties. In the last section of this paper the author verifies that if K_1 is a tamely imbedded subset, $H = N(K_1)$ and $K = K_1 - \{1\}$, then (1.5) and (2.2) are satisfied. Furthermore, in this situation the maps σ and τ coincide.

W. Feit (Ithaca, N.Y.)

Hering, Christoph 4024

Eine Charakterisierung der endlichen zweidimensionalen projektiven Gruppen.

Math. Z. **82** (1963), 152–175.

Let p be a prime; $q=p^n$. Let $PGL(3, q)$ denote the (full) projective group of the desarguesian projective plane of order q and let $N(q)$ be the centralizer in $PGL(3, q)$ of a non-trivial translation of that plane. Generalizing results of Suzuki [Trans. Amer. Math. Soc. **92** (1959), 191–219; MR **21** #7252] the author characterizes $PGL(3, q)$. Let G be a finite group containing an element π of order p with the following properties: (i) The centralizer $\mathfrak{C}_G\pi$ of π in G is isomorphic to $N(q)$; (ii) $\mathfrak{C}_G\pi$ contains two abelian normal subgroups of order q^2 whose elements $\neq 1$ are conjugate to π in G; (iii) If p is odd, the intersection of any $(q+1)^2$ mutually distinct conjugates of $\mathfrak{C}_G\pi$ is equal to 1. Then $G\simeq PGL(3, q)$ unless $q=2$ and $G\simeq A_6$. The converse also holds true. The proof is based on a detailed description of the group $N(q)$. *P. Scherk* (Toronto, Ont.)

Moser, W. O. J. 4025

Remarks on a paper by S. Trott.

Canad. Math. Bull. **7** (1964), 49–51.

"There are several statements in [3] which require clarification", [3] being a paper by S. M. Trott [same Bull. **5** (1962), 245–252; MR **25** #5113]. The clarification consists in showing how some of Trott's results could have been simply derived from results contained in the well-known book by H. S. M. Coxeter and the author [*Generators and relations for discrete groups*, Springer, Berlin, 1957; MR **19**, 527]; and claiming that a fact for which Trott had quoted a 1957 thesis by D. Beldin [Reed College, Portland, Ore., 1957] was already contained in the reviewer's 1932 thesis [Math. Ann. **107** (1932), 367–386]. {This latter claim is, however, not strictly justified; and the author's further claim "but Neumann gave a set of defining relations" is false. There are also several minor errors.} *B. H. Neumann* (Canberra)

Wong, W. J. 4026

On finite groups whose 2-Sylow subgroups have cyclic subgroups of index 2.

J. Austral. Math. Soc. **4** (1964), 90–112.

This paper is concerned with a finite group G whose Sylow 2-subgroup S is generated by two elements α, β with the relations

$$\alpha^{2^a} = \beta^2 = 1, \quad \beta^{-1}\alpha\beta = \alpha^{2^{a-1}+\varepsilon}, \quad \varepsilon = \pm 1, \quad a \geqq 3.$$

If $\varepsilon=1$, then G can easily be shown to possess a normal 2-complement. If $\varepsilon=-1$, the additional assumption is made that the 2-complement of the centraliser of $\alpha^{2^{a-1}}$ is abelian, and character-theoretic calculations are used to determine G, up to a normal subgroup of odd order. The only simple groups that arise are $PSL(3, 3)$ and the Mathieu group M_{11}. *Graham Higman* (Oxford)

Megibben, Charles 4027

A note on a paper of Bernard Charles (Étude sur les sous-groupes d'un groupe abélien).

Bull. Soc. Math. France **91** (1963), 453–454.

Soit G un p-groupe abélien, et soit donnée dans G la topologie (p-adique) dont les voisinages de 0 sont les sous-groupes p^iG (i nombre naturel). Soit H un sous-groupe de G, et soit K un sous-groupe pur minimal de G contenant H. Alors, tout élément de K d'ordre p est dans H si une des conditions suivantes est satisfaite: (1) l'ordre des éléments de H est $\leqq p$; (2) il y a un sous-groupe D de H qui est dense dans H (dans la topologie p-adique de G) et est pur dans G. On donne aussi un exemple d'un p-groupe abélien G sans éléments de hauteur infinie, contenant un sous-groupe H vérifiant (1), tel qu'il n'y a aucun sous-groupe pur minimal de G contenant H. Aussi T. J. Head [même Bull. **91** (1963), 109–112; MR **27** #201] avait donné un exemple analogue sous l'hypothèse que H vérifie (2) au lieu de (1). Ces deux résultats prouvent qu'un théorème de B. Charles [*ibid.* **88** (1960), 217–227; MR **22** #8055] n'est pas correct. *G. Zappa* (Florence)

Berkovič, Ja. G. 4028

Structure of insoluble finite groups of variance four. (Russian)

Dokl. Akad. Nauk BSSR **8** (1964), 85–86.

All groups mentioned in this review are finite. The notion of the variance $v(G)$ of a group G was introduced by W. E. Deskins [Illinois J. Math. **5** (1961), 306–313; MR **23** #A1708], and he proved there that any group with variance less than 4 is soluble. If G is any group, denote by $\lambda(G)$ the number of prime factors of the order of G; that is, if G has order $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ with obvious notation, then $\lambda(G)=\alpha_1+\cdots+\alpha_k$. The following results are announced. Theorem 1: Let G be an insoluble group such that $\lambda(H)\leqq 3$ for every soluble subgroup H of G. Then $\lambda(G)\leqq 6$. Further: (a) If $\lambda(G)=4$, then G is A_5; (b) If $\lambda(G)=5$, then G is either $LF(2, 11)$ or $LF(2, 13)$; (c) If $\lambda(G)=6$, then $G=LF(2, p)$, where, if $p\neq 19$, the numbers $(p-1)/2$, $(p+1)/2$ are square-free. Theorem 2: Let G be an insoluble group of variance 4. Then G is either $SL(2, 5)$ or one of the groups mentioned in Theorem 1, for which $p\not\equiv \pm 1 \pmod 5$. Theorem 3: Let G be an insoluble group all of whose fourth maximal subgroups are normal. Then $v(G)=4$. *J. Wiegold* (Cardiff)

Kargapolov, M. I. 4029

On generalized solvable groups. (Russian)

Algebra i Logika Sem. **2** (1963), *no.* 5, 19–28.

The first two sections of the paper contain proofs of results announced elsewhere [Uspehi Mat. Nauk **14** (1959), no. 5 (89), 223–224; Dokl. Akad. Nauk SSSR **127** (1959), 1164–1166; MR **21** #6392]. V. S. Čarin showed [Mat. Sb. (N.S.) **41** (**83**) (1957), 297–316; MR **19**, 385] that if G is a group with an invariant series in which the factors are abelian groups of boundedly finite ranks, then some term of the finite derived series of G has an ascending central series. In his doctoral dissertation Čarin left open the analogous question: If a group G has an invariant system where the factors are abelian groups of boundedly finite rank, must some term of the finite derived series of G have a central system? The third section of the paper under review provides a negative answer. The counterexample is an extension H of a finite elementary abelian p-group G by a free group F such that the automorphism group induced by H on G is a non-abelian simple group with order not divisible by p. *J. Wiegold* (Cardiff)

Merzljakov, Ju. I. 4030
On the theory of generalized solvable and generalized nilpotent groups. (Russian)
Algebra i Logika Sem. **2** (1963), *no.* 5, 29–36.

The paper consists of construction of two examples answering in the negative several questions about generalised soluble and generalised nilpotent groups. Example 1: There is a $\bar{Z}$-group (every factor-group has a central system) possessing a non-abelian free group as subgroup. Specifically, the group of all matrices of the form $\begin{pmatrix} 2a+1 & 2b \\ 2c & 2d+1 \end{pmatrix}$, where a, b, c, d are all rationals with odd denominator in reduced form, is a $\bar{Z}$-group; but $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ generate a free group of rank 2. This answers problem X of Kuroš and Černikov [Uspehi Mat. Nauk **2** (1947), no. 3 (19), 18–59; MR **10**, 677], which asks whether every subgroup of an $\overline{RI}$-group (every factor-group has an invariant soluble system) is likewise an $\overline{RI}$-group. Example 2: There exists a complete direct product G of finite nilpotent groups, a homomorphism ϕ and a subgroup H of G such that $\phi(H)$ is a finite non-nilpotent group. For the purposes of this review, let C_n denote a cyclic group of order n. Then G is the complete direct product of groups G_n, G_n being the wreath product of a C_{p^n} with a C_p. In this, one picks out a subgroup H which is the wreath product of a C_ω by a C_p; there is a homomorphism of G such that the image of H is the wreath product of a C_q by a C_p. (It is understood that p and q are distinct primes.) This means that none of the following properties is inherited by all complete direct products: $\bar{Z}$, ZA, $\bar{N}$ (through every subgroup there is a normal system), N (through every subgroup there is a fully ordered normal system), N_0 (through every subgroup there is a finite normal series).
J. Wiegold (Cardiff)

Mohamed, I. J. 4031
On the class of the stability group of a subgroup chain.
J. London Math. Soc. **39** (1964), 109–114.

This is a sequel to the author's earlier paper [Proc. London Math. Soc. (3) **13** (1963), 711–723; MR **27** #5836]. Let A be a group of automorphisms of the group G. In the holomorph put $G_0 = G$ for $i > 0$, $G_{i+1} = [G_i, A]$. If $G_n = 1$ but $G_{n-1} \neq 1$, G is A-nilpotent of A-class n. L. Kaloujnine [Bericht über die Mathematiker-Tagung (Berlin, 1953), pp. 164–172, Deutscher Verlag der Wiss., 1953; MR **17**, 456] showed that if each G_i is normal in G and G is A-nilpotent of A-class n, then A is nilpotent of class less than n. P. Hall [Illinois J. Math. **2** (1958), 787–801; MR **21** #4183] removed the condition of normality of the G_i and showed that A is still nilpotent, but of class not more than $n(n-1)/2$. The following is a generalization of Hall's result. Theorem 3: If G is A-nilpotent of A-class n, then $[G_{n-i}, U] \subseteq G_{n-i+j+1}$, where U stands for the $((i-2)(i-1)/2+j)$th term of the lower central series of A; this for $0 \leq j \leq i-1 \leq n-1$; and A is nilpotent of class at most $n(n-1)/2$. The author goes on to consider what happens if the bound $n(n-1)/2$ is actually attained. Theorem 4: Let G be an A-nilpotent of A-class n, $n \geq 3$. If A is of class exactly $n(n-1)/2$, then G_i/G_{i+1} $(1 \leq i \leq n-1)$ contains a subgroup of class $(n-i)(n-i-1)/2+1$. If the class of A exceeds $(n-1)(n-2)/2$, then G_1 has non-trivial centre.
J. Wiegold (Cardiff)

Tate, John 4032
Nilpotent quotient groups.
Topology **3** (1964), *suppl.* 1, 109–111.

Let p be a positive integer. For each finite group G define $G_0 = G$, $G_{n+1} = (G_n)^p[G, G_n]$ for $n > 0$, and $G_\infty = \bigcap_{n=0}^\infty G_n$. Theorem: Let G be a finite group and S a subgroup of index prime to p. If $S_1 = S \cap G_1$, then $S_n = S \cap G_n$ for all n and $S_\infty = S \cap G_\infty$. The special case in which p is a prime and S is a Sylow p-subgroup of G has been obtained by Huppert for $p \neq 2$ as a consequence of Thompson's thesis [Acta Sci. Math. (Szeged) **22** (1961), 46–61; MR **24** #A1310] and by Thompson for $p = 2$ (unpublished). However, the proof given here is shorter and more direct than the previous proofs. Aside from elementary properties of group cohomology it is necessary to use Shapiro's lemma relating the cohomology of a subgroup to that of a group and to insure the good behavior of the "transgression" map. The author also observes that the theorem can be dualized.
W. Feit (Ithaca, N.Y.)

Liebeck, Hans 4033
Locally inner and almost inner automorphisms.
Arch. Math. **15** (1964), 18–27.

A covering of a group is a set of subgroups which form a directed set under inclusion and whose union is the whole group. An automorphism σ of a group is called almost inner if for any covering Σ by finitely generated subgroups, there exists T in Σ such that σ induces an inner automorphism in every X in Σ containing T. The almost inner automorphisms form a normal subgroup of the automorphism group, containing the inner automorphisms, and contained in the group of locally inner automorphisms. Examples are given to show that these inclusions may be proper, and conditions are given in which they are not.
Graham Higman (Oxford)

Galyarskii, É. I. [Galjarskiĭ, È. I.]; Zamorzaev, A. M. 4034
Similarity symmetric and antisymmetric groups.
Kristallografija **8** (1963), 691–698 (*Russian*); *translated as Soviet Physics Cryst.* **8** (1964), 553–558.

Authors' summary: "The concept of groups of symmetry of similarity, introduced by Šubnikov [Kristallografija **5** (1960), 489–496; MR **23** #A941], is defined more precisely. The study and derivation of groups of symmetry and antisymmetry of similarity are related to the study of groups of symmetry and antisymmetry of stems. All nontrivial generalizations of two-dimensional crystallographic groups of similarity symmetry are examined, including the concept of antisymmetry of different types, and prospects for further studies are considered."

Bass, Hyman 4035
The stable structure of quite general linear groups.
Bull. Amer. Math. Soc. **70** (1964), 429–433.

Generalizing a classical result of Dieudonné, the reviewer recently showed [Amer. J. Math. **83** (1961), 137–153; MR **23** #A1724] that, for A any local ring, the normal subgroups of $GL(n, A)$, $n \geq 3$, are essentially the congruence subgroups with respect to the ideals of A. In the present paper, the author describes the beginnings of a global theory pertaining to the structure of $GL(n, A)$ for a quite general ring A. As is to be expected, the results are

effective only if n is sufficiently large compared with the dimension of the space X of maximal ideals in A, i.e., only if n is in the stable range. Put, for an ideal $\mathfrak{a}$ of A, $GL(n, A, \mathfrak{a}) = \ker(GL(n, A) \to GL(n, A/\mathfrak{a}))$. Let $E(n, A)$ be the subgroup of $GL(n, A)$ generated by the elementary matrices and let $E(n, A, \mathfrak{a})$ be the normal subgroup of $E(n, A)$ generated by the elementary matrices in $GL(n, A, \mathfrak{a})$. From the obvious inclusion $GL(n, A) \to GL(n+1, A)$, we have the inductive limits of these groups which will be denoted by $GL(A)$, $E(A)$, etc. The first theorem states that

$$E(A, \mathfrak{a}) = [E(A), E(A, \mathfrak{a})] = [GL(A), GL(A, \mathfrak{a})].$$

Moreover, if $H \subset GL(A)$ is normalized by $E(A)$, then, for a unique ideal $\mathfrak{a}$, $E(A, \mathfrak{a}) \subset H \subset GL(A, \mathfrak{a})$, and H is normal in $GL(A)$. Hence, the knowledge of the normal subgroups of $GL(A)$ is equivalent to a determination of the abelian groups $K^1(A, \mathfrak{a}) = GL(A, \mathfrak{a})/E(A, \mathfrak{a})$. When $\mathfrak{a} = A$, we write $K^1(A)$ instead; this is just the commutator quotient of $GL(A)$.

The second theorem gives information on the stable range whenever A is an algebra, finitely generated as a module, over a commutative ring whose maximal ideal spectrum is a Noetherian space of dimension d. In particular, for $n > d+1$, the homomorphism

$$f_n : GL(n, A, \mathfrak{a})/E(n, A, \mathfrak{a}) \to GL(n+1, A, \mathfrak{a})/E(n+1, A, \mathfrak{a}),$$

induced by the inclusion, is surjective. Topological considerations suggest that, for $n > d+2$, f_n is also injective. An affirmation of this conjecture which would have a number of important applications, constitutes, when A is a division algebra, the essential part of Dieudonné's theory of noncommutative determinants. Next follow finiteness theorems. E.g., let Σ be a semisimple, finite-dimensional algebra over $\mathbf{Q}$, and let A be an order in Σ and $\mathfrak{a}$ an ideal in A. Then $\ker(K^1(A, \mathfrak{a}) \to K^1(\Sigma))$ is finite and $K^1(A, \mathfrak{a})$ is finitely generated. If, in addition, Σ is simple, then center $E(n, A) =$ center $SL(n, A)$ ($SL(n, A)$ is the group of elements of reduced norm one in $GL(n, A)$). Moreover, for $n \geqq 3$, a normal subgroup of $E(n, A)$ is either finite and central or of finite index and, for all sufficiently large n, the same is true for $SL(n, A)$. Here the author remarks that these results, combined with a rather formidable cohomological calculation, form the basis of the proof of the fact that every subgroup of finite index in $SL(n, \mathbf{Z})$, $n \geqq 3$, contains a congruence subgroup (cf. the author, M. Lazard and J.-P. Serre [Bull. Amer. Math. Soc. **70** (1964), 385–392]). In the next paragraph the author announces results of a joint paper with A. Heller and R. Swan (to appear) on the structure of the homomorphisms $K^1(A) \to K^1(A[t])$ and $K^1(A) \to K^1(A[t, t^{-1}])$, t an indeterminate. Concerning the latter, Atiyah has pointed out that the results form an analogue of Bott periodicity for the unitary group. In the final section, it is shown that the previous results yield information on J. H. C. Whitehead's groups of simple homotopy types, results which extend some earlier work of G. Higman [Proc. London Math. Soc. (2) **46** (1940), 231–248; MR **2**, 5]: Let π be a finite group and put $K^1(Z\pi)/\pm\pi = Wh(\pi)$, where $\pm\pi$ denotes the image of $\pm\pi \subset GL(1, Z\pi)$ in $GL(Z\pi)$. Assume that π has q irreducible rational representations and r irreducible real representations. Then $Wh(\pi)$ is finitely generated of rank $r-q$. In particular, if π is free abelian, then $Wh(\pi) = 0$.

W. Klingenberg (Mainz)

Crowe, D. W. 4036

Generating reflections for $U(2, p^{2n})$. II. $p = 2$.
Canad. Math. Bull. **7** (1964), 213–217.

The author has previously shown [Proc. Amer. Math. Soc. **13** (1962), 500–502; MR **25** #1231] that the finite 2-dimensional unitary group $U(2, p^{2n})$ is generated by two reflections if p is different from 2. The present article gives two generating reflections in the case when $p = 2$ and $n > 1$. If $n = 1$, the group is not generated by unitary reflections. The author also gives defining relations in the case $p = 2$, $n = 2$, and in the case $p = 5$, $n = 1$. These are, respectively, $R^5 = (RS)^{15} = I$, $RSR = SRS$, and

$$R^6 = I, \quad R^2S^2R^2S^2R^2 = S^2R^2S^2R^2S^2,$$
$$RS = S^2R^{-2}S^{-2}R^2(S^2R^2)^{-10}.$$

R. G. Stanton (Waterloo, Ont.)

Vol′vačev, R. T. 4037

Sylow p-subgroups of the full linear group. (Russian)
Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 1031–1054.

In this important paper the p-subgroups of $GL(n, F)$ for commutative fields F are studied with respect to reducibility (this had been done before by Suprunenko for algebraically closed F). The many very interesting results in this direction are too detailed to be stated here in full. For p-Sylow subgroups, i.e., for maximal p-subgroups of $GL(n, F)$ the main result is that they form just one conjugate class of subgroups, unless $p = 2$ and the field F satisfies the following three conditions: (a) F is of characteristic $\neq 2$ and does not contain a primitive fourth root of unity; (b) -1 is the sum of two squares in F; (c) in the quadratic extension $F(i)$ ($i^2 = -1$) of F, every element $(\alpha + i\beta)$ ($\alpha, \beta \in F$) of order some power of 2 satisfies the equation $(\alpha + i\beta)(\alpha - i\beta) = 1$. In this exceptional case it is shown that there exist exactly $[n/2]+1$ different classes of conjugate 2-Sylow subgroups of $GL(n, F)$.

Finally, a survey is given of the structure of the p-Sylow subgroups of $GL(n, F)$ for F chosen to be the field of real numbers, the rationals, and an arbitrary finite field. As examples for the exceptional case the 2-Sylow subgroups of $GL(n, F)$ are discussed for $F = R(\sqrt{-5})$, where R is the field of rational numbers.

O. H. Kegel (Chicago, Ill.)

Srinivasan, Bhama 4038

A note on blocks of modular representations.
Proc. Cambridge Philos. Soc. **60** (1964), 179–182.

For p a prime, let the ring R be either a field of characteristic p, or a complete discrete valuation ring in which p is a non-unit. Let RG denote the group ring of a finite group G over R, and make RG into a right $(G \times G)$-module by means of

$$x(g, g') = g^{-1}xg', \quad x \in RG, \quad g, g' \in G.$$

Let E be a block idempotent of RG with defect group D, and let H be any subgroup of G such that $D \cdot C_G(D) \subset H \subset N_G(D)$. The Brauer correspondence assigns to E a central idempotent e of RH.

By using his theory of vertices, J. A. Green [Math. Z. **79** (1962), 100–115; MR **25** #5114] recently proved that $(RH)e$ is isomorphic to a direct summand of $(RG)E$,

viewed as $(H\times H)$-module. The present note gives a much simpler proof of this fact, using only a few basic properties of block idempotents and the Brauer correspondence.

The author also shows that when $H=N(D)$, the module $(RG)E$ is isomorphic to a direct summand of the induced $(G\times G)$-module $\{(RH)e\}^{G\times G}$.

I. Reiner (Urbana, Ill.)

Vislavskiĭ, M. N. 4039

On irreducible representations of finite metabelian p-groups. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 14–18.

A description is given of the non-linear complex irreducible representations of finite metabelian p-groups G whose commutator subgroups are cyclic or elementary abelian. (It should be noted that metabelian in the context of the article means nilpotent of class two.) The construction of the representations proceeds as follows: (i) explicit matrices are given for the case where G has two generators and cyclic center; (ii) central products yield the case where G has cyclic center; (iii) the general case follows from a description of the possible kernels of the representations together with the results of (ii).

P. Fong (Berkeley, Calif.)

Lamont, P. J. C. 4040

Approximation theorems for the group G_2.

Nederl. Akad. Wetensch. Proc. Ser. A **67** $=$ *Indag. Math.* **26** (1964), 187–192.

The group is the group $G(C)$ of linear automorphisms of a Cayley-Dickson algebra C over a number field k. Let p be a place of k, $C_p=C\otimes_k k_p$, $G(C_p)$ its automorphism group (over k_p, the completion of k relative to p), and $G(C)_A$ the "adèlized" group [cf., e.g., M. Kneser, J. Reine Angew. Math. **209** (1962), 96–97; MR **25** #3094]. Then $G(C)$ is embedded in each finite product $G(C_p)\times\cdots\times G(C_q)$ and in $G(C)_A$ via the diagonal map; moreover, since $G(C)_A$ is a subset of $\prod_p G(C_p)$, $G(C_p)$ is embedded in a canonical way in $G(C)_A$. The author's "weak approximation theorem" asserts the density of $G(C)$ in $G(C_p)\times\cdots\times G(C_q)$ (finite product, distinct places); his "strong approximation theorem" asserts the density in $G(C)_A$ of the set of products $G(C_q)G(C)$ whenever q is a place such that C_q is split. The "strong theorem" then applies to show that if C is indefinite (i.e., the associated norm form is indefinite), then any two maximal orders in C are conjugate under $G(C)$. An example shows that conjugacy of maximal orders fails in at least one case if C is definite.

G. B. Seligman (New Haven, Conn.)

Jakubík, J. 4041

Interval topology of an l-group.

Colloq. Math. **11** (1963), 65–72.

Once upon a time G. Birkhoff [*Lattice theory*, Amer. Math. Soc., New York, 1948; MR **10**, 673] asked whether or not a po-group is a topological group in its interval topology. Since that time several authors have found examples of lattice-ordered groups ("l-groups") for which the answer is no, and, in fact, large classes of l-groups with this property. The question now is whether an l-group must be totally ordered in order for it to be a topological group in its interval topology. In this note the author gives an elegant proof that the answer is yes for all l-groups that can be represented as a subdirect sum of totally ordered groups. This, of course, includes the class of all abelian l-groups.

P. F. Conrad (New Orleans, La.)

Yakabe, Iwao 4042

Equivalence of the Krull-Müller-Jaffard theorem and Ribenboim's approximation theorem.

Mem. Fac. Sci. Kyushu Univ. Ser. A **17** (1963), 145–152.

L'auteur démontre ici que le théorème d'approximation de Ribenboim [Math. Z. **68** (1957), 1–18; MR **19**, 1035] concernant l'existence d'un élément d'un corps ayant des valeurs données à l'avance pour des valuations de Krull est une conséquence de l'hypothèse de Krull [Colloq. Algèbre Supérieure (Bruxelles, 1956), pp. 45–77, Établissements Ceuterick, Louvain, 1957; MR **21** #4951] sur les groupes réticulés. Le rapporteur [Bull. Sci. Math. (2) **85** (1961), 1-ère partie, 127–135; MR **27** #5828] avait montré auparavant que l'hypothèse de Krull est une conséquence du théorème de Ribenboim. D. Müller [Math. Z. **77** (1961), 45–62; MR **23** #A3791] et le rapporteur avaient également donné des démonstrations de l'hypothèse de Krull indépendantes du théorème de Ribenboim.

P. Jaffard (Paris)

Bosbach, Bruno 4043

Arithmetische Halbgruppen.

Math. Ann. **144** (1961), 239–252.

In zwei früheren Arbeiten des Verfassers [dieselben Ann. **139** (1960), 184–196; MR **26** #3798; ibid. **141** (1960), 193–209; MR **26** #3799] wurden Holoide, Verbände und Ringe charakterisiert, in denen sich jedes Element in gewisser Weise eindeutig in (Halb-)Primelemente zerlegen lässt. Das Hauptziel der vorliegenden Behandlung ist die Beantwortung einer allgemeineren Frage nach den Bedingungen der halbeindeutigen Zerlegbarkeit. An die in den zitierten Arbeiten eingeführten Begriffe schliessen folgende an: Ein Holoid H heisst (halb-)schwachkanonisch (kurz: hsk. bzw. sk), wenn jedes $a\in H$ eine (Halb-)Primfaktorzerlegung besitzt und je zwei unverkürzbare (Halb-) Primfaktorzerlegungen desselben $a\in H$ dieselben (Halb-) Primfaktoren aufweisen (wenn auch nicht in jeweils derselben Potenz). Wir nennen H ein $A^{*''}$-Holoid [ein A''-Holoid], wenn neben den früher eingeführten Bedingungen $A_1{}^*$ und A_2 [A_1 und A_2] der folgenden $A_3{}'''$ [$A_3{}''$] genügt: $A_3{}'''$ [$A_3{}''$] ist p ein Halbprimelement und gilt $p\underline{\underline{\prec}}bc$ [$p|bc$] und sind p, c minimal fremd, so folgt $p\underline{\underline{\prec}}b$ [$p|b$]. Die Hauptresultate sind folgende: H ist hsk [sk] genau dann, wenn es ein $A^{*''}$- [A''-]Holoid ist. Die Eigenschaften eines Holoids (1) ein A-Holoid, (2) ein B-Holoid, (3) ein C-Holoid, (4) ein C'-Holoid, (5) ein C''-Holoid, (6) halbprimkanonisch und (7) primkanonisch zu sein sind äquivalent. Unter dem Quotientenideal $\mathfrak{b}:\mathfrak{a}$ zweier Ideale $\mathfrak{b}$, $\mathfrak{a}$ aus H verstehen wir das Ideal aller x mit $\mathfrak{a}x\subseteq\mathfrak{b}$. Wir nennen H ein E-Holoid, wenn $\mathfrak{a}:\mathfrak{b}$ stets ein Hauptideal ist und die Menge der endlichen Ideale die aufsteigende Teilerkettenbedingung bezüglich $\underline{\subseteq}$ erfüllt (wobei $\mathfrak{a}\underline{\subseteq}\mathfrak{b}$ bedeutet, dass ein endliches Ideal $\mathfrak{c}$ mit $\mathfrak{b}=\mathfrak{a}:\mathfrak{c}$ gibt). H ist ein E-Holoid genau dann, wenn H das Bedingungssystem (C_1 & C_2 & $C_\cap$) erfüllt oder äquivalent wenn H ein

primkanonisches +Holoid ist. Ein Halbverband mit Einselement ist genau dann kanonisch, wenn er relativ pseudokomplementär ist und C_2 erfüllt. Einen Ring R nennen wir hsk [sk], wenn R hsk [sk] ist. Ein Ring R ist ein Z-Ring genau dann, wenn eine der folgenden Aussagen erfüllt ist: (1) Jedes $a \in R$ lässt sich in Primelemente zerlegen; (2) R ist hsk; (3) R ist sk; (4) R ist halbkanonisch; (5) R ist kanonisch; (6) R ist halbprimkanonisch; (7) R ist primkanonisch. Die Verträglichkeit und Unabhängigkeit der aufgestellten Systeme wird nachgewiesen. *F. Šik* (Brno)

Endler, Otto 4044

Über multiplikative Strukturen und eudoxische Hüllen von archimedischen totalgeordneten Gruppen.

Math. Z. **77** (1961), 339–358.

Mit P sei eine kommutative und assoziative Halbgruppe, mit $\cdot P$ ein Halbring (d.h. eine mit einer assoziativen und hinsichtlich der Addition distributiven Multiplikation versehene Halbgruppe) bezeichnet. $\cdot P$ heisst ein Halbkörper, wenn er ein Einselement und zu jedem $r \in P$ ein Inverses besitzt. Die Menge aller Endomorphismen einer Halbgruppe P bildet hinsichtlich der Komposition eine Halbgruppe. Definiert man $(\sigma \circ \rho)r = \sigma(\rho r)$ $(r \in P;\ \rho, \sigma \in \mathfrak{P})$, so bekommt man einen Halbring $\circ \mathfrak{P}$. Ist $\circ \mathfrak{P}$ ein Halbkörper, so nennen wir P symmetrisch. Durch die Vorschrift $\Theta_r\rho = \rho r$ $(\rho \in \mathfrak{P}, r \in P)$ wird ein Homomorphismus Θ_r von $\mathfrak{P}$ in P definiert. Ist Θ_r eine Bijektion (d.h. eineindeutig und "auf"), so nennen wir r primitiv für P. Eine Injektion $\varphi: P \to P'$ (d.h. ein eineindeutiger Homomorphismus) heisst regulär, wenn ein Homomorphismus $f: \circ \mathfrak{P} \to \circ \mathfrak{P}'$ mit $\varphi \circ \sigma = (f\sigma) \circ \varphi$ $(\sigma \in \mathfrak{P})$ existiert. Eine Halbgruppe P, für die $\circ \mathfrak{P}$ kommutativ ist (1a), alle $\rho \in \mathfrak{P}$ injektiv sind (1b) und für jedes $r \in P$ Θ_r injektiv ist (1c), wollen wir eine A-Halbgruppe nennen. Eine Halbgruppe P heisst zusammenhängend, wenn für alle $r, s \in P$ stets $\rho, \sigma \in \mathfrak{P}$ mit $\rho r = \sigma s$ existieren. In einer A-Halbgruppe P wird durch die Vorschrift $r \sim_P s \Leftrightarrow \rho r = \sigma s$ für geeignete $\rho, \sigma \in \mathfrak{P}$ eine Äquivalenzrelation $\sim_P$ definiert; die zugehörigen Äquivalenzklassen nennen wir die Zusammenhangskomponenten von P. In der Paarmenge $P \times \mathfrak{P}$ (P ist eine A-Halbgruppe) wird durch die Vorschrift $(r, \rho) \approx_P (s, \sigma) \Leftrightarrow \sigma r = \rho s$ $(r, s \in P;\ \rho, \sigma \in \mathfrak{P})$ eine Äquivalenzrelation $\approx_P$ definiert; die durch (r, ρ) bestimmte $\approx_P$-Äquivalenzklasse bezeichnen wir mit r/ρ. Die Menge aller r/ρ $(r \in P;\ \rho \in \mathfrak{P})$ bildet hinsichtlich der durch die Vorschrift $r/\rho + s/\sigma = (\sigma r + \rho s)/(\rho \circ \sigma)$ $(r, s \in P;\ \rho, \sigma \in \mathfrak{P})$ definierten Addition eine Halbgruppe $\bar{P}$, die wir die Fraktionshalbgruppe von P nennen. Durch die Vorschrift $\eta r = r/\iota$ $(r \in P;\ \iota$ die identische Abbildung von $P)$ bzw. $(\mathfrak{f}\sigma)(r/\rho) = (\sigma r)/\rho$ $(r \in P;\ \rho, \sigma \in \mathfrak{P})$ werden eine (reguläre) Injektion $\eta: P \to \bar{P}$ und eine Injektion $\mathfrak{f}: \circ \mathfrak{P} \to \circ \bar{\mathfrak{P}}$ definiert. η ist genau dann eine Bijektion, wenn $\mathfrak{f}\mathfrak{P} = \bar{\mathfrak{P}}$ und dies trifft genau dann zu, wenn P symmetrisch ist. P heisst fraktionsregulär, wenn $(\mathfrak{f}\mathfrak{P})^L \circ \mathfrak{f}\mathfrak{P} = \bar{\mathfrak{P}}$ ist (für eine Bijektion φ_1 φ^L bedeutet die zu φ inverse Bijektion). Ist die Halbgruppe P fraktionsregulär, so ist $\bar{P}$ eine symmetrische A-Halbgruppe. Eine A-Halbgruppe P ist genau dann fraktionsregulär und $\bar{P}$ zusammenhängend, wenn P zusammenhängend ist und P der Bedingung (1c) genügt. Ein kommutativer Halbring (Halbkörper) $\cdot P$, dessen Additionshalbgruppe P eine A-Halbgruppe ist, heisst ein A-Halbring (ein A-Halbkörper). Die A-Halbgruppe P lässt sich genau dann zu einem Halbring (zu einem A-Halbkörper) machen, wenn ein für P primitives Element existiert (wenn P symmetrisch und zusammenhängend ist).

Eine kommutative reguläre Halbgruppe P nennen wir archimedisch, wenn die durch die Vorschrift $r > s \Leftrightarrow r = s + t$ für ein $t \in P$ $(r, s \in P)$ erklärte Relation $>$ den folgenden Bedingungen genügt: $r > s$ oder $s > r$ für jedes Paar $r, s \in P$, $r \neq s$ und für jedes Paar $r, s \in P$ gibt es eine natürliche Zahl n mit $nr > s$. Eine solche Halbgruppe ist die Halbgruppe aller positiven Elemente einer eindeutig bestimmten archimedischen totalgeordneten Gruppe. Der Halbring $\circ \mathfrak{P}$ einer archimedischen Halbgruppe ist kommutativ. Jede archimedische Halbgruppe ist eine A-Halbgruppe. Der Endomorphismenhalbring $\circ \mathfrak{P}$ einer beliebigen archimedischen Halbgruppe P ist ein kommutativer archimedischer Halbring mit dem Einselement ι. Die Fraktionshalbgruppe $\bar{P}$ einer archimedischen Halbgruppe P ist archimedisch, ebenso jede Zusammenhangskomponente P_μ von P und jede Halbgruppe $\bar{P}_{(\mu)} = \{r/\rho : r \in P_\mu;\ \rho \in \mathfrak{P}\}$. Es seien P und P' beliebige archimedische Halbgruppen. Für Elementepaare $(r, s) \in P \times P$ und $(r', s') \in P' \times P'$ schreiben wir $(r, s) \asymp (r', s')$ genau dann, wenn $kr > ls \Leftrightarrow kr' > ls'$ für natürliche Zahlen k, l. Die Relation $\asymp$ ist eine Äquivalenzrelation auf $P \times P$. Die durch $(r, s) \in P \times P$ bestimmte Äquivalenzklasse bezeichnen wir mit $r:_P s$ und nennen sie eine Proportion von P. Mit $P : P$ bezeichnen wir die Menge aller Proportionen von P. Die Proportion $r:_P s$ heisst regulär, wenn zu jedem $t \in P$ ein $u \in P$ mit $r:_P s = t:_P u$ existiert. Die Menge aller regulären Proportionen von P bezeichnen wir mit $R(P : P)$. Gilt $R(P : P) = P : P$, so heisst P eudoxisch. Insbesondere ist die Additionshalbgruppe P jedes archimedischen Halbkörpers $\cdot P$ eudoxisch. Wir schreiben $P \lesseqqgtr P'$, falls eine Injektion von P in P' existiert. Im Falle $P \lesseqqgtr P'$ wird durch die Vorschrift $\Pi_{P,P'} r:_P s = r':_{P'} s' \Leftrightarrow (r, s) \asymp (r', s')$ $(r, s \in P;\ r', s' \in P')$ eine eineindeutige Abbildung $\Pi_{P,P'}$ von $P : P$ in $P' : P'$ definiert, und für jede Injektion φ von P in P' gilt $\Pi_{P,P'} r:_P s = \varphi r :_{P'} \varphi s$ $(r, s \in P)$. Ist $P \lesseqqgtr P'$ und P' eudoxisch, so existiert zu jedem Paar $r \in P$, $r' \in P'$ genau eine Injektion φ von P in P' mit $\varphi r = r$. Die folgenden Bedingungen sind äquivalent: (α) P ist eudoxisch; (β) alle $r \in P$ sind primitiv für P; (γ) P ist symmetrisch und zusammenhängend. Wir sagen dass P regulär einbettbar in P' ist, in Zeichen $P \preccurlyeq P'$ wenn $P \lesseqqgtr P'$ und $\Pi_{P,P'} R(P : P) \subseteq R(P' : P')$ gelten. Eine archimedische Halbgruppe P^* soll symmetrische bzw. eudoxische Hülle von P heissen, wenn $P \preccurlyeq P^*$ gilt (A), P^* symmetrisch bzw. eudoxisch ist (B) und ist P' symmetrisch bzw. eudoxisch und $P \preccurlyeq P'$, so gilt $P^* \lesseqqgtr P'$ (C). Ist P fraktionsregulär, so ist $\bar{P}$ eine symmetrische Hülle von P, die in Verschärfung von (C) der folgenden Bedingung genügt: Ist P' symmetrisch und $P \preccurlyeq P'$, so ist $\bar{P} \preccurlyeq P'$. Ist P zusammenhängend, so ist $\bar{P}$ eine symmetrische und eudoxische Hülle von P. Für nichtfraktionsreguläre P lässt sich über die Fraktionshalbgruppe $\bar{P}$ von P keine allgemeingültige Aussage machen. Der Verfasser bemerkt, dass das Problem, ob zu jeder nicht-fraktionsregulären archimedischen Halbgruppe eine symmetrische Hülle existiert, ungelöst bleibt. Dagegen besitzt jede archimedische Halbgruppe eine euxodische Hülle. Im weiteren werden archimedische Halbringe untersucht. Die Behandlung wird durch Beispiele abgeschlossen. *F. Šik* (Brno)

Inasaridze, H. N. 4045
On simple semigroups. (Russian)
Mat. Sb. (*N.S.*) **57** (**99**) (1962), 225–232.

Eine Halbgruppe D heisst einfach, wenn jeder ihrer Homomorphismen ein Isomorphismus ist oder wenn er D auf ein Element abbildet. In der vorliegenden Arbeit werden alle einfachen Halbgruppen D mit Nullelement der zwei folgenden Typen beschrieben: (1) Zu jedem Elementepaar $a, b \in D$ existieren natürliche Zahlen r, s, t so dass $(ab)^r = a^s b^t = b^t a^s$ gilt. (2) Ohne nilpotente Elemente. Weiterhin wird eine Konstruktion (das sog. starke Produkt) vorgegeben, die aus einem System von einfachen Halbgruppen mit Nullelement eine weitere einfache Halbgruppe mit Nullelement konstruieren zulässt. Für die Einfachheit von Halbgruppen mit Einselement wird eine notwendige und hinreichende Bedingung festgestellt. Weitere Ergebnisse stellen Beispiele einfacher Halbgruppen dar, die einigen weiteren Bedingungen genügen: Zwei nichtisomorphe einfache Halbgruppen mit Nullelement und mit Elemente unendlicher Ordnung (eine von denen eine verallgemeinerte Gruppe ist) und zwei nichtisomorphe Halbgruppen ohne das Nullelement und mit Elementen unendlicher Ordnung (eine von denen auch eine verallgemeinerte Gruppe ist). *F. Šik* (Brno)

Lallement, Gérard 4046
Sur les homomorphismes d'un demi-groupe sur un demi-groupe complètement-0-simple.
C. R. Acad. Sci. Paris **258** (1964), 3609–3612.

Let ρ be a congruence on a semigroup D such that D/ρ is completely 0-simple. Let **K** be a family of ρ-classes such that the complexes from **K** are elements of D/ρ belonging to different **H**-class-groups [cf. G. B. Preston, Proc. London Math. Soc. (3) **11** (1961), 557–576; MR **24** #A2628], every **H**-class being considered exactly once. Then ρ is the maximal congruence having all complexes from **K** as its classes. There are found relations between two such families **K** and **K**′ of ρ-classes.

Let **K** be a family of complexes K_i of a semigroup D. Set $\mathbf{R}(\mathbf{K}) = \mathbf{R}_{\mathbf{K}} \cap {}_{\mathbf{K}}\mathbf{R}$, where $\mathbf{R}_{\mathbf{K}} = \bigcap_{K_i \in \mathbf{K}} \mathbf{R}_{K_i}$, $\mathbf{R}_{K_i}$ being the principal right equivalence associated with K_i [P. Dubreil, *Algèbre*, Tome I, 2ième éd., Gauthier-Villars, Paris, 1954; MR **16**, 328], ${}_{\mathbf{K}}\mathbf{R}$ is defined dually. **K** is called regular if $\mathbf{R}(\mathbf{K})$ is a congruence; **K** is called strong ("forte") if for every $K_i, K_j \in \mathbf{K}$, $K_i \cdot a \cap K_i \cdot b \neq \emptyset$, $K_j \cdot a \cap K_j \cdot b \neq \emptyset \Rightarrow a\mathbf{R}(\mathbf{K})b$; **K** is called neat ("nette") if $\bigcup K_i$ is "nette". A complex K is feebly unitary if for every $x \in D$, $k, k' \in K$, $xk, k'x \in K \Rightarrow x \in K$.

If **S** is a regular, strong and neat family of feebly unitary subsemigroups, then $D/\mathbf{R}(\mathbf{S})$ is completely simple. Conversely, if R is a congruence and D/R is completely simple, then $R = \mathbf{R}(\mathbf{S})$, where **S** is a regular, strong and neat family of subsemigroups of D, inverse images of idempotents of D/R, and every subsemigroup from **S** is feebly unitary.

There are given also two characterizations of completely 0-simple semigroups. *B. M. Schein* (*Šain*) (Saratov)

Saitô, Tôru 4047
Ordered completely regular semigroups.
Pacific J. Math. **14** (1964), 295–308.

The author determines the structure of all totally ordered semigroups which are unions of groups, in the sense of reducing such a semigroup to a set of ordered groups together with a system of order-preserving homomorphisms between them. Clifford [Proc. Amer. Math. Soc. **5** (1954), 449–504; MR **15**, 930] has shown that a semigroup S is a union of groups if and only if S is a semilattice of completely simple semigroups without zero. Thus there is a semilattice T_s and to each element α of T_s there corresponds a semigroup $S(\alpha)$ of S such that the $S(\alpha)$ are mutually disjoint and their union is S, each $S(\alpha)$ is a completely simple semigroup without zero, and $S(\alpha)S(\beta) \subseteq S(\alpha\beta)$, where $\alpha\beta$ is the product of α and β in T_s. If, in addition, S is ordered, then each $S(\alpha)$ is order isomorphic to a lexicographically ordered direct product $G(\alpha)E(\alpha)$ of an ordered group $G(\alpha)$ and an ordered semigroup $E(\alpha)$ consisting of all the idempotents of $S(\alpha)$. The structure of the $E(\alpha)$ has been determined by the author [J. Math. Soc. Japan **14** (1962), 150–169; MR **26** #2533].

It is shown that there exists a set of order-preserving homomorphisms $\varphi_\beta{}^\alpha$ of $G(\alpha)$ into $G(\beta)$ such that if $\alpha \geqq \beta \geqq \gamma$, then $\varphi_\beta{}^\alpha \varphi_\gamma{}^\beta = \varphi_\gamma{}^\alpha$. For each (g, f) in $S(\alpha)$ and (g', f') in $S(\beta)$, $(g, f)(g', f') = ((g\varphi_{\alpha\beta}{}^\alpha)(g'\varphi_{\alpha\beta}{}^\beta), ff')$. Moreover, $(g, f) < (g', f')$ if and only if $g\varphi_{\alpha\beta}{}^\alpha < g'\varphi_{\alpha\beta}{}^\beta$ or $\varphi_{\alpha\beta}{}^\alpha = g'\varphi_{\alpha\beta}{}^\beta$ and $f < f'$. Thus S is completely determined by T_s, the $S(\alpha)$ and the $\varphi_\beta{}^\alpha$. *P. F. Conrad* (New Orleans, La.)

Tully, E. J., Jr. 4048
The equivalence, for varieties of semigroups, of two properties concerning congruence relations.
Bull. Amer. Math. Soc. **70** (1964), 399–400.

A semigroup S is said to be disjunctive if for each s in S the only congruence relation for which $\{s\}$ is an equivalence class is the identity relation, and S is called congruence-permutable if $\alpha\beta = \beta\alpha$ for every pair of congruences α and β on S. A family V of semigroups is called a variety if V contains all subsemigroups, all homomorphic images, and all direct products of elements in V. The author states that for a variety V of semigroups the following are equivalent: each S in V is disjunctive, each S in V is congruence-permutable, each S in V is a group, each S in V is a periodic group. The equivalence of the last two properties is, of course, obvious.
P. F. Conrad (New Orleans, La.)

Vagner, V. V. 4049
Semigroups of partial mappings with a symmetric transitivity relation. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1957**, *no.* 1, 81–88.

Eine Halbgruppe G nennen wir eine verallgemeinerte Gruppe, wenn ihre Idempotente je zwei vertauschbar sind und zu jedem Elemente g ein durch die Vorschrift $gg^{-1} = g$, $g^{-1}gg^{-1} = g^{-1}$ definiertes verallgemeinertes Inverses g^{-1} gibt. Das Element g^{-1} wird eindeutig bestimmt. Die kanonische Darstellung einer Halbgruppe ist eine durch die Vorschrift $C(g) = \lambda_g$ definierte Darstellung, wobei λ_g die durch das Element g bestimmte rechte Translation bedeutet. Ein Sonderinteresse hat die Darstellung einer verallgemeinerten Gruppe mit Hilfe der sogenannten induzierten rechten Translationen, die jedem Elemente g die eineindeutige teilweise Abbildung der Menge G $\bar{\lambda}_g = \lambda_g \cap \lambda_{g^{-1}}^{-1}$ zuordnen. Das Hauptresultat der vorliegenden Arbeit ist der folgende Satz. Das Symmetrant der kanonischen Darstellung einer verallgemeinerten Gruppe

G ist identisch mit der Darstellung mit Hilfe der induzierten rechten Translationen $\bar{C}(g)=\bar{\lambda}_g$. Unter dem Symmetrant $\bar{r}$ einer Darstellung (d.h. einer homomorphen Abbildung) r einer Halbgruppe G in die Halbgruppe $\mathcal{F}(A\times A)$ aller binären Relationen auf einer Menge A verstehen wir die durch die Formel $\bar{r}=f_{\bar{\imath}_r(G)}\circ r$ definierte Darstellung. Dabei für eine beliebige Unterhalbgruppe $\Gamma\subset\mathcal{F}(A\times A)$ und ein beliebiges $\gamma\in\Gamma$ die Abbildung $f_{\bar{\imath}_r}:\Gamma\to\mathcal{F}(A\times A)$ wird durch die Vorschrift $f_{\bar{\imath}_r}(\gamma)=(\bigcup_{\delta\in\Gamma}\delta)^{-1}\cap\gamma$ definiert. *F. Šik* (Brno)

Yusuf, S. M. 4050
Semigroups with operators.
J. Natur. Sci. and Math. **3** (1963), 57–72.

Let S be an inverse semigroup and let E denote its set of idempotents. Let $N=\bigcup\{N_e: e\in E\}$ be a subsemigroup of S, where, for $e\in E$, N_e is a subgroup of S with e as its identity element. Suppose further that $aNa^{-1}\subseteq N$, for all $a\in S$. Then there exists a unique congruence on S with the property that, for $e\in E$, N_e is the congruence class containing e. N is said to be the kernel of this congruence. Conversely, if ρ is an idempotent-separating congruence on S, i.e., if no two distinct idempotents are ρ-equivalent, then, for each $e\in E$, the ρ-class containing e is a subgroup N_e of S and $N=\bigcup\{N_e: e\in E\}$ has the properties listed above [the reviewer, Proc. Glasgow Math. Assoc. **3** (1956), 1–9; MR **18**, 717]. The author begins by extending the above result to inverse semigroups with operators. Analogues of the first and second isomorphism theorems for groups are then obtained for inverse semigroups with operators. *G. B. Preston* (Clayton)

Hoehnke, Hans-Jürgen 4051
Zur Strukturtheorie der Halbgruppen.
Math. Nachr. **26** (1963), 1–13.

The objective of this paper is to base a structure theory of semigroups on the theory of their representations by transformations of a set, analogous to Jacobson's structure theory of rings [*Structure of rings*, Amer. Math. Soc., Providence, R.I., 1956; MR **18**, 373]. Some of the ideas and results were first found by E. J. Tully, Jr. [Amer. J. Math. **83** (1961), 533–541; MR **25** #135]. By a representation ρ of a semigroup S by transformations of a set M is meant a homomorphism $a\to\rho_a$ $(a\in S)$ of S into the semigroup of all transformations of M. Defining $xa=x\rho_a$ for each $x\in M$ and $a\in S$, the set M becomes an S-system (S-operand in Tully's terminology). The paper deals only with semigroups S having a zero element 0, and with S-systems M having exactly one element, also denoted by 0, such that $0a=0$ for all $a\in S$. A representation ρ, or the corresponding S-system M, is called irreducible if $MS\neq 0$ and $\{0\}$ is the only proper S-subsystem of M. Let I be the set of all irreducible S-systems for a given S. The 0-radical $\mathrm{rad}^0 S$ of S is defined to be the ideal $\bigcap\{M^{-1}0: M\in I\}$, where $M^{-1}0=\{a\in S: Ma=0\}$. S is called 0-radical-free if $\mathrm{rad}^0 S=0$. The Rees quotient $S/\mathrm{rad}^0 S$ is always 0-radical-free. S is called 0-primitive if there exists an irreducible S-system M such that $M^{-1}0=0$, and an ideal P of S is called 0-primitive if S/P is 0-primitive. $\mathrm{rad}^0 S$ is shown to be the intersection of all the 0-primitive ideals of S, and hence any 0-radical-free semigroup is a subdirect product of 0-primitive semigroups.

As Tully [loc. cit.] showed, an S-system M is strictly cyclic if and only if it is S-isomorphic with S/μ, where μ is a right congruence on S which is modular in the sense that there exists e in S such that $ea\,\mu\,a$ for all $a\in S$. Let K be the lattice of all right congruences on S, and, for each right ideal R of S, let K_R be the set of all μ in K such that R is the μ-class $[0]_\mu$ containing 0. For each R, the greatest element of K_R is $\mu_R=\{(a,b)\in S\times S: as\in R\Leftrightarrow bs\in R\ (s\in S^1)\}$; this is essentially the principal right regular equivalence relation of Dubreil [Mém. Acad. Sci. Inst. France (2) **63** (1941), no. 3; MR **8**, 15] determined by R. R is said to be modular if some μ in K_R, and hence μ_R, is modular. Every modular right congruence $\neq S\times S$ is contained in a maximal one. M is called totally irreducible if $M\cong S/\mu$, where μ is a maximal modular right congruence on S; Tully calls it primitive. If M is totally irreducible, it is irreducible. A right ideal R of S is called maximal modular if it is modular, and if $\mu_R\subseteq\mu_{R'}$ for some right ideal R' of S implies $R'=R$ or $R'=S$. R is maximal modular if and only if μ_R is maximal, hence if and only if S/μ_R is totally irreducible. $\mathrm{rad}^0 S$ is the intersection of all the maximal modular right ideals of S. The semigroup of S-endomorphisms of a totally irreducible M is a group G with zero, and the corresponding representation can be formulated as a representation of S by monomial matrices over G. *A. H. Clifford* (New Orleans, La.)

Hoehnke, Hans-Jürgen 4052
Zur Theorie der Gruppoide. I.
Math. Nachr. **24** (1962), 137–168.

The main purpose of this paper is to discuss homomorphisms of certain groupoids. The author considers four notions of algebraic systems related to semigroups with zero, "groupids", categories, groupoids in which each one is specialization of its preceding, categories introduced by Eilenberg and MacLane, and groupoids by Ehresmann. Let G be a semigroup with a two-sided zero 0. Two elements x and y of G are said to be mutually "right [left] related", if there is an element a $[b]$ such that $xa\neq 0$, $ya\neq 0$ $[bx\neq 0, by\neq 0]$; if x and y are mutually right and left related, then they are called mutually doubly related. A relation σ is defined in G: $x\sigma y$ if and only if $x=y=0$ or x and y are mutually doubly related. If σ is a congruence on G, G is called a $^+$groupid, and $^-G=G-\{0\}$ is called a groupid. If a non-zero element ε satisfies the condition that $x\varepsilon\neq 0$ $[\varepsilon x\neq 0]$ implies $x\varepsilon=x$ $[\varepsilon x=x]$, then ε is called a right [left] unit of G; if ε is a right [left] unit of G and $a\varepsilon=a$ $[\varepsilon a=a]$, ε is called a right [left] unit of a in G, denoted by $E_r(a)$ $[E_l(a)]$. A $^+$category is a semigroup K with zero satisfying the following: (1) $^-K=K-\{0\}$ is not empty, and for any $a\in{}^-K$, there exist $E_r(a)$ and $E_l(a)$ in K; (2) $xyz=0$, $x,y,z\in K$, implies $xy=0$ or $yz=0$. A $^+$category K is a $^+$groupid. A $^+$category in which $x\neq 0$ has a right inverse element $\hat{x}$, i.e., $x\hat{x}=E_l(x)$, is called an Ehresmann $^+$groupoid. Then a right inverse element coincides with a left inverse element, and is hence unique; it is called the inverse element of x, denoted by x^{-1}. If G is an Ehresmann $^+$groupoid, $^-G=G-\{0\}$ is called an Ehresmann groupoid. An Ehresmann $^+$groupoid is called a Brandt $^+$groupoid or $^+$groupoid if for two units e_{ii}, e_{jj} there exists x such that $e_{ii}=E_l(x)$, $e_{jj}=E_r(x)$. If G is a $^+$groupoid, ^-G is called a groupoid. Let G be any groupoid and let G_{ij} $(i,j\in I)$ be each σ-class. Then G_{ii} is a subgroup of G.

Let a_i be an element of G such that $E_r(a_i)=e_{11}$, $E_l(a_i)=$

e_{ii}. Then $G_{ij}=a_iG_{11}a_j^{-1}$. By using this, the author shows that any groupoid is isomorphic into a matrices groupoid which is a generalization of matrices semigroup due to Rees. When is a category embedded into a groupoid? The author mentions that this generalized problem is connected with that of finding all subsemigroups of a $^+$groupoid ^+G. A homomorphism φ of a groupoid G onto G' is a mapping of G onto G' such that if xy is defined, $\varphi(x)\varphi(y)$ exists and $\varphi(xy)=\varphi(x)\varphi(y)$. The restriction of φ to ^-G is a homomorphism of ^-G onto $^-G'$. A homomorphism of ^-G onto $^-G'$ is said to be of first kind if an additional definition $\varphi(0)=0\in G'$ gives a homomorphism of G onto G'; otherwise of second kind. A homomorphism φ is called subnormal if for any $x, y\in {}^-G$ with $\varphi(x)\varphi(y)\neq 0$, there are $x_1, y_1\in {}^-G$ such that $\varphi(x)=\varphi(x_1)$, $\varphi(y)=\varphi(y_1)$ and $x_1y_1\neq 0$. A homomorphism of first kind is subnormal. The author proves that any homomorphism of ^-G onto $^-G'$ is extended to a homomorphism of first kind of certain extension ^-V of ^-G onto $^-G'$. If G is a $^+$groupoid, then a homomorphic image G' of first kind of G is also a $^+$groupoid. Various properties with respect to groupoid-homomorphisms are shown, for example, the union of the inverse image of units of a groupoid is an Ehresmann groupoid. Let φ be a homomorphism of a groupoid G onto G' and let $\bar\varphi$ be a natural mapping of G onto the quotient groupoid $\bar G$ due to φ. Then $\bar\varphi$ is expressed as the product of a homomorphism of first kind of G onto certain G' and a homomorphism of G' onto $\bar G$, and also as the product of a normal homomorphism of G onto certain G'' and a homomorphism of G'' onto $\bar G$. By a normal homomorphism the author means a homomorphism such that the inverse image of a unit of G'' is a subgroup of G.

Let φ be a normal homomorphism of G onto $\bar G$. Then $\varphi(x)=\varphi(y)$ if and only if there are u, v in E such that $uxv=y$ where E is the set union of the inverse images of units of $\bar G$, and E is an Ehresmann groupoid; and $xEx^{-1}\subseteq E$ for all $x\in G$. If E satisfies $xEx^{-1}\subseteq E$ for all $x\in G$, E is called a normal divisor of G. Even if the inverse image set E of units of G satisfies $xEx^{-1}\subseteq E$, the homomorphism is not necessarily normal. The author, however, proves that for a given normal divisor E of G, there is a normal homomorphism. Also a normal homomorphism is characterized in the following way. φ is a normal homomorphism of a groupoid G onto a groupoid G' if and only if a maximal subgroup of G is mapped to a maximal subgroup of G' under φ.

The main theorem of this paper gives a way to determine all the groupoid-homomorphisms: A homomorphism of G onto G' is constructed by a homomorphism of G_{11} onto $G'_{\omega\omega}$ where G_{11} and $G'_{\omega\omega}$ are maximal subgroups of G and G', respectively. In addition, there exists a matrix semigroup U in the sense of Rees such that $\varphi=\psi\chi$, G is homomorphic onto U under χ, and U is homomorphic onto G' under ψ. *T. Tamura* (Davis, Calif.)

Hoehnke, Hans-Jürgen 4053

Zur Theorie der Gruppoide. II.

Math. Nachr. **24** (1962), 169–179.

This paper is a continuation of the paper above [#4052]. The principal purpose is to establish isomorphism theorems with respect to a factor-groupoid related to a group. Following Shoda [Osaka Math. J. **1** (1949), 182–225; MR **11**, 308], a meromorphism is considered as a many-to-many mapping preserving the binary operation in an algebraic system. The author defines a direct product of groupoids and a complex multiplication by which complexes of a groupoid form a groupoid. Now let H and U be two arbitrary subgroups of a group G, and consider the groupoid H/U composed of all the complexes of the form xUy, $x, y\in H$. Let $D=H\cap U$, N_U be a normalizer of U. Then H/U is isomorphic onto the direct product of $(H\cap N_U)/D$ and certain "plain" groupoid. A "plain" groupoid means a groupoid G in which σ [cf. #4052 above] is the equality relation of G. This result includes the usual second isomorphism theorem in the group theory. Let H and U be subgroups of a group G and let $D=H\cap U$. Then a mapping $xUy\to xDy$ is a subnormal homomorphism of H/U onto H/D. Let U be a subgroup of a group G. The system of all complexes of the form UzU, $z\in G$, is a hypergroup in the sense of Bruck [*A survey of binary systems*, p. 41, Springer, Berlin, 1958; MR **20** #76]. Finally the author gives two interesting theorems on the relationship between the factor-groupoid G/U and the hypergroup. *T. Tamura* (Davis, Calif.)

Bigard, Alain 4054

Sur quelques équivalences remarquables dans un groupoïde quasi-résidué.

C. R. Acad. Sci. Paris **258** (1964), 3414–3416.

Soit G un groupoïde ordonné, $a, b\in G$. On appelle quasi-résiduels de a par b les ensembles $\langle a.\cdot b\rangle=\{x|bx\leqq a\}$, $\langle a\cdot.b\rangle=\{x|xb\leqq a\}$. Si $\langle a.\cdot b\rangle$ et $\langle a\cdot.b\rangle$ ne sont pas vides, quels que soient a et b, G est dit quasi-résidué. Si $\langle a.\cdot b\rangle$ et $\langle a\cdot.b\rangle$ sont distincts de $\emptyset$ et filtrants supérieurement, G est dit quasi-résidué fort. L'auteur énonce quelques résultats concernant les équivalences régulières à gauche dans un groupoïde quasi-résidué, qui généralisent naturellement ceux obtenus par J. Querré dans le cas résidué [mêmes C. R. **252** (1961), 49–51; MR **23** #A235].

Soit G un groupoïde quasi-résidué fort et φ une fermeture de Moore dans G telle que l'équivalence associée R soit régulière. Si G/R est un groupe et si ε désigne l'élément maximum de la classe unité: (1) les résiduels de ε existent et, pour tout x, $\varepsilon.\cdot x=\varepsilon\cdot.x$; (2) pour tout x, $\varphi(x)=\varepsilon.\cdot(\varepsilon\cdot.x)$; (3) ε est le maximum commun des ensembles $\bigcup\langle a.\cdot a\rangle_{a\in G}$ et $\bigcup\langle a\cdot.a\rangle_{a\in G}$.

L'auteur généralise la notion de demi-groupe A-nomalement fermé [loc. cit.] pour le cas quasi-résidué, considère les éléments F-nomaloïdes et B-nomaloïdes [I. Molinaro, Thèse, Paris, 1956; J. Math. Pures Appl. (9) **39** (1960), 319–356; ibid. **40** (1961), 43–110; MR **26** #245] d'un demi-groupe quasi-résidué en connexion avec les groupes ordonnés homomorphes.

B. M. Schein (*Šain*) (Saratov)

Fischer, Bernd 4055

Distributive Quasigruppen endlicher Ordnung.

Math. Z. **83** (1964), 267–303.

A distributive quasi-group is a set D with a binary operation $\circ$ satisfying the following axioms: (i) $a\circ(b\circ c)=(a\circ b)\circ(a\circ c)$, $(a\circ b)\circ c=(a\circ c)\circ(b\circ c)$ for all a, b, c in D; (ii) If $a, b\in D$, then there exist unique elements x, y in D such that $a\circ x=b$ and $y\circ a=b$. A class of examples can be constructed as follows. Let F be a field and $\alpha\neq 0, 1$ a fixed element of F. Let $D=F$, where $a\circ b=a\gamma+b(1-\gamma)$. The author proves (though it is not explicitly stated) that a finite minimal distributive quasi-group is of the above

form. If $a \in D$, then the mapping $a^D: x \to x \circ a$ is an automorphism of D. Let $R(D)$ denote the group generated by all such automorphisms. Main theorem: If D is a finite distributive quasi-group, then $R(D)$ is solvable. In proving this theorem the following result of independent interest is needed. Theorem: Let G be a group which is generated by a conjugate class D of involutions. Let p be an odd prime. Assume that for x, y in D, xy is always a p-element. Then G' is nilpotent. At one point of the proof of this theorem it is necessary to consider the case where all involutions in G are in D and G' has odd order. The author handles this by using the fact that groups of odd order are solvable [the reviewer and J. G. Thompson, Pacific J. Math. **13** (1963), 775–1029]. However, this case can be dealt with directly as follows. Let $q_1, \cdots, q_n$ be all the primes distinct from p which divide $|G'|$. By Sylow's theorem there exist S_{2,q_i}-subgroups of G for $i = 1, \cdots, n$. By assumption such a subgroup is of the direct product of an S_{q_i}-subgroup and a group of order 2. Thus $|G : P|$ divides $C_G(v)$ for $v \in D$, P an S_p-group of G, and so $|D|$ is a power of p. The proof can now be completed as in the text. *W. Feit* (Ithaca, N.Y.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 4014, 4022, 4343, 4377b, 4384, 4386, 4484, 4850, 4857.

Johnson, B. E. 4056

Isometric isomorphisms of measure algebras.

Proc. Amer. Math. Soc. **15** (1964), 186–188.

Wendel [Pacific J. Math. **2** (1952), 251–261; MR **14**, 246] has shown any two locally compact groups with isometrically isomorphic L_1-algebras are isomorphic. The author now shows the same conclusion holds if each L_1 algebra is replaced by the algebra of all finite measures. The proof involves showing one L_1 maps onto the other, so that Wendel's result applies.

I. Glicksberg (Seattle, Wash.)

Pym, John S. 4057

A note on the Kawada-into theorem.

Proc. Edinburgh Math. Soc. (2) **13** (1962/63), 295–296.

This note contains a simple counter-example to a conjecture of the reviewer [same Proc. (2) **11** (1958/59), 71–77; MR **20** #5818]. The author constructs on a certain locally compact unimodular group G (with Haar measure m) a measure μ not having one-point support, and such that a function $f \in L_1(G)$ is non-negative a.e. (m) if and only if the convolution product $\mu * f$ is non-negative a.e. (m).

J. H. Williamson (Cambridge, England)

Macbeath, A. M. 4058

Groups of homeomorphisms of a simply connected space.

Ann. of Math. (2) **79** (1964), 473–488.

The main purpose of the paper is to study in a general topological setting the method of finding abstract definitions of discrete groups by considering the transformations which map a fundamental region of a manifold onto its neighbours. Similar investigations have been made by M. Gerstenhaber [Proc. Amer. Math. Soc. **4** (1953), 745–750; MR **15**, 397] and H. Behr [J. Reine Angew. Math. **211** (1962), 116–122; MR **26** #5554], but the method of proof is quite different.

In two concluding sections of the paper, the author connects his results with part of a paper by A. Weil [Ann. of Math. (2) **72** (1960), 369–384; MR **25** #1241] on discrete subgroups of Lie groups. He proves that if G is a discrete subgroup, with compact coset space, of a connected Lie group Λ, then G has a neighbourhood in the topological space of discrete subgroups of Λ, consisting entirely of groups isomorphic to G. Combining this with Weil's results, he establishes a local homeomorphism between the space of subgroups of Λ and the space of "representations" of a given abstract group into the Lie group. *F. A. Sherk* (Toronto, Ont.)

Guillemot-Teissier, Marianne 4059

Convolution des courants sur un groupe de Lie.

Ann. Sci. École Norm. Sup. (3) **79** (1962), 321–352.

L'auteur développe dans ce mémoire les résultats annoncés dans la première de deux notes consacrées à la convolution des courants sur un groupe de Lie [C. R. Acad. Sci. Paris **252** (1961), 3705–3707; MR **23** #A3203; ibid. **254** (1962), 4124–4126; MR **25** #3121]. La convolution envisagée particularise (au signe près) la convolution de première espèce étudiée dans des espaces plus généraux par le rapporteur [ibid. **240** (1955), 830–832; MR **16**, 857; Thèse, Fac. Sci. Paris, Paris, 1960; Ann. Inst. Fourier (Grenoble) **11** (1961), 1–82; MR **27** #6203] et la convolution des courants étudiée par J. Braconnier [C. R. Acad. Sci. Paris **252** (1961), 60–62; MR **23** #A3179] dans une situation également plus générale. L'exposé de quelques résultats déjà connus (convolution par un courant de Dirac, différentielle du produit de convolution, commutativité du produit de convolution) précède l'étude des relations entre la convolution et la dérivation (respectivement le produit intérieur) que l'on associe à un champ de vecteurs invariant à gauche ou à droite, entre la convolution et l'algèbre extérieure des multivecteurs à l'origine, entre la convolution et l'algèbre de Lie du groupe. L'auteur étudie encore la régularisation, l'opération transposée de la convolution, et prouve l'isomorphisme de l'algèbre des opérateurs différentiels invariants à droite, de l'algèbre de convolution des courants de support l'origine et de l'algèbre enveloppante universelle de l'algèbre de Lie du groupe. *F. Norguet* (Strasbourg)

Wolf, Joseph A. 4060

On the classification of hermitian symmetric spaces.

J. Math. Mech. **13** (1964), 489–495.

The author proves Cartan's classification theorem of hermitian symmetric spaces by reducing the problem to the determination of the largest root of a simple, semisimple complex Lie algebra, which is found in the work of Borel and de Siebenthal on subgroups of maximal rank in compact Lie groups [Comment. Math. Helv. **23** (1949), 200–221; MR **11**, 326]. *J. Hano* (St. Louis, Mo.)

van Est, W. T.; Korthagen, Th. J. 4061

Non-enlargible Lie algebras.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 15–31.

A Lie algebra L is called metric if it is a Banach space and if $\|[x, y]\| \leq A\|x\|\|y\|$, A a constant, x, y in L. The authors construct a metric Lie algebra L such that none of the local analytic groups generated by L by the Schur-Campbell-Hausdorff process [G. Birkhoff, Trans. Amer. Math. Soc. **43** (1938), 61–101] is a local subgroup of any (global) topological group. The existence of such an L shows that the global "converse of the third fundamental theorem of Lie" does not hold when the assumption of finite dimensionality is dropped. Let $\mathfrak{Q}$ be the Lie algebra of the multiplicative group of real unit quaternions and let $\Omega^1(\mathfrak{Q})$ be the space of differentiable loops on $\mathfrak{Q}$ based at 0, converted to a metric Lie algebra by $\|\lambda\| = \max|\lambda(t)| + \max|\lambda'(t)|$, $[\lambda_1, \lambda_2](t) = [\lambda_1(t), \lambda_2(t)]$. The Lie algebra in question is obtained from a certain extension of $\Omega^1(\mathfrak{Q})$ by R (regarded as a Lie algebra).
P. A. Smith (New York)

Levitan, B. M. 4062
Lie theorems for generalized translation operators. (Russian)
Trudy Moskov. Mat. Obšč. **11** (1962), 127–197.

This paper amplifies, extends, and proves statements announced previously [Dokl. Akad. Nauk SSSR **123** (1958), 32–35; MR **22** #9873]. Generalized translation operators on a space Ω are a collection of linear operators T^s (s in Ω) which act on $C(\Omega)$ and for which (1) there exists a neutral point 0 in Ω such that T^0 is the identity, (2) $T_s^r T^s f(t) = T_t^s T^r f(t)$ for all r, s, t in Ω and f in $C(\Omega)$. For example, if Ω is a Lie group, we may set $T^s f(t) = f(ts)$. If Ω is an analytic manifold with coordinates $s_1, \cdots, s_n$ about 0, one defines the infinitesimal generators by

$$L_{k_1 \cdots k_n;t}(f) = \left.\frac{\partial^{k_1+\cdots+k_n} u}{\partial s_1^{k_1} \cdots \partial s_n^{k_n}}\right|_{s=0},$$

where $u(s, t) = T^s f(t)$. The idea, now, is to duplicate the theory of local Lie groups in this more general situation, and the author does this with some success. He restricts himself, however, to certain situations in which the generalized translations are completely determined either by the infinitesimal generators of first order, or by those of second order. Only the latter is new. In both cases he finds a Lie algebra and examines to what extent the structure constants are arbitrary and to what extent they determine the generalized translations.
G. Hufford (Seattle, Wash.)

FUNCTIONS OF REAL VARIABLES

See also 4073, 4077, 4311, 4361.

Ivanov, L. D. 4063
Smooth mappings of the space R_n into R_1. (Russian)
Dokl. Akad. Nauk SSSR **155** (1964), 258–261.

Es sei f eine Abbildung von R_n in R_1, die außerhalb einer Kugel vom Radius ρ verschwindet und Ableitungen bis zu einer Ordnung s einschließlich hat, wobei die Ableitungen f_i der höchsten Ordnung Ungleichungen $|f_i(x) - f_i(y)| \leq M|x-y|^\alpha$ mit $0 \leq \alpha \leq 1$ erfüllen. Es bedeute G die Menge aller Punkte x mit der Eigenschaft, daß grad f in der den Punkt x enthaltenden zusammenhängenden Komponente der Menge $\{y : f(y) = f(x)\}$ nirgends verschwindet, und es seien K_i die Komponenten von G, h_i die Länge von $f(K_i)$, $l = s + \alpha$, $n_i = h_i^{n/l}$ und $k(t)$ die Anzahl der Komponenten von $f^{-1}(t)$. Der Verfasser beweist eine Abschätzung der Form $\sum_i n_i \leq a(n, l) M^{n/l} \rho^n$, die in den Spezialfällen $n = 1$ und $l = n$ von Vituškin hergeleitet worden war, und folgert hieraus $\int_{-\infty}^{\infty} (k(t))^{l/n}\, dt \leq (a(n, l))^{l/n} M \rho^l$, falls $n \leq l$.
K. Krickeberg (Heidelberg)

Ivanov, N. A. 4064
On Young's theorem. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1963**, *no.* 6 (37), 81–83.

The author extends a theorem of W. H. Young on the inversion of order of repeated partial derivations. For this purpose, he introduces a concept of almost differentiability at a point, which he illustrates by considering the function $|x + y|$ at the origin of the (x, y)-plane. He remarks further that the theorem can be interpreted to apply to a function defined in the Cartesian product of two normed linear spaces.
L. C. Young (Madison, Wis.)

Louhivaara, Ilippo Simo 4065
Applications of absolute analysis to vector analysis. (Finnish. German summary)
Arkhimedes **1961**, *no.* 2, 10–32.

Author's summary: "In diesem Artikel wird bei einigen grundlegenden Problemen des Infinitesimalkalküls ein Vergleich der Resultate der üblichen Differential- und Integralrechnung und der klassischen (3-dimensionalen) Vektoranalysis einerseits und des von F. und R. Nevanlinna in den letzten Jahren entwickelten koordinaten- und dimensionsfreien Infinitesimalkalküls, der sog. absoluten Analysis, andererseits gemacht. Im ersten Teil wird ein kurzer Überblick über einige zentrale Begriffe und Resultate der absoluten Analysis gegeben. Im zweiten Teil folgt zuerst, als eine Anwendung des in der Monographie [F. und R. Nevanlinna, *Absolute Analysis*, Springer, Berlin, 1959; MR **22** #12176] von Nevanlinna eingeführten erweiterten Begriffes des Rotors einer alternierenden multilinearen Operatorfunktion, eine koordinatenfreie Definition der Grundoperationen der Vektoranalysis (des Gradienten, der Divergenz und der Rotation), die gegenüber der üblichen auf Bildung der partiellen Ableitungen der zu operierenden Funktion fussenden Definition dieser Begriffe zu einer Verallgemeinerung führt: Falls die zu operierende Funktion stetig differenzierbar ist, sind die Ergebnisse der alten und der neuen Definition gleich; eine Vektorfunktion kann aber sogar eine stetige Divergenz oder Rotation in diesem verallgemeinerten Sinne besitzen, während die entsprechenden Ausdrücke im üblichen Sinne nicht zu bilden sind. Diese Definition stimmt mit einer von Claus Müller [Math. Ann. **124** (1952), 427–449; MR **14**, 374; *Grundprobleme der mathematischen Theorie elektromagnetischer Schwingungen*, Springer, Berlin, 1957; MR **19**, 209] eingeführten und systematisch benutzten Erweiterung dieser Begriffe überein. Im Anschluss an Nevanlinnas Monographie [F. und R. Nevanlinna, loc. cit.] werden dann die klassischen Integraltransformationsformeln von Gauss und Stokes aus dem allgemeinen Stokesschen Satz für alternierende Operatorfunktionen hergeleitet. Ein entsprechendes Ergebnis ist neulich auch von Paul Kustaanheimo [Soc. Sci. Fenn. Comment. Phys.-Math. **25** (1960), no. 3; MR

23 #A4092] in seinem Infinitesimalkalkül über Tensorringalgebra veröffentlicht worden. Zum Schluss wird die wohl auch früher bekannte Tatsache betont, dass man für die Existenz der Skalar- und Vektorpotentiale eines Vektorfeldes notwendige und hinreichende Bedingungen (Wirbelfreiheit bzw. Quellenfreiheit) sowie die Ausdrücke dieser Potentiale aus einem allgemeinen für alternierende Operatorfunktionen gültigen Prinzip (Lemma von Poincaré) erhalten kann."

Łojasiewicz, S. 4066

Une propriété topologique des sous-ensembles analytiques réels.

Les Équations aux Dérivées Partielles (Paris, 1962), pp. 87–89. Éditions du Centre National de la Recherche Scientifique, Paris, 1963.

The author's principal result may be stated as follows. If f is an analytic function in an open set $G \subset R^n$, $f^{-1}(0)$ is a deformation retract of an open neighborhood of $f^{-1}(0)$ in G. For earlier results see the author [Studia Math. **18** (1959), 87–136; MR **21** #5893; Rozprawy Mat. **22** (1961); MR **23** #A3369]. *H. A. Antosiewicz* (Los Angeles, Calif.)

Bahvalov, N. S. 4067

Imbedding theorems for classes of functions with a number of bounded derivatives. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1963**, *no.* 3, 7–16.

All functions dealt with in this paper are defined for $x=(x_1, \cdots, x_s) \in R^s$ and are of period 1 in each variable. Let $r=(r_1, \cdots, r_s)$, $r_k \geqq 0$, $r_k=\rho_k+\alpha_k$, where $0<\alpha_k \leqq 1$ and ρ_k is an integer. Let $\delta_h{}^k f$ denote the operator $f \to \delta_h{}^k f = (2i)^{-1}[f(x_1, \cdots, x_k+h, \cdots, x_s)-f(x_1, \cdots, x_k-h, \cdots, x_s)]$, let us suppose that for arbitrary $h_1, \cdots, h_s$ the Sobolev derivative $[(\prod_{\alpha_k \neq 0} \delta_{h_k}{}^k) f]^{(\rho_1, \cdots, \rho_s)}$ exists, let us set

$$\|f^{(r)}\|_{L_p} = \|f^{(r_1, \cdots, r_s)}\|_{L_p} = \sup_{h_1, \cdots, h_s} \left\| \left[\left(\prod_{\alpha_k \neq 0} h_k{}^{-\alpha_k} \delta_{h_k}{}^k \right) f \right]^{(\rho_1, \cdots, \rho_s)} \right\|_{L_p},$$

the norm on the right-hand side being the usual L_p-norm. Let $P_k{}^0 f = \int_0^1 f(x_1, \cdots, x_s)\, dx_k$ be the projection operator, and let $P_{1, \cdots, 1}=(E-P_1{}^0) \cdots (E-P_s{}^0)$. Theorem 1: Let $\varphi = P_{1, \cdots, 1}\varphi \in L_1$, $\|\varphi^{(r)}\|_{L_p} < \infty$ where $r_k>0$, $1 \leqq p \leqq \infty$. Then there exists a function $\tilde{\varphi}$ equal a.e. to φ such that for all $l=(l_1, \cdots, l_s)$ and q satisfying $r_k-1/p>l_k-1/q$, $r_k>l_k$ there exists a constant $C=C(r, l, p, q, s)$ such that $\|\tilde{\varphi}^{(l)}\|_{L_q} \leqq C\|\varphi^{(r)}\|_{L_p}$. If $r_k-1/p<l_k-1/q$ for at least one k, then there exists a function φ such that $\|\varphi^{(r)}\|_{L_p}=1$ and $\|\tilde{\varphi}^{(l)}\|_{L_q}=\infty$ for each function $\tilde{\varphi}$ equal a.e. to φ. Theorem 2: Let $\varphi=P_{1, \cdots, 1}\varphi \in L_1$, $\|\varphi^{(r)}\|_{L_p}<\infty$, $1 \leqq p \leqq \infty$, let $r_k-1/p> l_k \geqq 0$ for $k>\gamma$, and let $r_k>l_k$, $r_k-1/p>l_k-1/q$ for $k \leqq \gamma$. Then

$$\sup_{x_{\gamma+1}, \cdots, x_s} \sup_{h_1, \cdots, h_s} \left\{ \int_0^1 \cdots \int_0^1 \left| \left[\left(\prod_{\beta_k \neq 0} h_k{}^{-\beta_k} \delta_{h_k}{}^k \right) \varphi \right]^{(m)} \right|^q dx_1 \cdots dx_\gamma \right\}^{1/q} \leqq C(\gamma, r, l, p, q, s)\|\varphi^{(r)}\|_{L_p}.$$

Let R denote the class of functions f satisfying $\|f^{(r^i)}\|_{L_{p^i}} \leqq A$ for $i=1, \cdots, N$, where $r^i=(r_1{}^i, \cdots, r_s{}^i)$, let $Q^i=(r_1{}^i, \cdots, r_s{}^i, 1/p^i)$. For each system of $\varepsilon_k=0, 1$ $(k=1, \cdots, s)$ there exists a set $\Omega_{\varepsilon_1, \cdots, \varepsilon_s} \in R^{s+1}$ such that for $z=(z_1, \cdots, z_{s+1})=Q \in \Omega_{\varepsilon_1, \cdots, \varepsilon_s}$ and $f \in R$ one has $\|(P_{\varepsilon_1, \cdots, \varepsilon_s} f)^{(z_1, \cdots, z_s)}\|_{L_{1/z_{s+1}}} < \infty$, $P_{\varepsilon_1, \cdots, \varepsilon_s}$ being equal to $P_1{}^{\varepsilon_1} \cdots P_s{}^{\varepsilon_s}$, where $P_k{}^1=E$. Theorem 3: Let $f \in R$ and let $Q=(l_1, \cdots, l_s, 1/q) \in \bigcap_{\varepsilon_k \geq l_k} \Omega_{\varepsilon_1, \cdots, \varepsilon_s}$, then $\|f^{(l)}\|_{L_q} \leqq C(Q, R)A$. *A. Alexiewicz* (Poznań)

MEASURE AND INTEGRATION

See also 4056, 4339, 4345, 4556, 4557, 4573.

Erdős, P. 4068

On some properties of Hamel bases.

Colloq. Math. **10** (1963), 267–269.

It is shown (extending results of Sierpiński to whom this note is dedicated) that for any Hamel basis H of the real numbers the set H^* of all (finite) sums of the form $\sum n_\alpha a_\alpha$ ($a_\alpha \in H$, n_α integers) is non-measurable (H^* has inner measure 0 and for every interval (a, b) the outer measure of $H^* \cap (a, b)$ is $b-a$). Under the assumption of the continuum hypothesis the author proves furthermore that there exists H for which even the set H^+ of all $\sum r_\alpha a_\alpha$ ($a_\alpha \in H$, $r_\alpha \geqq 0$, r_α rational) has measure 0; in fact, the author constructs H such that for any nowhere dense perfect set S the intersection $S \cap H^+$ is countable.
E. Engeler (Zürich)

Roberts, G. T. 4069

Order continuous measures.

Proc. Cambridge Philos. Soc. **60** (1964), 205–207.

A measure μ on a locally compact, σ-compact space Ω, when regarded as a functional on the vector lattice $C(\Omega)$ of all continuous real-valued functions on Ω with compact support, is continuous for order convergence in $C(\Omega)$ if and only if all sets of the first category in Ω are of μ-measure zero. The proof is quite direct.
F. Cunningham, Jr. (Bryn Mawr, Pa.)

Rogers, C. A.; Taylor, S. J. 4070

Additive set functions in Euclidean space. II.

Acta Math. **109** (1963), 207–240.

In a previous paper [same Acta **101** (1959), 273–302, quoted as (I); MR **21** #6413] the authors investigated the decomposition of a completely additive finite set function F defined on all Borel subsets E of a closed interval I_0 in E^k with respect first to a single Hausdorff measure, then to a class of comparable Hausdorff measures. $\mathfrak{F}$ denotes the set of the functions F. One of the main theorems (Theorem 6 in (I, § 6)) is not satisfactory because there is an error in the proof of a preliminary lemma (Lemma 10 in (I, § 6)). In the present paper the analysis of (I) is corrected and extended. Unfortunately, the results are more complicated than those they replace. $\mathfrak{H}$ is the set of all positive functions $h(t)$, defined for $t>0$, continuous, increasing, and with $\lim h(t)=0$ for $t \to +0$. $h-m(E)$ denotes the Hausdorff measure of E corresponding to the measure function h. The relation $h \prec g$ is defined as in (I, § 6). $h \sim g$ means $0<\liminf g(t)/h(t) \leqq \limsup g(t)/h(t)<$

$+\infty$ for $t \to +0$. h and g are said to be comparable if $h \prec g$, or $h \sim g$, or $g \prec h$, monotonically comparable if, in addition, $h(t)/g(t)$ is monotonic for all sufficiently small positive t. $\mathfrak{L}$ denotes a set of functions of $\mathfrak{H}$ such that two elements $g, h \in \mathfrak{L}$ are monotonically comparable; the function t^k is in $\mathfrak{L}$; if $g, h \in \mathfrak{L}$ and α, β are real numbers, and $g^\alpha h^\beta \in \mathfrak{H}$, then $g^\alpha h^\beta \in \mathfrak{L}$; if $g, h \in \mathfrak{L}$ and $g \neq h$, then g is not equivalent to h (irreducibility); if $h \in \mathfrak{L}$ and h is comparable with each element of $\mathfrak{L}$, then h is equivalent to at least one element of $\mathfrak{L}$ (maximality). Two subsets L, R of $\mathfrak{L}$ are said to form a section s of $\mathfrak{L}$ if L has no maximal element, $L \cap R = \emptyset$, $L \cup R = L$, and $h_1 \in L$, $h_2 \in R$ implies $h_1 \prec h_2$. $R' = R_s'$ denotes the set obtained by removing from $R = R_s$ its least element, if it has one. $\mathfrak{C}_s^{**}$ is the class of functions of $\mathfrak{F}$, which are hyper-s-continuous, that is, those set functions which can be expressed as $F = \sum F_i$ for $i = 1, \cdots, n, \cdots$, with $\sum |F_i|(I_0)$ convergent, where each F_i is h_i-continuous (I, § 3), for some h_i in R_s'. $\mathfrak{S}_0^{**}$ is the class of functions of $\mathfrak{F}$ which are hypo-s-singular, that is, for each h in R_s', there is a Borel set E_0 in I_0 such that $h - m(E_0) = 0$ and $F(E) = F(E \cap E_0)$. The property that $\mathfrak{C}_s^{**}$ and $\mathfrak{S}_s^{**}$ form complementary bands in the vector lattice $\mathfrak{F}$ is deduced from a general theorem about collections of bands in a uniformly monotone Banach lattice. F is said to have the exact $\mathfrak{L}$-dimension s if F is s-continuous (I, § 6) and hypo-s-singular. F is said to have diffuse $\mathfrak{L}$-dimension spectrum if there is no function $G \in \mathfrak{F}$ with $0 < |G| \leq |F|$ having an exact $\mathfrak{L}$-dimension. Main theorem (replacing Theorem 6 in (I, § 6)): Given $\mathfrak{L}$ and F, there is a finite or enumerable sequence $s_1, s_2, \cdots$ of distinct sections of $\mathfrak{L}$, and a decomposition $F = F^{(d)} + F_2^{(s_1)} + F_3^{(s_1)} + F_2^{(s_2)} + F_3^{(s_2)} + \cdots$, where $F^{(d)}$ has diffuse $\mathfrak{L}$-dimension, $F_2^{(s_i)}$ is strongly s_i-continuous and s_i-singular, and $F_3^{(s_i)}$ is s_i-continuous and almost s_i-singular (I, § 6) for $i = 1, 2, \cdots$. The set of the sections s_i and the decomposition (apart from the order of its terms) are uniquely determined by F. The remainder of the paper is concerned with the further study of the decomposition; frequent reference is made to the authors' paper [Mathematika 8 (1961), 1–31; MR **24** #A200].

Chr. Y. Pauc (Nantes)

Lloyd, S. P. 4071

On finitely additive set functions.

Proc. Amer. Math. Soc. **14** (1963), 701–704.

Let $\mathscr{F}$ be a σ-field of subsets of a set X that separate the points of X, $\mathrm{ba}(X, \mathscr{F})$ the Banach space of all bounded real-valued finitely additive set functions on $\mathscr{F}$, X^* the Boolean space of the Boolean algebra $\mathscr{F}$; then $\mathrm{ba}(X, \mathscr{F})$ may be represented as the space $C^*(X^*)$ of all bounded real-valued signed Baire measures on X^*. The author's main result is that the purely finitely additive members of $\mathrm{ba}(X, \mathscr{F})$ are those whose correspondents in $C^*(X^*)$ live on a Baire F_σ-set of first category in X^*; this solves a problem raised by Yosida and Hewitt [Trans. Amer. Math. Soc. **72** (1952), 46–66; MR **13**, 543].

L. Gillman (Rochester, N.Y.)

Phakadze, Š. S. 4072a

Measure decomposition. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **31** (1963), 15–22.

Phakadze, Š. S. 4072b

Decomposition of measures of different types. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **31** (1963), 521–527.

The author denotes by F and F_1 two groups of automorphisms of a space X, either or both of which may reduce to the identity; and by M an F-invariant σ-algebra of subsets of X. The measures considered are F-invariant measures defined in M. The papers study certain transfinite decompositions $\mu = \sum \mu_\alpha$ in terms of partial measures μ_α $(0 \leqq \alpha\gamma)$, where γ is an initial ordinal, or, as the author prefers to write, $\mu = \mu_c + \sum \mu_\alpha$, where μ_c is an additional term written separately. The decompositions studied are those in which the partial measures μ_α satisfy special conditions: in the first paper they are taken to be "trivial measures", or more generally "elements", while in the second paper they are restricted by certain maximal, and other, conditions dependent on the choice of F_1.

No fewer than thirty-one theorems are announced, and the many concepts and definitions on which they depend, and which are introduced only in the briefest terms, are analogues or extensions of those in the author's previous work [Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze **25** (1958), 3–271; MR **21** #5001]. The detailed presentation of this material should be of considerable interest when it appears.

L. C. Young (Madison, Wis.)

Skvorcov, V. A. 4073

On integrating the exact Schwarzian derivative. (Russian)

Mat. Sb. (*N.S.*) **63** (**105**) (1964), 329–340.

The author considers an old question connected with the asymptotic version of the second derivative in the sense of H. A. Schwarz, namely, that this integration process can integrate any finite exact asymptotic Schwarzian second derivative. He shows that the desired integration can be effected by means of what he terms an AP^2-integral. The latter generalizes the P^2-integral of R. D. James [R. D. James and W. H. Gage, Trans. Roy. Soc. Canada Sect. III (3) **40** (1946), 25–35; MR **9**, 19; R. D. James, Canad. J. Math. **2** (1950), 297–306; MR **12**, 94]. He shows also that the Denjoy process T_{2S} does not achieve this, and he thus answers negatively a question raised by D. E. Menshov.

L. C. Young (Madison, Wis.)

Zakon, Elias 4074

A remark on the theorems of Lusin and Egoroff.

Canad. Math. Bull. **7** (1964), 291–295.

The author remarks in this note that Lusin's theorem (on the almost continuity of Borel measurable functions) can be proved for σ-finite regular measures by a direct application of an easy extension of Egoroff's theorem to this case, in which, however, the uniformity of convergence achieved is only local, on a neighborhood of each point.

F. Cunningham, Jr. (Bryn Mawr, Pa.)

Bondi, I. L. 4075

The A-integral and generalized functions. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 2 (116), 131–138.

If y, n are real numbers, put $[y]_n = y$ for $|y| \leqq n$ and $[y]_n = 0$ for $|y| > n$. If f is a function on $[a, b]$ such that $n \cdot \operatorname{mes}\{x;\ |f(x)| > n\} \to 0$ for $n \to \infty$ and such that the finite limit $L = \lim \int_a^b [f(x)]_n \, dx$ exists $(n \to \infty)$, we write $L = (A) \int_a^b f$. Let $A(a, b)$ be the family of all such f. The author proves the following assertion: Let P be a perfect non-dense subset of $[a, b]$, and let $F \in A(a, b)$. Then there exists a function f with the following properties: (1) $f(x) = 0$ for $x \in P$; (2) $(A) \int_a^b f = 0$; (3) $(A) \int_a^x f = F(x)$ for almost all $x \in [a, b]$; (4) if the function φ has a continuous derivative in $[a, b]$ and if $F \cdot \varphi' \in A(a, b)$, then $(A) \int_a^b f \cdot \varphi = -(A) \int_a^b F \cdot \varphi'$. Let, further, a be a positive number, let K be the family of all infinitely differentiable functions φ on $(-\infty, \infty)$ such that $\varphi(x) = 0$ whenever $x \notin [-a, a]$ and let T be a generalized function on $[-a, a]$. It is well known that there exists a continuous function F on $[-a, a]$ and a natural number p such that T is the generalized pth derivative of F. It follows that there exists a function f such that $(T, \varphi) = (A) \int_{-a}^a f \cdot \varphi$ for each $\varphi \in K$. *J. Mařík* (Prague)

Baiada, E.; Vinti, C. 4076

Generalizzazioni non Markoviane della definizione di perimetro.

Ann. Mat. Pura Appl. (4) **62** (1963), 1–58.

Let $f(r, x; s, y)$ be defined for $0 < r < s < T$, $x \in R$, $y \in R$, where $\{R, B, \mu\}$ is a measure space and $T > 0$ a given number. Moreover, let $f \geqq 0$,

$$\int_R f \, d\mu_y = 1 \text{ for every } (r, x, s),$$

$$\lim_{r \to s-0} \int_{R-U} f(r, x_0; s, y) \, d\mu_y = 0$$

for every x_0 and open U containing x_0. Let $F \geqq 0$ satisfy

$$\int_R F \, d\mu_x = 1 \text{ for every } (r, s, y),$$

$$\int_R F(r, x; \theta, y) f(\theta, y; s, z) \, d\mu_y = f(r, x; s, z).$$

If $F = f$, this says that f is a Markov transition density. Using these ideas, a "forward" definition of total variation of the gradient of a bounded, μ-measurable function g on R is given. If R is euclidean E^n, μ is Lebesgue measure, and $F = f$ is the transition density for n-dimensional Brownian motion, this is due to De Giorgi [same Ann. (4) **36** (1954), 191–213; MR **15**, 945]. If g is the characteristic function of a set E, then the total variation of the gradient is the perimeter of E. A "backward" definition is also given. *W. H. Fleming* (Providence, R.I.)

Král, Josef; Riečan, Beloslav 4077

Note on the Stokes formula for 2-dimensional integrals in n-space. (Russian summary)

Mat.-Fyz. Časopis Sloven. Akad. Vied **12** (1962), 280–292.

Es seien $f^1, \cdots, f^m$ Parameterdarstellungen von rektifizierbaren geschlossenen Kurven in der Ebene, C die Menge der Punkte, die auf wenigstens einer dieser Kurven liegen, $\omega(z)$ die Summe der Indizes von z in bezug auf $f^1, \cdots, f^m$ wenn $z \notin C$ und $G = \{z : \omega(z) \neq 0\}$, $G_p = \{z : \omega(z) = p\}$. Weiter mögen Φ und Ψ stetige Abbildungen von $C \cup G$ in R_n darstellen, wobei Ψ Lipschitzsch sei, und es existiere $\gamma = \operatorname{rot}(\Phi, \Psi)$ in G in dem Sinne, daß $\int_{f_K} \Phi \, d\Psi = \iint_K \gamma$ für jedes in G enthaltene Rechteck K mit der in positiver Richtung durchlaufenen Randkurve f_K. Dann gilt $\sum_{k=1}^m \int_{f^k} \Phi \, d\Psi = \sum_{l=1}^\infty [\sum_{p \geq l} (\iint_{G_p} \gamma - \iint_{G_{-p}} \gamma)]$ und zwar auch dann, wenn die Integrale rechterhand nur als uneigentliche Integrale, definiert mittels Ausschöpfungen von G_p bzw. G_{-p} durch Folgen von Vereinigungen von Rechtecken mit beschränkten Randlängen, existieren. Der Beweis stützt sich auf eine Reihe früherer Arbeiten des ersten Verfassers; eine wesentliche Rolle spielen die meßbaren Teilmengen A der Ebene, auf deren Rand $A^{\cdot}$ ein endliches Vektormaß P existiert, so daß $\int_{A^{\cdot}} v \, dP = \int_A \operatorname{div} v(z) \, dz$ für jede unendlich differenzierbare Vektorfunktion v mit kompaktem Träger. *K. Krickeberg* (Heidelberg)

Trjitzinsky, W. J. 4078

★Totalisations dans les espaces abstraits.

Mémor. Sci. Math., Fasc. CLV.

Gauthier-Villars & C^ie, Éditeur-Imprimeur-Libraire, Paris, 1963. iii + 131 *pp.*

L'auteur dans un mémoire précédent [*Théorie métrique dans les espaces où il y a une mesure*, Mémor. Sci. Math., Fasc. CXLIII, Gauthier-Villars, Paris, 1960; MR **24** #A3253] a généralisé dans le cadre abstrait de A. Denjoy [Amer. J. Math. **73** (1951), 314–356; MR **12**, 685] des théorèmes de dérivation. Dans le présent mémoire, s'inspirant de Romanovski [Mat. Sb. (N.S.) **9 (51)** (1941), 67–120; MR **2**, 354] il postule en outre une topologie à base dénombrable G' et une famille $\mathscr{M}$ d'ensembles R analogue à celle des intervalles euclidiens ouverts. Pour deux points x_1 et x_2, $\rho(x_1, x_2)$ désigne le minimum de la mesure $\phi(0)$ pour tous les ensembles O de G' qui contiennent x_1 et x_2, $S(x, r)$ la "sphère" lieu des points z tels que $\rho(x, z) \leqq r$. $\mathscr{M}$ permet l'introduction d'une intégrale de Burkill. $\tilde{\mathscr{M}}$ est un anneau booléen extension de $\mathscr{M}$. La totale ψ relative à l'intégrant f est définie sur $\tilde{\mathscr{M}}$ de manière semblable à Romanovski, toutefois elle doit être complètement additive dans $\tilde{\mathscr{M}}$. Il est montré qu'à une f totalisable correspond une seule totale ψ. La dérivée de ψ relativement aux ensembles de $\mathscr{M}$ inclus dans R contractant en mesure existe et est égale à f presque partout sur R. Les totales sont caractérisées, les fonctions totalisables classifiées. Partant du théorème de Denjoy-Vitali pour "sphères", l'auteur définit une "intégrale de Burkill au sens sphérique symétrique" et une nouvelle totale dite "totale-S". La nouvelle totale jouit de propriétés analogues à la précédente. Aucun exemple n'est donné; il n'est pas sûr que la théorie dépasse essentiellement le cadre euclidien. {Remarque du rapporteur: Dans Hypothèse 2.6, si la famille des enveloppes est prise comme famille d'ouverts comme semble le confirmer (2.13), on voit que si U est l'intervalle $[0, 1]$, $\mathfrak{F}$ la famille des sous-intervalles ouverts de $[0, 1]$, ϕ la mesure de Lebesgue, les ensembles uniponctuels sont des enveloppes, 3^{18} et l'axiome (2.5d) impliquent que tous les sous-ensembles de $[0, 1]$ sont mesurables-ϕ. L'exemple est à rejeter!}

Chr. Y. Pauc (Nantes)

Dyer, James A. 4079

The mean Stieltjes integral representation of a bounded linear transformation.

J. Math. Anal. Appl. **8** (1964), 452–460.

The author shows that a linear continuous functional on the space Q_L of functions f quasicontinuous on $[a, b]$, continuous at a and left continuous for $a < s \leqq b$, with the l.u.b. norm, is expressible in the form $\int_a^b f\,d\beta$, where β is of bounded variation and the integral is the mean Stieltjes integral by successive subdivisions. This result has been obtained by H. S. Kaltenborn [Bull. Amer. Math. Soc. **40** (1934), 702–708]. It then follows that a linear continuous transformation on Q_L to Q_L is expressible in the form $\int_a^b f(s)\,d_s\beta(s, t)$, where $\beta(s, t)$ is uniformly of bounded variation in s, with t on $[a, b]$ and $\beta(s, t)+\beta(s+, t)$ is in Q_L for each s. Following F. Riesz and B. v. Sz. Nagy [*Functional analysis*, p. 221, Ungar, New York, 1955; MR **17**, 175], it is proved that a necessary condition that such a transformation be completely continuous is that $\lim_{t\to t_0-0}\int_a^b |d_s\beta(s, t)-d_s\beta(s, t_0)|=0$, for all $a<t_0\leqq b$, together with a similar right continuity condition at $t=a$. The author's proof of the sufficiency of these conditions for complete continuity is faulty, since the Borel theorem is not usable. The statement that the adjoint transformation of the completely continuous transformation $\int_a^b f(s)\,d_s\beta(s, t)$ is $\int_a^b \beta(s, t)\,d\theta(t)$, θ in Q_L^*, depends on the reasoning of the preceding theorem.

T. H. Hildebrandt (Ann Arbor, Mich.)

Chacon, R. V. 4080

Linear operators in L_1.

Ergodic Theory (*Proc. Internat. Sympos., Tulane Univ., New Orleans, La.*, 1961), *pp.* 75–87. *Academic Press, New York*, 1963.

There has developed a considerable literature on extensions of the original ergodic theorems of von Neumann and Birkhoff. This paper is a discussion of various generalizations given by Doob, Kakutani, Hopf, Dunford and Schwartz, and Chacon and Ornstein, which extend Birkhoff's pointwise ergodic theorem in separate directions. References to this material are to be found in the paper. The author's purpose is to state a theorem bringing together all these different results. The setting of this theorem is the Banach space L' of complex-valued integrable functions defined on a σ-finite measure space $(S, \mathscr{F}, \mu)$. The norm $\|f\|$ of a function $f\in L'$ is defined as usual by $\|f\|=\int_S |f(s)|\,d\mu$ and the norm $\|T\|$ of a linear operator T of L_1 into itself by $\|T\|=\sup_{f\in L_1}\|Tf\|/\|f\|$. Theorem: Let $\{p_n\}$ be a sequence of non-negative measurable functions and let T be a linear operator of L_1 to L_1 with $\|T\|\leqq 1$ such that $|Tg(s)|\leqq p_{n+1}(s)$ a.e. whenever $|g(s)|\leqq p_n(s)$ a.e. and $g\in L_1$. Then

$$\lim_{n\to\infty} \sum_{k=0}^{n} T^k f(s) \Big/ \sum_{k=0}^{n} p_k(s)$$

exists a.e. on $\{s: 0<\sum_{k=0}^{\infty} p_k(s)\leqq +\infty\}$.

R. L. Adler (Yorktown Heights, N.Y.)

Chacon, R. V. 4081

Convergence of operator averages.

Ergodic Theory (*Proc. Internat. Sympos., Tulane Univ., New Orleans, La.*, 1961), *pp.* 89–120. *Academic Press, New York*, 1963.

In this paper the author supplies a proof of the theorem stated in the article above [#4080]. Among the lemmas required for the proof is a characteristic maximal lemma.

R. L. Adler (Yorktown Heights, N.Y.)

Ionescu Tulcea, Alexandra 4082

Random series and spectra of measure-preserving transformations.

Ergodic Theory (*Proc. Internat. Sympos., Tulane Univ., New Orleans, La.*, 1961), *pp.* 273–292. *Academic Press, New York*, 1963.

Let $(X, \mathscr{B}, \mu)$ be a totally σ-finite measure space; let $\mathscr{T}$ be the set of measurable, measure-preserving, invertible transformations τ of X onto X; and let U_τ denote the unitary transformation corresponding to τ. There are obtained results on the spectra of U_τ and the singularities of power series $\sum_{n=0}^{\infty} f(\tau^n(x))z^n$ and vector-valued power series $\sum_{n=0}^{\infty} (U_\tau^n f)z^n$, where $\tau\in\mathscr{T}$, $f\in L^2(X, \mathscr{B}, \mu)$, $x\in X$, $|z|<1$. It is shown in particular that the spectrum of U_τ is the boundary of the unit circle if and only if $U_\tau^n\neq I$ (the identity operator on $L^2(X, \mathscr{B}, \mu)$) for every positive integer n.

C. R. Putnam (Lafayette, Ind.)

Lloyd, S. P. 4083

An adjoint ergodic theorem.

Ergodic Theory (*Proc. Internat. Sympos., Tulane Univ., New Orleans, La.*, 1961), *pp.* 195–201. *Academic Press, New York*, 1963.

Let X be a Banach space, let X^* be its conjugate, let $B(X)$ be the collection of bounded linear operators on X, let W^* be the adjoint of an operator W, and let T be an additive semigroup of positive real numbers. Let $\{P(t),\ t \text{ in } T\}$ be a semigroup of elements of $B(X)$ and let α be a complex number such that

$$M = \limsup \|e^{-\alpha t}P(t)\| < \infty.$$

The author shows that the closed convex hull of $\{P^*(t),\ t \text{ in } T\}$ (in a suitable topology) contains Q^* such that $Q^*P^*(t)=P^*(t)Q^*=Q^*e^{\alpha t}$ for all t in T. Such a Q^* is a projection onto the subspace of members of X which are invariant under $e^{-\alpha t}P^*(t)$ for all t in T, and satisfies $\|Q^*\|\leqq M$. The theorem is applied to Markov processes.

D. L. Hanson (Columbia, Mo.)

Oxtoby, John C. 4084

On two theorems of Parthasarathy and Kakutani concerning the shift transformation.

Ergodic Theory (*Proc. Internat. Sympos., Tulane Univ., New Orleans, La.*, 1961), *pp.* 203–215. *Academic Press, New York*, 1963.

Let $\{X_n,\ n=0, \pm 1, \cdots\}$ be a bilateral sequence of replicas of a compact metric space X_0, and let T be the shift operator defined on the Cartesian product space $X=\prod_{-\infty}^{\infty} X_n$. Let $\mathfrak{M}$ be the class of invariant probability measures defined on the natural σ-algebra of subsets of X. Finally, let $\mathfrak{M}_E$ be the ergodic elements of $\mathfrak{M}$, $\mathfrak{M}_P$ those corresponding to periodic points of X, and $\mathfrak{M}_Q$ those corresponding to quasi-regular points of X. The author presents a new and unified way to prove the following three results, which he states are not new: (1) $\mathfrak{M}_E$ is a dense G_δ in $\mathfrak{M}$; (2) $\mathfrak{M}_P$ is dense in $\mathfrak{M}$; (3) $\mathfrak{M}_Q=\mathfrak{M}$. It is also shown that the compactness condition may be relaxed.

J. R. Blum (Albuquerque, N.M.)

Sucheston, L. 4085

Remarks on Kolmogorov automorphisms.

Ergodic Theory (*Proc. Internat. Sympos., Tulane Univ., New Orleans, La.*, 1961), *pp.* 251–258. *Academic Press, New York*, 1963.

Let $\{\Omega, \mathscr{A}, P\}$ be a measure space with $P(\Omega)=1$. The following theorem is proved. A Kolmogorov automorphism of Ω is mixing in all degrees. Here, an invertible, measure-preserving transformation T of Ω is called a Kolmogorov automorphism if there exists a σ-field $\mathscr{S} \subset \mathscr{A}$ such that: (i) $\mathscr{S} \subset T\mathscr{S}$ mod 0 (i.e., modulo P-null sets); (ii) the σ-field generated by the $T^n\mathscr{S}$, $n=0, \pm 1, \cdots$, is equal to $\mathscr{A}$ mod 0; (iii) the intersection of the $T^n\mathscr{S}$, $n=0, \pm 1, \cdots$, contains only null sets and their complements. T is called mixing of degree r if for every system of $r+1$ measurable sets $(A_i)_{0 \le i \le r}$ and every sequence $(k_n^i)_{0 \le i \le r}$, $n=1, 2, \cdots$ (k_n^i integers), such that $\lim_{n\to\infty} \min_{0 \le i < j < r} |k_n^j - k_n^i| = \infty$, we have $\lim_{n\to\infty} P(\bigcap_{i=0}^r T^{k_n^i} A_i) = \prod_{i=0}^r P(A_i)$.

N. Dinculeanu (Bucharest)

Tsurumi, Shigeru 4086

A random ergodic theorem.

Ergodic Theory (*Proc. Internat. Sympos., Tulane Univ., New Orleans, La.*, 1961), *pp.* 259–271. *Academic Press, New York*, 1963.

A very general random ergodic theorem is proved, extending the theorem of Beck and Schwartz, which in turn generalizes results of Kakutani, extending the theorem of the reviewer and J. von Neumann. The author's formulation extends also the theorem of Gładysz which deals with functions with values in the reflexive Banach space. The proof involves interesting constructions on elementary functions on products of measure spaces and the result of Chacon [Bull. Amer. Math. Soc. **67** (1961), 186–190; MR **23** #A525].

S. M. Ulam (La Jolla, Calif.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 3992, 4132, 4136, 4234, 4275, 4416.

Fuchs, B. A. [Fuks, B. A.]; Shabat, B. V. [Šabat, B. V.] 4087

★Functions of a complex variable and some of their applications. Vol. I.

Original translation by J. Berry; revised and expanded by J. W. Reed.

Pergamon Press, Oxford-London-New York-Paris; Addison-Wesley Publishing Co., Inc., Reading, Mass., 1964. xvi + 431 *pp.* $10.00.

This is an introductory text that is particularly detailed on special conformal mappings and their application to plane potential problems. The first Russian edition [GITTL, Moscow, 1949] was reviewed in MR **12**, 87. The present version has been extensively polished by J. W. Reed.

R. P. Boas, Jr. (Evanston, Ill.)

Teodorescu, N. 4088

Opérateurs de projection en théorie des primitives aréolaires bornées.

Ann. Mat. Pura Appl. (4) **60** (1962), 1–27.

Soient Δ un domaine borné du plan complexe, γ une courbe de Jordan rectifiable quelconque dans Δ, dont l'intérieur δ est compris dans Δ. On considère la fonctionnelle additive de domaine $F(\delta) = (1/2i)\int_\gamma f(z)\,dz$, f continue dans Δ. La dérivée $\varphi(v)$ de F au sens de Lebesgue, si elle existe, est appelée dérivée aréolaire de f et désignée par $Df/D\omega$. Si F est absolument continue, f est appelée primitive aréolaire de sa dérivée φ et on a alors $(1/2i)\int_\gamma f(z)\,dz = \int_\delta \varphi(v)\,d\omega$. Si la dérivée φ est bornée, f est appelée primitive aréolaire bornée. Après avoir re-évoqué l'histoire de ces notions, l'auteur prouve que les primitives aréolaires bornées dans $\bar\Delta$ admettent la décomposition directe $f(z) = h(z) + T\varphi(z)$ avec h holomorphe dans Δ et continue dans $\bar\Delta$ et $T\varphi(z) = -(1/\pi)\int_\Delta \varphi(v)/(v-z)\,d\omega$. Les opérateurs linéaires de projection correspondants sont

$$T\frac{D}{D\omega} = -\frac{1}{\pi}\int_\Delta \frac{D/D\omega}{v-z}\,d\omega, \qquad C = 1 - T\frac{D}{D\omega};$$

si Δ est limité par un système Γ fini de courbes de Jordan rectifiables, $Cf(\zeta) = (1/2\pi i)\int_\Gamma f(z)/(z-\zeta)\,dz$.

K. Strebel (Zürich)

Teodorescu, N. 4089

Fonctions monogènes (α) et fonctions analytiques généralisées.

Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (*N.S.*) **4 (52)** (1961), *no.* 1–2, 129–147 [1963].

Der Artikel gibt den Inhalt eines Vortrags wieder, den der Autor auf dem Kolloquium über partielle Differentialgleichungen im September 1959 in Bukarest hielt.

Der Autor geht zunächst auf die von D. Pompeiu eingeführte areolare Ableitung für Funktionen der Klasse C^1 ein, die später vom Autor auf den Fall stetiger Funktionen verallgemeinert wurde. Eine Funktion $f(z)$, die areolare Ableitungen besitzt, wurde vom Autor bereits in früheren Arbeiten α-monogen, und im Falle stetiger areolarer Ableitungen α-holomorph genannt. Eine Funktion $f(z) \in C^0(\Delta)$ heißt areolare Stammfunktion der Funktion $\varphi(z) \in L_1(\Delta)$ auf einer Menge $E \subset \Delta$, falls $f(z)$ auf E α-monogen ist und die areolare Ableitung $Df/D\omega = \varphi$ auf E besitzt. Die von I. N. Vekua eingeführten Funktionen der Klasse C_t sind auch α-monogen.

Unter Verwendung der verallgemeinerten Ableitung im Sobolevschen Sinne werden verallgemeinerte areolare Ableitungen definiert. Die so definierte Funktionenklasse ist der von I. N. Vekua eingeführten Klasse D_t^p $(p>1)$ äquivalent.

Wie gewöhnlich werden formal areolare Ableitungen höherer Ordnung definiert. Es wird gezeigt, daß jede verallgemeinerte analytische Funktion im Vekuaschen und Bersschen Sinne α-monogen ist. Sie stellen eine spezielle Klasse der α-monogenen Funktionen dar, besitzen areolare Stammfunktionen und laßen sich wie folgt darstellen:

$$f(\zeta) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\zeta}\,dz - \frac{1}{\pi}\int_\Delta \frac{\varphi(v)}{v-\zeta}\,d\omega.$$

Anschließend werden einige Sätze aus der Theorie der verallgemeinerten analytischen Funktionen, wie zum Beispiel der Liouvillesche Satz, die Cauchysche Integralformel, angeführt.

A. Derjugin and W. Schmidt (Simferopol)

Leont'ev, A. F. 4090

On the growth of functions defined by Dirichlet series and by certain other more general series. (Russian)

Mat. Sb. (*N.S.*) **63 (105)** (1964), 227–237.

Let $\lambda_1 < \lambda_2 < \cdots$ be a sequence of positive numbers. Let

$D(x)$ be the number of $\lambda_k < x$. It is assumed that $\limsup_{x\to\infty} D(x)/x < \infty$. Put $\Delta = \limsup (1/\lambda) \int_0^\lambda D(x)\,dx$. Let $F(s) = \sum_{n=1}^\infty d_n e^{-\lambda_n s}$ be an everywhere convergent Dirichlet series. Write $M(a, t_0, \sigma) = \sup_{|t-t_0|<a} |F(\sigma+it)|$, $\rho(a, t_0) = \sup_{\sigma\to\infty} (\log\log M(a, t_0, \sigma))/\sigma$, $\rho = \rho(\infty, 0) = \rho(\infty, t_0)$. Write $L(z) = \prod (1-(z/\lambda_n)^2)$,

$$q = \limsup_{n\to\infty} (-\log|L'(\lambda_n)|)/(\lambda_n \log \lambda_n).$$

The number q is always non-negative.

The author shows that for $a > \pi\Delta$, $\rho/(1+\rho q) \leqq \rho(a, t_0) \leqq \rho$ and that these limits on $\rho(a, t_0)$ cannot be replaced by narrower ones. Similar results are given for series of the type $\sum d_n g(\lambda_n s)$, where $g(z)$ is an entire function satisfying certain conditions. *W. H. J. Fuchs* (Ithaca, N.Y.)

Lick, D. R. 4091

Sets of non-uniform convergence of Taylor series.

J. London Math. Soc. **39** (1964), 81–85.

Using Fejér polynomials as building blocks, the author constructs, corresponding to each open set G on the unit circle C, a continuous function on $|z| \leqq 1$ whose power series converges everywhere on C and converges uniformly on every closed arc in G and on no closed arc meeting $C-G$. He uses the construction to prove several related propositions, including a theorem published with an incorrect proof by Herzog and the reviewer [Duke Math. J. **20** (1953), 41–54 (see Theorem 5 on p. 51); MR **14**, 738]. Finally, he shows how all his results can be transferred to the theory of Fourier series of continuous functions. *G. Piranian* (Ann Arbor, Mich.)

Walsh, J. L. 4092

A theorem of Grace on the zeros of polynomials, revisited.

Proc. Amer. Math. Soc. **15** (1964), 354–360.

Theorem 1: Suppose $|\alpha_k| \leqq 1$ for $k = 1, 2, \cdots, n$, with $\sum \alpha_k = 0$; we set $p(z) \equiv \prod (z-\alpha_k) - C$, where the constant C is arbitrary. Then for $|z| \leqq 1$ all zeros of $p(z)$ lie in the n circles $|z - C^{1/n}| \leqq 1$ and for $|z| > 1$ in the n lemniscate regions $|z(z-C^{1/n})| \leqq 1$, where $C^{1/n}$ takes all n values. Theorem 2: Suppose we have $|\alpha_k - a| \leqq r_1$ and $|\beta_k - b| \leqq r_2$, $k = 1, 2, \cdots, n$, with $\sum \alpha_k = na$, $\sum \beta_k = nb$. We set $p(z) \equiv \prod (z-\alpha_k) - A \prod (z-\beta_k)$, where A is an arbitrary constant. Then if $A \neq 1$, all zeros of $p(z)$ lie in the n loci

$$(*)\quad |z-(a-bA^{1/n})(1-A^{1/n})^{-1}| \leqq |1-A^{1/n}|^{-1}[r_1 \min(1, r_1/|z-a|) + r_2|A|^{1/n} \min(1, r_2/|z-b|)],$$

where $A^{1/n}$ is in turn each nth root of A. If $A = 1$ and we have

$$(**)\quad r_1 \min(1, r_1/|z-a|) + r_2 \min(1, r_2/|z-b|) > |a-b|,$$

then all zeros of $p(z)$ lie in the $n-1$ loci (*), where $A^{1/n}$ is in turn each nth root of unity except unity. If $A = 1$ and (**) is false, we draw no conclusion concerning the location of z.

Theorems 1 and 2 are proved by means of the following basic lemma. If we have $m_k > 0$, $\sum m_k = 1$, $|\alpha_k| \leqq 1$, $\sum m_k \alpha_k = 0$, $|z| > 1$ (where $k = 1, 2, \cdots, n$), then there exists an α such that $|\alpha| \leqq |z|^{-1}$, with $\sum m_k \log(1-\alpha_k z^{-1}) = \log(1-\alpha z^{-1})$, the logarithms being appropriately taken. *O. Shisha* (Dayton, Ohio)

Walsh, J. L.; Landau, H. J. 4093

On canonical conformal maps of multiply connected regions.

Trans. Amer. Math. Soc. **93** (1959), 81–96.

In a previous paper [same Trans. **82** (1956), 128–146; MR **18**, 290] the first author proved the following theorem: Let D be a region of the extended Z-plane whose boundary consists of mutually disjoint Jordan curves $B_1, \cdots, B_\mu$; $C_1, \cdots, C_\nu$, $\mu\nu \neq 0$. There exists a conformal map of D onto a region Δ of the extended Z-plane, one-to-one and continuous in the closures of the two regions, where Δ is defined by

$$1 < |T(Z)| < \exp(\tau^{-1}),$$

$$T(Z) = A \prod_{j=1}^{\mu} (Z-a_j)^{M_j} \prod_{k=1}^{\nu} (Z-b_k)^{-N_k}$$

with M_j, N_k, $\tau > 0$, $\sum_{j=1}^{\mu} M_j = \sum_{k=1}^{\nu} N_k = 1$. The locus $|T(Z)| = 1$ consists of μ mutually disjoint Jordan curves B_j^*, respective images of the B_j, which separate Δ from the a_j; the locus $|T(Z)| = \exp(\tau^{-1})$ consists of ν mutually disjoint Jordan curves C_k^*, respective images of the C_k, which separate Δ from the b_k. The number τ is a conformal invariant of D and represents the period of the conjugate function of the harmonic measure of the set $\bigcup_{k=1}^{\nu} C_k$ with respect to D around the set of curves $\bigcup_{j=1}^{\mu} B_j$.

In this paper the authors show that this theorem extends to the case of regions in which the sets $\bigcup_{j=1}^{\mu} B_j$ and $\bigcup_{k=1}^{\nu} C_k$ are made up of Jordan curves that are not necessarily disjoint, thus allowing multiple points. The method of proof involves approximating the domain D by a sequence D_n of domains bounded by mutually disjoint Jordan curves. By the main theorem of the previous paper cited above, for every D_n there is a canonical conformal map $Z = f_n(z)$ taking D_n onto a domain Δ_n of the Z-plane defined by $1 < |T_n(Z)| < \exp(\tau_n^{-1})$, where

$$T_n(Z) = A_n \prod_{j=1}^{\mu} (Z-a_{jn})^{M_{jn}} \prod_{k=1}^{\nu} (Z-b_{kn})^{-N_{kn}},$$

$$M_{jn}, N_{kn}, \tau_n > 0, \quad \sum M_{jn} = \sum N_{kn} = 1.$$

By approximating the harmonic measure of the set $\bigcup_{k=1}^{\nu} C_k$ with respect to D with a corresponding harmonic measure with respect to D_n, the authors are able to prove that M_{jn}, N_{kn}, τ_n approach the desired limits and that the limit loci $|T(Z)| = 1$ and $|T(Z)| = \exp(\tau^{-1})$ are the proper bounding curves for D. The limiting case where the curves B_j are allowed to shrink to points and the case where D is simply connected are studied.

W. C. Royster (Lexington, Ky.)

Fufaev, V. V. 4094

Conformal mappings of domains with corners and differential properties of solutions of the Poisson equation in domains with corners. (Russian)

Dokl. Akad. Nauk SSSR **152** (1963), 838–840.

Let G_1, G_2 be Jordan domains with corners, and let $f(z)$ map G_1 conformally onto G_2. Various hypotheses involving the angle at the corner and the Hölder continuity of rth derivatives of the boundary curves are used to imply that $f^{(r)}(z)$ is bounded in G_1, and satisfies a Hölder condition with stated exponent there. Proofs are not given, but the results are related to theorems of Kellogg, Lichtenstein, and Warschawski [cf. Warschawski, Pacific

J. Math. **5** (1955), 835–839; MR **17**, 357]. As an application it is stated that the solutions to the Dirichlet problem for a domain with corners belong to certain function classes depending on the boundary values, extending previous work of the author [Dokl. Akad. Nauk SSSR **131** (1960), 37–39; MR **22** #9750]. Further theorems deal with the differential properties of solutions of Poisson's equation with null boundary conditions. *E. Reich* (Stanford, Calif.)

Gaier, Dieter 4095

Konforme Abbildung mehrfach zusammenhängender Gebiete durch direkte Lösung von Extremalproblemen.
Math. Z. **82** (1963), 413–419.

The analytic functions yielding the conformal mappings of a given multiply connected domain D onto various canonical domains can be constructed by computing their Fourier coefficients with respect to orthonormal sets of functions which are closed in certain subclasses of the class of functions whose moduli are square-integrable over D [S. Bergman, *The kernel function and conformal mapping*, Amer. Math. Soc., New York, 1950; MR **12**, 402]. In the case in which D is simply-connected and the canonical domain is a disk, a well-known alternative—but not essentially different—procedure is based on a classical minimum property of this mapping which makes it possible to approximate it by means of a Rayleigh-Ritz process. The author points out that the same approach may be used in the case of all the canonical mappings which can be characterized by suitable minimum properties. This is illustrated in the case of the parallel-slit mappings, and for the mapping of a doubly-connected domain onto a ring. While the integrals to be evaluated are the same as those required in the orthonormalization of the given sequence of functions, the author says experiments indicate that his procedure leads to a considerable saving of algebraic labor.
Z. Nehari (Pittsburgh, Pa.)

Clunie, J.; Keogh, F. R. 4096

Addendum to a note on schlicht functions.
J. London Math. Soc. **39** (1964), 63–64.

The original note appeared in same J. **35** (1960), 229–233 [MR **22** #1682]. The authors prove that if $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ is univalent in $|z|<1$ and maps $|z|<1$ onto a domain whose boundary consists of a closed continuous rectifiable curve, then $a_n=o(1/n)$ as $n\to\infty$. This is best possible because, given any sequence $\{\eta(n)\}$ which converges to 0 as $n\to\infty$, there is an $f(z)$ of the type described such that $n|a_n|>\eta(n)$ for infinitely many n.
A. W. Goodman (Lexington, Ky.)

Jenkins, James A. 4097

Some area theorems and a special coefficient theorem.
Illinois J. Math. **8** (1964), 80–99.

Let D be a domain of finite connectivity in the complex plane containing the point ∞. Let $\Sigma(D)$ denote the class of univalent functions $f(z)$ in D which are normalized by $f'(\infty)=1$. Grunsky [Math. Z. **45** (1939), 29–61] derived necessary and sufficient conditions for the Laurent coefficients of $F(z)$ at ∞ in order that $f\in\Sigma(D)$; these are the very useful Grunsky inequalities which estimate quadratic forms in terms of hermitian forms. Golusin [Mat. Sb. (N.S.) **8** (50) (1940), 277–284; MR **2**, 185] showed that for every polynomial $Q_m(t)$ of degree m and every $f(z)$ univalent in $|z|>1$ the Laurent series $Q_m[f(z)]=\sum_{-m}^{\infty} C_\nu z^{-\nu}$ satisfies the inequality $\sum_{-m}^{\infty} \nu|C_\nu|^2\leqq 0$. Wolibner [Colloq. Math. **2** (1951), 249–253; MR **14**, 35] proved that if these conditions hold for fixed $f(z)$ and all $Q_M(t)$, they are conversely sufficient for f to be univalent in $|z|>1$. In the present paper the relations between the two types of inequalities are studied and their common root in the area principle is stressed. Golusin's estimate is extended to the case that $Q(t)$ is an integral function and to univalent mappings into Riemann surfaces. Various simple Grunsky type inequalities for univalent functions are given in case that D is a circular region. The methods used are always modifications of the area principle. Finally, the author discusses the connection between the area principle and the method of extremal metrics. *M. Schiffer* (Stanford, Calif.)

Kazdan, Jerry L. 4098

A boundary value problem arising in the theory of univalent functions.
J. Math. Mech. **13** (1964), 283–303.

Let D be a plane domain and let

$$z^* = z+\varepsilon F(z,\bar z) \tag{1}$$

give a univalent map of the domain D onto a slightly varied domain D^*. It is never very clear what $F(z,\bar z)$ is. The author states (p. 285) that $F(z,\bar z)$ maps D onto D^*, but this can hardly be true. Various variational formulae for Dirichlet integrals, Green's functions and coefficients of mapping functions are derived and necessary conditions are set up for a solution of Loewner's differential equation to be extremal for the nth coefficient of a univalent function.

The method has been used to set up a numerical procedure using the IBM 7090 for finding a counterexample to the Bieberbach conjecture, but apparently the results are inconclusive so far.

The paper is obscure to the point of being unintelligible. Thus, in addition to the point about $F(z,\bar z)$ in (1) above, an obviously basic formula (2) on p. 286 is said to follow by "a direct computation" which the reviewer, at least, was unable to carry out. *W. K. Hayman* (London)

Libera, Richard J. 4099

Some radius of convexity problems.
Duke Math. J. **31** (1964), 143–158.

Let $\mathscr{S}$ be the class of functions regular and univalent in the unit disc with the usual normalisation, let $\mathscr{R}\subset\mathscr{S}$ be the subclass of $\mathscr{S}$ consisting of those functions whose derivatives have positive real parts, and let $\mathscr{S}_\sigma{}^*$ be the subclass of $\mathscr{S}$ consisting of functions for which $\operatorname{Re}(zF'(z)/F(z))\geqq\sigma$, $0\leqq\sigma\leqq 1$.

The author considers the classes $C(\lambda,\sigma)$ of functions $f(z)\in\mathscr{S}$ such that there exist $F(z)\in\mathscr{S}_\sigma{}^*$ and ε, $|\varepsilon|=1$, for which $\operatorname{Re}(zf'(z)/\varepsilon F(z))\geqq\lambda$ holds. He obtains the radius of convexity, exact estimates of $|f'(z)|$, $|f(z)|$, $|a_n|$, and finds the exact value $1/(4-\sigma-\lambda)$ of the Koebe constant for the class $C(\lambda,\sigma)$. He also evaluates the radius of convexity for the class $\mathscr{R}$. *Z. Lewandowski* (Lublin)

Popov, V. I. 4100
Starshapedness of arcs of level curves for univalent conformal mappings. (Russian)
Dokl. Akad. Nauk SSSR **155** (1964), 757–760.

The author considers a paper by Bazilevič and Korickiĭ [Mat. Sb. (N.S.) **58 (100)** (1962), 249–280; MR **26** #2600] in which it is shown that if $f(z)$ is a member of the class S of normalized univalent functions regular in $|z|<1$ and $\underline{R}=r(1+r)^{-2}$, $\bar{R}=r(1-r)^{-2}$, $|z|\leqq r<1$, then there exists a constant $\alpha_s(0.1005<\alpha_s<0.134)$ such that each arc of the image L_r of $|z|=r$ under $w=f(z)\in S$ is starlike in $\alpha_s\bar{R}\leqq |w|\leqq \bar{R}$ for each $f(z)\in S$ and is not starlike in a wider ring for some $f(z)\in S$. It is shown that if $r_s<r<1$ $(r_s=\tanh \pi/4)$, $f(z)\in S$, then each arc of the image curve is starlike in $\lambda\underline{R}\bar{R}\leqq|w|\leqq\bar{R}$, where $\lambda=\frac{5}{4}(5^{1/2}e^{-4\arctan 2})$. It is shown that this result includes the previous one. The methods used involve Loewner's equations.

W. C. Royster (Lexington, Ky.)

Wilf, Herbert S. 4101
Calculations relating to a conjecture of Pólya and Schoenberg.
Math. Comp. **17** (1963), 200–201.

The conjecture is that if $f(z)=\sum_1^\infty a_\nu z^\nu$ and $g(z)=\sum_1^\infty b_\nu z^\nu$ are regular in $|z|<1$ and map it onto a schlicht convex domain, then the same is true of the function $h(z)=\sum_1^\infty a_\nu b_\nu z^\nu$ [see Pólya and the reviewer, Pacific J. Math. **8** (1958), 295–334; MR **20** #7181]. 175 independent computations were carried out on the ILIAC computer with pairs $f(z)$, $g(z)$ generated by a randomized procedure using the Schwarz-Christoffel mapping formula. The verification of the conjecture for the composite function $h(z)$ consisted in verifying the non-negativity of the first ten Toeplitz determinants for the expansion of $zh''(z)(h'(z))^{-1}+1$ in powers of z. "No counterexample was found, which would seem to enhance the plausibility of the conjecture, particularly in view of the fact that the test functions tended to have large, non-real coefficients, and therefore provided a reasonably stern test." The reviewer agrees with this statement.

I. J. Schoenberg (Princeton, N.J.)

Lewittes, Joseph 4102
Automorphisms of compact Riemann surfaces.
Amer. J. Math. **85** (1963), 734–752.

The author first points out that the representation of the group $H(S)$ of automorphisms of a compact Riemann surface S of genus $g\geqq 2$ as permutations of the set of Weierstrass points leads to the Hurwitz theorem that $H(S)$ is finite. He then proceeds to consider the representations of $H(S)$ as linear transformations of the spaces D_m and A_m, where D_m is the vector space of meromorphic differentials of type m on S, $m\geqq 0$, and A_m is the subspace of D_m consisting of analytic differentials. He first shows that the representation R_m of $H(S)$ as linear transformations of A_m is faithful if $m\geqq 1$ except when $g=2$ and m is even, in which case the kernel of R_m is (I, J), where I is the identity automorphism and J is the automorphism of order 2 corresponding to an interchange of the two sheets of S (considered as a two-sheeted covering of the sphere). For a subgroup $H\subset H(S)$, let $D_m{}^H$ denote the subspace of D_m consisting of differentials which are invariant under the elements of H, and let $\bar{D}_m$ denote the meromorphic differentials of type m on the quotient Riemann surface $\bar{S}=S/H$. It is shown that $D_m{}^H$ is isomorphic to $\bar{D}_m$, and the dimension of $A_m{}^H=A_m\cap D_m{}^H$ is then determined. If $h\in H(S)$ has prime order, the matrix representation of $R_1(h)$ is given explicitly. More generally, he obtains explicitly a matrix representation of $R_m(h)$ for any $h\in H(S)$ as a diagonal matrix with respect to a normalized basis of A_m. An application of this matrix representation yields the theorem that if a non-Weierstrass point is fixed, then there are only 2, 3, or 4 fixed points. More specific information is obtained about the automorphisms of hyperelliptic surfaces and of surfaces having $(g-1)g(g+1)$ fixed points.

G. Springer (Lawrence, Kans.)

Patt, Charles 4103
Variations of Teichmueller and Torelli surfaces.
J. Analyse Math. **11** (1963), 221–247.

For compact Riemann surfaces of genus $g>1$ the author studies the effect of a Schiffer-Spencer type variation [Schiffer and Spencer, *Functionals of finite Riemann surfaces*, Princeton Univ. Press, Princeton, N.J., 1954; MR **16**, 461] at points $P_1, \cdots, P_n$ on the kind of local Teichmüller coordinates used by Rauch [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 42–49; MR **17**, 251] and the reviewer [*Analytic functions*, pp. 45–66, Princeton Univ. Press, Princeton, N.J., 1960; MR **23** #A1798].

The notion of variation matrix is introduced. Its rank may vary with the location of $P_1, \cdots, P_n$ in a manner that depends on the Wronskian. If the reviewer's interpretation of the main result is correct, the rank is always maximal if $n<2g-2$ or $n>4g-4$, but need not be if $2g-2\leqq n\leqq 4g-4$.

L. V. Ahlfors (Cambridge, Mass.)

Royden, H. L. 4104
Riemann surfaces with the AB-maximum principle.
Ann. Acad. Sci. Fenn. Ser. A I No. 336/16 (1963), 7 pp.

Let (W, Γ) be a bordered Riemann surface with compact border Γ. Consider an algebra A of AB-functions on (W, Γ) that take their maxima on Γ. The author establishes the following result: There is an analytic mapping τ of (W, Γ) into a Riemann surface with compact closure of $\tau(W, \Gamma)$ and such that every $f\in A$ is of the form $g\circ\tau$, with g an AB-function on $\overline{\tau(W, \Gamma)}$.

If (W, Γ) satisfies the AB-maximum principle (every AB-function takes its maximum on Γ), then the class of all AB-functions on (W, Γ) consists precisely of the composites $g\circ\tau$. This corollary is an extension of the theorem of M. H. Heins [Ann. of Math. (2) **55** (1952), 296–317; MR **13**, 643]: Every parabolic Riemann surface with one boundary component has an end W and a mapping τ of W into $|z|<1$ with the above property. Further specialization gives the classical example of P. J. Myrberg [Ann. Acad. Sci. Fenn. Ser. A I Math.-Phys. No. 58 (1949); MR **10**, 441]: Every AB-function on a 2-sheeted covering of $0<|z|<1$ with branch points accumulating at 0 assumes identical values on the two sheets.

L. Sario (Los Angeles, Calif.)

Stout, E. L. 4105
Some theorems on bounded holomorphic functions.
Bull. Amer. Math. Soc. **70** (1964), 419–421.

Let R be a finite open Riemann surface, and $H^\infty(R)$ the sup-normed algebra of all bounded holomorphic functions

on R. In a natural way, R is embedded in the maximal ideal space of $H^\infty(R)$. The author announces a proof of the "corona conjecture" for R: R is dense in the maximal ideal space of $H^\infty(R)$. This theorem, due to L. Carleson when R is the unit disk, has been obtained independently by Alling.

The author uses this result to obtain interpolation theorems, again due to Carleson when R is the unit disk. Theorem: The set $\{z_n\}$ is an interpolating set for $H^\infty(R)$ if and only if there exists $\delta > 0$ such that for all n we have

$$\sup\{|f(z_n)| : f \text{ in } H^\infty(R),\ \|f\| \leqq 1,\ f(z_k) = 0 \text{ for } k \neq n\} \geqq \delta.$$

Carleson's condition in terms of Blaschke products is also generalized. Theorem: The set $\{z_n\}$ is an interpolating set if and only if there exists M such that for every n, $\sum_{k\neq n} g(z_k, z_n) \leqq M$. Here $g(\cdot, z)$ denotes the Green's function for R with singularity at z.

Let U be the open unit disk, and Φ the covering map of U onto R. Then the subset S of R is an interpolating set for $H^\infty(R)$ if and only if $\Phi^{-1}(S)$ is an interpolating set for $H^\infty(U)$.

The proofs will appear elsewhere.

A. Browder (Berkeley, Calif.)

Rodin, Burton 4106

The sharpness of Sario's generalized Picard theorem.

Proc. Amer. Math. Soc. **15** (1964), 373–374.

By construction of a particular Riemann surface as a covering surface of the complex plane, the author establishes the sharpness of Sario's generalized Picard theorem [L. Sario, Pacific J. Math. **12** (1962), 1079–1098, in particular, p. 1092; MR **27** #286; Trans. Amer. Math. Soc. **106** (1963), 521–533, in particular, p. 530; MR **26** #2603].

F. Huckemann (Giessen)

Gehring, F. W. 4107

The Carathéodory convergence theorem for quasiconformal mappings in space.

Ann. Acad. Sci. Fenn. Ser. A I No. 336/11 (1963), 21 *pp.*

The paper gives an analogue of Carathéodory's theorem on convergence of conformal mappings of variable regions for the case of q.c. (quasiconformal) mappings in space. The results are what one would expect to be proved, but the lemmas on which the proof has to be based are of course less explicit and less familiar.

The theorem in space has many interesting applications. For instance, it is found that the kernel of a convergent sequence of regions, each K-q.c. equivalent to a sphere, is itself K-q.c. equivalent to a sphere.

Most of the paper is rather technical and many results seem to be intended for use in other connections. There is in particular an interesting characterization of q.c. homeomorphisms: The Hurwitz property is the analogue of the usual Hurwitz lemma for schlicht conformal mappings. If a family $\mathscr{F}$ of q.c. homeomorphisms is normal and complete with respect to composition with similarities, then it turns out that the mappings are K-q.c. for some fixed K if and only if $\mathscr{F}$ has the Hurwitz property. Such results are, of course, very valuable because they often eliminate the need for explicit estimates.

L. V. Ahlfors (Cambridge, Mass.)

Accola, R. 4108

Invariant domains for Kleinian groups.

Bull. Amer. Math. Soc. **70** (1964), 412–413.

The author announces results about Kleinian groups and invariant domains with respect to them. A domain O is called invariant with respect to Γ if each element of Γ maps O onto itself. In particular, the author gives the following theorems. Theorem 1. If Γ possesses three disjoint invariant domains, then Γ is cyclic. Theorem 3. If Γ is a Kleinian group with two disjoint invariant domains, then there exists a maximal pair of disjoint invariant domains each of which is simply connected. If Ω_1 is a component of the set of discontinuities of Γ, and Ω_1 is not invariant under Γ, then it is invariant under no element of Γ (except *id*).

H. L. Royden (Stanford, Calif.)

Heins, Maurice 4109

Fundamental polygons of Fuchsian and Fuchsoid groups.

Ann. Acad. Sci. Fenn. Ser. A I No. 337 (1964), 30 *pp.*

C. L. Siegel proved, a long time ago, that the metric fundamental polygons of a Fuchsian group have a finite number of sides if and only if the noneuclidean area is finite. The primary aim of this paper is to extend Siegel's result to groups of the second kind. Naturally, the theorem has to be reformulated. The group should be considered in the whole plane, and the noneuclidean area becomes the area of the quotient space with respect to the Poincaré metric with constant negative curvature. Better still, the Poincaré area is finite if and only if the group is finitely generated. This theorem is proved with painstaking attention to all details.

These results were announced in Bull. Amer. Math. Soc. **69** (1963), 747–751 [MR **27** #5907], and the reviewer, swayed by the author's modesty, had described them as easy to prove. This should be taken with a grain of salt, for the success of the author's elementary method should not be taken for granted.

L. V. Ahlfors (Cambridge, Mass.)

Knopp, Marvin Isadore 4110

On generalized abelian integrals of the second kind and modular forms of dimension zero.

Amer. J. Math. **86** (1964), 430–440.

In a previous paper the author treated abelian integrals in the uniformizing plane, i.e., functions $f(\tau)$ satisfying $f(V\tau) = f(\tau) + C(V)$, where $V \in \Gamma(j)$, the principal congruence subgroup of level j, and $C(V)$, independent of τ, is an additive character on $\Gamma(j)$ [same J. **84** (1962), 615–628; MR **26** #3901]. In the present paper he treats generalized abelian integrals, functions satisfying

$$f(V\tau) = v(V)f(\tau) + C(V),$$

where $v(V)$, independent of τ, is a multiplicative character. He makes the restriction that v is a congruence character, i.e., $v \equiv 1$ on some group $\Gamma(nj)$.

The results are similar to those of the previous paper. Since the methods are also similar, the author contents himself with a sketch of the proofs.

The restriction to congruence characters v is made so that the Kloosterman-type sums which occur can be estimated nontrivially by a theorem of Petersson. Also the author uses Petersson's generalization of the Weierstrass gap theorem; in the previous paper the classical theorem was sufficient.

Explicit bases are constructed for the integrals of the

first and second kinds and for modular forms. These functions are all exhibited by their Fourier series, the Fourier coefficients being infinite series of the Petersson-Rademacher type. *J. Lehner* (College Park, Md.)

Bowen, N. A. 4111

On canonical products whose zeros lie on a pair of radii.
Quart. J. Math. Oxford Ser. (2) **15** (1964), 89–95.

Let $P(z, a, b, \gamma, q)$ be a canonical product of genus q with zeros $-a_n e^{-i\gamma}$, $-b_n e^{i\gamma}$ $(n = 1, 2, \cdots)$. The author's results (too complicated to be given in detail) determine $\lim_{r\to\infty} r^{-\rho} n(r)$, where $q < \rho < q+1$, from the hypotheses

$$\operatorname{Im}\{\log P(re^{i\alpha})\} \sim A_1 r^\rho, \quad \operatorname{Im}\{\log P(re^{-i\alpha})\} \sim A_2 r^\rho,$$

or

$$\log|P(re^{i\alpha})| \sim B_1 r^\rho, \quad \log|P(re^{-i\alpha})| \sim B_2 r^\rho$$

with suitable restrictions on α, γ, ρ. In the former case he also shows that, in certain cases, four asymptotic relations determine the asymptotic distributions of the zeros on the two half lines $\operatorname{Ang} z = \pi \pm \gamma$ separately.

The proofs use earlier results of the author and A. J. Macintyre [Trans. Amer. Math. Soc. **70** (1951), 114–126; MR **12**, 689]. *M. E. Noble* (Nottingham)

Brickman, L. 4112

Non-negative entire and meromorphic functions.
J. London Math. Soc. **39** (1964), 15–18.

The following three representation theorems for entire and meromorphic functions which are non-negative on a segment of the real axis are proved. Theorem 1: Let $f(z)$ be an entire (meromorphic) function non-negative for $-\infty < z < \infty$. Then there exist real entire (meromorphic) functions $p(z)$ and $q(z)$ such that $f(z) = p^2(z) + q^2(z)$. Theorem 2: Let $f(z)$ be an entire (meromorphic) function non-negative for $z \geqq 0$. Then there exist real entire (meromorphic) functions $p(z)$ and $q(z)$ such that $f(z) = p^2(z) + zq^2(z)$. Theorem 3: Let $f(z)$ be an entire (meromorphic) function non-negative on the finite interval $a \leqq z \leqq b$. Then there exist real entire (meromorphic) functions $p(z)$ and $q(z)$ such that $f(z) = (z-a)p^2(z) + (b-z)q^2(z)$. Here the phrase "$f(z)$ is real" means that $f(z)$ is real on the real axis. The corresponding theorems for polynomials are known, references for which are given in the paper.
Hari Shankar (Athens, Ohio)

Gol'dberg, A. A. 4113

An example of an entire function of finite order with a non-asymptotic defective value. (Russian)
Užgorod. Gos. Univ. Naučn. Zap. **18** (1957), 191–194.

The order of the example described in the title is $\rho = \frac{3}{2}$. The author asserts that the method of construction will yield another such function of order $1+\varepsilon$, $\varepsilon > 0$.

Gol'dberg, A. A. 4114

On the lower order of an entire function with a finite defective value. (Russian)
Sibirsk. Mat. Ž. **5** (1964), 54–76.

Construction of an example of an entire function of assigned order $\rho > 1$ whose lower order λ is arbitrarily close to 1 for which the Nevanlinna deficiency $\delta(0, f)$ is positive. (This demolishes the conjecture that, under the hypothesis $\delta(0, f) > 0$, $\lambda \geqq \frac{1}{2}\rho$.) The Riemann surface of the function in question has all its branch points over 0, 1, ∞; the proof of the assertion is based on very delicate estimates of the conformal mapping of this Riemann surface on the z-plane. *W. H. J. Fuchs* (Ithaca, N.Y.)

Hillion, Pierre 4115

Espaces de Hilbert de fonctions entières périodiques sur l'axe réel.
C. R. Acad. Sci. Paris **258** (1964), 3967–3969.

A Hilbert space whose elements are entire functions is defined in such a way that differentiation is a symmetric transformation in the space. The space has the functions $(k!)^{-1/2}\exp(\pm ikz)$, $k = 1, 2, 3, \cdots$, as an orthonormal basis. *L. de Branges* (Lafayette, Ind.)

Liht, M. K. 4116

A remark on the theorem of Paley and Wiener on entire functions of finite type. (Russian)
Uspehi Mat. Nauk **19** (1964), *no.* 1 (115), 169–171.

The author proves the following theorem. Let $f(z)$ be an entire function of finite exponential type whose indicator diagram is a circle, and let $h(\theta)$ be its indicator function. Then a necessary and sufficient condition for $f(z)$ to have a representation

$$f(z) = \int_\Gamma e^{\lambda z}\psi(\lambda)\, d\lambda \quad (\psi(\lambda) \in L_2(\Gamma)),$$

where Γ is the boundary of the conjugate indicator diagram, is

$$\int_0^{2\pi} d\theta \int_0^\infty x^{1/2} e^{-2h(\theta)x} |f(xe^{i\theta})|^2\, dx < \infty.$$

I. N. Baker (London)

Satô, Daihachiro 4117

On the rate of growth of rapidly increasing integral functions. (Japanese)
Sûgaku **15** (1963), 101–105.

Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ be an integral function with finite order ρ and type σ. It is well known that, putting $M(r) = \operatorname{Max}_{|z|=r}|f(z)|$,

$$(1) \quad \rho = \limsup_{r\to+\infty} \log\log M(r)/\log r = \limsup_{n\to+\infty} n\cdot\log n/\log(1/|a_n|),$$

$$(2) \quad \sigma = \limsup_{r\to+\infty} \log M(r)/r^\rho = 1/e\rho \cdot \limsup_{n\to+\infty} n|a_n|^{\rho/n}.$$

The author attempts to generalize these formulas to the case of integral functions of infinite order. For this purpose, he defines

$$\lambda_{(q)} = \limsup_{r\to+\infty} \log^{[q]} M(r)/\log r, \qquad \kappa_{(q)} = \limsup_{r\to+\infty} \log^{[q-1]} M(r)/r^{\lambda_{(q)}}, \qquad (q = 2, 3, \cdots)$$

where $\log^{[q]}x$ is the qth iterated logarithm. If $\lambda_{(q-1)} = +\infty$ and $\lambda_{(q)} < +\infty$, then $f(z)$ is called an integral function of genus number q. His generalizations of (1) and (2) read

as follows. If $f(z)=\sum_{n=0}^{+\infty} a_n z^n$ is an integral function of genus number $q<+\infty$, then

$$\lambda_{(q)} = \limsup_{n\to+\infty} n \log^{[q-1]} n/\log(1/|a_n|) \quad (q = 2, 3, \cdots);$$

$$\kappa_{(2)} = 1/e\lambda_{(2)}\cdot \limsup_{n\to+\infty} n|a_n|^{\lambda_{(2)}/n} \quad (q = 2);$$

$$\kappa_{(q)} = \limsup_{n\to+\infty} \log^{[q-2]} n\cdot|a_n|^{\lambda_{(q)}/n} \quad (q = 3, 4, \cdots).$$

C. Tanaka (Tokyo)

Gončar, A. A. 4118

Examples of non-uniqueness of analytic functions. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 1, 37–43.

If $\{z_i\}$ and $\{r_i\}$ are a complex and a positive sequence, respectively, with $|z_i|<1$, $|z_i|\to 1$, $r_i<1-|z_i|$, and $r_i+r_j<|z_i-z_j|$ for $i\neq j$, the author calls the complement $P(\{z_i\},\{r_i\})$ relative to $|z|\leq 1$ of the union of the disks $|z-z_i|<r_i$ a closed region of type (L). He constructs nontrivial functions of the form (*) $f(z)=\sum A_i/(z-z_i)$ with the following two properties. (1) For each $\beta>2$ and each $\varepsilon>0$, there exists a sequence $\{r_i\}$ such that $\sum|\log r_i|^{-\beta}<\varepsilon$ and such that in the closed region $P(\{z_i\},\{r_i\})$ of type (L) the series in (*) converges uniformly, even after n term-by-term differentiations ($n=0, 1, \cdots$). (2) $f(z)\equiv 0$ on $|z|=1$.

G. Piranian (Ann Arbor, Mich.)

Linden, C. N. 4119

The representation of regular functions.

J. London Math. Soc. **39** (1964), 19–30.

Let $f(z)$ be regular in $|z|<1$. The M-order $\alpha(f)$ of f is defined to be

$$\limsup_{r\to 1-0} \log^+\log^+ M(r,f)/(-\log(1-r)).$$

It is proved that if the M-order $\alpha(f)\geq 1$, then $f(z)=p(z)q(z)$, where $p(z)$ is a canonical product displaying the zeros of $f(z)$, $q(z)$ is non-zero and both $p(z)$ and $q(z)$ are regular and of M-order at most $\alpha(f)$ in $|z|<1$. When $\alpha(f)<1$, an additional hypothesis concerning the zeros of $f(z)$ gives the following theorem. Let $f(z)$ have zeros $\{a_n\}$ in $|z|<1$, define $n_1(r,\theta,f)$ to be the number of these zeros satisfying $r\leq|a_n|<(1+r)/2$, $|\theta-\arg a_n|\leq\pi(1-r)$, and set $N(r,f)=\max_{0\leq\theta<2\pi} n_1(r,\theta,f)$,

$$\gamma = \limsup_{r\to 1-0} \log^+ N(r,f)/(-\log(1-r)).$$

Then $f(z)=p(z)q(z)$, where

$$p(z) = \prod_1^\infty \frac{\bar a_n(a_n-z)}{1-z\bar a_n}\exp\sum_{r=1}^{s+1}\frac{1}{r}\left(\frac{1-|a_n|^2}{1-z\bar a_n}\right)^r$$

with $s=[\gamma]$, $q(z)$ is non-zero, and both $p(z)$ and $q(z)$ are regular and of M-order at most $\max(\alpha,\gamma)$ in $|z|<1$.

S. M. Shah (Lawrence, Kans.)

Petrenko, V. P. 4120

The growth of meromorphic functions along a ray. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 281–284.

Let $f(z)$ be a meromorphic function of lower order $\lambda>\frac{1}{2}$, and $T(r)$ its Nevanlinna characteristic. The author proves that for all real α (a) $\liminf \log^+|f(re^{i\alpha})|/T(r)\leq\pi\lambda$.

For $\lambda\leq\frac{1}{2}$ A. A. Gol'dberg and I. V. Ostrovskiĭ proved, for $f(z)$ entire, that the left-hand side of (a) is at most $\pi\lambda/\sin\pi\lambda$ [Har'kov. Gos. Univ. Učen. Zap. **115** Zap. Mat. Otd. Fiz.-Mat. Fak. Har'kov. Mat. Obšč. (4) **27** (1961), 3–37].

W. H. J. Fuchs (Ithaca, N.Y.)

Bagemihl, Frederick 4121

Some boundary properties of normal functions bounded on nontangential arcs.

Arch. Math. **14** (1963), 399–406.

A meromorphic function $f(z)$ defined in the unit disk D is normal there provided the family $\{f_a(z)\}$ is normal, $f_a(z)=f((z+a)/(1+\bar az))$, $a\in D$. The author's principal results are concerned with properties of bounded functions (or variants thereof) which are possessed by normal functions. He shows that if a holomorphic, normal function f is bounded on a nontangential Jordan curve γ (γ is a one-one continuous image of $(0,1)$ with $\gamma(0)=\gamma(1)=e^{i\theta}$), then f is bounded in the domain $\gamma^*\subset D$ determined by γ. Another result states that if f is holomorphic, normal and $E\subset\{z\mid |z|=1\}$ with the property that to every $\zeta\in E$ there corresponds a nontangential arc on which f is bounded, then f has finite angular limit a.e. on E. Further, for $\zeta\in E-M$ every chordal cluster set of f at ζ coincides with the global cluster set, where M is a set of first category.

The author shows the above results are not valid for normal meromorphic functions. Also he shows that the nontangentiality of the arc is necessary in the second result. There is also a uniqueness theorem proven for normal functions.

J. A. Cima (Stanford, Calif.)

Cantor, David G. 4122

A simple construction of analytic functions without radial limits.

Proc. Amer. Math. Soc. **15** (1964), 335–336.

Corresponding to any increasing sequence $\{n_k\}$ of positive integers, let $\{r_k\}=\{(1-2^{-k})^{1/n_k}\}$. The author shows that if $\{n_k\}$ grows fast enough so that

$$(1) \qquad r_k^{n_{k+1}} \leq (1-r_k)2^{-k} \quad (k = 1, 2, \cdots),$$

then for each bounded sequence $\{a_k\}$ of complex numbers the existence of the radial limit at $e^{i\theta}$ of the function $f(z)=\sum a_k z^{n_k}$ implies the convergence of the power series $\sum a_k z^{n_k}$ at $z=e^{i\theta}$. It follows immediately that if $\{n_k\}$ satisfies (1) and $\{a_k\}$ is bounded but does not converge to 0, then f has no radial limit.

In particular, let ϕ denote any increasing, unbounded function on $[0,1)$ (with $\phi(0)>0$). Choose $\{n_k\}$ so that it satisfies (1) and $r^{n_{k+1}}\leq 2^{-k-1}\phi(r)$ ($0<r<1$; $k=1,2,\cdots$). Then the function $\sum z^{n_k}$ has no radial limit and satisfies the growth condition $|f(z)|\leq\phi(|z|)$. A more complicated construction of a holomorphic function of arbitrarily slow growth and without radial limit was achieved by G. R. MacLane [Michigan Math. J. **9** (1962), 21–24; MR **25** #203b].

G. Piranian (Ann Arbor, Mich.)

Collingwood, E. F. 4123

Cluster set theorems for arbitrary functions with applications to function theory.

Ann. Acad. Sci. Fenn. Ser. A I 336/8 (1963), 15 *pp.*

The topology of Euclidean space E^n imposes certain restrictions on the pathology of complex-valued functions in E^n. The author surveys the relation between these restrictions and recent developments in the theory of analytic functions. The topics range over questions of symmetry of left and right cluster sets, maximality of specialized cluster sets (such as the intersection of all Stoltz cluster sets at a point), the theory of prime ends, and the set of radial limits of a function meromorphic in a disk. *G. Piranian* (Ann Arbor, Mich.)

Kuran, Ü. 4124

Two theorems related to the class $H^p \log^+ H$.

J. London Math. Soc. **39** (1964), 35–40.

Generalizing results of Flett for the case $p=2$ [Proc. Cambridge Philos. Soc. **55** (1959), 31–50; MR **21** #2878] the author proves Theorem 1: Let $\varphi(z)$ be analytic and of class H^1 in $|z|<1$; let $\Phi(\theta)=\lim_{r\to 1-0}\varphi(re^{i\theta})$. If $1<p\leq 2$ and $\int_{-\pi}^{\pi} d\theta \int_0^{\pi} |\Phi(\theta+t)-\Phi(\theta-t)|^p t^{-1}\,dt<\infty$, then

$$(1)\qquad \int_{-\pi}^{\pi} |\Phi(\theta)|^p \log^+|\Phi(\theta)|\,d\theta < \infty.$$

Theorem 2: Let $\varphi(z)$ be analytic in $|z|<1$. If $p\geqq 2$, and if $\int_{-\pi}^{\pi} d\theta\{\int_0^1 (1-\rho)\log[e/(1-\rho)]|\varphi'(\rho e^{i\theta})|^2\,d\rho\}^{p/2}<\infty$, then Φ exists and (1) holds. *W. H. J. Fuchs* (Ithaca, N.Y.)

Thompson, Maynard 4125

Approximation by polynomials whose zeros lie on a curve.

Duke Math. J. **31** (1964), 255–265.

Let R and D be, respectively, a closed and an open set in the complex plane. R is called a polynomial approximation set for D if and only if the set of all functions holomorphic and $\not\equiv 0$ in D whose zeros in D lie in R equals the set of all functions ($\not\equiv 0$ in D), each being throughout D a limit of a sequence of polynomials whose zeros belong to R, the convergence being uniform in every compact subset of D. A result due to G. R. MacLane [same J. **16** (1949), 461–477; MR **11**, 20] states that if R is a rectifiable Jordan curve and D its interior, then R is a polynomial approximation set for D. The author gives further cases in which R is a polynomial approximation set for D. For instance, if (z_p) is a sequence of points satisfying $\operatorname{Im} z_p\to+\infty$, the set consisting of the real axis and the z_p's is a polynomial approximation set for the upper half-plane. Also, let $0<\beta<\pi/2$. Then the pair of rays emanating from 0 and forming an angle β with the positive x-axis forms a polynomial approximation set for the (smaller) sector bounded by these rays. *O. Shisha* (Dayton, Ohio)

Dračinskiĭ, A. È. 4126

The Riemann-Privalov boundary-value problem in the class of summable functions. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **32** (1963), 271–276.

The author considers the boundary problem $\phi^+(t)=G(t)\phi^-(t)+g(t)$, where the solution function $\phi(z)$, representable by an integral of Cauchy type, has a jump across the open Liapunov curve Γ at t and $G(t)$ has the form

$$G(t)=\left[\prod_{i=1}^{m}(t-a_i)^{\lambda_i}\prod_{k=1}^{l}\ln^{\nu_k}(t-c_k)\Big/\prod_{j=1}^{n}(t-b_j)^{\mu_j}\right]G_1(t),$$

where a_i, b_j, c_k are given points of Γ; λ_i, μ_j, ν_k are arbitrary complex numbers with $0\leqq\alpha_i=\operatorname{Re}\lambda_i<1$, $0\leqq\beta_j=\operatorname{Re}\mu_j<1$ $(i=1,\cdots,m;\ j=1,\cdots,n)$. The function G_1, non-vanishing on Γ, has discontinuities on Γ at d_s, $s=1,\cdots,h$, for which $G_1(d_s-0)/G_1(d_s+0)=e^{2\pi i\sigma_s}$, with $-1<\operatorname{Re}\sigma_s\leqq 0$. For the function g it is assumed that

$$g(t)\in L_p\left(\Gamma;\prod_{i=1}^{m}|t-a_i|^{-\alpha_i}\prod_{s=1}^{h}|t-d_s|^{(p-1)|\operatorname{Re}\sigma_s|}\times\prod_{j=1}^{n}|t-b_j|^{(p-1)\beta_j}\right),\quad p>1.$$

This problem was considered previously by Hvedelidze [Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze **23** (1956), 3–158; MR **21** #5873] on the assumption that $\phi(z)$ on Γ belongs to the same class as g. In the present paper Hvedelidze's work is extended by considering solution functions $\phi(z)$ summable on Γ.
J. F. Heyda (King of Prussia, Pa.)

Niczyporowicz, E. 4127

Sur un problème aux limites discontinues dans la théorie des fonctions analytiques.

Ann. Polon. Math. **14** (1963/64), 269–288.

The author employs methods of functional analysis to solve the problem of finding a set of functions $\phi_\nu(z)=u_\nu(x,y)+iv_\nu(x,y)$, $\nu=1,\cdots,n$, holomorphic in the domain D^+ interior to the unit circle L and having limiting values $\phi_\nu^+(t)=u_\nu(t)+iv_\nu(t)$ for $t\in L'=L-\sum_{j=1}^{p}c_j$, where the c_j are points of discontinuity in whose vicinity the $\phi_\nu(z)$ satisfy the inequalities $|\phi_\nu(z)|<\text{const}/\prod_{j=1}^{p}|z-c_j|^{\theta_j\rho}$, $\nu=1,\cdots,n$; $0\leqq\rho<1$. Here the θ_j are zero or one according as j is or is not one of the set of $q<p$ integers $k_1,\cdots,k_q$. The functions $\phi_\nu^+(t)$ for $t\in L'$ are required to satisfy the conditions

$$\operatorname{Re}\{[a_\nu(t)+ib_\nu(t)]\phi_\nu^+(t)\}=f_\nu(t)+\lambda F_\nu[t,u_1(t),\cdots,u_n(t),v_1(t),\cdots,v_n(t)]+\int_L\frac{\varphi_\nu[t,u_1(\tau),\cdots,u_n(\tau),v_1(\tau),\cdots,v_n(\tau)]}{|\tau-t|^\delta}\,d\tau$$

$$(\nu=1,\cdots,n;\ 0\leqq\delta<1).$$

Specifications on the given functions $a_\nu, b_\nu, f_\nu, F_\nu, \varphi_\nu$ involve considerable detail and will not be given here.
J. F. Heyda (King of Prussia, Pa.)

Nitsche, Johannes C. C. 4128

Zum Heinzschen Lemma über harmonische Abbildungen.

Arch. Math. **14** (1963), 407–410.

Let H be the class of harmonic mappings $x=x(u,v)$, $y=y(u,v)$ of the unit disc $u^2+v^2\leqq 1$ onto the unit disc $x^2+y^2\leqq 1$. Let $\mu=\inf_H[x_u^2+x_v^2+y_u^2+y_v^2]$. E. Heinz [Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1952**, 51–56; MR **14**, 885] first established the positivity of μ by proving $\mu>.3577$. This had been improved to $\mu>.8691$ by H. L. de Vries [J. Math. Mech. **11** (1962), 469–471; MR **26** #338] who refined the method of Heinz. The author improves further the direction taken by de Vries to obtain $\mu>.8952$.
J. L. Ullman (Ann Arbor, Mich.)

Voichick, Michael 4129

Ideals and invariant subspaces of analytic functions.

Trans. Amer. Math. Soc. **111** (1964), 493–512.

This paper contains a generalization of the closed ideal theory of the algebra of analytic functions on $K=\{|z|<1\}$ with continuous boundary values due to Beurling; cf. also Rudin [Canad. J. Math. **9** (1957), 426–434; MR **19**, 641]. Let R be a relatively compact domain on an open Riemann surface, and suppose R is bounded by finitely many analytic curves Γ. Let A be the algebra of continuous functions on Γ which have analytic extensions to R. The universal covering surface of R is the unit disk K; let S be the group of covering transformations of R. The desired generalization to R is obtained by relating the structure of R with that of K and using the factorization theory for H^p functions on K. We have the following definition. $H_p(R)$ $(1\leqq p<\infty)$ is the class of analytic functions F on R such that $|F|^p$ has a harmonic majorant. $H_\infty(R)$ is the class of bounded analytic functions on R. A holomorphic function on K with S-invariant modulus is called a multiplicative function on R. A multiplicative function on R which is inner on K (i.e., has boundary values of modulus one a.e. $d\theta$) is called an inner function on R. Let ν be a measure on Γ which is obtained on each component γ of Γ by pulling back $d\theta$ via an analytic coordinatization of γ as the unit circle. Then, if F is inner on R, $|F|$ has boundary value 1 a.e. (ν). The main theorem is: If I is a closed ideal in A, there is an inner function F and a closed set E of ν-measure zero on Γ such that $I=\{f\in A\,;f=0 \text{ on } E \text{ and } |fF^{-1}| \text{ is bounded on } R\}$. Beurling's theorem on invariant subspaces of H_2 is also generalized: if C is a closed invariant subspace of $H_2(R)$, then there is an inner function F such that $C=\{f\in H_2(R)\,;\,|fF^{-1}|^2 \text{ has a harmonic majorant on } R\}$. *H. Rossi* (Princeton, N.J.)

POTENTIAL THEORY

See also 4087, 4094, 4128, 4163, 4245, 4246, 4760.

Havpačev, S. K. 4130

A harmonic function representable by the Poisson integral of a function of bounded second variation. (Russian)

Soobšč. Akad. Nauk Gruzin. SSR **29** (1962), 263–268.

A real measurable function is defined to be of bounded second variation on $[a,b]$ if

$$V(f;2;a,b) = \sup\sum_{k=0}^{m-1}|f(x_k)+f(x_{k+1})-f[(x_k+x_{k+1})/2]|$$

is finite, where the supremum is taken over all partitions of $[a,b]$; see F. I. Haršiladze [Dokl. Akad. Nauk SSSR **79** (1951), 201—204; MR **13**, 121; Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze **20** (1954), 145–156; MR **16**, 806] for properties of these functions. In the present paper, necessary and sufficient conditions are given in order that a function u, harmonic on $\{|z|<1\}$, be the Poisson integral of a function with bounded second variation. The conditions are (1) $|u(r,\theta)|\leqq M$ on $\{|z|<1\}$, (2) $V(u;2;0,2\pi)\leqq M$ for $0<r<1$, and (3) except for a countable set of values of θ on $[0,2\pi]$ the following is satisfied: for each $\varepsilon>0$ there is a $\delta>0$ and $r_0<1$ such that $|u(r,\theta+h)-u(r,\theta)|<\varepsilon$ for $0\leqq h\leqq\delta$ and $r_0\leqq r\leqq 1$.

G. Johnson (Houston, Tex.)

Kleiner, W. 4131

Une condition de Dini-Lipschitz dans la théorie du potentiel.

Ann. Polon. Math. **14** (1963/64), 117–130.

Etant donné un arc simple compact Γ de classe C^1 dans le plan, soit A l'ensemble des arcs partiels γ de Γ: l'espace des mesures σ sur Γ peut être muni de la norme $[\sigma]=\sup_{\gamma\in A}|\sigma(\gamma)|$, celui des différences de mesures positives d'énergie finie de la norme $\|\sigma\|=\sqrt{\int U^\sigma d\sigma}$. Pour majorer $[\sigma]/\|\sigma\|$, on considère les $\sigma=\sigma^+-\sigma^-$ telles que $\sigma^++\sigma^-\leqq\nu$ donnée et $\int d\sigma^+=\int d\sigma^-$: il existe alors des constantes $c>0$ et $M\in\,]0,1[$, ne dépendant que de Γ, telles que $[\sigma]\leqq c\|\sigma\|\sqrt{\log(1/\delta)}$, où δ est la longueur maxima de γ pour que $\nu(\gamma)\leqq M[\sigma]$. *R. M. Hervé* (Nancy)

Kleiner, W. 4132

Sur l'approximation de la représentation conforme par la méthode des points extrémaux de M. Leja.

Ann. Polon. Math. **14** (1963/64), 131–140.

Les notations sont celles de l'analyse précédente [#4131], et en outre: η est la mesure d'équilibre sur Γ, de masse 1, η_n une mesure discrète de masse 1 répartie également entre n points $\eta_{n,k}\in\Gamma$, $1\leqq k\leqq n$, rendant maximum le produit des $|\eta_{n,j}-\eta_{n,k}|$, $1\leqq j<k\leqq n$. On précise $\eta=\lim\eta_n$ par $[\eta-\eta_n]\leqq c\log n/\sqrt{n}$, où la constante c ne dépend que de l'arc Γ, moyennant une condition de Hölder vérifiée par la fonction $\varphi(z)=\arg dz$, $z\in\Gamma$. *R. M. Hervé* (Nancy)

Durand, Émile 4133

Développement en série du potentiel au voisinage d'une droite indéfinie portant une distribution de charges.

C. R. Acad. Sci. Paris **255** (1962), 1879–1881.

The author works out, by recurrence, formulas for the potentials and gradients produced by densities z^n $(n=0,1,2,\cdots)$ on the z-axis in three-space. It appears to the reviewer that almost all the integrals involved are divergent, so that the significance of the formulas derived is questionable. *Bernard Epstein* (New York)

Mitchell, Josephine 4134

Integral theorems for harmonic vectors in three real variables.

Math. Z. **82** (1963), 314–334.

The author considers a class of harmonic functions in three variables generated by the Bergman-Whittaker operator [S. Bergman, *Integral operators in the theory of linear partial differential equations*, Springer, Berlin, 1961; MR **25** #5277],

$$H(\mathbf{X}) = \mathbf{B}[f] \equiv \frac{1}{2\pi i}\int_{\mathscr{L}} f(v,\zeta)\frac{d\zeta}{\zeta},$$

$$v(\mathbf{X},\zeta) = Z\zeta^2+x\zeta+Z^*,$$

$Z=\frac12(iy+z)$, $Z^*=\frac12(iy-z)$, $\mathbf{X}=(x,y,z)$, where $\mathscr{L}$ is a regular Jordan curve in the ζ-plane not passing through the origin. The class of harmonic functions discussed are those which correspond to an $f(u,\zeta)$, $u=\zeta^{-1}v$, which is analytic in u for $u\in\mathscr{N}\equiv\{|u-u_0|<\varepsilon\}$ for $\forall\zeta\in\mathscr{L}$, and a measurable function of ζ dominated by a Lebesgue integrable function of ζ on $\mathscr{L}$ for $\forall u\in\mathscr{N}$.

The author also considers a certain class of line integrals of harmonic vectors $\mathbf{H}(\mathbf{X})$, $\int_{\mathscr{J}^1}\mathbf{H}(\mathbf{X})\,d\mathbf{X}$, where $\mathscr{J}^1$ is a

differentiable Jordan curve in $\mathbf{E}^3$, and the harmonic vectors have components of the form

$$H_1(\mathbf{X}) = \mathbf{B}[\zeta f],$$
$$H_2(\mathbf{X}) = \mathbf{B}[\tfrac{1}{2}i(\zeta^2+1)f]+\operatorname{Im} g(x+iy),$$
$$H_3(\mathbf{X}) = \mathbf{B}[\tfrac{1}{2}(\zeta^2-1)f]+\operatorname{Re} g(x+iy).$$

Such integrals have been studied previously for the case where f is a rational function of v, ζ [S. Bergman, Trans. Amer. Math. Soc. **68** (1950), 461–507; MR **12**, 25] and for the case where f is an algebraic function [the author, Arch. Rational Mech. Anal. **3** (1959), 439–459; MR **21** #7357]. The author considers here the case where f is a complex function of three variables ζ, v, and S, where v and S are connected by the relation $S^2=\prod_{k=0}^{\infty}(1-v/e_k)$ $(0<e_1<e_2<\cdots)$ and where the series $\sum_{k=0}^{\infty}(1/e_k)<\infty$. Furthermore, it is assumed that f is a Lebesgue square integrable function of (ζ, v), which for fixed $\zeta\in\mathscr{L}$ is an analytic function of v except for isolated singularities. Under these conditions it is possible for the author to generalize the previous results mentioned above for rational and algebraic associate functions to the present case. She uses results due to P. J. Myrberg [C. R. Dixième Congrès Math. Scandinaves 1946, pp. 77–96, Gjellerups, Copenhagen, 1947; MR **8**, 509] and H. Hornich [Monatsh. Math. **53** (1949), 187–201; MR **11**, 510; ibid. **54** (1950), 37–44; MR **12**, 493] concerning normal integrals on the Riemann surface $\mathscr{R}\equiv\{S=0\}$ to obtain representations for the integrals $\int_{\mathscr{I}} \mathbf{H}(\mathbf{X})\,d\mathbf{X}$.

Finally the author also obtains an inequality for integrals over a surface in $\mathbf{E}^3$, which corresponds to a Dirichlet integral in the case of a plane.

R. P. Gilbert (College Park, Md.)

Tung, Shih-Hsiung 4135

Harnack's inequality and theorems on matrix spaces.
Proc. Amer. Math. Soc. **15** (1964), 375–381.

In the space of complex matrices $z=(z_{jk})$ consider the "disk" D of those z for which $I-zz^*$ is positive definite. A real-valued function $\phi(z)$ is called harmonic in D if $\Delta\phi(z)=0$, where

$$\Delta = \sum_{j,k,p,q}\left(\delta_{jp}-\sum_r z_{pr}\bar{z}_{jr}\right)\left(\delta_{kq}-\sum_s z_{sq}\bar{z}_{sk}\right)\frac{\partial^2}{\partial\bar{z}_{jk}\partial z_{pq}}$$

is an elliptic operator. For these harmonic functions a Harnack inequality of the type $c^{-1}\phi(0)\le\phi(z)\le c\phi(0)$ is derived for positive harmonic functions $\phi(z)$ in D. The constant c, which depends on z only, is evaluated explicitly. The proof proceeds by estimating the Poisson kernel, which for this problem is explicitly given by

$$P(z,u) = \frac{1}{V}\det(I-zz^*)^n|\det(I-zu^*)|^{-2n}.$$

Also, Harnack's theorems on monotone sequences of harmonic functions are generalized.

J. Moser (New Rochelle, N.Y.)

Arsove, Maynard G. 4136

The Lusin-Privalov theorem for subharmonic functions.
Proc. London Math. Soc. (3) **14** (1964), 260–270.

The theorem of Lusin and Privalov to which the title refers asserts that if a function f, analytic in the unit disk Ω, has the radial limit 0 on a set of second category that is metrically dense (i.e., locally of positive measure) on some arc of the unit circle, then f is constant in Ω. The author proves the analogues: If u is subharmonic in Ω, and if the set of values θ for which $\limsup_{r\uparrow 1} u(re^{i\theta})<\infty$ is residual in some interval $I:\alpha<\theta<\beta$, then the set of values θ for which $\liminf u(re^{i\theta})=-\infty$ is not metrically dense in I. If on a boundary region of Ω the function u is subharmonic and admits a positive harmonic majorant, then its radial limit exists almost everywhere on the corresponding boundary arc of Ω. If u and v are a subharmonic and a positive superharmonic function, respectively, in a boundary region of Ω, and $\limsup u(re^{i\theta})/v(re^{i\theta})<\infty$ for a residual set of values $e^{i\theta}$ on the corresponding boundary arc I of Ω, then the radial limit of u exists (and is finite) almost everywhere on some subarc of I. For the proofs, the author defines several new classes of functions; for example, a function u defined in a subregion ω of Ω belongs to $\mathscr{U}_0$ if for each z_0 in ω, each neighborhood N of z_0, and each $c>f(z_0)$, the inequality $f(re^{i\theta})<c$ holds for a set of values $re^{i\theta}$ in N whose θ-coordinates cover an interval. The proof invokes the extension to one of these classes of the following theorem of Collingwood [Ann. Acad. Sci. Fenn. Ser. A I No. 250/6 (1958); MR **20** #2451]: If f is continuous in Ω, then the cluster set of f at $e^{i\theta}$ coincides with the radial cluster set at $e^{i\theta}$, except possibly on a set of first category.

G. Piranian (Ann Arbor, Mich.)

Curtiss, J. H. 4137

Interpolation by harmonic polynomials.
J. Soc. Indust. Appl. Math. **10** (1962), 709–736.

Let D be a finite region bounded by a Jordan curve C. The author studies the possibility of representing the solution of the harmonic Dirichlet problem for D as the limit of a sequence of harmonic polynomials found by interpolation of the given boundary data at suitably chosen points. This question had originally been raised by Walsh who showed that, if the boundary data satisfy certain smoothness requirements, the solution of the Dirichlet problem by this procedure is possible if D is a circle [Proc. Nat. Acad. Sci. U.S.A. **18** (1932), 514–517] or, more generally, an ellipse [J. Math. Mech. **9** (1960), 193–196; MR **22** #4891]. In the present paper the author establishes the possibility of this type of approximation by means of interpolating harmonic polynomials for a wide class of regions. In the case of the ellipse, a special choice of the interpolation points leads to explicit formulas for the harmonic interpolation polynomials.

Z. Nehari (Pittsburgh, Pa.)

Hiromi, Genkō 4138

On a criterion for regularity.
Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **7**, 263–268 (1963).

Let Ω be a domain exterior to the unit circle bounded by an infinite number of Jordan arcs which cluster only at ∞. Let P be the class of positive harmonic functions in Ω with vanishing boundary values on each of the Jordan arcs. For any $u\in P$ set $M(r)=\max\{u(z):|z|<r\}$ and $T(r)=\int u(r,\theta)\,d\theta$ over $\Omega\cap\{|z|=r\}$. The author gives criteria for the regularity of the point at infinity for the Dirichlet problem for Ω in terms of the growth of $M(r)$ and $T(r)$.

H. L. Royden (Stanford, Calif.)

Williams, W. Elwyn 4139

Integral representations in two dimensional potential theory. (German summary)

Z. Angew. Math. Phys. **14** (1963), 675–681.

The author shows that a variety of mixed boundary-value problems in two-dimensional potential theory can, by using suitable integral representations for the potential function, be reduced to the solution of Abel integral equations. The appropriate integral representation to be used varies with the type of boundary-value problem considered, and the different types of representation available are illustrated by application to four different problems, these being (a) steady flow past a thin aerofoil, (b) steady flow of a conducting fluid past a thin aerofoil, (c) a crack in an elastic medium, and (d) unsteady flow past a thin aerofoil. These problems can all be solved by using Cauchy integrals, but the use of these new representations gives an elegant and useful alternative method. Representations of the type used by the author have, however, already been introduced by England and Green [Proc. Cambridge Philos. Soc. **59** (1963), 489–500; MR **27** #1004] and used by them in two-dimensional punch and crack problems in classical elasticity including the author's problem (c). Problem (b), however, gives an interesting extension of the method of England and Green, while problems (a) and (d) illustrate its usefulness in aerofoil theory.

W. D. Collins (Manchester)

Widder, D. V. 4140

Functions of three variables which satisfy both the heat equation and Laplace's equation in two variables.

J. Austral. Math. Soc. **3** (1963), 396–407.

Several theorems are proved here, dealing with types of integral representations of functions $u(x, y, t)$ which satisfy the heat equation or Laplace's equation in two of the variables. Let $k(x, t) = (4\pi t)^{-1/2} \exp[-x^2/(4t)]$ and let $\alpha(r)$ be non-decreasing. Then necessary and sufficient conditions that

$$u(x, y, t) = \operatorname{Re} \int_{-\infty}^{\infty} k(t-r, x+iy)\, d\alpha(r), \quad x > 0,$$

are: (a) for each real t, u is the real part of a function of $x+iy$ which is analytic for $x>0$ and real on the real axis, and (b) $u(x, 0, t) \geqq 0$ and $u_x(x, 0, t) = u_{tt}(x, 0, t)$ whenever $x>0$. Another theorem shows that

$$u(x, y, t) = \int_{-\infty}^{\infty} \exp\,(ixr - y|r| - tr^2)\phi(r)\, dr,$$

where ϕ is positive definite, if and only if $u \geqq 0$ and $u_{xx} = -u_{yy} = u_t$ for $y>0$, $t>0$ and $\int_{-\infty}^{\infty} u(x, y_0, t_0)\, dx < \infty$ for some $y_0 > 0$, $t_0 > 0$. Conditions for integral representations whose kernels are other exponential and trigonometric functions are also established.

R. V. Churchill (Ann Arbor, Mich.)

SEVERAL COMPLEX VARIABLES

See also 4554.

Matsuura, Syozo 4141

On the theory of pseudo-conformal mappings.

Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **7**, 231–253 (1963).

From the author's introduction: "It is known that Bergman's metric $ds^2 = \sum_{i,j=1}^{n} T_{ij} dz_i d\bar{z}_j$, where $T_{ij} = \partial^2 \log k_D(z, \bar{z})/\partial z_i \partial \bar{z}_j$ and $k_D(z, \bar{z})$ is the kernel function of the domain D in the n-dimensional complex Euclidean $z \equiv (z_1, z_2, \cdots, z_n)$ space, is invariant under pseudo-conformal mappings. The objects of this paper are to discuss some invariant forms under pseudo-conformal mappings, and as their applications, to treat representative domains, normal domains, and Riemannian curvature, etc."

A. Haimovici (Iaşi)

Spallek, Karlheinz 4142

Verallgemeinerung eines Satzes von Osgood-Hartogs auf komplexe Räume.

Math. Ann. **151** (1963), 200–218.

Der Verfasser untersucht Verallgemeinerungsmöglichkeiten des klassischen Satzes von Osgood-Hartogs (d.h. komplexwertige Funktionen sind holomorph, wenn sie auf allen achsenparallelen 1-dimensionalen Ebenen holomorph sind).

Nach einigen Hilfsüberlegungen werden zunächst Verallgemeinerungen für kohärente analytische Garben in (schlichten) Gebieten G des $\mathbf{C}^n$ bewiesen. Das Hauptergebnis ist das folgende: Es seien $\mathfrak{O}$ die Strukturgarbe von G, $\mathfrak{A}$ eine kohärente analytische Untergarbe der freien Garbe $\mathfrak{O}^k$ und F ein k-Tupel komplexwertiger Funktionen. Zu jedem $x \in G$ gebe es eine in der Umgebung von x offene Parallelenschar S q-dimensionale Ebenenkeime, so daß für jedes $E_z^q \in S$ gilt: Die Beschränkung $F|E_z^q$ von F auf den Keim E_z^q in z ist eine Schnittfläche in der Beschränkung $\mathfrak{A}|E_z^q$ von $\mathfrak{A}$ auf E_z^q. Dann gibt es eine nur von $\mathfrak{A}$ (nicht von F) abhängige höchstens $(n-q-1)$-dimensionale analytische Menge A in G, so daß $F|G-A \in H^0(G-A, \mathfrak{A})$. Eine offene Parallelenschar q-dimensionale Ebenenkeime in einem Gebiet U in G kann dabei definiert werden durch Parallelverschiebung in U der Keime von T_0, wobei T_0 eine offene Menge im Raum T der q-dimensionale Ebenen durch ein $x_0 \in U$ ist. $T_0 = T$ wäre für den Satz eine unnötig einschränkende Voraussetzung, wie Beispiele zeigen. Der Satz von Osgood-Hartogs geht nur in den Beweis ein, um sicherzustellen, daß die Komponenten von F holomorph sind. Durch Beispiele wird gezeigt, daß der Satz ohne weitere Voraussetzungen an $\mathfrak{A}$ nicht verschärft werden kann. Ist aber die homologische Dimension der Halme von $\mathfrak{A}$ "klein", dann kann man schließen, daß A niederdimensionaler ist.

Nach einer Verallgemeinerung des Begriffs der offenen Parallelenschar von Ebenenkeimen können die Überlegungen auf komplex-analytische Räume ausgedehnt werden. Verfasser erhält u.a. eine genaue Übertragung des Satzes von Osgood-Hartogs für alle diejenigen komplexen Räume, die lokal dem zweiten Riemannschen Hebbarkeitssatz genügen (beispielsweise: normale Räume).

G. Scheja (Münster)

Bergman, Stefan 4143

Value distribution of meromorphic functions of two complex variables.

Bull. Soc. Sci. Lettres Łódź **13** (1962), *no.* 14, 14 *pp.*

For fixed $r \geqq r_0$ let $D(r)$ be a domain which is the transform of the bicylinder $[(Z, t)|\ |Z| \leqq r,\ |t| \leqq S(r)]$ ($S = S(r)$ an increasing analytic function of r) under the one-to-one

and continuous mapping $z_1=Z$, $z_2=h(Z,t,l)$, h a continuously differentiable function of its arguments. $D(r)$ has a Bergman-Šilov boundary $B(r)$ which is the transform of $Z=re^{i\phi}$, $t=Se^{i\psi}$, $0\leq\phi$, $\psi\leq 2\pi$. A generalization of Nevanlinna's first theorem is proved: Let $f(z_1,z_2)$ be a meromorphic function for $|z_k|<\infty$, $k=1, 2$, and holomorphic in a sufficiently small neighborhood of the origin. Then for an arbitrary constant $a\neq f(0,0)$

$$T(r,S,(f-a)^{-1}) \leq T(r,S,f)+\log^+|a|+\log 2 + |\log|f(0,0)-a||,$$

where

$$T(r,S,f) = (1/4\pi^2)\int_0^{2\pi}\int_0^{2\pi}\log^+|F|\,d\phi d\psi + (1/2\pi)\int_0^{2\pi}\log\prod r|a_{2\mu}(S,\psi)|^{-1}\,d\psi+\log\prod S|A_{2\mu}(0)|^{-1},$$

F is the value of f on $D(r)$, $a_{1\mu}$, $a_{2\mu}$ are the zeros and poles respectively of F in $|Z|\leq r$ and $A_{1\mu}$, $A_{2\mu}$ those of $F(0,z_2)$ in $|z_2|\leq S$. Also see Compositio Math. **3** (1936), 136–173, and other papers by the author.

J. Mitchell (University Park, Pa.)

Röhrl, H. 4144

Berichtigung zu der Arbeit: "Über das Riemann-Privalovsche Randwertproblem".

Math. Ann. **153** (1964), 350.

Minor corrections to the paper of the title [same Ann. **151** (1963), 365–423; MR **28** #249].

Abhyankar, Shreeram 4145

A remark on the nonnormal locus of an analytic space.

Proc. Amer. Math. Soc. **15** (1964), 505–508.

One finds a criterion for an analytic space to be normal along an analytic set. To be precise, let X be a reduced analytic space (over any algebraically closed complete valued field), $N(X)$ the set of points at which X is not normal, and p a point of X. Let R be the local ring of X at p, Y an irreducible germ of analytic set in X at p, and Q the prime ideal of Y in R. The criterion says that Y is not contained in $N(X)$ if and only if the localization R_Q is normal (i.e., integrally closed in its ring of fractions by all non-zero divisors, which is then necessarily a field). The non-trivial part of the proof is done elsewhere, by K. Oka in the complex case and by the author in the non-archimedean case. The fact is that $N(X)$ is an analytic set in X and its ideal in the local ring R is equal to the radical of the conductor of R for integral closure.

{Misprints in the statements of criteria, p. 506, lines 7 and 10: $R(p,X)$ should be replaced by $R(p,Y,X)$.}

H. Hironaka (Waltham, Mass.)

Eickel, Jürgen 4146

Glatte starre Holomorphiegebiete vom topologischen Typ der Hyperkugel.

Schr. Math. Inst. Univ. Münster No. 20 (1961), ii+58 pp.

Der Verfasser behandelt ausführlich die Konstruktion von Holomorphiegebieten im $\mathbf{C}^n$, $n\geqq 2$, deren Gruppe holomorpher Automorphismen nur aus der Identität besteht (sogenannte starre Holomorphiegebiete; Beispielmaterial für die Abbildungstheorie im Mehrdimensionalen). Hauptresultat der Untersuchung ist eine Methode, ein starres Gebiet G bisher bekannten Typs zu glätten, d. h. Kanten des Randes durch streng-pseudokonvexe neue Randflächen so abzuschneiden, daß das neue Gebiet G^* einen unendlich oft differenzierbaren Rand bekommt. (Da die neuen Randstücke keine nichttrivialen analytischen Mengen enthalten, bekommt G^* gegenüber G keine neuen analytischen Zerlegungen und der Selbstabbildungscharakter bleibt erhalten.) Es gilt sogar: Die entstehenden Gebiete besitzen keine eigentlichen holomorphen Selbstabbildungen außer der Identität. Die Methoden machen wesentlichen Gebrauch von den Sätzen von R. Remmert und K. Stein über eigentliche holomorphe Abbildungen [Math. Z. **73** (1960), 159–189; MR **23** #A1840].

G. Scheja (Münster)

Kajiwara, Jyoji 4147

Cousin problems and their applications. (Japanese)

Sûgaku **15** (1963), 82–96.

This paper is a nice exposition of the Cousin problems and several of their applications to differential equations. The author first gives a formulation of the Cousin problems using the notion of sheaves, and some results on a Stein space. In order to discuss the problem on a general (not necessarily Stein) analytic space, he mentions the results on the extension of the cohomology classes. Some results concerning the domain of holomorphy are referred to; for example, he mentions his results on the generalized tube [Kōdai Math. Sem. Rep. **15** (1963), 106–110; MR **27** #1621].

The Cousin problems are, roughly speaking, the method of obtaining global solutions under the assumption of existence of local solutions. In the final part of the present paper, the author mentions some applications of the Cousin problems to the existence of global solutions of differential equations, especially for the case of the system of ordinary linear differential equations with coefficients meromorphic over an open Riemann surface. These are suggested by a work by Ehrenpreis [Proc. Amer. Math. Soc. **7** (1956), 1131–1138; MR **19**, 36] and the main parts are published in the author's paper [Kōdai Math. Sem. Rep. **15** (1963), 94–105; MR **27** #1642].

S. Hitotumatu (Tokyo)

Kuhlmann, Norbert 4148

Über eine Verallgemeinerung eines Satzes von H. Hopf.

Arch. Math. **11** (1960), 431–436.

Der Verfasser verallgemeinert Sätze von H. Hopf und A. Aeppli [A. Aeppli, Comment. Math. Helv. **33** (1959), 1–22; MR **22** #7152] und bestätigt eine Vermutung von H. Grauert und R. Remmert, indem er beweist: X und Y seien komplexe Mannigfaltigkeiten mit einer eigentlichen holomorphen Modifikationsabbildung p: $X\to Y$, deren Entartungsmenge E nicht-leer und irreduzibel sei; $p(E)$ liege singularitätenfrei in Y; dann kann (X,p,Y) nur eine sogenannte σ-Modifikation mit der Basis $p(E)$ sein (Einsetzen projektiver Räume in den Punkten der Basis). Ausgangspunkt für den Beweis bildet eine analoge Ziele in der algebraischen Geometrie verfolgende Methode von J. P. Murre [Nederl. Akad. Wetensch. Proc. Ser. A **62** (1959), 129–134; MR **22** #9491].

G. Scheja (Münster)

Meynieux, Robert 4149

Sur une propriété locale des ensembles analytiques réels, et les domaines d'analyticité de certaines fonctions continues; application à une équation fonctionnelle.

C. R. Acad. Sci. Paris **255** (1962), 28–30.

Mehrere Noten des Autors liegen über analytische Lösungen der Funktionalgleichung $F(f(u), g(v), h(u+v))=0$ vor, wobei F eine reell-analytische Abbildung ist und f, g, h zunächst stetige Abbildungen sind [siehe dieselben C. R. **200** (1935), 201–203; ibid. **200** (1935), 892–894; ibid. **254** (1962), 1726–1728; MR **25** #225; ibid. **254** (1962), 3301–3303; MR **26** #6630; ibid. **254** (1962), 4413–4414; MR **29** #388]. In obiger Note sind f, g, h stetige Abbildungen von Gebieten U, V, W des R^q in den R^l, R^m, R^n, es ist Δ ein Gebiet im $R^q \times R^q$ und W ist die Menge $\{u+v\,;(u,v)\in\Delta\}$. D ist ein Gebiet im $R^l\times R^m\times R^n$, so daß $(f(u), g(v), h(u+v))\in D$ ist, falls $(u,v)\in\Delta$ gilt. F ist eine reell-analytische Abbildung von D in den R^n, $J(x,y,z)$ die zugehörige Jacobische Determinante in z gebildet. Auf Δ gilt die Identität

$$F(f(u), g(v), h(u+v)) = 0.$$

$$\Delta' := \{(u,v)\,; J(f(u), g(v), h(u+v)) \neq 0\}$$

liegt überall dicht in Δ. Dann folgt aus [der Autor, ibid. **254** (1962), 4413–4414; MR **29** #388], daß h reell-analytisch auf $W' := \{u+v\,;(u,v)\in\Delta'\}$ ist und aus Theorem 2 obiger Note ergibt sich: Ist c nicht aus W', so gibt es eine fallende Folge von c-Umgebungen W_j, derart, daß $W_j\cap W'$ in endlich viele Gebiete zerfällt, für die c erreichbarer Randpunkt ist.

Der Autor erhält Theorem 2 aus einer Ergänzung zu einer Note von F. Bruhat und H. Cartan [ibid. **244** (1957), 988–990; MR **19**, 125]: Es sei E eine lokal-analytische Menge im R^n, O aus E und E erzeuge in O einen analytischen Keim, der eine p-dimensionale Komponente enthält. Eine Teilmenge E' bestehe aus p-dimensionalen regulären Punkten von E (nicht notwendig aus allen), so daß O Berührpunkt von $E-E'$ ist und $E-E'$ in O einen Mengenkeim induziert, der in einem höchstens $(p-1)$-dimensionalen analytischen Keim enthalten ist. Dann gibt es eine fallende Folge U_i von O-Umgebungen des R^n, so daß $U_i\cap E'$ in endlich viele Komponenten zerfällt, für die O erreichbarer Randpunkt ist.

W. Fensch (Münster)

SPECIAL FUNCTIONS

See also 4272, 4391, 4615, 4616, 4676, 4759.

Pickard, William F. 4150

Tables of the generalized Stirling numbers of the first kind.

J. Assoc. Comput. Mach. **11** (1964), 70–78.

Author's summary: "The generalized Stirling numbers of the first kind are defined, certain of their basic properties are discussed, and tables are given for the square grid $k=0(1)10$ and $j=0(1)10$ with $l=-10(1)10$."

Gokhale, D. V. 4151

On an inequality for gamma functions.

Skand. Aktuarietidskr. **1962**, 213–215 (1963).

It is shown that

$$\frac{\Gamma(\lambda+2\alpha)\Gamma(\lambda)}{\Gamma^2(\lambda+\alpha)} > 1+\frac{\alpha^2(\lambda-2)}{(\lambda+\alpha-1)^2}$$

for all real λ and α such that $\lambda>2$, $\lambda+2\alpha>0$, $\alpha\neq0$, $\alpha\neq1$. The result follows from an application of the inequality

$$\operatorname{Var} T \geqq \frac{[E(\partial T/\partial x)]^2}{E[(\partial \log f/\partial x)^2]},$$

where $f=e^{-x}x^{\lambda-1}/\Gamma(\lambda)$, $0\leqq x<\infty$, and $T=x^\alpha$.

H. Hochstadt (Brooklyn, N.Y.)

Epstein, Leo F.; Hubbell, J. H. 4152

Evaluation of a generalized elliptic-type integral.

J. Res. Nat. Bur. Standards Sect. B **67B** (1963), 1–17.

This paper gives the integrals

$$\Omega_j(k) \equiv \int_0^\pi (1-k^2\cos\phi)^{-j-1/2}\,d\phi$$

in terms of complete elliptic integrals for $j=0$, 1 and gives a recurrence relation and other formulae and series when j is an integer exceeding unity.

Eight-figure numerical values are tabulated for $j=0(1)9$, $k^2=0(.01)1$. *J. C. P. Miller* (Cambridge, England)

Carlitz, L. 4153

A Saalschützian theorem for double series.

J. London Math. Soc. **38** (1963), 415–418.

In this paper the author proves summation formulae for a general terminating double Appell series under various conditions connecting the parameters. Using these results, he deduces a well-known transformation formula. He gives an alternative proof of this transformation and restates his theorems in terms of the more general $S_{m,n}[a;b,b';c;d,d']$ functions.

L. J. Slater (Cambridge, England)

Džrbašjan, V. A. 4154

On a theorem of Whipple. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 348–351.

Whipple [Proc. London Math. Soc. (2) **23** (1924), 104–114] has given a formula for

$${}_3F_2\left[\begin{matrix}a,\,1-a,\,c\,;\\ f,\,2c+1-f\end{matrix}\right].$$

The author points out that Whipple's derivation is incorrect if c is a negative integer, and shows that

$${}_3F_2\left[\begin{matrix}a,\,1-a,\,-n\,;\\ f,\,1-2n-f\end{matrix}\right] = 2^{2n}\frac{(\frac12a+\frac12f)_n(\frac12a-n-\frac12f+\frac12)_n}{(f)_n(1-2n-f)_n}.$$

P. G. Rooney (Toronto, Ont.)

Nagaraja, K. S. 4155

A note on certain integrals.

J. Math. and Phys. **43** (1964), 55–59.

The integrals

$$I_n = \int_{\nu/t}^{+\infty} x^n e^{-\nu t/x-\beta x^2}\,dx \quad (n = 0, 1, 2, \cdots),$$

which appear in certain physical problems, are considered. The evaluation of I_n is reduced to the evaluation of E-functions defined by $E_m(x)=\int_x^{+\infty} e^{-u}u^{-m}\,du$. The latter are tabulated for several positive and negative, integral and fractional values of m. *D. Ž. Đjoković* (Belgrade)

Curtiss, C. F. 4156

Expansions of integrals of Bessel functions of large order.

J. Mathematical Phys. **5** (1964), 561-564.

Let $f(x)$ have an expansion in a uniformly convergent series of Laguerre functions, $0\leqq x<\infty$. The following expansion is obtained:

$$(1)\quad \int_0^\infty f(x)J_\nu(\nu x/y)\,dx = \sum_{n,j}(-1)^j\nu^{-n-1}A_{nj}y^{j+1}f^{(j)}(y),$$

the constants A_{nj} determined by a recursive process. Similar expansions are obtained for the integrals of the product of $f(x)$ and $J_\nu^2(\nu x/y)$ and the product of $f(x)$ and $J_\nu(\nu x/y)J_{\nu+1}(\nu x/y)$.

It follows from (1) that

$$J_\nu(\nu x/y) = \sum_{n,j}\nu^{-n-1}A_{nj}y^{j+1}\delta^{(j)}(x-y);$$

$\delta^{(j)}$ is the jth derivative of the "delta function" normalized to unity. *A. E. Danese* (Buffalo, N.Y.)

Meligy, A. S.; El Gazzy, E. M. 4157

Expansion of the generalized exponential integral in terms of Bessel functions.

J. Math. and Phys. **43** (1964), 60-64.

In a previous paper [same J. **42** (1963), 157-161; MR **27** #358], the authors gave an expansion of $\int_z^\infty e^{-t}t^{-n}\,dt$ for $n=1$ in series of the modified Bessel functions of the first kind of half odd order. Here the result is extended to cover the case for n any positive integer.

In the derivation of this formula it is necessary to differentiate a certain ${}_3F_2$ of unit argument with respect to a parameter. Here the ${}_3F_2$ is first transformed into another ${}_3F_2$ of unit argument and then the differentiation is performed. If the ${}_3F_2$, as is, is differentiated, then one is led to a similar expansion also given previously by the authors [Proc. Cambridge Philos. Soc. **59** (1963), 735-737; MR **27** #3843]. The authors do not make reference to this in their present paper. *Y. L. Luke* (Kansas City, Mo.)

Ragab, F. M. 4158

The Laplace transform of the modified Bessel function $K(t^{\pm m}x)$ where $m=1,2,3,\cdots,n$.

Proc. Edinburgh Math. Soc. (2) **13** (1962/63), 325-329.

Let $K_n(x)$ denote the modified Bessel function of the second kind. The author evaluates in this paper the following integrals (with suitable restrictions on the parameters m, n, k, p):

$$\int_0^\infty e^{-t}t^{k-1}K_n(t^m x)\,dt,\quad \int_0^\infty e^{-pt}K_n(t^m x)\,dt,$$

where m is an integer. The result is expressed as a finite sum of generalized hypergeometric functions when m is positive or as a sum of MacRobert's E-functions when m is negative. *W. A. Al-Salam* (Durham, N.C.)

Hayakawa, Michikazu 4159

On Legendre functions with non-integer indices.

Math. Japon. **7** (1962), 137-139.

This paper discusses the zeros of the Legendre function derivative $P_\nu'(\cos\alpha)$ for fixed α and varying ν.

As an illustration, the first 10 roots ν_i, $i=1(1)10$, exceeding zero for $\alpha=\pi/6$ are given to 2 decimals, and, it is stated, corresponding values of $P_{\nu_i}(\cos\theta)$ for $\theta=k\pi/120$, $k=1, 2(2)20$.

It is not clear why or how 7-decimal values are given with 2-decimal zeros.

J. C. P. Miller (Cambridge, England)

Saxena, R. K. 4160

Integrals involving Legendre functions. II.

Math. Ann. **154** (1964), 181-184.

This is a sequel to a previous paper [same Ann. **147** (1962), 154-157; MR **25** #2240]. By means of operational calculus (Laplace transform) the author evaluates four different definite integrals, in which the integrand involves an associated Legendre function of the first or the second kind, in terms of Appell's function F_4.

C. J. Bouwkamp (Eindhoven)

Arscott, F. M. 4161

Integral equations and relations for Lamé functions.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 103-115.

A kernel for integral relations among the solutions of a family of linear differential equations may often be obtained as a solution of a partial differential equation so chosen that separation of variables leads back to the original ordinary equations. This idea, which is traced back to Liouville, has been exploited by E. T. Whittaker and many later writers, among them S. L. Malurkar, J. Meixner and F. Schäfke, A. Erdélyi, the author, and the reviewer. In the present work, such relations are obtained for the first time connecting two linearly independent solutions of the same equation, specifically the Lamé polynomials and second solutions of the corresponding Lamé equations. The author suggests that these relations may be of value in conjunction with his recently published book [the author and Khabaza, *Tables of Lamé polynomials*, Macmillan, New York, 1962; MR **27** #2655] for the effective computation of non-polynomial Lamé functions. *I. Marx* (Lafayette, Ind.)

Anastassiadis, Jean 4162

Définition fonctionnelle des polynomes de Bernoulli et d'Euler.

C. R. Acad. Sci. Paris **258** (1964), 1971-1973.

The author shows that the polynomials of Bernoulli can be defined by means of the functional equation, $f(x)=m^{k-1}\sum_{s=0}^{m-1}f((x+s)/m)$, where m is a positive integer. He also shows that the polynomials of Euler can be defined by means of the equation

$$f(x) = m^k\sum_{s=0}^{m-1}(-1)^s f\left(\frac{x+s}{m}\right),$$

where $m\geqq 3$ is odd. *T. Fort* (Columbia, S.C.)

Salihov, G. N. 4163

Harmonic polynomials and spherical functions in a four-dimensional space. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 6, 21–24.

En prenant comme coordonnées sphériques dans l'espace à quatre dimensions, r, θ, φ, ψ données par $x_1 = r\cos\theta\cos\varphi$, $x_2 = r\cos\theta\sin\varphi$, $x_3 = r\sin\theta\cos\psi$, $x_4 = r\sin\theta\sin\psi$, on trouve une certaine famille de polynômes sphériques ayant la forme générale

$$r^n P_n^{(m,k)}(\sin\theta)\begin{cases}\cos(m\varphi + k\psi)\\ \sin(m\varphi + k\psi).\end{cases}$$

On démontre ensuite qu'une fonction continue sur la sphère est limite uniforme d'une suite de tels polynômes, et que cette famille de fonctions coïncide avec l'ensemble des fonctions propres de $\Delta u + \lambda u = 0$, régulières sur la surface de la sphère. *A. Haimovici* (Iași)

ORDINARY DIFFERENTIAL EQUATIONS

See also 4066, 4147, 4300, 4389, 4400, 4442, 4682, 4700, 4702, 4962, 4970.

Boisvert, Louis-Mozart 4164

À propos de l'oscillation sous-harmonique d'un système de Duffing.

C. R. Acad. Sci. Paris **258** (1964), 1131–1134.

Author's summary: "L'équation de Duffing avec amortissement peut avoir une solution sous-harmonique dans certains cas. La présente Note a pour but de définir les zones d'amorçage et les zones d'existence de cette solution. Les résultats expérimentaux vérifient bien les conditions d'existence définies analytiquement." *J. T. Day* (Zürich)

Grunsky, Helmut 4165

Über die Umkehrung eines linearen Differentialoperators zweiter Ordnung im Komplexen.

Arch. Math. **14** (1963), 247–251.

Let L be the second-order linear differential operator defined by $Lu = u'' + p_1(z)u' + p_2(z)u$, where the $p_k(z)$ are regular analytic functions in the regions considered, and denote by u_1 and u_2 two linearly independent solutions of $Lu = 0$. Let D be the Wronskian $D = u_1u_2' - u_2u_1'$, and write $w = u_1/u_2$. The author shows that the solution of the differential equation $Ly = g(z)$ which vanishes at the points $z = z_1$ and $z = z_2$ is given by the formula

$$y(z) = \tfrac{1}{2}(w-w_1)(w-w_2)u_1\iint_T \bar{w}'u_1 g\,dxdy\left[D\iint_T |w'|^2\,dxdy\right]^{-1}$$

($z = x + iy$), where $w_k = w(z_k)$, and T is the triangle with the vertices z, z_1, z_2. This generalizes an earlier result of the author for the case $Lu = u''$ [Math. Z. **63** (1955), 320–323; MR **17**, 601]. *Z. Nehari* (Pittsburgh, Pa.)

Iwano, Masahiro 4166

On a singular point of Briot-Bouquet type of a system of ordinary non-linear differential equations.

Comment. Math. Univ. St. Paul. **11** (1963), 37–78.

Let $f_i(x, y_1, \cdots, y_n)$ $(i = 1, 2, \cdots, n)$ be regular analytic functions of the variables x, y_k $(k = 1, \cdots, n)$ for $|x| < a$, $|y_k| < b$, and let $f(0, 0, \cdots, 0) = 0$. The author studies in detail the behavior of the solutions of the system of differential equations $x\,dy_i/dx = f_i(x, y_1, \cdots, y_n)$ in the neighborhood of its singular point at $x = 0$, under the additional assumption that the matrix $(\partial f_i/\partial y_k)_{x=y_1=\cdots=y_n=0}$ has precisely one vanishing eigenvalue, say λ_1. The results, which are too complicated to be reproduced here, depend critically on the largest number ν with the following property: There exists an angle ω $(|\omega| < \frac{1}{2}\pi)$ such that $|\arg\lambda_i - \omega| < \frac{1}{2}\pi$ for $i = 2, \cdots, \nu$, $|\arg\lambda_i - \omega - \pi| < \frac{1}{2}\pi$ for $i = \nu + 1, \cdots, n$, where the λ_i are the eigenvalues of the above matrix. *Z. Nehari* (Pittsburgh, Pa.)

Rosati, Francesco 4167

Formule di interpolazione per funzioni olomorfe di una variabile e per funzioni reali di due variabili. (English summary)

Rend. Accad. Sci. Fis. Mat. Napoli (4) **29** (1962), 106–115.

An interpolation formula is derived whose fundamental functions form a fundamental system of solutions of a given linear differential equation with coefficients holomorphic in a given domain. The main discussion concerns the remainder which is so expressed that it can be estimated. The point of view and method correspond to M. Picone's ideas on interpolation [Ann. Scuola Norm. Super. Pisa (3) **5** (1951), 193–244; MR **14**, 144] to whose results the present ones reduce if the differential equation is $(d^n/dz^n)u = 0$. Similar ideas are applied to derive a formula interpolating a function of two real variables in an unequally spaced grid of points (x_i, y_j).

I. J. Schoenberg (Princeton, N.J.)

Coles, W. J. 4168

Some boundary value problems for linear differential systems.

Proc. Amer. Math. Soc. **14** (1963), 956–960.

Let $A(t)$ be an $(n\times n)$-matrix and $f(t)$ an n-vector, both continuous on an interval $[a, b]$. The author considers the system of linear differential equations $y' = Ay + f$ with the boundary conditions $y_i(a) = \beta_i$ $(1 \leqq i < n)$, $y_n(b) = \beta_n$ and, more generally, $y_1 = \beta_1$, $y_i(c) = \beta_i$ $(1 < i < n,\ a < c < b)$, $y_n(b) = \beta_n$. He obtains conditions, stated in terms of the elements of $A(t)$, which guarantee the uniqueness of the solutions of these boundary-value problems. These conditions strengthen earlier results of J. B. Garner and L. P. Burton [same Proc. **12** (1961), 100–106; MR **22** #12269]. *Z. Nehari* (Pittsburgh, Pa.)

Lewis, Robert M. 4169

An atypical problem for linear ordinary differential equations.

Comm. Pure Appl. Math. **17** (1964), 93–100.

The author proves the following theorem: "Let the n-vector $y(t)$ and the $n\times n$ matrix $A(t)$ be analytic functions of t in some neighborhood of the real interval I. Let $n - r$ be the dimension of the space X of all solutions $x(t)$ of $\dot{x} = A(t)x$, which also satisfy $\overline{y(t)}x(t) \equiv 0$ on I. Let $D = d/dt + A^*$. Then the set of vectors $\{y, Dy, \cdots, D^{r-1}y\}$

is linearly independent in I except at discrete points, and the set $\{y, Dy, \cdots, D^r y\}$ is linearly $\dot{x}$-dependent at every t in I." He then shows that any element in the space of solutions can be written as $x = C\mu$, where μ is a solution to an $(n-r)$-dimensional system, $\dot{\mu} = B\mu$. The matrices C and B can be expressed in terms of the components of the known vectors $\{D^j y\}$, $0 \leq j \leq r-1$. These results were motivated by a problem in geometrical optics where the constraint $\bar{y}x \equiv 0$ expresses the requirement that the rays and wave fronts be everywhere orthogonal.

W. T. Kyner (Los Angeles, Calif.)

Gasymov, M. G. 4170

On the inverse problem for a Sturm-Liouville equation. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 254–257.

The author studies the problem when two preassigned sequences $\{\lambda_n(h_1)\}$ and $\{\lambda_n(h_2)\}$ are the eigenvalues of one and the same equation of the type $-y'' + q(x)y = sy$ $(0 \leq x < \infty)$ with the respective boundary conditions $y'(0) - h_1 y(0) = 0$ and $y'(0) - h_2 y(0) = 0$. A necessary and sufficient condition in terms of the functions

$$\sigma_i(\lambda) = \frac{1}{h_2 - h_1} \sum_{\lambda_n(h_i) < \lambda} (\lambda_n(h_2) - \lambda_n(h_1))$$

$(i = 1, 2)$ is obtained for the case when q is restricted to a certain class of functions.

Ky Fan (Evanston, Ill.)

Gasymov, M. G.; Levitan, B. M. 4171

Sturm-Liouville differential operators with discrete spectrum. (Russian)

Mat. Sb. (N.S.) **63 (105)** (1964), 445–458.

Let L_h be the self-adjoint operator in $L^2[0, c)$ determined by the differential operator $y'' - q(x)y$ and the boundary condition $y'(0) - hy(0) = 0$, where h is a real number and q is a real function defined for $0 \leq x < c \leq \infty$ and summable in each interval $[0, b]$, $b < c$. (If needed, a boundary condition at c is introduced, which is one and the same for all h.) Suppose that for each h, L_h has a discrete spectrum with the only limit point being at infinity. Let $\{\lambda_n\}_{n=-\infty}^{\infty}$ and $\{\mu_n\}_{n=-\infty}^{\infty}$ be the spectra of L_{h_1} and L_{h_2}, respectively. The present article is devoted to the problem of constructing q, h_1 and h_2 in terms of $\{\lambda_n\}$ and $\{\mu_n\}$. (The corresponding problem for the regular case has already been considered by a number of authors.) There are derived certain necessary conditions that the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ be spectra of L_{h_1} and L_{h_2}, and there is determined a formula for the spectral function of L_{h_1} in terms of $\{\lambda_n\}$ and $\{\mu_n\}$. Using these results and the work of I. M. Gel'fand and B. M. Levitan [Izv. Akad. Nauk SSSR Ser. Mat. **15** (1951), 309–360; MR **13**, 558], it is then possible to construct q, h_1 and h_2. There is also derived a necessary condition that $\{\lambda_n\}$ and $\{\mu_n\}$ be spectra of two singular problems in which the second is obtained from the first by perturbation of both q (by a finite function) and the boundary condition at zero. Most of the results of the present article were previously reported by the authors [Dokl. Akad. Nauk SSSR **151** (1963), 1014–1017; MR **27** #4978]. M. G. Gasymov [#4170 above] has used the results of the article to state for a certain class of potentials q necessary and sufficient conditions that $\{\lambda_n\}$ and $\{\mu_n\}$ be spectra of L_{h_1} and L_{h_2}.

R. C. Gilbert (Fullerton, Calif.)

Kreith, Kurt 4172

Comparison theorems for constrained rods.

SIAM Rev. **6** (1964), 31–36.

This paper deals with an extension of Sturm's comparison theorem for second-order ordinary differential equations to the fourth-order equations

$$(1) \qquad y^{\mathrm{iv}} = p(x)y,$$

$$(2) \qquad y^{\mathrm{iv}} = q(x)y,$$

where $0 < p(x) \leq q(x)$. The problem arises from considerations of the small vibrations of an elastic rod with boundary conditions of constraint

$$(3) \qquad \alpha_{ij} y^{(j-1)}(x_i) + (-1)^{i+j} y^{(4-j)}(x_i) = 0 \qquad (i = 1, 2; j = 1, 2).$$

The main theorem then states that if $u(x)$ and $v(x)$ are solutions, respectively, of (1) and (2), if $u(x)$ satisfies (3) with $u(x) \not\equiv 0$, and if $v(x)$ satisfies

$$(4) \qquad \beta_{ij} y^{(j-1)}(x_i) + (-1)^{i+j} y^{(4-j)}(x_i) = 0$$

with $0 < \beta_{ij} < \alpha_{ij}$, then $v(x)$ has a zero in (x_1, x_2).

F. M. Stein (Fort Collins, Colo.)

Stein, F. Max 4173

Sturm-Liouville systems that generate families.

SIAM Rev. **6** (1964), 12–19.

A family of Sturm-Liouville boundary-value problems is defined to be the collection of all Sturm-Liouville boundary-value problems which can be obtained from a given one by repeatedly differentiating or integrating the differential equation and applying the given boundary conditions. The author gives a necessary and sufficient condition that a Sturm-Liouville problem generate a family. He also shows that the derivatives of the eigenfunctions of a Sturm-Liouville problem which generates a family are orthogonal with respect to a weight function which can be calculated explicitly.

F. Brauer (Madison, Wis.)

McKelvey, Robert 4174

Symmetric differential operators.

Amer. Math. Monthly **71** (1964), 119–129.

This article was adapted from a lecture delivered in 1962 in Vancouver, B.C. The author reviews in it the results concerning the theory of ordinary differential symmetric operators. The following topics are discussed: Naĭmark's extension of symmetric operators to self-adjoint operators, spectral representation of ordinary differential self-adjoint operators, explicit form of the spectral matrix function for such operators, and properties of dissipative operators. The paper contains key bibliographical material.

W. Bogdanowicz (Washington, D.C.)

Moroney, Richard M. 4175

A class of characteristic-value problems.

Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics, pp. 224–230. *Academic Press, New York*, 1963.

The author continues his study of the question as to when the differential system $x'' + xF(x, \lambda, t) = 0$, $x(0) = 0$, $x'(0) = 1$, $x(1) = 0$, has a sequence of solutions x_k corresponding to a sequence of characteristic values λ_k for

which $\lambda_k \to \infty$ as $k \to \infty$ [Trans. Amer. Math. Soc. **102** (1962), 446–470; MR **26** #6478]. The assumptions concerning $F = F(x, \lambda, t)$ are: (1) F is continuous for $-\infty < x < \infty$, $\alpha \leqq \lambda < \infty$, $0 \leqq t \leqq 1$; (2) For fixed λ, F satisfies a Lipschitz condition in x (uniformly in t); (3) F is positive if $x \neq 0$ and zero for $x = 0$; (4) For fixed λ and t, F is monotonically non-decreasing for $x \geqq 0$, and monotonically non-increasing for $x \leqq 0$. Finally, there is an assumption concerning the dependence of F on λ, too complicated to be stated here, which, together with the conditions just mentioned, guarantees the existence of the solutions in question. *Z. Nehari* (Pittsburgh, Pa.)

Rofe-Beketov, F. S. 4176

A finiteness test for the number of discrete levels which can be introduced into the gaps of the continuous spectrum by perturbations of a periodic potential. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 515–518.

The discrete part of the spectrum is studied for a self-adjoint equation $y'' + \{\lambda - p(x) - q(x)\}y = 0$, $-\infty < x < \infty$, when $q(x)$ is periodic of period 1 and $p(x)$ is small in the sense that $\int_{-\infty}^{+\infty} (1 + |x|)|p(x)|\,dx < \infty$. It is known that the equation has a purely continuous spectrum when $p(x) = 0$, and that this part of the spectrum is unchanged by a change of $p(x)$. Write the real complement of the continuous spectrum as a union of disjoint intervals. Each such interval is now shown to contain only a finite number of eigenvalues, and at most two eigenvalues if the interval is sufficiently far from the origin.
L. de Branges (Lafayette, Ind.)

Greguš, M. [Greguš, Michal] 4177

Oscillatory properties of solutions of a third-order differential equation of the type

$$y''' + 2A(x)y' + [A'(x) + b(x)]y = 0.$$

(Slovak. Russian and German summaries)

Acta Fac. Nat. Univ. Comenian. **6**, 275–300 (1961).

Given the third-order differential equation

$$\text{(a)} \qquad y''' + 2A(x)y' + [A'(x) + b(x)]y = 0, \qquad x \in (-\infty, \infty) \equiv j;$$

assume further that $A(x) > \rho^2 = \text{const} > 0$, $A'(x)$ and $b(x)$ are continuous with $0 \not\equiv b(x) \geqq 0$ in every subinterval of j. Let $x_1 \in j$ be an arbitrary number. The set of all integrals of (a) vanishing at x_1 will be called the integral pencil (x_1) of (a). Let ω be an integral of the differential equation $z''' + 2A(x)z' + [A'(x) - b(x)]z = 0$ which is adjoint to (a) and which satisfies the initial conditions $\omega(x_1) = \omega'(x_1) = 0$. Then $\omega(x) \neq 0$ for $x > x_1$ and all integrals of the pencil (x_1) of (a) have an infinite number of zeros at x_1 and satisfy the linear second-order equation

$$(y'\omega^{-1})' + (2A\omega^{-1} + \omega''\omega^{-2})y = 0.$$

This is possible because the dispersion theory developed for linear second-order equations by the reviewer [Czechoslovak Math. J. **3** (**78**) (1953), 199–255; MR **15**, 706] can be extended to the integral pencil of (a). Within the framework of this new theory the author investigates, in particular, reciprocal transformations of specific integral pencils of (a). *O. Borůvka* (Brno)

Greguš, M. [Greguš, Michal] 4178

Über einige Eigenschaften der Lösungen der Differentialgleichung dritter Ordnung. (Slovak and Russian summaries)

Acta Fac. Nat. Univ. Comenian. 7, 585–595 (1963).

Es sei (a) $y''' + 2A(x)y' + [A'(x) + b(x)]y = 0$ eine lineare Differentialgleichung 3. Ordnung mit stetigen Koeffizienten $A(x)$, $A'(x)$, $b(x)$, $x \in (-\infty, \infty) \equiv j$. Der Verfasser zeigt: Ist $2A(x) + 1 \leqq 0$ und $0 \leqq b(x) \leqq -2A(x) + A'(x) - 1$ für $x \in j$, wobei b und $-2A + A' - 1$ in keinem Teilintervall von j identisch verschwinden, so hat jedes Integral von (a) höchstens zwei einfache oder eine zweifache Nullstelle. Im zweiten Teil der Arbeit werden auf Grund von Eigenschaften gewisser Systeme von Integralen der Differentialgleichung (a), der sogenannten Büschel von der 1., 2. und 3. Art, mehrere Randwertprobleme für die Differentialgleichung (a) im Bezug auf drei Punkte untersucht.
O. Borůvka (Brno)

Greguš, Michal 4179

Über einige Randwertprobleme dritter Ordnung. (Russian summary)

Czechoslovak Math. J. **13** (**88**) (1963), 551–560.

In this paper the author considers oscillation theorems for the equation $y''' + 2Ay' + [A' + b]y = 0$ subject to conditions $y(a) = 0$, $\alpha_1 y(b) - \alpha y'(b) = 0$, and $\beta_1 y(c) - \beta y'(c) = 0$ $(-\infty < a < b < c < \infty)$. He allows A, b, β_1, β, α_1, and α to depend on the parameter λ in a certain interval, and demonstrates the existence of an infinite number of eigenvalues under certain conditions. *R. R. D. Kemp* (Kingston, Ont.)

Ezeilo, J. O. C. 4180

A boundedness theorem for a differential equation of the third order. (Russian summary)

Qualitative methods in the theory of non-linear vibrations (Proc. Internat. Sympos. Non-linear Vibrations, Vol. II 1961), pp. 513–538. *Izdat. Akad. Nauk Ukrain. SSR Kiev*, 1963.

The author continues his investigation of ultimate boundedness of solutions of third-order differential equations [Proc. London Math. Soc. (3) **13** (1963), 99–124 MR **26** #418]. He considers the equation (1) $\dddot{x} + a\ddot{x} + \varphi_2(\dot{x}) + \varphi_3(x) = p(t)$ and proves the following theorem. Assumptions: $a > 0$ constant; $\varphi_2(0) = \varphi_3(0) = 0$; $\varphi_2(y)/y \geqq \delta_2 > 0$ for $y \neq 0$; $\delta_3(x)/x \geqq \delta_3' > 0$ for $|x|$ sufficiently large; $\varphi_3(x)$ is monotone non-decreasing for $|x|$ sufficiently large; $\varphi_3'(x) \leqq \delta_3$ for all x, where $\delta_3 > 0$, $a\delta_2 - \delta_3 > 0$; $\varphi_2(y)$ is monotone non-decreasing for $|y|$ sufficiently large; p is bounded for all t. Then the solutions and their first and second derivatives are uniformly ultimately bounded; the ultimate bound depends only on the constants mentioned, the bound of p, and the numbers involved in the "sufficiently large" conditions, plus the way in which $|\varphi_2(y)|$ tends to infinity. *J. J. Schäffer* (Montevideo)

Lasota, A. 4181

Sur la distance entre les zéros de l'équation différentielle linéaire du troisième ordre.

Ann. Polon. Math. **13** (1963), 129–132.

Let $a_k(t)$ $(k = 1, 2, 3)$ be continuous in $[0, h]$, and assume that the differential equation $x''' + a_1(t)x'' + a_2(t)x' + a_3(t)x = 0$ has a solution $x(t)$ which possesses three zeros in $[0, h]$

The author shows that this assumption implies the inequality $1 < \frac{1}{2}A_1h + A_2\pi^{-2}h^2 + \frac{1}{2}A_3\pi^{-2}h^3$, where $A_k = \max |a_k(t)|$ in $[0, h]$. This strengthens an earlier inequality of the same type by de la Vallée Poussin [J. Math. Pures Appl. (9) 8 (1929), 125–144].

Z. Nehari (Pittsburgh, Pa.)

Baxter, W. E.; Pellicciaro, E. J. 4182

Cyclicly related differential equations.

Duke Math. J. **31** (1964), 229–234.

An $n \times n$ matrix (a_{ij}) is called a circulant if $a_{ij} = a_{j-i+1} \bmod n$, where $a_1, \cdots, a_n, a_n = a_0$, are n functions of a given class. The matrix is called an anticirculant if $a_{ij} = a_{j+i-1} \bmod n$. The paper studies the matrix differential systems $Y' = A(x)Y$, $Y' = B(x)Y$, $Y' = (A+B)Y$, where A is a circulant of functions which are continuous on $a \leqq x \leqq b$ while B is an anticirculant of similar functions on this interval. Cases where integrability replaces continuity in the hypotheses are treated. For the equation $Y' = AY$, it is shown that a solution matrix $Y(x)$, and its inverse, are circulants if $Y(a)$ is the identity matrix. If $Y(x)$, where $Y(a)$ is equal to the identity matrix, is a solution of $Y' = BY$, then $Y(x)$ is not an anticirculant if $n \geqq 3$. In this case, Y is a circulant if and only if B is a circulant. Explicit expressions are found for the solutions of the three equations listed above.

W. M. Whyburn (Chapel Hill, N.C.)

Chandra, Jagdish 4183

On the asymptotic behaviour of solutions of non-linear differential equations.

Proc. Nat. Acad. Sci. India Sect. A **32** (1962), 148–151.

The author uses a Lyapunov-like function of the type considered by the reviewer and S. Sternberg [Amer. J. Math. **80** (1958), 421–430; MR **20** #1806] to obtain a bound for the difference between two solutions of a system of differential equations. He then applies this bound to obtain results on asymptotic behaviour and comparison of solutions of two different systems. The results are closely related to some results of V. Lakshmikantham [Arch. Rational Mech. Anal. **10** (1962), 119–126; MR **25** #5237].

F. Brauer (Madison, Wis.)

Chandra, Jagdish 4184

On uniqueness of solutions of a non-linear system of differential equations.

Proc. Nat. Acad. Sci. India Sect. A **32** (1962), 187–190.

The bound established in the paper reviewed above [#4183] is used to obtain a uniqueness theorem.

F. Brauer (Madison, Wis.)

Dragilev, A. V. 4185

On non-linear second-order systems. (Russian)

Mat. Sb. (N.S.) **63 (105)** (1964), 309–320.

A non-linear system of the form (1) $\dot{x} = -f(x, y)$, $\dot{y} = -g(x)$ is considered under the assumption that the solution of the initial-value problem is unique and depends continuously on the initial data. Other standing assumptions are: $f(0, y) = 0$ only for $y = 0$; $\operatorname{sgn} g(x) = \operatorname{sgn} x$. A comparison theorem between (1) and a system (1′) $\dot{x} = -f_1(x, y)$, $\dot{y} = -g_1(x)$ of the same type is given. If (u, v) is the image of (x, y) under the mapping $\int_0^x g(\xi)\, d\xi = \int_0^u g_1(\eta)\, d\eta$, $\operatorname{sgn} x = \operatorname{sgn} u$, $y = v$, the inequality $f(x, y) \operatorname{sgn} x \geqq f_1(u, v) \operatorname{sgn} u$ insures that the trajectories of (1′) are crossed from left to right, as t increases, by the images of the trajectories of (1). Several applications of this result are made, in particular to the case $f(x, y) = -\varphi(y, -x) - \psi(y)$, $g(x) = x$, $f_1(x, y) = -\varphi_1(y, -x) - \psi(y)$, $g_1(x) = x$. Such applications include the behavior of trajectories near the singular point $(0, 0)$, ultimate boundedness, and the existence of limit cycles.

R. Conti (Baltimore, Md.)

Halvorsen, Sigbjørn 4186

Bounds for solutions of second order linear differential equations, with applications to L^2-boundedness.

Norske Vid. Selsk. Forh. (Trondheim) **36** (1963), 36–40.

On a half-open t-interval of the form $(0, b]$ or $[a, \infty)$ let there be defined a real-valued, locally integrable function $f(t)$ having a singularity at the open end. There is investigated the behavior of the solutions of the differential equation (1) $x'' + f(t)x = 0$ near the singularity. An estimate of Levinson [Duke Math. J. **8** (1941), 1–10; MR **2**, 287] for the bound of solutions of (1) is used to obtain sufficient conditions in order that (1) be of the limit circle type (Weyl) at either 0 or ∞. In particular, it is shown that (1) is of this type at the origin if there exists a positive constant δ such that $(-3/4+\delta)t^{-2} \leqq f(t) \leqq (13/4-\delta)t^{-2}$, a relation representing an improvement of a criterion due to Sears [J. London Math. Soc. **24** (1949), 207–215; MR **11**, 360].

C. R. Putnam (Lafayette, Ind.)

Jablonskiĭ, A. I. 4187

Asymptotic expansion of regular solutions of certain classes of differential equations. (Russian)

Dokl. Akad. Nauk BSSR **8** (1964), 77–80.

Asymptotic expansions of the regular solutions of the first and second Painlevé transcendent differential equations are presented; the coefficients are specified by recursive formulas. The equations are of the class $y'' = f(x, y)$ and the regular solutions are singled out by conditions of the form: $\lim_{x\to\infty} (y(x) - \phi(x)) = 0$ (either from above or below $\phi(x)$), where $\phi(x)$ is a branch of a solution of $f(x, y) = 0$, and $\lim_{x\to\infty} y'(x) = 0$. These asymptotic expansions diverge except for isolated cases. Most results are stated without proof.

M. Weisfeld (San Bernardino, Calif.)

Kukles, I. S.; Hasanova, M. 4188

On the global behaviour of the characteristics of a differential equation on planes. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1964**, no. 1, 6–12.

The authors study the critical points of the system $\dot{x} = xy$, $\dot{y} = P(x, y)$, where P is a polynomial of the second degree. The global behaviour of the paths is not considered.

W. A. Coppel (Canberra)

Lim, Y. S.; Kazda, L. F. 4189

A study of second order nonlinear systems.

J. Math. Anal. Appl. **8** (1964), 423–444.

The authors consider the equation of the form

$$\ddot{x} + f(\dot{x}) + g(x) = e(t)$$

and study the ultimate bounds of $x(t)$ and $\dot{x}(t)$, and the

convergence of all solutions to one (steady-state) solution. The following results are obtained. Theorem I: Let $f(y)$, $g(x)$ and $e(t)$ be continuous, $g(x)$ a monotonically increasing function that satisfies a Lipschitz condition, $g(0)=f(0)=0$, and if there exist positive constants b, c and E such that $f(y)/y \geqq b$, $g(x)/x \geqq c$, $|e(t)| \leqq E$, then for any solution $|\dot{x}(t)| \leqq J = 2E/b$ when $t > t_0$ (t_0 depending on the particular solution) and $|x(t)| \leqq I$, where

$$I = E/c \text{ for } c/b^2 < \tfrac{1}{4},$$

$$= \frac{E}{c}\left[1+\sqrt{\left(\frac{4c}{b^2}-1\right)}\cdot \exp\left(-\frac{\pi}{2\sqrt{(4c/b^2-1)}}\right)\right] \text{for } c/b^2 \geqq \tfrac{1}{4}.$$

Theorem II: Let $f(y)$, $g(x)$ and $e(t)$ be continuous, $f'(y)$ exists and $0 < p_1 \leqq f'(y) \leqq p_2$, and $g'(x)$ exists and $0 < q_1 \leqq g'(x) \leqq q_2$, then all solutions converge to a unique (steady-state) solution provided that the constants p_1, p_2, q_1 and q_2 satisfy conditions (31) and (32) or (37) and (38) of the paper. Theorem III: In addition to the assumptions in Theorem I, if $g'(x)$ is continuous and $g'(x) \geqq q_1 > 0$ in $|x| \leqq I$, $g''(x)$ exists and $|g''(x)| \leqq C_2$ in $|x| \leqq I$, and $f'(y) \geqq p_1 > 0$ in $|y| \leqq J$, then all solutions converge to a unique solution provided $p_1^2 > 2EC_2/q_1$.

The authors also show that if $e(t)$ is periodic and the assumptions in Theorems II or III are satisfied, all solutions approach a limit cycle.

R. Liu (Notre Dame, Ind.)

Olver, F. W. J. 4190

Error bounds for first approximations in turning-point problems.

J. Soc. Indust. Appl. Math. **11** (1963), 748–772.

Let x be a real variable, and let $E(x) = \exp(\frac{2}{3}x^{3/2})$. With Ai (x) and Bi (x) standard Airy functions, write

$$A_i(x) = E^{-1}(x)M(x)\sin\chi(x),$$
$$B_i(x) = E(x)M(x)\cos\chi(x),$$
$$A_i'(x) = E^{-1}(x)N(x)\sin\psi(x),$$
$$B_i'(x) = E(x)N(x)\cos\psi(x).$$

The main result of the paper is that if f is a piecewise continuous real or complex valued function on $[a, b]$, then on $[a, b]$

$$(*) \qquad \frac{d^2w}{dx^2} = [x+f(x)]w$$

has solutions w_1 and w_2 such that $w_1 = A_i(x)+\varepsilon_1$, $w_2 = B_i(x)+\varepsilon_2$, $w_1' = A_i'(x)+\eta_1$, $w_2' = B_i'(x)+\eta_2$, where

$$|\varepsilon_1| \leqq \lambda_1^{-1}[e^{\lambda_1 F_1(x)}-1]E^{-1}(x)M(x) \equiv G(x),$$
$$|\eta_1| \leqq G(x)N(x)/M(x),$$
$$|\varepsilon_2| \leqq (\lambda_2/\lambda_1)[e^{\lambda_1 F_2(x)}-1]E(x)M(x) \equiv H(x),$$
$$|\eta_2| \leqq H(x)N(x)/M(x),$$

$\lambda_1 = 1.430\cdots$, $\lambda_2 = 1.315\cdots$, $F_1 = \int_x^b |t^{-1/2}f(t)|\,dt$, and $F(x) = \int_a^x |t^{-1/2}f(t)|\,dt$. The interval (a, b) may be infinite provided the integrals for F_i converge. The proof is similar to that of Theorem 1 of the author's paper [Proc. Cambridge Philos. Soc. **57** (1961), 790–810; MR **24** #A313]. There is then derived a corresponding result for an equation

$$(**) \qquad \frac{d^2w}{dx^2} = [u^2p(u, x)+q(u, x)]w$$

(u is a large, complex parameter) that has a simple turning point on (a, b). This is done by use of Langer's transformation, which reduces (**) to an equation of the form (*).

In the remainder of the paper the error bounds obtained are discussed and sharpened, and the results are then applied to Hermite polynomials.

N. D. Kazarinoff (Ann Arbor, Mich.)

Petty, C. M.; Leitmann, G. 4191

A boundedness theorem for a nonlinear, nonautonomous system.

Monatsh. Math. **68** (1964), 46–51.

The system $x''+b(t)x'+c(t)f(x)=0$ ($'=d/dt$) is considered under precisely stated hypotheses which include $c(t)>0$ and $xf(x) \geqq 0$. By use of the Lyapunov function $V(t, x) = x'^2+2c(t)\int_0^x f(s)\,ds$ it is proved that for $t_0 \leqq t < T$ (where in certain cases T may be ∞) $x(t)$ will remain within prescribed bounds provided $V(t_0, x_0)$ is beneath a precisely stated bound. A more general result for a system of arbitrary order is given, and the proof is concentrated on the latter.

T. M. Cherry (Melbourne)

Perov, A. I. 4192

On multi-dimensional differential equations with constant coefficients. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 1266–1269.

On établit six théorèmes. Le théorème 1 concerne le problème de Cauchy

$$\frac{dy}{dx} = Ay, \quad y(\xi) = \eta \quad (\xi \in E_x, \eta \in E_y),$$

E_x, E_y étant des espaces sur les corps commutatifs P_x, P_y, et A l'opérateur expliqué dans le texte. Le théorème 2 traite le problème de Cauchy relatif à l'équation conjuguée

$$\frac{d\varphi}{dx} = -A'\varphi, \quad \varphi(\xi) = \theta \quad (\xi \in E_x, \theta \in E_\varphi).$$

Le théorème 3 concerne le problème de Cauchy relatif aux équations des opérateurs:

$$\frac{dY}{dx} = \widetilde{AY}, \quad Y(\xi) = H; \quad \frac{d\Phi}{dx} = -\widetilde{A'\Phi}, \quad \Phi(\xi) = \theta.$$

On introduit dans le théorème 4 la notion de vecteur caractéristique de l'opérateur A ($\tilde{A}$) et la notion de fonctionnelle caractéristique du même opérateur. Le théorème 5 établit les conditions nécessaires et suffisantes pour que la fonction exp $\tilde{A}x$ soit périodique. Le théorème 6 donne une égalité sous l'hypothèse que E_y soit normé.

M. Bertolino (Belgrade)

Krasnosel'skiĭ, M. A.; Perov, A. I. 4193

Some criteria for existence of periodic solutions of systems of ordinary differential equations. (Russian. English summary)

Qualitative methods in the theory of non-linear vibrations (Proc. Internat. Sympos. Non-linear Vibrations, Vol. II, 1961), pp. 202–211. Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963.

This survey of recent work on the subject described in the title emphasizes the use of fixed-point methods, especially in function space. There are 38 items in the list of references, all but four of them Russian.
D. Bushaw (Pullman, Wash.)

Das, K. M. 4194
Properties of solutions of certain nonlinear differential equations.
J. Math. Anal. Appl. **8** (1964), 445–451.
The real differential equation considered is

$$(r(t)u')' + p(t)f(u) = 0,$$

where $r(t) > 0$, $p(t) > 0$, $uf(u) > 0$ for $u \neq 0$ and

$$\lim_{u \to +\infty} \int_0^u f(x)\, dx = +\infty.$$

For this equation the author proves a boundedness theorem of C. T. Taam [Proc. Amer. Math. Soc. **6** (1955), 377–385; MR **17**, 366], a non-oscillation theorem of J. J. Jones [ibid. **9** (1958), 586–588; errata, ibid. **10** (1959), 1000; MR **21** #3629] and an oscillation theorem of P. Waltman [Monatsh. Math. **67** (1963), 50–54; MR **26** #5214].
W. R. Utz (Columbia, Mo.)

Bader, W. 4195
Ein neues Verfahren zur mathematischen Behandlung gewisser nichtlinearer Schwingungen. (Russian summary)
Analytic methods in the theory of non-linear vibrations (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. I,* 1961), *pp.* 73–86. *Izdat. Akad. Nauk Ukrain. SSR, Kiev,* 1963.
Having assumed $\varphi(\tau)$ is a solution of $\varphi^{(2)} + a \sin \varphi = b \sin \tau$, where $\varphi^{(2)} = d^2\varphi/d\tau^2$ is of the form

$$\varphi(\tau) = \sum_1^N B_{2\nu-1} \sin(2\nu - 1)\tau,$$

the author computes $\varphi^{(k)}(0)$ for $k = 0, 1, \cdots, 9, 11$ for $N = 5$ and applies the result.
S. P. Diliberto (Berkeley, Calif.)

de Pater, A. D. 4196
The vibrations of nonlinear mechanical systems with rigid stops. (Russian summary)
Applications of the methods of non-linear vibrations to the problems of physics and technology (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. III,* 1961), *pp.* 326–346. *Izdat. Akad. Nauk Ukrain. SSR, Kiev,* 1963.
The method of Krylov and Bogoliubov is used to investigate the motion of a mechanical system in which the displacements of certain parts are limited by rigid stops, the system being otherwise linear, and being subjected to a driving force which is a sinusoidal function of the time. A mass-spring system with rigid stops is discussed first, and then an extensive study is made of certain vibratory motions of a railway vehicle. There is no critical examination of the approximate solutions from the mathematical standpoint, but the author indicates that the results are in agreement with various results obtained previously by other methods.
L. A. MacColl (New York)

Cartwright, M. L. 4197
From non-linear oscillations to topological dynamics.
J. London Math. Soc. **39** (1964), 193–201.
An expository paper containing the substance of the author's Presidential Address delivered to the London Mathematical Society on 21 November 1963.

Andronova-Leontovič, E. A.; Beljustina, L. N. 4198
The theory of bifurcation of dynamical systems of second order and its application to the study of non-linear problems in the theory of oscillations. (Russian. English summary)
Qualitative methods in the theory of non-linear vibrations (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. II,* 1961), *pp.* 7–28. *Izdat. Akad. Nauk Ukrain. SSR, Kiev,* 1963.
The paper gives a condensed exposition of the theory of bifurcation supplemented by recent contributions of the Soviet authors; it is assumed that the systems under study have analytic right-hand parts, although in the discussion it is mentioned that similar conclusions hold also for the non-analytic case if a sufficient number of partial derivatives is available. The authors append an extensive bibliography of Russian papers beginning with A. A. Andronov who initiated these studies in connection with physical problems [A. A. Andronov and C. E. Chaikin, *Theory of oscillations*, Princeton Univ. Press, Princeton, N.J., 1949; MR **10**, 535] and gave (together with L. S. Pontrjagin) the definition of non-critical and critical topological structures [A. A. Andronov and L. S. Pontrjagin, Dokl. Akad. Nauk SSSR **14** (1937), 247–250] which have been in extensive use ever since; in the process of these developments one has to note the contribution of American mathematicians [H. F. DeBaggis, *Contributions to the theory of nonlinear oscillations*, Vol. II, pp. 37–59, Princeton Univ. Press, Princeton, N.J., 1952; MR **14**, 557].

It is regrettable that the paper does not mention the first discoverer of the bifurcation properties of differential equations, H. Poincaré [Acta Math. **7** (1885/86), 1–32; ibid. **7** (1885/86), 259–380] who in his studies of differential equations containing a parameter λ introduced the following definition: If the variation of λ in $\lambda_0 \leq \lambda \leq \lambda_1$ results only in quantitative changes of the solution, such variations are called ordinary; if, however, for a special value $\lambda = \lambda^\kappa$ the solution undergoes a qualitative change, such a value is called the bifurcation value.

Andronov was the first to take advantage of the Poincaré theory for explaining a number of bifurcation phenomena in non-linear systems. At present most of these phenomena can be classified in two types [the reviewer, *Nonlinear oscillations*, Chap. 7, Van Nostrand, Princeton, N.J., 1962; MR **25** #1339]: (1) appearance (or disappearance) of limit cycles branching off from a focus (Andronov), and (2) appearance (or disappearance) of limit cycles (Poincaré).

There are also some other cases of bifurcations, such as branching off of a limit cycle from a separatrix of a saddle point and occasionally a few others. These other cases are investigated much less extensively than the two cases mentioned above; some of these cases are mentioned in the paper.

The paper gives a broad presentation of these various topological occurrences, illustrated by numerous figures,

but omitting the intermediate calculations which (especially in some of these pathological cases) are rather long.

This paper is recommended to those who wish to acquire a general grasp of this branch of non-linear analysis prior to undertaking its detailed study.

N. Minorsky (Aix en Provence)

Hsu, C. S. 4199

On the parametric excitation of a dynamic system having multiple degrees of freedom.

Trans. ASME Ser. E. J. Appl. Mech. **30** (1963), 367–372.

For $d^2x/dt + Ax = \varepsilon(B(t)x + C(t)dx/dt)$, where x is an n-vector and A is a diagonal matrix with positive elements $w_1^2 \leqq w_2^2 \leqq \cdots \leqq w_n^2$, with $B(t)$ and $C(t)$ periodic of period ω, the author obtains inequalities involving the Fourier coefficients of B and C which represent a first approximation to regions in which solutions are either stable or unstable. These results, of interest in the resonance cases, are new but heuristic.

S. P. Diliberto (Berkeley, Calif.)

Huaux, Aimé 4200

Sur la stabilité des solutions constantes de l'équation autonome de Liénard $\ddot{x} + a(x)\dot{x} + \varphi(x) = 0$.

C. R. Acad. Sci. Paris **258** (1964), 2003–2006.

Take the Liénard equation (1) $\ddot{x} + a(x)\dot{x} + \varphi(x) = 0$ or its equivalent system (2) $\dot{x} = y$, $\dot{y} = -a(x)y - \varphi(x)$. The critical points of (2) are the points $(x_i, 0)$ of the x-axis where x_i is any zero of $\varphi(x)$. The author studies the stability of one isolated critical point or of two isolated such points. The attack is by means of rather simple Liapunov functions. The assumptions are rather simple, and at least for one critical point the various possible cases are fully discussed.

To form an idea of the treatment let $\varphi(0) = 0$, and $x\varphi(x) > 0$ for $|x|$ small. Set also $\Phi(x) = \int_0^x \varphi(x)\,dx$; $A(x) = \int_0^x a(x)\,dx$. Reasonable assumptions on continuity and uniqueness of solutions are made. The Liapunov function $v = \frac{1}{2}(y + A(x))^2 + \frac{1}{2}y^2 + 2\Phi(x)$ is introduced. Its derivative along the solutions near zero is $-\dot{v} = a(x)y^2 + A(x)\varphi(x)$. By means of v, $\dot{v}$ and through the Liapunov theory it is shown that if $a(x) > 0$ near zero, the origin is asymptotically stable.

The case of two zeros of $\varphi(x)$ is dealt with through behavior properties of $a(x)$ and $\varphi(x)$ over an interval including the origin. *S. Lefschetz* (Baltimore, Md.)

Jataev, M. 4201

On the existence of a Lyapunov function. (Russian. Kazak summary)

Vestnik Akad. Nauk Kazah. SSR **1964**, *no.* 3 (228), 78–79.

Let $x' = f(x)$ with $f(0) = 0$ be such that the trivial solution is stable and every non-trivial solution is of class C^1 with respect to the initial conditions. The author shows that there exists a Lyapunov function which is a function of x alone. *H. A. Antosiewicz* (Los Angeles, Calif.)

Lochak, Georges 4202

Sur le comportement d'un mouvement asymptotiquement stable soumis à des perturbations aléatoires.

C. R. Acad. Sci. Paris **258** (1964), 1999–2002.

Consider the dynamical system (1) $dx_i/dt = X_i(x_1, \cdots, x_n, t)$, $i = 1, \cdots, n$; where X_i satisfy certain restrictions. Assume that for $t > t_0$ and $\|x\|_2 < H$, $x_1 = x_2 = \cdots = x_n = 0$ is solution of (1) and is uniformly asymptotically stable. The author considers the system

$$(2) \qquad \frac{dx_i}{dt} = X_i(x_1, \cdots, x_n, t) + R_i(x_1, \cdots, x_n, t), \qquad i = 1, \cdots, n$$

under restrictions on $R_i(x_1, \cdots, x_n, t)$ of the form $|R_i(x_1, \cdots, x_n, t)| \leqq \varphi(t)$ $(i = 1, \cdots, n)$ for $\|x\|_2 \leqq H$, $(1/T)\int_t^{t+T} \varphi(\theta)\,d\theta \leqq \eta$ for all $t > 0$ and $T > \tau$, $(1/T)\int_t^{t+T} \varphi(\theta)\,d\theta \leqq \omega$ for all $t > 0$ and $0 \leqq T \leqq \tau$. He then shows that given ε and τ, one may find $\delta(\varepsilon)$, $\eta(\varepsilon)$ and $\omega(\varepsilon, \tau)$ such that if $x(x_0, t_0, t)$ is any solution of (2) which satisfies $\|x_0\|_2 < \delta$, then $\|x(x_0, t_0, t)\|_2 < \varepsilon$ for all $t \geqq t_0$.

J. C. Lillo (W. Lafayette, Ind.)

Pliss, V. A. 4203

Stability of motion in the doubtful case of two zero roots. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 268–270.

A problem which considerably preoccupied Liapunov was the stability of the origin for a real analytical system

$$(1) \qquad \begin{aligned} \dot{x} &= y + X(x, y, z), \\ \dot{y} &= Y(x, y, z), \\ \dot{z} &= Pz + Z(x, y, z), \end{aligned}$$

where x, y are scalars, z is an n-vector, X, Y, Z are power series beginning with degree at least two, and P is a constant stable matrix. In 1893, one year after his Opus Magnum [Thesis, Har'kov Univ., Kharkov, 1892] he published a full treatment of the plane case: $z = Z = 0$ [*Collected works* (Russian), Vol. II, pp. 272–331, Izdat. Akad. Nauk SSSR, Moscow, 1956; MR **27** #5980]. In Liapunov's Nachlass there was found recently an extensive mémoire in which he settled all but one case of the general problem. This mémoire has just been published by the Academy of Sciences of the U.S.S.R., as a monograph in 116 pages under the same title as the above paper with an excellent introduction by V. P. Basso[v] [*A study of one of the special cases of the problem of stability of motion* (Russian), Izdat. Leningrad. Univ., Leningrad 1963]. In the present paper the author settles the case which baffled Liapunov (and probably discouraged him from publishing the complete mémoire).

S. Lefschetz (Baltimore, Md.)

Rosenbrock, H. H. 4204

On the stability of a second-order differential equation.

J. London Math. Soc. **39** (1964), 77–80.

It is shown that the null solution of the equation $\ddot{x} + a(t, x, \dot{x})\dot{x} + b(t, x, \dot{x})x = 0$ is uniformly asymptotically stable in the large if there exist constants α_i, β_i $(i = 1, 2)$ such that $0 < \alpha_1 \leqq \alpha \leqq \alpha_2 < \beta_1 \leqq \beta \leqq \beta_2$, where α, β are the roots of the equation $\lambda^2 - a\lambda + b = 0$. Most previous results apply either to linear time-dependent equations or to nonlinear autonomous equations [W. Hahn, *Theory and application of Liapunov's direct method*, Prentice-Hall, Englewood Cliffs, N.J., 1963; MR **26** #5230].

W. A. Coppel (Canberra)

Rozenvasser, E. N. 4205

The absolute stability of nonlinear systems.

Avtomat. i Telemeh. **24** (1963), 304–313 (*Russian. English summary*); *translated as Automat. Remote Control* **24** (1963), 283–291.

Consider the non-linear system

$$(1) \qquad \dot{x} = Bx + hf(\sigma), \quad \sigma = d^T x,$$

where x is the variable n-vector, B a constant $n \times n$ matrix, h and d constant n-vectors. σ and $f(\sigma)$ are scalars satisfying, for some constant c, the relation

$$(2) \qquad 0 < \sigma f(\sigma) < c\sigma^2.$$

Absolute stability of the solution $x = 0$ of (1) in the range $(0, c)$ (i.e., asymptotical stability in the large for all functions $f(\sigma)$ satisfying (2)) is usually studied with the aid of a Lyapunov function of the "quadratic form plus a non-linearity integral" type [see A. I. Lur'e and the author, Proc. First Internat. Congr. Internat. Federation Automatic Control (Moscow, 1960), pp. 709–722, Izdat. Akad. Nauk SSSR, Moscow, 1961]. In the present paper, conditions for the existence of such a Lyapunov function given earlier by the author [Izv. Akad. Nauk SSSR Otd. Tehn. Nauk Meh. Mašinostr. **1960**, no. 2, 7–16; MR **25** #2294] are written in matrix form and shown to be equivalent to V. M. Popov's criterion of absolute stability [Avtomat. i Telemeh. **22** (1961), 961–979; MR **24** #A3394]. The interesting point is that Popov's criterion was obtained by an analysis of the system without using Lyapunov's second method, and that the relation between both types of approach to the problem was not known before.

The results mentioned are also generalized for equations of the type

$$\dot{x} = Bx + hf(\sigma), \quad \sigma = d^T x - pf(\sigma),$$

where $p > 0$ and $f(\sigma)$ is assumed differentiable with $df(\sigma)/d\sigma > -p$. Finally, the non-autonomous case with $f = f(\sigma, t)$ is discussed. *E. O. Roxin* (Baltimore, Md.)

Samoĭlenko, A. M. 4206

A proof of the averaging principle for differential equations with discontinuous right-hand side. **(Russian)**

Approximate methods of solving differential equations, pp. 90–95. *Izdat. Akad. Nauk Ukrain. SSR, Kiev,* 1963.

The author has made a significant contribution to A. F. Filippov's [Mat. Sb. (N.S.) **51** **(93)** (1960), 99–128; MR **22** #4846] theory of differential equations with discontinuous elements by establishing the validity of the principle of averaging for a class of such equations. An important example to which his theorem applies is $x'' + x = \varepsilon g(x - x_0) + \varepsilon f(x, x', t)$, where ε is a small parameter, g is a step function, and f is a smooth function with period T in t. His proof parallels that of I. I. Gihman [Ukrain. Mat. Ž. **4** (1952), 215–219; MR **17**, 738].

W. T. Kyner (Los Angeles, Calif.)

Pavlov, V. V. 4207

Sufficient conditions for invariance of non-linear systems. **(Ukrainian)**

Avtomatika **1963**, *no.* 4, 65–67.

On considère dans ce travail les conditions d'invariabilité des systèmes aux larges classes des fonctions non-linéaires, figurant dans n équations du système

$$[a_{110}p^{m_1}x_1 + F_{11}(x_1, t)] + [a_{120}p^{m_2}x_2 + F_{12}(x_2, t)] + \cdots + [a_{1n0}p^{m_n}x_n + F_{1n}(x_n, t)] = f_1(t),$$

$$[a_{210}p^{m_1}x_1 + F_{21}(x_1, t)] + [a_{220}p^{m_2}x_2 + F_{22}(x_2, t)] + \cdots + [a_{2n0}p^{m_n}x_n + F_{2n}(x_n, t)] = f_2(t),$$

$$\cdots\cdots\cdots$$

$$[a_{n10}p^{m_1}x_1 + F_{n1}(x_1, t)] + [a_{n20}p^{m_2}x_2 + F_{n2}(x_2, t)] + \cdots + [a_{nn0}p^{m_n}x_n + F_{nn}(x_n, t)] = f_n(t),$$

liant la variable indépendante t, n fonctions $x_1, x_2, \cdots, x_n$ et leurs dérivées jusqu'aux ordres $m_1, m_2, \cdots, m_n$ inclusivement. Les coefficients $a_{110}, \cdots, a_{nn0}$ sont constants et $F_{11}(x_1, t), \cdots, F_{nn}(x_n, t)$ désignent $F_{11}(p^{m_1-1}x_1, p^{m_1-2}x_1, \cdots, x_1, t), \cdots, F_{nn}(p^{m_n-1}x_n, p^{m_n-2}x_n, \cdots, x_n, t)$. On désigne ensuite $p = d/dt$. La forme canonique du système permet d'établir les conditions suffisantes d'invariabilité. On considère alors le changement de ces conditions sous l'hypothèse de changement du type des fonctions figurant dans le système. *M. Bertolino* (Belgrade)

Klimušev, A. I. 4208

Asymptotic stability of a system of differential equations with aftereffect and with a small parameter. **(Russian)**

Sibirsk. Mat. Ž. **4** (1963), 611–621.

The author considers the following system, which has the character of a system of ordinary differential equations in the variable y and a system of differential-difference equations in the variable x:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + By(t) + Ex(t-\theta), \\ (1\text{-}\mu) \qquad \mu\dot{y}(t) &= Cx(t) + Dy(t) + Fx(t-\theta), \\ x &= g(t) \quad (t_0 - \theta \leqq t \leqq t_0), \quad y(t_0) = p_0. \end{aligned}$$

Here x is an unknown m-vector, y is an unknown n-vector, A, B, C, D, E, F are given constant matrices, θ is a positive number, p_0 a given vector, and g a given initial vector function. Placing $\mu = 0$, one obtains a degenerate problem (1-0), in which the initial condition $y(t_0) = p_0$ is dropped. If D is nonsingular, the degenerate system can be written

$$(2) \qquad \dot{x}(t) = (A - BD^{-1}C)x(t) + (E - B^{-1}DF)x(t-\theta).$$

Theorem: Let $x(t, \mu)$, $y(t, \mu)$ denote the solution of problem (1-μ), and let $x(t)$, $y(t)$ denote the solution of (1-0). Assume that (2) is asymptotically stable and that D is a stability matrix. Then given any $\varepsilon > 0$, there exists a μ_0 such that

$$|x(t, \mu) - x(t)| < \varepsilon, \qquad |y(t, \mu) - y(t)| < \varepsilon, \qquad 0 \leqq \mu < \mu_0,$$

for $t > t_1$, and system (1-μ) is asymptotically stable for $\mu < \mu_0$. Also, for a given solution of (1-0) and any $Q > 0$, $\delta > 0$, there is a μ_0 such that $\mu < \mu_0$ implies $|t_1 - t_0| < \delta$ for all initial conditions for which $|y(t_0, \mu) - y(t_0)| < Q$.

The same theorem for systems of form (1-μ) in which A, B, C, D, E, F are bounded continuous functions of t, $|D(t)| \geqq \beta > 0$, has been proved elsewhere by the author [Prikl. Mat. Meh. **26** (1962), 52–61; MR **26** #2689]. In both papers the proof uses Liapunov functionals.

K. L. Cooke (Baltimore, Md.)

Livartovskiĭ, I. V. 4209

A generalization of the second method of Lyapunov. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 87–95.

Der Verfasser betrachtet das Stabilitätsproblem in dem Fall, wenn die Gleichung der gestörten Gleichung nur stückweise stetig ist, und formuliert hinreichende Stabilitätskriterien mit Hilfe von Ljapunovschen Funktionen, die Sprünge aufweisen. Die ungestörte Bewegung werde durch $x=0$ (x ein n-dimensionaler Vektor) beschrieben, und die Störungsgleichung laute $x'=\varphi(x,t)$, wobei ein Zylinder $|x|\leqq H$ betrachtet wird. Dieser sei durch eine unendliche Folge stetiger Flächen $\Phi_\alpha(x,t)=0$ in Bereiche H_α zerlegt; jede Fläche soll die t-Achse an genau einer Stelle M_α schneiden, und es gelte für die Schnittkoordinaten $t=t_\alpha$ die Beziehung $t_{\alpha+1}-t_\alpha\geqq T>0$. Die rechte Seite $\varphi(x,t)$ der Gleichung darf auf den Flächen $\Phi_\alpha=0$ sowie auf den Ebenen $t=t_\alpha$ Sprünge haben. Den Bereich H_α teilt man in Winkelbereiche (zwischen den Sprungflächen) sowie in einen Zentralbereich ein; in M_α kann die Funktion φ drei Werte annehmen: 0, wenn man sich aus dem Zentralbereich nähert; $+\xi_\alpha$ oder $-\xi_\alpha$, wenn man vom Winkelbereich unter oder über der Ebene $t=t_\alpha$ kommt. Die Werte ξ_α seien beschränkt: $|\xi_\alpha|\leqq m$; die zuerst genannten Winkelbereiche seien als $t_\alpha-t=h_\alpha^{-}x+o(x)$ darstellbar, und man definiert für sie die Matrizen $S_\alpha^{(k)}=(\delta_{ij}+k\xi_{\alpha i}h_{\alpha j}^{-})$. Man betrachtet nun Vergleichsfunktionen $V(x,t)$; $t\geqq 0$, $|x|\leqq H$ mit $V(0,t)=0$, die in den Zentralbereichen stetig differenzierbar sind und auf den Sprungflächen der Funktion φ unstetig sein können, wobei auf den unteren Begrenzungen der Winkelbereiche $V^+/V^-=V(x,t+0)/V(x,t-0)\leqq N$, $N>1$ gelte; die Funktionen V heißen positiv-definit, falls man $V(x,t)\geqq W(x)$ [$W(0)=0$, $\inf W(x)_{0<l\leqq|x|\leqq H}>0$] hat. Es wird sodann der Satz aufgestellt: Die ungestörte Bewegung $x=0$ ist stabil, wenn es eine positiv-definite Funktion $V(x,t)$ gibt, für die in jedem Zentralbereich $V'(x,t)/V(x,t)\leqq-\mu_\alpha$ ($\mu_\alpha>0$), aber in den Punkten (x_α',t_α') der Sprungflächen, die die Winkelbereiche von unten begrenzen,

$$V(S_\alpha^{(k)}x_\alpha'+\cdot,\ t_\alpha'+k(t_\alpha-t_\alpha'))\leqq NV^+(x_\alpha',t_\alpha'),\qquad k\in[0,1]$$

beliebig, und

$$V^+(S_\alpha^{(1)}x_\alpha'+\cdot,\ t_\alpha)\exp(-\mu_\alpha\vartheta T)\leqq V^-(x_\alpha',t_\alpha'),\qquad 0<\vartheta<1,$$

gilt. Unter der stärkeren Bedingung

$$V^+(S_\alpha^{(1)}x_\alpha'+\cdot,\ t_\alpha)\exp(-\mu_\alpha\vartheta T)\leqq \exp(-\nu_\alpha)V^-(x_\alpha',t_\alpha'),\qquad \nu_\alpha>0,\ \sum\nu_\alpha=\infty,$$

herrscht sogar asymptotische Stabilität. Der Verfasser betrachtet noch das Problem der Stabilität nach der ersten Näherung und zerlegt in den Zentralbereichen $\varphi(x,t)=P(t)x+R(x,t)$, wobei $P(t)$ für $t_\alpha\leqq t\leqq t_{\alpha+1}$ stetig sei und $|R(x,t)|<a|x|$ gelten soll. Als lineare Näherung betrachtet er die linearen Gleichungen $x'=P(t)x$ in den Intervallen $t_\alpha\leqq t\leqq t_{\alpha+1}$ sowie die Sprungbedingungen für $t=t_\alpha$: $x(t_\alpha+0)=S_\alpha^{(1)}x(t_\alpha-0)$. Es wird bewiesen: Die ungestörte Bewegung (des nichtlinearen Systems) ist asymptotisch stabil, wenn die Konstante a, die den Grad der Nichtlinearität charakterisiert, genügend klein ist und wenn eine gleichmäßig kleine, den Bedingungen $V(S_\alpha^{(1)}x,t_\alpha+0)=V(x,t_\alpha-0)$ unterworfene, positiv-definite Funktion $V(x,t)$ existiert, deren Ableitung auf Grund der linearisierten Gleichung negativ-definit ist. *R. Reissig* (Berlin)

Ličko, Imrich; Švec, Marko 4210

Le caractère oscillatoire des solutions de l'équation $y^{(n)}+f(x)y^\alpha=0$, $n>1$. (Russian summary)

Czechoslovak Math. J. **13** (**88**) (1963), 481–491.

The equation of the title is considered, with $n>1$, $f(x)$ continuous and positive-valued on $[a,\infty)$ and $\alpha\neq 1$ the quotient of two odd positive integers. The authors generalize conditions for the case $n=2$ given by Atkinson [Pacific J. Math. **5** (1955), 643–647; MR **17**, 264] and Belohorec [Mat.-Fyz. Časopis Sloven. Akad. Vied **11** (1961), 250–255] and proves the following result: Assume that $0<\alpha<1$ [$\alpha>1$], and consider the condition

$$(*)\quad \int^\infty x^{\alpha(n-1)}f(x)\,dx=\infty\quad\left[\int^\infty x^{n-1}f(x)\,dx=\infty\right].$$

If n is even, (*) is necessary and sufficient for all the solutions to be oscillatory; if n is odd, (*) is necessary and sufficient for each solution to be either oscillatory or monotone with limit 0, together with all derivatives through order $n-1$. *J. J. Schäffer* (Montevideo)

Seifert, George 4211

Periodic integral surfaces for periodic systems of differential equations.

Contributions to Differential Equations **2** (1963), 341–350.

If the sphere S: $|\chi|=1$ in E^n is carried into itself after one period by the solution-induced transformation of $\mathring{\chi}=f(\chi,t)=f(\chi,t+\omega)$ (of class $C^{(1)}$) and if this transformation is locally a contraction in the strong sense, the author establishes for smooth $y=g(\chi,t,\eta)=g(\chi,t+\omega,\eta)$, where $g(\chi,t,0)=f(\chi,t)$, that the equation $\mathring{\chi}=y$ has an invariant sphere S_η smooth in η and $S_0=S$. This is a special case of the corresponding semi-published result for manifolds by J. McCarthy [Stanford Univ. Appl. Math. and Statist. Lab. Tech. Rep. No. 36 (1955)]. *S. P. Diliberto* (Berkeley, Calif.)

Bellman, Richard 4212

★Perturbation techniques in mathematics, physics, and engineering.

Holt, Rinehart and Winston, Inc., New York, 1964. viii+118 *pp.* $3.75.

For some years now there has been an ever-increasing need for books and/or monographs which attempted surveys of perturbation techniques in ordinary differential equations. The present work, despite its many faults, will prove extremely useful. This utility arises from two counts: The book contains more information about perturbation techniques for ordinary differential equations than any recent text in the field; and secondly, because of the informal or casual presentation, the basic ideas will be more accessible to the engineers and applied scientists.

The informal approach used in this book is one that makes the subject essentially non-mathematical; the author simply outlines a manipulation via a specific example and seldom indicates whether there is a theorem justifying the procedure.

The two largest areas of subjects consist of material concerning perturbations of periodic solutions and then asymptotics for $x'' + a(t)x = 0$. Other topics include singular-point expansion and boundary-value problems. The book (114 pages) is divided into parts called sections (64 pages), exercises (30 pages), comments and bibliography (20 pages). The material in the exercises consists of a few easy exercises, some non-problems (e.g., "Discuss ..."), some difficult problems, and simply the statements of results not in the text.

The 35 bibliographies are independent of each other and have unnecessary duplications.

The comment on page 86 (line 12) is incorrect ("The lemma was first developed", etc.). The inequality referred to (i.e., the lemma of page 84) was first established in a more general form by T. H. Gronwall [Ann. of Math. (2) **20** (1919), 292–296]. Today the lemma is frequently used and most authors now refer to it as Gronwall's lemma.

The elementary nature of much of the material makes a standard text on ordinary differential equations a more suitable reference source than "research" papers. Since the book includes material not completable to known (mathematical) results, it would have been convenient (for the uninitiated) to have labeled "standard" techniques as such and "proposed" techniques as such. Unless the reader is widely read, he will be unable to determine from the text what techniques are mathematically justified. *S. P. Diliberto* (Berkeley, Calif.)

Cenacchi, Anna 4213

Sulla teoria dell'eccitazione "dura".

Atti Sem. Mat. Fis. Univ. Modena **12** (1962/63), 1–16.

Author's summary: "Si determinano condizioni sufficienti ad assicurare l'esistenza di una ed una sola soluzione periodica stabile della equazione di Liénard nel caso dell'eccitazione dura."

S. P. Diliberto (Berkeley, Calif.)

Kudakova, R. V. 4214

Quasiperiodic solutions of non-linear systems of differential equations containing a small parameter. (Russian. Kazak summary)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 3, 48–52.

The author obtains a sufficient condition for the existence of a doubly periodic solution of the system

$$(*)\qquad \left(\frac{\partial}{\partial u_1}+\frac{\partial}{\partial u_2}\right)x_1 = x_2, \qquad \left(\frac{\partial}{\partial u_1}+\frac{\partial}{\partial u_2}\right)x_2 = Q(u_1, u_2)+\varepsilon P(u_1, u_2, x_1, x_2, \varepsilon),$$

where P and Q have period p_i in u_i, and ε is a small parameter. With such a solution one can construct an almost periodic solution of $x''=Q(t,t)+\varepsilon P(t,t,x,x',\varepsilon)$ by setting $u_1=u_2=t$. The sufficient condition is an integral equation to be satisfied by a solution of the homogeneous equation obtained by setting $Q=P=0$ in (*).

W. T. Kyner (Los Angeles, Calif.)

Loud, W. S. 4215

Locking-in in perturbed autonomous systems. (Russian summary)

Qualitative methods in the theory of non-linear vibration (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. I,* 1961), *pp.* 223–232. *Izdat. Akad. Nauk Ukrain. SSR, Kiev,* 1963.

Using standard techniques (there are many versions th author uses [the author, Ann. of Math. (2) **70** (1959), 490 529; MR **21** #5786]) for existence in a perturbatio problem of periodic solutions, the author establishes th existence of subharmonics. These subharmonics lie on torus; hence, for the example given it is established tha the rotation number on the surface is constant for a interval of the perturbation parameter.

S. P. Diliberto (Berkeley, Calif.

Myrzaliev, D. 421

The existence of a toroidal invariant surface in th neighbourhood of a family of weakly stable solution (Russian. Kazak summary)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nau **1963**, *no.* 3, 45–47.

The author proves the existence of a family of periodi invariant manifolds for the equations $ds/dt=1+\varepsilon F(s,t,z)$, $dz/dt=-z^3+z^4L(s,t,z)+\varepsilon R(s,t,z)$, where F, L, R ar finite trigonometric polynomials in s and t with period p_1 and p_2, respectively. The coefficients are analyti functions of z. His proof consists of utilizing a standar substitution to reduce the problem to one solved b M. Marcus [*Contributions to the theory of nonlinear oscillations*, Vol. 3, pp. 261–268, Princeton Univ. Press, Princeton, N.J., 1956; MR **18**, 395].

W. T. Kyner (Los Angeles, Calif.

Faĭzibaev, È. F. 421

On the solution of a type of non-linear differentia equation. (Russian)

Approximate methods of solving differential equation *pp.* 126–128. *Izdat. Akad. Nauk Ukrain. SSR, Kie* 1963.

On obtient l'intégrale générale de l'équation

$$\frac{d^2x}{dt^2}+k(x)\left(\frac{dx}{dt}\right)^2+F(x) = B.$$

Le cas

$$\frac{d^2x}{dt^2}-\frac{3}{2}\frac{d\ln x}{dt}\frac{dx}{dt}+(\alpha+\gamma_1x^2)x = 0$$

(où α, γ_1 sont des constantes) est considéré pour $\alpha<0$ $\alpha>0$; $\alpha<0$, $C_1^2+16\alpha\gamma_1<0$; $\alpha>0$, $C_1^2+16\alpha\gamma_1>0$ (où C est la constante d'intégration). *M. Bertolino* (Belgrade

Diliberto, S. P. 421

On stability of linear mechanical systems. (Russia summary)

Analytic methods in the theory of non-linear vibration (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. I* 1961), *pp.* 189–203. *Izdat. Akad. Nauk Ukrain. SSR Kiev,* 1963.

In this paper linear Hamiltonian systems (1) $\dot{x}=A(t,\varepsilon)x$ are studied when the $2n$-by-$2n$ matrix $A=A_0+\varepsilon A_1(t)$ ha period T in t and is symmetric with respect to the bilinea

form $\sum_{\nu=1}^{n}(x_\nu x'_{\nu+n}-x_\nu' x_{\nu+n})$. Moreover, it is assumed that A_0 belongs to the set H_Λ of matrices

$$\begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix}, \qquad \Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$$

with $\lambda_\nu > 0$ $(\nu=1, \cdots, n)$. It is shown (Theorem 13) that the normalizing transformation $x = T(t, \varepsilon)y$ which reduces (1) to constant coefficients can be chosen as a symplectic matrix which depends analytically on ε (if $|\varepsilon|$ is sufficiently small). This includes the case of multiple roots λ_ν, in which case one would expect branch points in general. The above system is precisely in the parametrically stable case for $\varepsilon = 0$. *J. Moser* (New Rochelle, N.Y.)

Reĭzin', L. È. [Reiziņš, L. È.] 4219

A homeomorphism of systems of differential equations in neighbourhoods of closed trajectories. (Russian)

Mat. Sb. (*N.S.*) **63** (**105**) (1964), 392–408.

Consider two autonomous vector fields of class C^1 in regions of real n-space, each with a periodic orbit. If the two periodic orbits have the same number of multipliers of modulus less than 1 among which are the same number of negative multipliers, if they have the same number of multipliers of modulus greater than 1 among which are the same number of negative multipliers, and if they have no multipliers of modulus equal to 1, then it is proved that there exists an orbit-preserving homeomorphism between neighborhoods of the two periodic orbits. Here a multiplier is an eigenvalue of the linear part of a cross-sectional map. *W. H. Gottschalk* (Middletown, Conn.)

Błaż, J.; Zima, K. 4220

Über eine Differentialungleichung mit Verzögerung.

Ann. Polon. Math. **14** (1963/64), 311–319.

For the system of differential equations

$$y_i'(t) = \int_0^\infty F_i(t, y_1(t-s), \cdots, y_n(t-s))\, dr_i(t, s) + f_i(t) \qquad (t \in \langle t_0, \alpha);\ i = 1, \cdots, n)$$

and the initial conditions $y_i(t) = \varphi_i(t)$ $(t \leqq t_0;\ i = 1, \cdots, n)$ under some assumptions (too complicated to be quoted here) on the functions F_i, r_i, f_i and φ_i, the authors prove the existence of a maximal solution $g_1(t), \cdots, g_n(t)$ and the theorem: If the continuous functions $z_1(t), \cdots, z_n(t)$ satisfy the inequalities $z_i(t) \leqq \varphi_i(t)$ $(t \leqq t_0;\ i = 1, \cdots, n)$ and

$$D_- z_i(t) \leqq \int_0^\infty F_i(t, z_1(t-s), \cdots, z_n(t-s))\, dr_i(t, s) + f_i(t) \qquad (t \in \langle t_0, \alpha);\ i = 1, \cdots, n),$$

then $z_i(t) \leqq g_i(t)$ $(i = 1, \cdots, n)$ for $t \in \langle t_0, \alpha)$. Thus they obtain a generalization of the classical theorems of T. Ważewski [Ann. Soc. Polon. Math. **23** (1950), 112–166; MR **12**, 705].

An analogous generalization of the reviewer's theorem on integral inequalities [Ann. Polon. Math. **3** (1957), 200–209; MR **19**, 271] and its application to a theorem of G. M. Ždanov [Uspehi Mat. Nauk **16** (1961), no. 1 (97), 143–148] are given. *Z. Opial* (Kraków)

PARTIAL DIFFERENTIAL EQUATIONS

See also 4066, 4135, 4308, 4315, 4359, 4388, 4389, 4390, 4392, 4394, 4395, 4396, 4397, 4398, 4400, 4402, 4704, 4717, 4906.

Hörmander, Lars 4221

★Linear partial differential operators.

Die Grundlehren der mathematischen Wissenschaften, Bd. 116.

Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963. vii + 287 *pp.* $10.50.

This book contains some of the recent developments in the theory of linear partial differential equations, in particular, those topics close to the interests of the author. There are three parts. The first deals with background material in functional analysis and develops the spaces in which the equations are embedded. The second deals with constant coefficient operators considering existence of solutions and interior regularity. The last and largest part discusses operators with variable coefficients.

The first chapter gives the elements of distribution theory. This is done for the convenience of the reader unfamiliar with the theory. Much useful information is developed in a concise, elementary manner. The second chapter defines the function spaces employed throughout the remainder of the book. These all stem from the spaces $\mathscr{B}_{p,k}$ of distributions $u(x)$ in R_n such that $k(\xi)\hat{u}(\xi)$ is in L_p, where $k(\xi)$ is a weight function and $\hat{u}(\xi)$ denotes the Fourier transform of $u(x)$. Concerning the weight function it is assumed that

$$k(\xi+\eta) \leqq (1+C|\xi|)^N k(\eta)$$

for some C, N and all $\xi, \eta \in R_n$. This is done to assure that $\mathscr{B}_{p,k}$ is a module over C_0^∞, the set of test functions in R_n. The most familiar weight function is $k_s(\xi) = (1+|\xi|^2)^{s/2}$. The only values of p which are actually needed in the book are $p = 1, 2, \infty$. The spaces $\mathscr{B}_{2,k_s}$ are denoted by $\mathscr{H}_{(s)}$. {Remark: The distinction between the spaces $\mathscr{B}_{p,k_s}$ and the spaces $H^{s,p}$ of Bessel potentials should be noted. The latter consist of those distributions $u(x)$ in R_n for which the inverse Fourier transform of $k_s(\xi)\hat{u}(\xi)$ is in L_p. They are equivalent only when $p = 2$.} It is commendable that the author develops most of his results using only the spaces $\mathscr{B}_{p,k}$.

Slight variations of the spaces $\mathscr{H}_{(s)}$ prove to be very useful. The spaces $\mathscr{H}_{(m,s)}$ having weight functions

$$k_{m,s} = (1+|\xi|^2)^{m/2}(1+|\xi'|^2)^{s/2}$$

are considered, where $\xi' = (\xi_1, \cdots, \xi_{n-1})$. These latter spaces were introduced by Lions and Magenes [Ann. Scuola Norm. Sup. Pisa (3) **14** (1960), 269–308; MR **24** #A3409] and Peetre [Comm. Pure Appl. Math. **14** (1961), 711–731].

Chapter 3 is concerned with existence and approximation of solutions of differential equations with constant coefficients. It is first proved that every constant coefficient operator $P(D)$ has a fundamental solution E such that for every $\varepsilon > 0$ $\tilde{P}(\xi)[E/\cosh \varepsilon|x|]^\wedge$ is bounded by a constant depending only on ε, n and the degree of P, where $\tilde{P}(\xi)^2 = \sum_\alpha |P^{(\alpha)}(\xi)|^2$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index of non-negative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$P^{(\alpha)}(\xi) = \partial^{|\alpha|}P(\xi)/\partial\xi_1^{\alpha_1}\cdots\partial\xi_n^{\alpha_n}.$$

This result is due to Ehrenpreis [Amer. J. Math. **76** (1954), 883–903; MR **16**, 834] and Malgrange [Ann. Inst. Fourier (Grenoble) **6** (1955/56), 271–355; MR **19**, 280]. It follows from this that if f has compact support and is in $\mathscr{B}_{p,k}$, then $P(D)u=f$ has a solution in $\mathscr{B}^{\text{loc}}_{p,k\tilde{P}}$. Moreover, $Q(D)u \in \mathscr{B}^{\text{loc}}_{p,k}$ for every Q weaker than P, i.e., such that $\tilde{Q}(\xi)/\tilde{P}(\xi) \leq$ const for real ξ.

This leads to a discussion on the comparison of differential operators. Q weaker than P is denoted by $Q \prec P$. The symbol $Q \prec\prec P$ (read P dominates Q) signifies that $\sup_\xi \tilde{Q}(\xi, t)/\tilde{P}(\xi, t) \to 0$ as $t \to \infty$, where

$$\tilde{P}(\xi, t)^2 = \sum_\alpha |P^{(\alpha)}(\xi)|^2 t^{2|\alpha|},$$

$\xi \in R_n$, $t \in R_1$. It results that $P \prec P + aQ \prec P$ for all complex a if and only if $Q \prec\prec P$. Theorems of Malgrange [loc. cit.] on approximation of solutions of $P(D)u=0$ by means of exponentials or by solutions defined in a larger set (of the Runge type) are then proved.

The next few sections are devoted to the question when does $P(D)u=f$ have a solution u belonging to the space X for all $f \in X$, where $X = C^\infty(\Omega)$, $\mathscr{D}_F'(\Omega) = \bigcup_{p,k} \mathscr{B}^{\text{loc}}_{p,k}$, and $\mathscr{D}'(\Omega)$, respectively, and Ω is an open set in R_n. For the first two cases, a necessary and sufficient condition is that Ω be P-convex, i.e., such that for every compact set $K \subset \Omega$ there exists another compact set $K' \subset \Omega$ such that $P(-D)\phi = 0$ outside K implies $\phi = 0$ outside K' for all $\phi \in C_0^\infty(\Omega)$. For the third it is necessary and sufficient that Ω be strongly P-convex, a condition slightly more complicated to state. Every open convex set is both P-convex and strongly P-convex. Every open set is P-convex when P is elliptic. These results are due to Malgrange [loc. cit.] and the author [Ann. of Math. (2) **76** (1962), 148–170; MR **25** #5379]. The geometry of P-convex sets is then discussed with special attention given to operators with real principal parts. The last section of the chapter deals with determined systems of equations with application of the methods of Fuglede [Acta Math. **105** (1961), 177–195; MR **25** #4232].

Chapter 4 considers interior regularity of solutions. Hypo-elliptic operators are defined and their properties discussed. Examples are given. A treatment of partially hypo-elliptic operators is given following Gårding and Malgrange [Math. Scand. **9** (1961), 5–21; MR **23** #A3367]. The chapter concludes with estimates of the growth of derivatives of solutions of hypo-elliptic equations with the order of differentiation; these latter results are due to the author [Acta Math. **94** (1955), 161–248; MR **17**, 853; Ark. Mat. **3** (1958), 527–535; MR **21** #3673].

Chapter 5 deals with the Cauchy problem. A generalization of the Cauchy-Kovalevsky and Darboux-Goursat-Beudon theorems is proved. When the equation has constant coefficients, non-uniqueness is proved for the characteristic Cauchy problem. Holmgren's uniqueness theorem is included. Next is proved Gårding's theorem that for constant coefficient operators hyperbolicity is necessary and sufficient for the non-characteristic Cauchy problem to have a solution for all C^∞ data. Algebraic properties of hyperbolic polynomials are discussed. Also a global uniqueness theorem of F. John is proved which shows that a non-characteristic Cauchy problem cannot be solved for data with compact support unless some factor has a hyperbolic principal part. A theorem of Brodda [Math. Scand. **9** (1961), 55–68; MR **28** #322] giving information on the support of the solution of a hyperbolic Cauchy problem is proved. The last section deals with hypo-elliptic operators for which the characteristic Cauchy problem always can be solved in a half-space.

In the next chapter the non-existence of solutions is discussed. The ideas are mainly due to the author [Math. Ann. **140** (1960), 124–146; MR **24** #A434; ibid. **140** (1960), 169–173; MR **26** #5279]. Chapter 7 considers variable coefficient operators of constant strength, i.e. those for which $\tilde{P}(x, \xi)/\tilde{P}(y, \xi)$ is bounded in ξ for each x, y. Such operators can be dealt with locally as perturbations of constant coefficient operators. Existence of fundamental solutions, regularity for hypo-elliptic operators and analyticity of solutions of elliptic equations are studied.

Chapter 8 gives generalizations of Calderón's results on unique continuation. These introduce the concepts of principally normal operators and pseudo-convex surfaces. The method employs energy integrals. {Remark: Similar results have recently been obtained by Calderón [Fluid Dynamics and Applied Mathematics (Proc. Sympos., Univ. Maryland, 1961), pp. 147–195, Gordon and Breach, New York, 1962; MR **27** #6010].} Examples of non-uniqueness due to P. Cohen and A. Pliś are also given. Chapter 9 extends the existence theory for the Cauchy problem to operators with variable coefficients which are strictly hyperbolic at each point. The results are due to Petrovskiĭ [Mat. Sb. (N.S.) **2 (44)** (1937), 815–870]. Estimates are proved by a method of Leray [*Hyperbolic differential equations*, Inst. Advanced Study, Princeton, N.J., 1953; MR **16**, 139].

The final chapter deals with elliptic boundary-value problems. The method employed is to construct a local solution (called a "fundamental solution" by the author) in the neighborhood of each boundary point. This is then employed to give regularity theorems. {Remark: The nature of the formal adjoint boundary-value problem is not discussed.} The index of a boundary-value problem is defined and a few perturbation theorems proved. Examples are given. In the last section systems are discussed briefly.

An appendix gives some algebraic lemmas concerning polynomials which are used in the text.

The book will be invaluable to many. The researcher in the field will find it a ready reference. For others it will serve as a readable account of current research. It certainly does not cover all major trends in partial differential equations, but comes as close as any book can.

On the other hand, it can hardly be recommended for the novice. The introduction to each topic is good, but little is done in the way of motivation. The exposition is essentially of the form: definition, lemma, theorem. References for the methods employed in the book are very good. However, it is difficult to trace the origin of a theorem.

Mistakes are rare and consist usually of misplaced letters. The printing is somewhat cramped.

In content and exposition the author has done an excellent job. The book cannot be praised too highly.

M. Schechter (New York)

Palczewski, B.; Pawelski, W. 4222

Some remarks on the uniqueness of solutions of the Darboux problem with conditions of the Krasnosielski-Krein type.

Ann. Polon. Math. **14** (1963/64), 97–100.

Let $f(x, y, u)$ be bounded and continuous for $0 \leq x \leq a$, $0 \leq y \leq b$, $-\infty < u < \infty$ $(a, b > 0)$ and suppose $|f(x, y, u) -$

$f(x, y, \bar{u})| \leqq k|u-\bar{u}|/xy$, $|f(x, y, u)-f(x, y, \bar{u})| \leqq C|u-\bar{u}|^\alpha$, $0<\alpha<1$, $k(1-\alpha)^2<1$. Then there exists at most one solution of the Darboux problem for the partial differential equation $\partial^2 u/\partial x \partial y = f(x, y, u)$. This result is analogous to a theorem of Krasnosel'skiĭ and Kreĭn [Uspehi Mat. Nauk **11** (1956), no. 1 (67), 209–213; MR **18**, 38] for ordinary differential equations.

F. Brauer (Madison, Wis.)

Palczewski, B. 4223

On the uniqueness of solutions and the convergence of successive approximations in the Darboux problem under the conditions of the Krasnosielski and Krein type.

Ann. Polon. Math. **14** (1963/64), 183–190.

The uniqueness theorem of the paper reviewed above [#4222] is generalized slightly, and it is shown that the hypotheses of this uniqueness theorem suffice to assure the convergence of successive approximations.

F. Brauer (Madison, Wis.)

Kan, V. L.; Kel'zon, A. S. 4224

Exact solutions of the partial navigation equations. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1962**, *no.* 1 (26), 50–56.

Finn, Robert 4225

On partial differential equations (whose solutions admit no isolated singularities).

Scripta Math. **26**, 107–115 (1963).

Suppose the operator

$$L[u] = (A(u_x, u_y))_x + (B(u_x, u_y))_y$$

is elliptic in a region Ω of the uv-plane. The author proves that if u is a solution of the equation $L[u]=H$, H constant, in a deleted neighborhood of the origin and is such that $(u_x, u_y) \in \Omega$, then u has a removable singularity at the origin. Examples are given by the equation of minimal surfaces, surfaces of constant mean curvature and the gasdynamic equation. Uniformization and quasi-conformal mappings are used to show that u_x and u_y are Hölder-continuous at the singular point. This allows the reduction of the problem to the case of a linear equation.

The motion of removability is extended to equations for which uniformization does not apply. Suppose

$$L[u] = \sum_{i=1}^{n} [A_i(u_{x_1}, \cdots, u_{x_n})]_{x_i}$$

has a convex domain of ellipticity, Ω. Let u be a solution of $L[u]=H(u)$ in a deleted neighborhood N with gradient in Ω and such that $\partial A_1/\partial u_{x_1} > 0$, $H'(u) \geqq 0$, A_i bounded for $x \in N$. Then u has a removable singularity.

R. C. MacCamy (Pittsburgh, Pa.)

Gerver, M. L. 4226

On the possible rate of decrease of a solution of an elliptic equation. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 13–16.

L'auteur généralise un théorème de E. M. Landis [Uspehi Mat. Nauk **14** (1959), no. 1 (85), 21–85; MR **22** #3870] de la manière suivante: Dans la demibande $x>0$, $-h<y<h$, $h<1$, on considère une solution faible $u \in W_2^{(1)}$ de l'équation

$$\frac{\partial}{\partial x}\left(A\frac{\partial u}{\partial x}+B\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(B\frac{\partial u}{\partial x}+C\frac{\partial u}{\partial y}\right) = 0,$$

avec A, B, C mesurables, les valeurs absolues $|A|$, $|B|$, $|C|$ bornées par 1, $A\xi^2+2B\xi\eta+C\eta^2>\alpha(\xi^2+\eta^2)$, $\alpha>0$. On suppose $|u|<1$, $u(0, y)>a$, $0<a<2^{-8}$. Si $M(x)=\sup_{x'\geqq x, -h<y<h}|u(x', y)|$, on a pour $x>1$:

$$(*) \qquad M(x) \geqq 2^{-2^{kx/h+\log_2\log_2 a^{-1}}},$$

où k ne dépend que de α. L'inégalité (*) est la meilleure possible.

J. Nečas (Prague)

Kordzadze, R. A. 4227

A general boundary-value problem with shift for a second-order equation of elliptic type. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 739–742.

Let S^+ be a bounded region in $\mathbf{C}$ containing the origin and bounded by a simple closed Lyapunov curve Γ. Let $\alpha(t)$ map Γ on itself homeomorphically, preserving orientation, with $\alpha' \in H$, $\alpha'(t) \neq 0$ on Γ, and for some integer n let $\alpha_n(t)=\alpha[\alpha_{n-1}(t)]=t$ ($\alpha_0(t)\equiv t$, $t \in \Gamma$). Consider the equation $\Delta u + a(x, y)\partial u/\partial x + b(x, y)\partial u/\partial y + c(x, y)u = 0$, where a, b, and c are real analytic functions in some region containing S^+. Given m, it is required to find a smooth solution of this equation satisfying:

$$\sum_{\nu=0}^{n-1}\sum_{j,k=0}^{j+k\leq m}\{a_\nu^{j,k}(t_0)u_{j,k}^+[\alpha_\nu(t_0)] + \int_\Gamma b_\nu^{j,k}(t_0, \tau)u_{j,k}^+(\tau)\,d\sigma\} = f(t_0)$$

($u_{j,k}^+ = (\partial^{j+k}u/\partial x^j \partial y^k)^+$) under certain conditions on the given coefficients and f. The problem is reduced to a singular integral equation $T\mu = f(t_0)$ for T of the form

$$T\mu = \sum_{\nu=0}^{n-1}\left(A_\nu(t_0)\mu[\alpha_\nu(t_0)] + \frac{1}{\pi i}\int_\Gamma\left(\frac{k_\nu(t_0, t)\mu(t)}{t-\alpha_\nu(t_0)}\right)dt\right),$$

using a method of Vekua. A necessary and sufficient condition is given such that $H/R(T)$ is finite-dimensional and $N(T)$ is finite-dimensional. A formula for the index of the problem is determined.

R. Carroll (New Brunswick, N.J.)

Littman, W.; Stampacchia, G.; Weinberger, H. F. 4228

Regular points for elliptic equations with discontinuous coefficients.

Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 43–77.

From the well-known results of De Giorgi and Nash (simplified greatly by Moser) it follows that the solutions of uniformly elliptic equations of the form

$$(*) \qquad Lu \equiv -\sum_{\alpha,\beta}\frac{\partial}{\partial x^\alpha}(a^{\alpha\beta}u_{x^\beta}) = 0,$$

$$\lambda^{-1}\sum \xi_\alpha^2 \leqq \sum a^{\alpha\beta}(x)\xi_\alpha\xi_\beta \leqq \lambda\sum\xi_\alpha^2, \quad \lambda \geqq 1,$$

satisfy Hölder conditions on domains Ω' with $\bar{\Omega}' \subset \Omega$ even when the coefficients $a^{\alpha\beta}$ are merely bounded and measurable; in this latter case the equation is to be understood in the distribution sense:

$$\int_\Omega \sum_{\alpha,\beta} v_{x^\alpha}a^{\alpha\beta}u_{x^\beta}\,dx = 0, \quad v \in C_c^1(G).$$

The maximum principle holds for such equations. In case $h \in H_2^1(\Omega) \cap C^0(\bar{\Omega})$, it is well known that there exists a unique solution u of (*) such that $u-h \in H_{20}^1(\Omega)$ and u is bounded on Ω by the bound of h on $\bar{\Omega}$. Since each h, continuous on $\partial\Omega$, can be approximated uniformly by functions in $C^1(\bar{\Omega})$, it follows that a generalized solution u ($\in H_2^1$ loc) can be associated with each h continuous on $\partial\Omega$. A boundary point P is regular with respect to L if and only if, for each function h, continuous on $\partial\Omega$, $u(Q)\to u(P)$ as $Q\to P$ in Ω, u being the corresponding generalized solution. The authors prove the remarkable theorem that a boundary point P is regular for such an operator L if and only if it is regular for the Laplacian. In the course of proving this the authors carry over much of potential theory. The L-Ω-capacity of compact subsets of Ω is defined along with the "capacitory potential", potential functions of arbitrary finite measures on Ω are shown to exist and belong to $H_{q0}^1(\Omega)$ for each q with $1\leq q<n/(n-1)$, and many of the usual properties of the Green's function are established.

C. B. Morrey, Jr. (Berkeley, Calif.)

Magenes, E. 4229

Sur les problèmes aux limites pour les équations linéaires elliptiques.

Les Équations aux Dérivées Partielles (Paris, 1962), pp. 95–111. Éditions du Centre National de la Recherche Scientifique, Paris, 1963.

On donne un bref exposé des résultats obtenus par J. L. Lions et l'auteur dans une série des travaux [Ann. Inst. Fourier (Grenoble) **11** (1961), 137–178; MR **26** #4047; Ann. Scuola Norm. Sup. Pisa (3) **14** (1960), 269–308; MR **24** #A3409; ibid. (3) **15** (1961), 41–103; MR **26** #4048; ibid. (3) **16** (1962), 1–44; MR **26** #4049].

J. Nečas (Prague)

Lions, J. L.; Magenes, E. 4230

Problèmes aux limites non homogènes. VI.

J. Analyse Math. **11** (1963), 165–188.

In the preceding papers of this series [cf., e.g., part V, Ann. Scuola Norm. Sup. Pisa (3) **16** (1962), 1–44; MR **26** #4049], the authors have developed a rather complete theory of general boundary-value problems for elliptic differential equations of the form $Au=f$ on Ω and $B_ju=g_j$ on $\partial\Omega$, $j=1,\cdots,m$, in which A and the B_j satisfy the usual conditions of Agmon, Douglis, and Nirenberg, but f and the g_j are allowed to be distributions. In these previous papers u was required to be in some interpolated Sobolev type space $W^{r,p}(\Omega)$, where $1<p<\infty$ and $0\leq r\leq 2m$ (r real) the order of A. In the present case the authors extend their previous results to cases where u may be in some space $W^{-r,p}(\Omega)$, where $r>0$. To do this, the authors restrict themselves to normal sets of boundary conditions, in which case there is an adjoint problem of the same type (the coefficients and domain are assumed of class C^∞). Their idea is to dualize certain problems of the type already treated and to work directly in certain quotient spaces. As an example we quote the following one of their theorems: Suppose that r is real and ≥ 0, that $r+1/p$ is not an integer, and that A and B_j satisfy the conditions above. Then the operator (A, B) is an isomorphism from $D_A^{-r,p}(\Omega)/N$ onto the space

$$\{L^p(\Omega)\times\prod_{j=0}^{m-1} W^{-r-m_j-1/p,p}(\partial\Omega); N^*, CN^*\}.$$

Here $D_A^{-r,p}(\Omega)$ is the set of $u\in W^{-r,p}(\Omega)$ with $Au\in L^p(\Omega)$ and the N^* and CN^* are certain null-space factors to be removed from the first two spaces in the brace. The proof of this involves a proof of the theorem that the mapping $u\to Bu$ from $C^\infty(\bar{\Omega})$ into $[C^\infty(\partial\Omega)]^m$ can be extended to be a bounded linear map from $D_A^{-r,p}(\Omega)$ into the product space above (in the brace).

C. B. Morrey, Jr. (Berkeley, Calif.)

Muramatu, Tosinobu 4231

On the uniqueness of the Cauchy problem for elliptic systems of partial differential equations.

Sci. Papers College Gen. Ed. Univ. Tokyo **11** (1961), 13–23.

The author deals with some classes of elliptic systems $\sum_k P_{jk}(x, D)u^k=0$ (for the statements of the further natural conditions on the coefficients and the characteristics, see the paper). By following Hörmander's methods he proves that every solution u of such a system which is identically zero in the intersection of a neighborhood of the origin and the domain in R^ν space defined by $x^1<(x^2)^2+\cdots+(x^\nu)^2$ must be identically zero in a whole neighborhood of the origin. The author proves the inequalities of Trèves in a slightly different, but more natural, way from that given originally by Trèves. Following Hörmander, he derives further inequalities and uses them to obtain uniqueness.

L. Nachbin (Rochester, N.Y.)

Narčaev, A. 4232

The first boundary-value problem for elliptic equations which degenerate at the boundary. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 28–31.

Let $L=L_0+A$ be a differential operator in n variables of fourth order with

$$L_0=\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}\left(A_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}\right)+b\frac{\partial^3}{\partial x_n^3},$$

$$A=\sum_{i/4+s/3<1} a_{(i,s)}(x)\frac{\partial^s}{\partial x_n^s}D^i\quad(s\geq 0,\ i\geq 0),$$

where $x=(x_1,\cdots,x_n)$, $i=i_1+\cdots+i_{n-1}$,

$$D^i=\partial^i/\partial x_1^{i_1}\cdots\partial x_{n-1}^{i_{n-1}},$$

$b=\pm 1$ and the coefficients $A_{ij}(x)=A_{ji}(x)$, $a_{(i,s)}(x)$ are sufficiently smooth in $\bar{Q}$. Further, it is assumed that (1) $\sum_{i,j=1}^{n-1} A_{ij}(x)\xi_i^2\xi_j^2\geq\theta^2>0$ for arbitrary ξ_i real satisfying $\sum_{i=1}^{n-1}\xi_i^2\neq 0$; (2) $C_{i1}^2x_n^{\alpha_i}\leq A_{in}(x)\leq C_{i2}^2x_n^{\alpha_i}$ $(i=1,\cdots,n)$ where the α_i are non-negative real numbers.

The author considers the following two boundary problems for the operator L in a bounded domain $Q\subset R^n$ contained in the half-space $x_n>0$ and with boundary Γ where a part Γ_0 of it lies on the hyperplane $x_n=0$. (The operator L is elliptic in Q but becomes "quasiparabolic" on Γ_0, as in the case of the Tricomi operator.)

Problem I: Find a solution u of the equation $Lu=h$ in Q vanishing together with its first derivatives on Γ, where one has the following two cases: (a) $b=-1$ and α_n is arbitrary, (b) $b=+1$ and $\alpha_n<1$. Problem II: Find a solution u of the equation $Lu=h$ in Q, vanishing together with its first derivatives on $\Gamma_1=\Gamma-\Gamma_0$ and u alone vanishing on Γ_0, in the case $b=+1$, $\alpha_n\geq 1$.

The author proves existence and uniqueness of the

strong solutions for Problem II and uniqueness of the weak solutions for Problem I. These results hold also for operators of higher order which are elliptic in a domain and degenerate on the boundary.
J. Nieto (College Park, Md.)

Wienholtz, Ernst 4233
Zur Regularität schwacher Lösungen für elliptische Systeme partieller Differentialoperatoren.
Math. Z. **83** (1964), 85–118.
A proof of the differentiability of (weak) solutions of systems elliptic in the sense of Douglis and Nirenberg with Hölder-continuous coefficients is given. The main difference from known proofs is, according to the author, the use of a parametrix rather than a complete fundamental solution. *J. Peetre* (Lund)

Ševčenko, V. I. 4234
Construction of local and global homeomorphisms for a class of quaternion equations. (Russian)
Dokl. Akad. Nauk SSSR **153** (1963), 300–302.
Special elliptic systems $\sum_{k=1}^{4} A_k(x)\partial u/\partial x_k = 0$ are considered, for which homeomorphic solution vectors $u(x)$ can be shown to exist. The A_k are 4×4 real matrices of the standard form used to represent the quaternions.
R. Osserman (Stanford, Calif.)

Adler, G. 4235
Majoration du gradient des solutions de l'équation $\Delta u - au_t' = f$. I.
Acta Math. Acad. Sci. Hungar. **15** (1964), 137–152.
Let Ω be a domain in n-dimensional Euclidean space, and let Σ be the frontier of Ω. In the equation of the title, a is a non-negative constant, and f is a given function of the coordinates x_i and of t. Solutions of the equation are considered which satisfy boundary and initial conditions of the usual forms. Thus the general problem includes many classical problems as special cases. The author's immediate objective is to obtain estimates of $|\text{grad } u|$ in terms of the geometrical properties of Σ and of the initial and boundary data. The present paper is the first part of an extensive article. In it, the necessary general preliminaries are developed, and then an upper bound is found for $|\text{grad } u|$, for the case in which u is independent of t, $\Delta u = 0$ in Ω, and $u = u_0(x_1, \cdots, x_n)$ on Σ, $u_0(x_1, \cdots, x_n)$ being a prescribed function. Since an elaborate terminology and symbolism is involved, no concise summary of the results can be given here. *L. A. MacColl* (New York)

Chen, Y. W. 4236
On the solutions of the wave equation in a quadrant of R^4.
Bull. Amer. Math. Soc. **70** (1964), 172–177.
The equation $w_{tt} = w_{xx} + w_{yy} + w_{zz}$ is considered in $Q: |t| \leqq x$. Let C^+ be the characteristic plane $t = x$, $x > 0$. The following uniqueness theorem, surprising in view of the 1-dimensional situation, is given. If $w = 0$ on C^+ and the first derivatives of w tend to 0 like $(x^2+y^2+z^2)^{-1/2-\varepsilon}$ on $t = 0$, then $w \equiv 0$ in Q.
A corresponding well-posed problem is also given, but it is a bit involved. *P. Ungar* (New York)

Gal'pern, S. A.; Kondrašov, V. E. 4237
The Cauchy problem for a differential operator decomposing into wave factors. (Russian)
Dokl. Akad. Nauk SSSR **154** (1964), 757–759.
Viene studiato il problema di Cauchy

$$(*)\qquad \prod_{k=1}^{l}\left(\frac{\partial^2}{\partial t^2} - \frac{1}{a_k^2}\Delta\right)^{r_k} u(x,t) = 0, \qquad \left.\frac{\partial^s u}{\partial t^s}\right|_{t=0} = f_s(x), \quad s = 0, 1, \cdots, 2m-1,$$

dove $a_1 > a_2 > \cdots > a_l > 0$, $\Delta = \sum_{j=1}^{n} \partial^2/\partial x_j^2$, $x = (x_1, \cdots, x_n)$, $m = r_1 + \cdots + r_l$. Viene enunciato un teorema che assicura l'esistenza di soluzioni classiche per il problema (*) se $f_s(x) \in C_{2m-1-s+q}$, $s = 0, \cdots, 2m-1$ dove

$$q = \max_{1\leq k\leq l} r_k + \tfrac{1}{2}n$$

se n è pari e $q = \max_{1\leq k\leq l} r_k + \frac{1}{2}(n-1)$ se n è dispari. Tale teorema viene dimostrato costruendo una formula di rappresentazione delle soluzioni del problema (*) in cui sia $f_s(x) = 0$, $s = 0, \cdots, 2m-2$, mediante medie sferiche di funzioni elementari. La formula di rappresentazione viene anche usata per studiare le lacune e le lacune generalizzate situate sulla base del cono caratteristico con vertice nel punto (x, t). *G. Geymonat* (Pavia)

Kruskal, Martin D.; Zabusky, Norman J. 4238
Stroboscopic-perturbation procedure for treating a class of nonlinear wave equations.
J. Mathematical Phys. **5** (1964), 231–244.
The authors study various aspects of a certain class of nonlinear hyperbolic partial differential equations typical of a wide range of physical phenomena. A perturbation procedure is presented for treating initial-value problems involving these differential equations in which the characteristic variables and functions of these variables are expanded in powers of ε, the formal solution being uniformly valid over time intervals $O(1/\varepsilon)$. In particular, the uniform first-order solution is evaluated for the equation $y_{tt} = (1+\varepsilon y_x)y_{xx}$, subject to standing-wave initial conditions. This solution is found to break down after a certain time, and a detailed study is made in the vicinity of this breakdown region. A Fourier decomposition of the analytic wave form gives good agreement with available numerical computations.
An argument is given which suggests that the time of description can be extended beyond the breakdown time.
G. Power (Nottingham)

Levine, Leo M. 4239
A uniqueness theorem for the reduced wave equation.
Comm. Pure Appl. Math. **17** (1964), 147–176.
This paper contains a generalization of the well-known uniqueness theorem for solutions of the Helmholtz equation in exterior domains due to F. Rellich [Jber. Deutsch. Math.-Verein. **53** (1943), 57–65; MR **8**, 204]. The author defines a class of "regular closed surfaces" which includes certain surfaces having sharp edges and vertices. The definition is complicated and somewhat obscure, and is not reproduced here. The main result of the paper is the following Theorem 1. Let G be the exterior of a regular closed surface B in 3-dimensional Euclidean space. Let $u(x)$ be defined in $\bar{G}$, the closure of G, and satisfy

(a) $u \in C^0(\bar{G})$, (b) u solves $\Delta u + k^2 u = 0$ in G, where $\mathrm{Re}\, k \geq 0$, $\mathrm{Im}\, k \geq 0$ and $k \neq 0$, (c) B has a subdivision into a finite number of surface elements F which are in the Hölder class $C^{2+\lambda}$, such that either $u = 0$ on F or ∇u has boundary values at interior points of F and $\partial u/\partial n + \beta u = 0$ there (where β is a non-negative function of class $C^{1+\lambda}$ on F), (d) u satisfies Sommerfeld's radiation condition. Then $u(x) = 0$ for $x \in \bar{G}$.

The proof of Theorem 1 is based on the following lemma. Let G, B and u satisfy hypotheses (a), (b) and (c) of Theorem 1. Then if $G_\rho = G \cap \{x : |x| \leq \rho\}$,

$$(1) \qquad \int_{G_\rho} \bar{u} \Delta u \, dV + \int_{G_\rho} |\nabla u|^2 \, dV = \int_{\dot{G}_\rho} \bar{u} \frac{\partial u}{\partial n} \, dS,$$

each of the integrals being finite.

After (1) is established the proof of Theorem 1 is completed by means of an argument due to the reviewer [Proc. Amer. Math. Soc. 7 (1956), 271–276; MR 17, 1261]. The author proves (1) without involving any auxiliary edge conditions at edges and vertices. Instead, it is shown that hypotheses (a), (b) and (c) imply an edge condition described by the following Theorem 2. Let G, B and u satisfy hypotheses (a), (b) and (c) of Theorem 1, and let $\mathscr{E}$ be the set of edge points of the surface elements in (c). Then $\nabla u(x) = O(1/d(x, \mathscr{E}))$ when $d(x, \mathscr{E}) \to 0$, uniformly for $x \in G$ (where $d(x, \mathscr{E})$ is the distance from x to $\mathscr{E}$). Theorem 2 is then used to prove (1).

The proof of Theorem 2 makes use of "a priori estimates" of the Schauder type for the derivatives of solutions of second-order elliptic equations.

C. H. Wilcox (Madison, Wis.)

Weinacht, R. J. 4240

Fundamental solutions for a class of singular equations.

Contributions to Differential Equations **3** (1964), 43–55.

Let

$$L_k[u] = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + (k/x_n) \frac{\partial u}{\partial x_n}, \quad n \geq 2,$$

where k is a real constant. The author obtains integral representations for fundamental solutions of the equation

$$(L_k + \lambda^2)^S[u] = 0, \quad S = 1, 2, \cdots.$$

These represent the polyharmonic equation if $\lambda = 0$ and the iterated Helmholtz equation when $\lambda \neq 0$. It is shown that in the case $\lambda = 0$ the fundamental solutions can be expressed in terms of hypergeometric functions. The formulas are obtained by using a theorem which indicates that under certain conditions one can obtain the fundamental solution of $A = A_1 A_2$ by solving the equation $A_2[E] = E_1$, where E_1 is the fundamental solution for A_1.

R. C. MacCamy (Pittsburgh, Pa.)

Mosolov, P. P. 4241

Riesz's theorem and an analogue of the first boundary-value problem for differential equations in an unbounded region. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1963**, *no.* 2, 13–19.

Using Riesz's representation theorem for bounded linear functionals in Hilbert space, the author defines what he means by a generalized solution of the first boundary-value problem in a region Ω for a partial differential equation $Au = f$ and shows that the problem of existence and uniqueness of a generalized solution reduces to the problem of continuity of the functional $(f, v)_2 = \int_\Omega f\bar{v}\, d$ in v with respect to the norm $\|u\|_1 = (Pu, u)$. Here P is positive differential operator formed from the symmetric parts of the real and imaginary parts of A. This result allows him to use some theorems of L. Hörmander and J. L. Lions [Math. Scand. 4 (1956), 259–270; MR 19, 42 to prove a theorem giving necessary and sufficient conditions on f and Ω for existence and uniqueness of generalized solution.

R. C. Gilbert (Fullerton, Cali

Oskolkov, A. P. 42

Hölder continuity of the generalized solutions of a cla of quasi-linear systems. (Russian)

Trudy Mat. Inst. Steklov. **70** (1964), 116–132.

Let $u(x) = (u^1(x), \cdots, u^N(x))$ be a generalized solution of the quasilinear system

$$\frac{\partial}{\partial x_i}(A_i^k(x, u, u_{x_l})) + A^k(x, u, u_{x_l}) = 0,$$

$k = 1, \cdots, N$, $x = (x_1, \cdots, x_n)$. Suppose that

$$\mu_1 |\nabla u|^p \leq A_i^k \cdot u_{x_i}{}^k \leq \mu_2[|\nabla u|^p + 1],$$

$$\sum_{i,k} |A_i^k| \leq \mu_3[|\nabla u|^{p-1} + 1],$$

$$\sum_k |A^k| \leq \mu_4[|u|^\alpha + 1][|\nabla u|^{p-\delta} + 1],$$

where $\mu_1, \cdots, \mu_4$ are positive constants, $1 < p \leq n$, and

$$\frac{n-p}{n} < \delta < p, \quad 0 < \alpha < \frac{n}{n-p}\left(\delta - \frac{n-p}{n}\right).$$

The author proves that if p is sufficiently near n (i particular, if $p = n$), then there is a Dirichlet growt estimate for $\int_{K_r} |\nabla u|^p \, dx$, where K_r is a spherical ball wit center x_0. From that a uniform Hölder estimate is foun for $u(x)$ in interior subdomains by a lemma of Morre [Trans. Amer. Math. Soc. 43 (1938), 126–166]. A prio estimates for the $W_p^{(1)}$ norm of generalized solutions c boundary problems are obtained, under the weake assumption $p > 1$.

W. H. Fleming (Providence, R.I.

Portnov, V. R. 424

Two embedding theorems for the space $L_{p,b}^{(1)}(\Omega \times R_+$ and their applications. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 761–764.

Si consideri in $R_+{}^n = \{x = (x_1, \cdots, x_n) = (x', x_n); x_n > 0\}$ i problema ai limiti

$$(1) \quad -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} b_{ij}(x) \frac{\partial u}{\partial x_j} + \sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f(x),$$

(2) $\lim_{x_n \to 0} u(x', x_n) = 0$ quasi ovunque in R^{n-1}. Tal problema viene studiato con il metodo variazionale in u opportuno spazio con peso (la cui definizione è tropp complicata per essere riportata qui) in modo che l'equazione (1) che è ellittica al finito possa eventualment degenerare all'infinito. Prima di studiare il problema (1) (2) l'autore enuncia un teorema di tracce per lo spazio co peso introdotto che permette di dare senso alla (2).

G. Geymonat (Pavia

Wloka, Joseph 4244

Korrekte Anfangswertaufgaben für inhomogene, partielle, lineare Differentialgleichungssysteme.

Math. Ann. **154** (1964), 103–115.

Continuing an investigation begun in an earlier paper [same Ann. **152** (1963), 351–409; MR **28** #1486], the author investigates inhomogeneous partial differential equations of the form

$$(1)\qquad \frac{\partial}{\partial t}u(x,t)-P\left(i\frac{\partial}{\partial x},t\right)u(x,t)=f(x,t)$$

by the methods of generalized function (distribution) theory. In (1), u, s, and f are finite-dimensional variables, t is a single parameter, and $P(\xi, t)$ is a polynomial in ξ depending continuously on t. By making a Fourier transformation, the equation (1) is replaced by an equation

$$(2)\qquad \frac{\partial}{\partial t}\hat{u}(s,t)-\hat{P}(s,t)\hat{u}(s,t)=\hat{f}(s,t).$$

The well-known near explicit form for the solutions of (2) enables one to give a fairly complete analysis of (1), which turns out to depend on the rate of growth of the solutions of (2), and hence on the location of the eigenvalues of the variable matrix $\hat{P}$.

J. T. Schwartz (New York)

Combet, Edmond 4245

Construction des solutions élémentaires du laplacien d'une variété riemannienne de dimension impaire et de type hyperbolique normal.

C. R. Acad. Sci. Paris **255** (1962), 2891–2893.

Der Verfasser bestimmt auf der analytischen normalen hyperbolischen Mannigfaltigkeit V_m ungerader Dimension die elementaren Lösungen $E_{x'}^{\pm}$ der Laplaceschen Differentialgleichung $\Delta=-g^{\alpha\beta}\nabla\partial\beta$ im Punkt x' und in der Umgebung desselben, mit der Methode der analytischen Fortsetzung. Es wird eine Methode von J. Hadamard verwendet, und die erwähnten elementaren Lösungen ergeben sich als Distributionen. Die Kerne $E^{\pm}(x,x')$ sind gegenüber den Isometrien von V_m invariant, und $E^{+}(x,x')=E^{-}(x,x')$ sowie $G(x,x')=E^{+}(x,x')-E(x,x')$ bedeuten den sogenannten Propagator, welcher von A. Lichnerowicz in die Quantentheorie der relativistischen Felder eingeführt wurde. *A. Rapcsák* (Debrecen)

Kim, Ju. C. 4246

On a method of solving the boundary value problem for the polyharmonic equation. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1959**, *no.* 3 (10), 65–73.

Aruffo, Giulio 4247

Su un caso eccezionale del problema di Goursat per un'equazione iperbolica.

Ricerche Mat. **9** (1960), 106–117.

Bobkov, V. V. 4248

Error estimates of the method of integral relations in solving certain problems for hyperbolic equations. (Russian)

Vesci Akad. Navuk BSSR Ser. Fiz.-Tèhn. Navuk **1964**, *no.* 1, 18–22.

The author considers Picard's problem:

$$u_{xy}=a(x,y)u_x+b(x,y)u_y+c(x,y)u+f(x,y),$$
$$u(x,0)=\varphi(x),\quad -l_1\leqq x\leqq l_2,\quad l_1,l_2\geqq 0,$$
$$u(g(y),y)=\psi(y),\quad 0\leqq y\leqq Y,\quad \varphi(0)=\psi(0),$$

where the function g satisfies the conditions: $g(0)=0$, $\min_{0\leq y\leq Y} g(y)\geqq -l_1$, $\max_{0\leq y\leq Y} g(y)\leqq l_2$, $|g'(y)|\leqq G<\infty$ and the solution is sought in the rectangle: $-l_1\leqq x\leqq l_2$, $0\leqq y\leqq Y$. Using the method of "integral relations" proposed by A. A. Dorodnicyn [Trudy Tret'ego Vsesojuznogo Mat. S″ezda (Moscow, 1956), Tom 3, pp. 447–453, Izdat. Akad. Nauk SSSR, Moscow, 1958; MR **20** #6973c], error estimates are obtained for the approximations to the solution of the given problem. These error estimates supply also a proof of the convergence of such approximations. These results can be generalized to systems of the form:

$$\frac{\partial^2 u_i}{\partial x\partial y}=a_i(x,y)\frac{\partial u_i}{\partial x}+\sum_{j=1}^{m}\left(b_{ij}(x,y)\frac{\partial u_j}{\partial y}+c_{ij}(x,y)u_j\right)+f_i(x,y),\quad i=1,\cdots,m.$$

J. Nieto (College Park, Md.)

Blagoveščenskiĭ, A. S. 4249

The characteristic problem for the ultrahyperbolic equation. (Russian)

Mat. Sb. (*N.S.*) **63** (**105**) (1964), 137–168.

In der vorliegenden Arbeit wird eine Lösung der ultrahyperbolischen Differentialgleichung $\Delta_x u=\Delta_y u$ im Gebiet $|x|<|y|$ gesucht, die der Randbedingung $u|_{|x|=|y|}=\varphi(x,y)$, wo $x=(x_1,\cdots,x_m)$, $y=(y_1,\cdots,y_n)$, $|x|=(\sum_{i=1}^m x_i^2)^{1/2}$, $|y|=(\sum_{i=1}^n y_i^2)^{1/2}$, genügt.

Über die in der Umgebung des Kegels $x^2=y^2$ vorgegebene Funktion $\varphi(x,y)$ wird vorausgesetzt, daß sie beliebig oft differenzierbar sei und alle ihre Ableitungen im Unendlichen von höherer Ordnung verschwinden als eine beliebige Potenz von x^2+y^2. Wie in der Arbeit bemerkt wird, werden derartig einschränkende Voraussetzungen lediglich zur Vereinfachung der Darstellungen benötigt. Die ermittelten Resultate bleiben aber auch dann gültig, wenn die Voraussetzungen über $\varphi(x,y)$ wesentlich abgeschwächt werden.

Die Lösung wird in der Klasse der im Gebiet $|x|<|y|$ unendlich oft differenzierbaren Funktionen gesucht, die jedoch im Koordinatenursprung eine von der Dimension des betrachteten Raumes abhängige Singularität besitzen können.

Es wird gezeigt, daß die formulierte Randwertaufgabe auf Grund des geforderten Verhaltens im Unendlichen stets eine eindeutige Lösung besitzt. Wird auf die Forderungen im Unendlichen verzichtet, so existieren unendlich viele derartige Lösungen. Zur Beweisführung wird zunächst unter Annahme ihrer Existenz eine solche Lösung konstruiert und ihre Eindeutigkeit gezeigt. Durch Nachweis der Lösungseigenschaften für die konstruierte Funktion ergibt sich dann auch ihre Existenz. Wird auf die Voraussetzungen im Unendlichen verzichtet, so geht die Eindeutigkeit der Lösung verloren. Die Lösungen der entsprechenden homogenen Aufgabe laßen sich leicht in Form von Reihen darstellen. *W. Schmidt* (Simferopol)

Èĭdel'man, S. D.; Ivasišen, S. D. 4250

Continuation of the solution of the Cauchy problem for parabolic systems. (Russian)

Dokl. Akad. Nauk SSSR **149** (1963), 1274–1277.

The authors consider the Cauchy problem for nonlinear parabolic equations of arbitrary order. First, local existence theorems of a very precise nature are announced in which the time interval over which existence holds is estimated in terms of the functions appearing in the differential equation and the initial data. Next, these local estimates allow a theorem on the extension of local solutions to solutions on a greater time interval. Essentially, the main result is that, with respect to a rather complicated norm, such an extension is always possible.

Several corollaries of these results are given. Extension theorems with respect to natural norms are given. An application is also given to a global existence theorem for a quasi-linear equation of second order, assuming that the coefficients of the equation satisfy a certain growth condition.

The notations of the paper are too complicated to be listed here. *B. Frank Jones, Jr.* (Houston, Tex.)

Matiĭčuk, M. I. 4251

Fundamental matrices of solutions of parabolic and elliptic systems whose coefficients satisfy an integral Hölder condition. (Russian)

Dokl. Akad. Nauk SSSR **150** (1963), 480–483.

The author constructs fundamental (matrix) solutions for linear parabolic and elliptic systems, under the assumption that the coefficients of the systems satisfy Dini conditions. The development is a straightforward generalization of the case in which the coefficients satisfy Hölder conditions, as given, for example, by Èĭdel'man [Mat. Sb. (N.S.) **53 (95)** (1961), 73–136; MR **24** #A925].

B. Frank Jones, Jr. (Houston, Tex.)

Ventcel', T. D. 4252

Quasi-linear parabolic systems with increasing coefficients. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1963**, *no.* 6, 34–44.

Consider the system

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}\operatorname{grad}\Phi(u), \qquad u = (u_1, \cdots, u_n)$$

in the strip $x_1 \leqq x \leqq x_2$, $t \geqq 0$, together with the initial-boundary data

$$u(x, 0) = u^{(0)}(x), \qquad u|_{x=x_1} = u|_{x=x_2} = 0.$$

Let Φ be convex with $\Phi(u) \geqq 0$, $\Phi(0) = 0$. If u is a solution, then $\int_{x_1}^{x_2} \Phi(u)\,dx$ is a non-increasing function of t. From this energy estimate follow a priori estimates for u and u_x, provided

$$(*) \qquad \max_{|u| \leq M} [|D^2\Phi| + M|D^3\Phi|] = o\left(f\left(\frac{M}{2\sqrt{n}}\right)\right)$$

as $M \to \infty$, where $D^i\Phi$ denotes any ith order partial derivative of Φ and $f(a) = \min_{|u| \geq a} \Phi(u)$. This implies a global existence theorem. If Φ is not convex, then there is a similar result provided on the right side of (*) one puts $o(M)$. *W. H. Fleming* (Providence, R.I.)

Has'minskiĭ, R. Z. 4253

The averaging principle for parabolic and elliptic differential equations and Markov processes with small diffusion. (Russian. English summary)

Teor. Verojatnost. i Primenen. **8** (1963), 3–25.

In this paper the author provides the proof of theorems announced earlier [Dokl. Akad. Nauk SSSR **143** (1962), 1060–1063; MR **26** #5296]. By using probabilistic methods, he derives an estimate for solutions of parabolic equations which is employed to prove a theorem about the continuous dependence of a solution on a parameter. From this, as is standard in the theory of ordinary differential equations, he proves an analogue of the averaging principle of N. N. Bogoljubov [Bogoljubov and Mitropol'skiĭ, *Asymptotic methods in the theory of nonlinear oscillations* (Russian), 2nd ed., Fizmatgiz, Moscow, 1958; MR **20** #6812]. This result is used to verify a conjecture of A. N. Kolmogorov concerning the limiting transition of an invariant measure of a Markov process into an invariant measure of a dynamical system on a torus. In a closely related paper, S. D. Èĭdel'man has also established an averaging principle for partial differential equations [Sibirsk. Mat. Ž. **3** (1962), 302–307; MR **27** #1713].

W. T. Kyner (Los Angeles, Calif.)

Seyferth, Carl 4254

Die Eindeutigkeit von Lösungen der eindimensionalen Diffusionsgleichung mit konzentrationsabhängigem Diffusionskoeffizienten.

Math. Nachr. **24** (1962), 13–32.

Under general initial and boundary conditions, it is shown that if the partial differential equation

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial}{\partial x} D(u) \frac{\partial u(x, t)}{\partial x}$$

has a bounded solution for finite t, the solution is unique. The uniqueness theorem is based on regions for the solution which always involve $0 < t < t^* \leqq \infty$, with other possible boundaries defined by curves

$$x = \alpha(t), \; x = \beta(t),$$

where $\alpha(t) < \beta(t)$. With respect to the x-coordinate, there are four acceptable regions: $-\infty < x < \infty$, $\alpha(t) < x < \infty$, $-\infty < x < \beta(t)$ and $\alpha(t) < x < \beta(t)$.

Associated with the curve $x = \alpha(t)$ are three functions $\phi_\alpha(t)$, $\psi_\alpha(t)$ and $\chi_\alpha(t)$. Similarly, $\phi_\beta(t)$, $\psi_\beta(t)$ and $\chi_\beta(t)$ are associated with the curve $x = \beta(t)$. By constructing a function

$$J\{u[\xi(t), t]\} = -D\{u[\xi(t), t]\}u_x[\xi(t), t] - u[\xi(t), t]\dot{\xi}(t),$$

the boundary condition at $x = \xi(t) = \alpha(t)$ is

$$\phi_\alpha(t)u[\alpha(t), t] + \psi_\alpha(t)J\{u[\alpha(t), t]\} = \chi_\alpha(t)$$

and at $x = \xi(t) = \beta(t)$ is

$$\phi_\beta(t)u[\beta(t), t] - \psi_\beta(t)J\{u[\beta(t), t]\} = \chi_\beta(t).$$

The initial condition is given by $u(x, 0) = f(x)$, where $f(x)$ is piecewise continuous. The discontinuities of $f(x)$ occur at the denumerable set of points $x = \cdots, s_{-2}, s_{-1}, s_0, s_1, s_2, \cdots$ which do not have a limit point but may include $\alpha(0)$ and $\beta(0)$.

The constraints placed on $u(x, t)$, $f(x)$, $\gamma(t)$, $\phi_\gamma(t)$, $\psi_\gamma(t)$

and $\chi_\gamma(t)$, $\gamma=\alpha$ and β, are those to be expected for applications where u represents the concentration in diffusion problems or the temperature in heat conduction problems.
G. W. Evans, II (Menlo Park, Calif.)

Arima, Reiko; Hasegawa, Yōjirō 4255
On global solutions for mixed problem of a semi-linear differential equation.
Proc. Japan Acad. **39** (1963), 721–725.
An equation of the form $u_{tt}=u_{txx}-f(u)u_t+g(u)$ with $-K(u^2+1)\leqq f(u)\leqq K$, $|g(u)|\leqq K(u^2+|u|)$, $\int_0^u g\leqq Ku^2$, has been proposed as a model for a neuron. Here existence and uniqueness are proved when u, u_t are prescribed on $t=0$ and u is given on $x=0$. *P. Ungar* (New York)

Yamaguti, Masaya 4256
The asymptotic behaviour of the solution of a semi-linear partial differential equation related to an active pulse transmission line.
Proc. Japan Acad. **39** (1963), 726–730.
Two special cases of the problem in the previous review [#4255] are studied. In one it is shown that $u\to 0$ as $t\to\infty$; in the other this need not hold. *P. Ungar* (New York)

Khoan, Vo-Khac 4257
Q-solutions faibles des équations différentielles opérationnelles linéaires.
C. R. Acad. Sci. Paris **258** (1964), 47–50.
Cette note prolonge la note précédente du même auteur [mêmes C. R. **257** (1963), 3800–3803; MR **28** #1480].

Soient H un espace de Hilbert séparable, V un sous-espace de H dense dans H, $a(u,v)$ une forme sesquilinéaire sur $V\times V$. On définit un opérateur A de H dans H par $a(u,v)=\langle Au|v\rangle$ (produit scalaire dans H). On considère l'équation fonctionnelle $du(t)/dt+Au(t)=f(t)$. On cherche des solutions faibles $u(t)$ appartenant, ainsi que leur dérivée, à l'espace $\tau^{\#}(f;\mathscr{P},V)$ des fermetures des combinaisons linéaires des $\mathscr{A}f(t+h)$, pour tous les opérateurs linéaires fermés $\mathscr{A}$ de H dans V, avec $u(0)=u_0\in H_0$. Le mot "faible" doit être interprété au sens de la topologie dans $\tau^{\#}$: on se ramène à l'équation

$$\mathscr{M}\{\langle u'|\phi\rangle+a(u,\phi)\}=\mathscr{M}\langle f|\phi\rangle,$$

où $\mathscr{M}$ est l'opérateur de moyenne temporelle, et où ϕ appartient à $Q(0,\infty,V)$ et est comparable à tous les éléments de $\tau^{\#}(f;\mathscr{P},V)$. On démontre, à l'aide de la méthode de Galerkine et de la théorie spectrale, que le problème a une solution et une seule. Ce résultat s'applique en particulier aux équations de Navier-Stokes linéarisées.
J. Bass (Paris)

Sobolevskiĭ, P. E. 4258
Application of the method of fractional powers of operators to the study of the Navier-Stokes equations. (Russian)
Dokl. Akad. Nauk SSSR **155** (1964), 50–53.
In previous notes [same Dokl. **128** (1959), 45–48; MR **22** #1763; ibid. **131** (1960), 758–760; MR **25** #3282] the author proved existence and regularity theorems for weak solutions of the mixed problem (in bounded domains) for the Navier-Stokes equations, a weak solution meaning a solution $\mathbf{v}(t)\in H$ of

$$\mathbf{v}(t)=e^{-t\nu A}\mathbf{v}(0)+\int_0^t e^{-(t-s)\nu A}P\mathbf{f}\,ds-\int_0^t e^{-(t-s)\nu A}P\frac{\partial}{\partial x_k}(v_k\cdot\mathbf{v})\,ds.$$

Here H is the closure in $L_2(\Omega)$ of smooth solenoidal vector fields vanishing near the boundary, P is orthogonal projection onto H, and A is the Friedrichs self-adjoint extension of $-P\Delta$. In the present paper it is shown that a change of variables $\mathbf{w}(t,\mu,\gamma)=t^{\mu-\gamma}A^{\mu}\mathbf{v}(t)$, for appropriate numbers μ,γ, reduces the above equation to another with a weaker kind of nonlinearity. Considered as an operator equation in the Banach space $C[0,\tau;L_2(\Omega)]$, the new equation can be solved by successive approximations for small enough τ (and sometimes even for arbitrarily large τ), thus yielding a local existence theorem. Finally, an a priori estimate for $\mathbf{w}$ is established, from which global existence follows.
P. C. Fife (Minneapolis, Minn.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 3976, 4149, 4162.

Paasche, Ivan 4259
Zu den Bernoullizahlen nach Nörlund und Adrian. (French, English and Italian summaries)
Mitt. Verein. Schweiz. Versich.-Math. **63** (1963), 41–43.
The author remarks that the familiar recurrence for the Bernoulli numbers $B_n=(-1-B)^n$ $(n=0,1,2,\cdots)$, $B_0=1$, is equivalent to

$$(*)\qquad (A+B)^n=0\qquad (n=1,2,3,\cdots),$$

where $A_n=(n+1)^{-1}$. By means of (*), B_n can be exhibited as a determinant of order n in the A_k; conversely, the A_n can be exhibited as the determinant in the B_k. He also discusses some matrix identities that involve the Pascal and Stirling matrices of the first and second kinds.
L. Carlitz (Durham, N.C.)

Borok, V. M.; Myškis, A. D. 4260
Solvability of difference equations in the entire space. (Russian)
Dokl. Akad. Nauk SSSR **154** (1964), 1007–1010.
Consider linear difference equations of the forms

$$(1)\qquad Lu=\sum_{\lambda=1}^{\Lambda}a_\lambda u(n_1+k_{\lambda 1},\cdots,n_m+k_{\lambda m})=0,$$

$$(2)\qquad Lu=1 \text{ for } n_1=\cdots=n_m=0,\quad =0 \text{ otherwise},$$

$$(3)\qquad Lu=\Psi(n_1,\cdots,n_m),$$

where $u(n_1,\cdots,n_m)$ is sought and $\Psi(n_1,\cdots,n_m)$ is a specified function. $m\geqq 1$ is an integer-valued parameter, the numbers $k_{\lambda 1},\cdots,k_{\lambda m}$, $1\leqq\lambda\leqq\Lambda$, $\Lambda\geqq 2$, are constants, and the a_λ's are arbitrary nonzero complex constants. The authors discuss and present conditions for the solvability of such systems when $n_1+\cdots+n_m$ is allowed to take on arbitrarily large values. *G. S. Jones* (Baltimore, Md.)

Godunov, S. K.; Rjaben'kiĭ, V. S. 4261
Spectral criteria for the stability of boundary-value problems for non-selfadjoint difference equations. (Russian)
Uspehi Mat. Nauk **18** (1963), *no.* 3 (111), 3–14.
The authors are concerned with the difference equation

$$\sum_k B_k u_{n+k}^{m+1} = \sum_k A_k u_{n+k}^m \quad (n = 0, \pm 1, \cdots; m = 0, 1, \cdots),$$

where A_k and B_k ($k=0, \pm 1, \pm 2, \cdots, \pm k_0$) are scalar ($p\times p$)-matrices; for fixed n, $u_n{}^m$ and $u_n{}^{m+1}$ are p-vectors. This equation can be written in the form $u^{m+1}=Ru^m$, where R is a bounded operator. In applications to the boundary-value problems one gets $u^{m+1}=R_h u^m$. The first chapter contains a stability criterion in spectral terms which coincides with that given by K. I. Babenko and I. M. Gel'fand, whose results were presented by O. V. Lokucievskiĭ at the Conference on Functional Analysis in Moscow in 1956. In the second chapter the spectrum structure of the family of operators R_h is investigated. These results are due to the second of the authors of the present paper. *M. Altman* (Warsaw)

Mira, Christian 4262
Condition de décroissance des solutions d'un système d'équations aux différences non linéaire.
C. R. Acad. Sci. Paris **258** (1964), 410–411.
Author's summary: "Une loi de décroissance des solutions d'un système de deux équations aux différences du premier ordre non linéaire étant fixée, on détermine le domaine des conditions initiales pour que la loi soit vérifiée." *T. Fort* (Columbia, S.C.)

Artiaga, Lucio 4263
Finite reciprocities.
Canad. Math. Bull. **7** (1964), 283–290.
The author examines when

$$g(x) = af(A/x)/x+bf(B/x)/x+cf(C/x)/x$$

is a solution of the functional equation

$$f(x) = ag(A/x)/x+bg(B/x)/x+cg(C/x)/x$$

{it seems to the reviewer that the enumeration of cases here might not be complete}, and when

$$g(x) = \sum_{k=0}^{n-1} a_k \exp(\pi ki/n) f(\exp(2\pi ki/n)/x)/x$$

is a solution of the functional equation

$$f(x) = \sum_{k=0}^{n-1} a_k \exp(\pi ki/n) g(\exp(2\pi ki/n)/x)/x.$$

J. Aczél (Gainesville, Fla.)

Dufresnoy, J.; Pisot, Ch. 4264
Sur la relation fonctionnelle $f(x+1)-f(x)=\varphi(x)$.
Bull. Soc. Math. Belg. **15** (1963), 259–270.
The authors consider the functional equation (*) $f(x+1)-f(x)=\phi(x)$, where $\phi(x)$ is a given function. Let $P_k(x)$ be a polynomial of degree $k+1$, defined by the relation

$$\frac{e^{zx}-1}{e^z-1} = 1+\sum_{k=0}^{\infty} P_k(x)\frac{z^k}{k!}.$$

It is proved that if $\phi(x)$ is k times differentiable and the kth-order derivative is non-increasing for $x\geqq 0$, then the equation (*) possesses the solution

$$f(x) = f(0)+x\{\phi(0)-S(1)\}+\{\phi'(0)-S'(1)\}\frac{P_1(x)}{1!}+\cdots$$
$$+\{\phi^{(k-1)}(0)-S^{(k-1)}(1)\}\frac{P_{k-1}(x)}{(k-1)!}+S(x)$$

where

$$S(x) = \sum_{n=0}^{\infty}\left\{\phi(n)+\frac{x}{1!}\phi'(n)+\cdots+\frac{x^k}{k!}\phi^{(k)}(n)-\phi(n+x)\right\}$$

and this solution is the only one for which $f(x)$ is k times differentiable, its kth-order derivative is non-increasing for $x\geqq 0$, and $f(0)$ is given arbitrarily.

The case when ϕ is an entire function of mean type order one at most, is also considered.
S. M. Shah (Lawrence, Kans.)

Kucharzewski, M.; Kuczma, M. 4265
Determination of geometric objects of the type [2, 2, 1] with a linear homogeneous transformation formula.
Ann. Polon. Math. **14** (1963), 29–48.
The problem at hand is equivalent to finding all non-trivial group homomorphisms $h: GL(2, \mathbf{R})\to GL(2, \mathbf{R})$, under an unusual equivalence relation. This problem is first (without any regularity hypotheses on h) reduced to the classification of solutions of certain functional equations in one variable. Then only measurable solutions are considered, which are classified, and these are all analytic in their proper domains. Let I be the identity, $\delta(A)=\det A$, $|\delta|(A)=|\delta(A)|$, $\sigma(A)=\operatorname{sgn}\delta(A)$, $\lambda(A)=\log|\delta(A)|$ and let $e_c(A)$ be the rotation in $\mathbf{R}^2$ over an angle $c\lambda(A)$. Then h is equivalent to, respectively, I, $I\otimes|\delta|^p$, $I\otimes\sigma\otimes|\delta|^p$, $I\otimes\sigma$, $|\delta|\oplus|\delta|$, $\delta\oplus\delta$, $\sigma\oplus\sigma$, $I\otimes e_c$, and $I\otimes\sigma\otimes e_c$. The last two representations are not given in the classification by Penzov [Dokl. Akad. Nauk SSSR **80** (1951), 537–540; MR **13**, 788] who used Lie group methods.

The definition of equivalence is (unfortunately) formulated so that, e.g., $\delta\oplus|\delta|$ is equivalent to $\delta\oplus\delta$ (replace $\omega_1\oplus\omega_2$ by $\omega'\oplus\omega_2$ with $\omega'=|\omega_1\omega_2|^{1/2}\operatorname{sgn}\omega_1\omega_2$). Also, the terminology does not allow, e.g., a density to be "accidentally" zero: it is then called the (zero) scalar. The authors' broad claim that any use of Lie group methods presupposes analyticity everywhere is unjustified. *A. Nijenhuis* (Amsterdam)

Kucharzewski, M.; Kuczma, M. 4266
On a system of functional equations occurring in the theory of geometric objects.
Ann. Polon. Math. **14** (1963), 59–67.
The classification of homomorphisms $h: GL(2, \mathbf{R})\to A(2, \mathbf{R})$ (the semi-group of affine maps $\mathbf{R}^2\to\mathbf{R}^2$) is reduced to the classification of solutions of the functional equations $\phi(\xi\eta)=\phi(\xi)\phi(\eta)$, $\alpha(\xi\eta)=\alpha(\xi)+\alpha(\eta)$ and the system $\kappa(\xi\eta)=\kappa(\xi)\kappa(\eta)-\sigma(\xi)\sigma(\eta)$, $\sigma(\xi\eta)=\kappa(\xi)\sigma(\eta)+\sigma(\xi)\kappa(\eta)$.
A. Nijenhuis (Amsterdam)

Persidskaja, L. 4267
Construction of a bounded solution of a certain functional equation. (Russian. Kazak summary)
Vestnik Akad. Nauk Kazah. SSR **1964**, *no.* 1 (226), 77–80.

On étudie l'équation

(1) $D^n u + q_1 D^{n-1}u + q_2 D^{n-2}u + \cdots + q_n u = \varphi(t, x, u)$,

où (a) t est une variable réelle, x et $u(t, x)$ sont des éléments de certains espaces de Banach M et N, $f(t, x, u)$ et $\varphi(t, x, u)$ sont des éléments des mêmes espaces M et N respectivement, satisfaisant encore à d'autres conditions restrictives, (b) l'opérateur Du est défini par

$$Du = \lim_{\Delta t \to 0} \frac{u(t+\Delta t, x+\Delta t f(t, x, u)) - u(t, x)}{\Delta t}.$$

On construit la solution de (1), appartenant à une famille de fonctions définie par des inégalités pour u et $D^k u$. *A. Haimovici* (Iaşi)

Rufener, Ernst 4268

Zur Charakterisierung von Makehams Gesetz durch eine Funktionalgleichung. (French, English and Italian summaries)

Mitt. Verein Schweiz. Versich.-Math. **63** (1963), 27–32.

The author finds the continuous monotonic solutions M of the functional equation $Q(x+t, y+t) = Q(x, y) + t$, where $Q(x, y) = M^{-1}[\alpha M(x) + \beta M(y)]$, $\alpha + \beta = 1$. He was apparently unaware that the same equation (in different notation) is solved by the same method in Aczél's book [*Vorlesungen über Funktionalgleichungen und ihre Anwendungen*, Birkhäuser, Basel, 1961; MR **23** #A1959]; the equation appears on p. 122. *R. P. Boas, Jr.* (Evanston, Ill.)

Bellman, Richard 4269

Stochastic transformations and functional equations.

Proc. Sympos. Appl. Math., Vol. XVI, pp. 171–177. *Amer. Math. Soc., Providence, R.I.*, 1964.

The equations of the title are of the form $x_{n+1} = g(x_n, r_n)$ where the r_i are independent. The author reviews three examples: the first concerns functional equations generalizing those of Abel-Schröder; the second involves a use of quasi-linearization, the third arises in connection with the product of random matrices. *J. Kiefer* (Ithaca, N.Y.)

SEQUENCES, SERIES, SUMMABILITY

See also 4153, 4300, 4305, 4310.

Petersen, G. M.; Thompson, Anne C. 4270

On a theorem of Pólya.

J. London Math. Soc. **39** (1964), 31–34.

Pólya [Comment. Math. Helv. **11** (1939), 234–252] established the following. Consider the infinite system of linear equations (1) $\sum_j a_{ij} u_j = b_i$ $(i = 1, 2, \cdots)$, where $\{b_i\}$ is an arbitrary sequence of real numbers. If the real matrix $A = (a_{ij})$ satisfies (i) the first row of A contains infinitely many non-zero elements, and (ii) $\liminf_{j\to\infty} [|a_{1j}| + |a_{2j}| + \cdots + |a_{i-1,j}|]/|a_{ij}| = 0$ $(i = 1, 2, \cdots)$, then there exists a sequence $\{u_j\}$ satisfying (1) such that all of the left-hand sides of the linear equations of (1) are absolutely convergent. In the present paper the authors show that, in order for solutions of (1) to exist for all sequences $\{b_i\}$ such that the left-hand sides are absolutely convergent, it is necessary that condition (ii) be satisfied for infinitely many i. They then provide an example of a matrix which satisfies (ii) for infinitely many i but not for all i. *B. E. Rhoades* (Berkeley, Calif.)

Tret'jakov, V. P. 4271

Multiplication of double series in the case when the partial sums of one of the series are unbounded. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 122–124.

L'auteur donne une généralisation du théorème de Mertens relative au produit de deux séries doubles $\sum a_{mn}$ et $\sum b_{mn}$ où $\sum b_{mn}$ converge absolument, mais les sommes partielles A_{mn} de $\sum a_{mn}$ ne sont pas bornées. Il indique des conditions supplémentaires imposées à la série absolument convergente $\sum b_{mn}$ pour en deduire "λ-convergence" du produit C_{mn}, c'est-à-dire pour en concluire $|C_{mn} - AB| \to 0$, lorsque $(m, n)_\lambda \to \infty$, où A et B désignent les sommes de $\sum a_{mn}$ et $\sum b_{mn}$, $C_{mn} = \sum_{i=0}^m \sum_{k=0}^n A_{ik} b_{m-i,n-k}$ et $\lambda^{-1} \leq m/n \leq \lambda$, $\lambda \geq 1$, avec m et n suffisamment grands. Pour les résultats semblables voir V. G. Čelidze [Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze **19** (1953), 135–151; MR **16**, 237] et la note de l'auteur [Izv. Vysš. Učebn. Zaved. Matematika **1961**, no. 5 (24), 78–85; MR **25** #4274]. *M. Tomić* (Belgrade)

Gould, H. W. 4272

A new series transform with applications to Bessel, Legendre, and Tchebycheff polynomials.

Duke Math. J. **31** (1964), 325–334.

In a series of earlier papers [see, in particular, same J. **29** (1962), 393–404; MR **25** #3333] the author has studied the transform

$$F(n, a, b, f) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a+bk}{n} f(k),$$

where $f(k)$ is independent of n. In the present paper he considers the transform

$$(*) \qquad G(n, a, b, f) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a+n+bk}{n} f(k)$$

and shows that this implies

$$\binom{a+n+bn}{n} f(n) = \sum_{k=0}^{n} (-1)^k \frac{a+k+bk+1}{a+n+bn+1} \binom{a+n+bn+1}{n-k} G(k, a, b, f).$$

Applications are made to the Bessel and Legendre polynomials. It is also proved that (*) implies

$$(**) \qquad \sum_{k=0}^{n} \binom{a+k+bk}{k} \binom{c+(b+1)(n-k)}{n-k} f(k) g(n-k) = \sum_{k=0}^{n} (-1)^k A_{n-k}(a+c+2+k+bk, b+1) K_k(a, c, b, f, g),$$

where

$$K_k(a, c, b, f, g) = \sum_{j=0}^{n} G(j, a, b, f) G(n-j, c, b, g)$$

and

$$A_k(a, b) = \frac{a}{a+bk}\binom{a+bk}{k}.$$

Several applications of (**) are given.

L. Carlitz (Durham, N.C.)

Ishiguro, Kazuo; Kuttner, Brian 4273

On the Gibbs phenomenon for quasi-Hausdorff means.

Proc. Japan Acad. **39** (1963), 731–735.

The regular quasi-Hausdorff transformation $\{h_n^*\}$ of the sequence $\{s_n\}$ is defined by the equation

$$h_n^* = \sum_{k=n}^{\infty} \binom{k}{n} s_k \int_0^1 x^{n+1}(1-x)^{k-n}\, d\psi(x),$$

where $\psi(x)$ is a function of bounded variation in [0, 1] and $\psi(1)-\psi(+0)=1$. Extending an earlier result of Ishiguro [Math. Z. **76** (1961), 288–294; MR **24** #A2196] concerning the Gibbs phenomenon for the (special quasi-Hausdorff) circle method, the authors prove the following theorem: For the regular quasi-Hausdorff method, $\lim_{n\to\infty} h_n^*(t_n) = \int_0^1 d\psi(x) \int_0^\tau y^{-1} \sin(y/x)\, dy$, provided $\psi(+0)=\psi(0)$, $nt_n\to\tau$ and $nt_n^2\to 0$, where $h_n^*(t)$ denotes the quasi-Hausdorff transform of the sequence $s_k(t)=\sum_{n=1}^k n^{-1}\sin nt$. Similar results are also given for the methods (s^*, ψ) of the reviewer.

M. S. Ramanujan (Ann Arbor, Mich.)

Ahmad, Z. U. 4274

Absolute summability factors of infinite series by Rieszian means.

Rend. Circ. Mat. Palermo (2) **11** (1962), 91–104.

The following theorem is established. Suppose κ is positive and non-integral. If the following three conditions hold: (1) $\varphi(t)$ and $\psi(t)$ are $(k+1)$st indefinite integrals for $t \geqq 0$; (2) there exists a function $\gamma(t) \geqq 0$ such that for $t \geqq h > 0$

(i) $$\gamma(t) = O(t),$$

(ii) $$t^n\psi^{(n)}(t) = O((\gamma(t)/t)^{\kappa+1-n}), \qquad n = 0, 1, \cdots, k+1,$$

(iii) $$\{\gamma(t)\}^n\varphi^{(n)}(t) = O(\varphi(t)), \qquad n = 1, 2, \cdots, k+1;$$

(3) uniformly in $0<\nu<1$ and $s>0$,

$$s^\kappa\left\{\frac{(1-\nu)t}{\varphi(s+t)-\varphi(s+\nu t)}\right\}^{[\kappa]+1-\kappa} \frac{\{\varphi^1(s+\nu t)\}^{[\kappa]+1}}{\{\varphi(s+t)\}^\kappa}\, \psi(s+\nu t)$$

is of bounded variation in $(0, \infty)$; then $|R, \lambda, \kappa|$-summability of $\sum a_n$ implies $|R, \varphi(\lambda), \kappa|$-summability of $\sum a_n\psi(\lambda_n)$. This is an extension of a theorem of B. N. Prasad and T. Pati [Math. Ann. **140** (1960), 187–197; MR **22** #5843], and at the same time a generalisation of a theorem of U. C. Guha [J. London Math. Soc. **31** (1956), 300–311; MR **19**, 135] and G. D. Dikshit [Thesis, Allahabad Univ., Allahabad, 1960].

K. Endl (Salt Lake City, Utah)

Alam, M. Abrar 4275

A note on the asymptotic behaviour of a power series near its circle of convergence.

Proc. Japan Acad. **39** (1963), 736–740.

Four theorems are proved of which Theorem 3 is typical. Let $f(z)=\sum a_n z^n$, $g(z)=\sum b_n z^n$ converge for $|z|<1$ and diverge for $z=1$; let S_n^α and T_n^β, their nth Cesàro sums of orders α and β, respectively, be non-negative. If $\sum S_n^\alpha$ and $\sum T_n^\beta$ diverge and $S_n^\alpha \sim CT_n^\beta$, then as $z\to 1$, $f(z)=C(1-z)^{\alpha-\beta}g(z)$. All the four results are in the folklore of summability theory (though, perhaps, not explicitly stated) and are easy consequences of a known theorem of Appell [Hardy, *Divergent series*, Theorem 57, Clarendon, Oxford, 1949; MR **11**, 25].

M. S. Ramanujan (Ann Arbor, Mich.)

Lal, Shiva Narain 4276

On the absolute harmonic summability of the factored power series on its circle of convergence.

Indian J. Math. **5** (1963), 55–66.

Let $\{S_n\}$ denote the nth partial sums of a given infinite series $\sum a_n$. The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{\nu=0}^{n} \frac{1}{n-\nu+1} S_\nu,$$

where $P_n=\sum_{\nu=0}^n (\nu+1)^{-1} \sim \log n$, defines the harmonic means of the sequence $\{S_n\}$. The series $\sum a_n$ is said to be absolutely harmonic summable if $\{t_n\}$ is of bounded variation, i.e., $\sum |t_n-t_{n-1}|<\infty$. If for some finite s, $\sum_{\nu=1}^n |S_\nu - s|/\nu = o(\log n)$ as $n\to\infty$, then $\sum a_n$ is said to be strongly summable by the logarithmic means with index 1, or summable $[R, \log n, 1]$, and if $\sum_{\nu=1}^n |S_\nu|/\nu = O(\log n)$, then $\sum a_n$ is said to be strongly bounded by the logarithmic means with index 1, or summable $[R, \log n, 1]$. In this paper, the following theorems are proved. Theorem 1: If $f(z)=\sum c_n z^n$ is a power series of complex class L, such that $\int_0^t |f(e^{i\theta})|\, d\theta = O(|t|)$, as $t\to 0$, and $\{\lambda_n\}$ is a complex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then $\sum c_n \log(n+1)\lambda_n/n$ is absolutely harmonic summable. Theorem 2: If $\sum a_n$ is bounded $[R, \log n, 1]$, then the series $\sum \log(n+1)\lambda_n a_n/n$, where $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, is absolutely harmonic summable.

F. C. Hsiang (Taipei)

Pati, T.; Ramanujan, M. S. 4277

On iteration products preserving absolute convergence.

Boll. Un. Mat. Ital. (3) **17** (1962), 385–393.

If A and B are two means, write $|A| \subset |A\cdot B|$ if any sequence $|A|$-limitable is also $|A\cdot B|$-limitable. The authors discuss special pairs of means. For the $[J, f]$-method introduced by A. Jakimovski [Bull. Res. Council Israel Sect. F **8F** (1960), 135–154; MR **23** #A3391]:

$$t(s_n, x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^m}{m!} f^{(m)}(x)\cdot s_n, \qquad x \geqq x_0 > 0,$$

it is shown (Theorem 1) that if $[J, f]$ is convergence-preserving, if the Hausdorff mean H is regular, and if $\{s_n\}$ is such that

(i) $$\sum_{n=0}^{\infty} \frac{x^n}{n!} |f^{(n)}(x)| \left\{\max_{0\leq m\leq n} |s_m|\right\} < \infty, \qquad x > 0,$$

(ii) $$t(s_n, x) \in \mathrm{BV}[0, \infty],$$

then the H-transform of $\{s_n\}$ is $|J, f|$-limitable. For the $[K, c_n]$-method, again defined by A. Jakimovski [Tech. Rep. No. 2, Contract No. AF 61(052)-187 (1959)],

$$t(s_n, x) = x^{-1}\sum_{n=0}^{\infty} s_n \sum_{m=0}^{n} (-1)^m \binom{n}{m} c_m x^{-m}, \qquad x \geqq x_0 > 0,$$

and the quasi-Hausdorff means H^*, it is shown (Theorem 2)

that if $[K, c_n]$ and H^* are convergence-preserving, and if $\{s_n\}$ is bounded and $|K, c_n|$-limitable, then its H^*-transform is also $|K, c_n|$-limitable. Similar results are established for Wiener methods, Hausdorff function-to-function methods and the $[M, \alpha(u)]$ methods, the last ones due to A. Jakimovski in the 1959 report cited above.

K. Endl (Salt Lake City, Utah)

Syrmus, T. [Sõrmus, T.] 4278

Some Mercer-type theorems for restrained convergence. (Russian. Estonian and German summaries)

Tartu Riikl. Ül. Toimetised No. 129 (1962), 264–273.

Let $\{s_{mn}\}$ be a double sequence of real numbers which converges to s. Suppose that $\lim_{m\to\infty} s_{mn}/(m+1)=0$ for every $n \geqq 1$ and $\lim_{n\to\infty} s_{mn}/(n+1)=0$ for every $m \geqq 1$. Let

$$\left(\binom{m}{k}\binom{n}{l}\Delta^{m-k,n-l}\mu_{kl}\right), \qquad k \leqq m,\ l \leqq n,$$

be the matrix of a two-dimensional regular Hausdorff-summability method satisfying the conditions

$$\limsup_{n\to\infty} n\binom{n}{l}\Delta^{0,n-l}\mu_{0l} < \infty$$

and

$$\limsup_{m\to\infty} m\binom{m}{k}\Delta^{m-k,0}\mu_{k0} < \infty.$$

Then $\{s_{mn}\}$ is restrained summable to s by this Hausdorff method. Furthermore, the following result is proven. Let be

$$t_{mn} = \alpha s_{mn} + (1-\alpha)\sum_{k,l=0}^{m-1,n-1}\binom{m-1}{k}\binom{n-1}{l}\Delta^{m-1-k,n-1-l}\mu_k\nu_l s_{kl},$$

where α is a real number >1 and where $\{\mu_k\}$ and $\{\nu_l\}$ are strictly monotone sequences satisfying the conditions $\mu_0\nu_0=1$, $\lim_{m\to\infty}\Delta^m\mu_0=\lim_{n\to\infty}\Delta^n\nu_0=0$. Suppose that $\lim_{m\to\infty} t_{mn}/(m+1)=0$ and $\lim_{n\to\infty} t_{mn}/(n+1)=0$ and that $\lim_{m,n\to\infty} t_{mn}=s\neq\infty$. Then s_{mn} converges restrainedly to s.

L. Schmetterer (Vienna)

Rhoades, B. E. 4279

Some Hausdorff matrices not of type M.

Proc. Amer. Math. Soc. **15** (1964), 361–365.

An infinite matrix A is said to be of type M if the null sequence $\alpha_n=0$ $(n=1,2,\cdots)$ is the only absolutely convergent sequence orthogonal to the columns of A. The author gives an example of a regular Hausdorff matrix which has no zero elements on the main diagonal but which is not of type M. This answers a question posed by J. D. Hill [Duke Math. J. **3** (1937), 702–714].

D. S. Greenstein (Evanston, Ill.)

Wlodarski, L. 4280

On the regularity of iteration products of matrix transformations.

Proc. London Math. Soc. (3) **14** (1964), 342–352.

Given the matrices A, B and the sequences x, y, z, the transformation $Z=By$, where $y=Ax$, defines the iteration product $B[A]$ of the matrices B and A. The composition product $C=BA$, as distinguished from $B[A]$, is defined by the matrix $C=\{c_{n,v}\}$, where $c_{n,v}=\sum_{k=1}^\infty b_{nk}a_{kv}$. {The notations are the same as those of R. G. Cooke [*Infinite matrices and sequence spaces*, p. 121, Macmillan, London, 1950; MR **12**, 694].}

The following questions are raised and answered by the author. What are the necessary and sufficient conditions on the elements $b_{n,m}$ and $a_{k,v}$ of B and A, respectively, in order that the above transformation: (i) may be defined on the set T_c of all convergent sequences (or on the set T_{c0} of all sequences convergent to zero or on the set T_b of all bounded sequences); (ii) transforms the set T_c (or T_{c0} or T_b) into the set T_b; (iii) transforms the set T_c (or T_{c0} or T_b) into the set T_c.

The results proved are generalizations of known theorems, since by taking B as the unit matrix we obtain a matrix transformation. Ten theorems are proved of which a typical one is as follows. Theorem II.2: The transformation $Z=B(Ax)$ transforms the set T_c into the set T_b if and only if the following conditions are satisfied:

(a) $$\sum_{v=1}^\infty |a_{m,v}| < \infty \qquad \text{for } m=1,2,3,\cdots;$$

(b) $$c_n = \sup_p \sum_{v=1}^\infty \left|\sum_{m=1}^p b_{n,m}a_{m,v}\right| < \infty \quad \text{for } n=1,2,3,\cdots;$$

(c) $$\sup_n \sum_{v=1}^\infty \left|\sum_{m=1}^\infty b_{n,m}a_{m,v}\right| < \infty;$$

(d) $$\sup_n \left|\sum_{m=1}^\infty \left(b_{n,m}\sum_{v=1}^\infty a_{m,v}\right)\right| < \infty.$$

Theorem III.2 is, in a sense, a generalization of a certain theorem of Mazur [Math. Z. **28** (1928), 599–611, p. 602]. In the last section of the paper two examples are given, one of non-regular matrices B and A for which the product $B[A]$ is regular and differs from both the composition products BA and AB and the others of a regular iteration product which is not equivalent to any ordinary matrix transformation. The latter is a modification of an example given by Mazur [W. Orlicz, Colloq. Théorie des Suites (Bruxelles, 1957), pp. 131–147, esp. p. 146, Gauthier-Villars, Paris, 1958; MR **21** #1469].

S. K. Basu (Calcutta)

Grepačevskaja, L. V. 4281

On absolute summability by the methods of Cesàro, Riesz and Zygmund. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 517–520.

Let $T=(a_{nk})$ be a Toeplitz matrix; a series $\sum_{k=1}^\infty u_k$ is called absolutely T-summable (shortly $|T|$-summable) if $\sum_{n=1}^\infty |\sigma_{n+1}-\sigma_n|<\infty$, where $\sigma_n=\sum_{k=0}^\infty a_{nk}\sum_{v=1}^k u_v$. The author gives necessary and sufficient conditions for a sequence (a_n) in order that all orthogonal series (*) $\sum_{k=1}^\infty a_k\varphi_k(x)$ be $|T|$-summable a.e. by certain methods T. Let C, α denote the Cesàro method of order α. Let $E_n=(\sum_{v=1}^n a_v^2)^{1/2}$. Theorem 1: The series (*) is $|C,\alpha|$-summable a.e. in $[0,1]$ if and only if

(a) $$\sum_{n=2}^\infty n^{-1}(\log n)^{-1/2}E_n < \infty$$

in the case $\alpha>\frac12$, (b) $\sum_{n=1}^\infty n^{-1}E_n<\infty$ in the case $\alpha=\frac12$, (c) $\sum_{n=1}^\infty n^{-1/2-\alpha}E_n<\infty$ in the case $-1<\alpha<\frac12$. Further on, this result is sharpened in the case when $A_n\downarrow 0$, and when $A_n\downarrow$ and $a_n\downarrow 0$. An analogous condition for Zygmund summability $|Z,\alpha|$ $(\alpha>0)$ is that $\sum_{n=1}^\infty(\sum_{v=2^n+1}^{2^{n+1}} a_v^2)^{1/2}<$

∞. Let R, λ_n denote the Riesz method determined by the sequence $\lambda_n \uparrow \infty$, let $\lambda(x)$ be a differentiable increasing function such that $\lambda(n)=\lambda_n$. Theorem 2: Let $\lambda(x)$ be either convex or concave and satisfy $\lambda(x+1)/\lambda(x)<q$ $(q>1)$, and let $\lambda'(x)/\lambda(x)\downarrow$. Then (a) all orthogonal series (*) are summable $|R, \lambda_n|$ a.e. in $[0, 1]$ if

$$(**) \qquad \sum_{n=1}^{\infty} \lambda'(n)/(\lambda(n)\sqrt{\log \lambda(n)})E_n < \infty;$$

(b) let $l(x)=\lambda^{-1}(x)$, $k(n)=l(2^n)$; if $A_n=(\sum_{\nu=k(n)+1}^{k(n+1)} a_\nu^2)^{1/2}\downarrow 0$, then the condition (**) is also necessary. Theorem 3: Let $\mu(x)\uparrow\infty$ and let λ_n satisfy the conditions of Theorem 2. If $\lambda(x)\geqq\mu(x)$, then (**) is sufficient for $|R, \mu(n)|$-summability a.e. in $[0, 1]$ of the series (*). Further theorems exhibit counterexamples. No proofs.

A. Alexiewicz (Poznań)

Kwee, B. 4282

The relation between Abel and Riemann summability.

J. London Math. Soc. **39** (1964), 5–11.

By construction the author proves that for any $k>2$, there is a series which is Riemann summable of order k, but not Abel summable.

J. Mayer-Kalkschmidt (Albuquerque, N.M.)

APPROXIMATIONS AND EXPANSIONS

See also 4125, 4137, 4167, 4666.

Walsh, J. L. 4283

Padé approximants as limits of rational functions of best approximation.

J. Math. Mech. **13** (1964), 305–312.

A rational function is said to be of type (n, ν) if it can be written as

$$\frac{s_0+s_1z+\cdots+s_nz^n}{t_0+t_1z+\cdots+t_\nu z^\nu}, \quad \sum|t_k| \neq 0.$$

If $f(z)=a_0+a_1z+a_2z^2+\cdots$, $a_0\neq 0$, then the Padé approximant to f, $P_{n\nu}(z)$, is that rational function of type (n, ν) with the highest order contact at the origin with $f(z)$. Let $R_{n\nu}(\varepsilon, z)$ be the rational function of type (n, ν) which is the best Chebyshev approximation to the analytic function $f(z)$ in the disc δ: $|z|\leqq\varepsilon$.

The author shows: (1) If n and ν are fixed, then under suitable conditions, as $\varepsilon\to 0$, $R_{n\nu}(\varepsilon, z)\to P_{n\nu}(z)$; (2) For ε fixed, the geometric degrees of convergence to $f(z)$ of $P_{n\nu}(z)$ and $R_{n\nu}(\varepsilon, z)$ as $n\to\infty$ are the same on certain natural sets. Thus it appears that Padé approximations may be interpreted as limiting cases of the Chebyshev rational approximations. *P. J. Davis* (Providence, R.I.)

Schoenberg, I. J. 4284

On best approximations of linear operators.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 155–163.

The author states in the introduction that he hopes spline interpolation "will flourish in the glory of its many virtues". He then describes some of its many virtues and the reader can only conclude that it should indeed flourish. First, there is a neat account of known properties of spline interpolation. Consider a linear operator $\mathscr{L}$ defined for $f\in C^{m-1}[I]$, $I=[a, b]$ which is of the form

$$(1) \qquad \mathscr{L}f = \sum_{\nu=0}^{m-1}\int_I f^{(\nu)}(x)\, d\mu_\nu(x).$$

Given $\{x_\nu \mid \nu=0, 1, \cdots, n\}$, we approximate $\mathscr{L}$ by $\sum B_\nu f(x_\nu)$, i.e., $\mathscr{L}f=\sum_0^n B_\nu f(x_\nu)+R(f)$, and require that $R(f)\equiv 0$ if f is a polynomial of degree $m-1$. Then $(m-1)!R(f)=\int_I K(t)f^{(m)}(t)\,dt$. If $m\leqq n$, we say that we have a best approximation if $J=\int_I K(t)^2\,dt=$ minimum.

The main result is the following. Theorem: The best approximation to $\mathscr{L}$ of the form $\sum B_\nu f(x_\nu)$ is obtained by operating with $\mathscr{L}$ on the spline function $S(x)$ which interpolates $f(x)$ at the points x_ν. Thus if $L_\nu(x)$ are the cardinal functions of spline interpolation, we have $B_\nu=\mathscr{L}L_\nu(x)$. Since the spline interpolation function $S(x)$ is the best approximation to $\mathscr{L}f=f(x)$, we have the remarkable fact that the best approximation to any operator of the form (1) is obtained by merely applying the given operator to the best approximation to the interpolation linear operator. This result includes many of the best interpolation and quadrature formulas as defined and studied by Sard [*Linear approximation*, Amer. Math. Soc. Providence, R.I., 1963; MR **28** #1429].

J. R. Rice (Warren, Mich.)

Bellman, Richard 4285

A note on differential approximation and orthogonal polynomials.

Boll. Un. Mat. Ital. (3) **18** (1963), 363–366.

The author considers the problem of approximating a Lebesgue square-integrable function by a sum of exponentials $\sum_{i=1}^N b_ie^{\lambda_it}$ with b_i and λ_i both retained as free parameters. If $f(t)$ were exactly expressible in this way, f would satisfy a linear ordinary differential equation of the form

$$f^{(N)}+c_1f^{(N-1)}+\cdots+c_Nf = 0.$$

Hence, a criterion of fit is provided by

$$\int_{-\infty}^{\infty}(f^{(N)}+c_1f^{(N-1)}+\cdots+c_Nf)^2\,dt,$$

with the c_i determined so as to minimize this integral; the corresponding b_i and λ_i can be calculated. Such a fit is called differential approximation.

By applying the Plancherel-Parseval formula, the author shows that the selection of the c_i reduces to the problem of finding orthogonal polynomials associated with weight functions defined in terms of the Fourier transform of $f(t)$.

W. Langlois (San Jose, Calif.)

Nachbin, Leopoldo 4286

Weighted approximation over topological spaces and the Bernstein problem over finite dimensional vector spaces.

Topology **3** (1964), *suppl.* 1, 125–130.

Let E be a topological space, W a subspace of the space $C(E)$ of continuous functions on E, A a sub-algebra of $C(E)$ containing the unit function 1 and such that $AW\subset W$.

The author considers the problem of weighted approximation: Given an upper semicontinuous, positive weight function $v(x)$, is it possible to find, to a given $f(x)$, a $w\in W$ such that $v(x)|f(x)-w(x)|<\varepsilon$? It is assumed that $f(x$

and all functions $u \in W$ are continuous and are such that the sets $|v(x)f(x)| \geqq \delta$ and $|v(x)u(x)| \geqq \delta$ are compact for every $\delta > 0$. An obvious necessary condition for the approximability of $f(x)$ by functions of W is the following: Let X be a set of points such that $a(X) = \text{const}$ for all $a \in A$. Then it must be possible to find a $w = w_X \in W$ such that $v(x)|f(x) - w(x)| < \varepsilon$ for all $x \in X$. The author calls the approximation problem localisable if this necessary condition is also sufficient.

He gives the following condition for localisability: Let B_N be the class of positive, upper semicontinuous functions $b(t_1, \cdots, t_N)$ of N real variables such that $b(t_1, \cdots, t_N) \cdot P(t_1, \cdots, t_N) \to 0$ uniformly as $(t_1, \cdots, t_N) \to \infty$ for every polynomial P. Suppose that to every $w \in W$ and to every finite set $a_1, a_2, \cdots, a_n$ of elements of A there is an $N \geqq n$, elements $a_{n+1}, \cdots, a_N \in A$ and $ab \in B_N$ such that $v(x)|w(x)| < b(a_1(x), a_2(x), \cdots, a_N(x))$ for all $x \in E$. Then the approximation problem with weight $v(x)$ is localisable. *W. H. J. Fuchs* (Ithaca, N.Y.)

FOURIER ANALYSIS

See also 4090, 4276, 4285, 4386, 4391, 4401, 4402, 4667b, 4762.

Kokilašvili, V. M. 4287

On the best approximation of a function and the Fourier-Lebesgue coefficients. (Russian)

Soobšč. Akad. Nauk Gruzin. SSR **30** (1963), 265–272.

For convenience, the *Trigonometrical series* [Dover, New York, 1955; MR **17**, 361] of A. Zygmund (cf. the author's reference (1) [Russian transl., Moscow, 1939]) is denoted by TS. Chapters 4 and 9 contain the reference material.

The Hardy-Littlewood theorems (TS 9) concerning the orders of magnitude of the coefficients and the partial sums of Fourier series for functions in the L^p classes have been extended to approximations by the series to the functions in the Lipschitz classes by A. A. Konjuškov [Mat. Sb. (N.S.) **44 (86)** (1958), 53–84; MR **20** #2571] and others. Beginning with a generalization of the Lipschitz conditions, the author proves eleven analogously related theorems concerning the order of magnitude of the coefficients and the partial sums, using a more general modulus of continuity.

The form of the generalized fractional integral suggested by Weyl (TS 9) is given in the introduction, together with the conditions on the continuity modulus studied by Orlicz (TS 4). Orlicz's definition of the Banach space L_{Φ^*} corresponding to Φ, one of a conjugate pair of functions in the sense of Young (TS 4), replaces the usual definition of L^p spaces. For example, Theorem 6 gives alternate order-of-magnitude conditions corresponding to the cases $2 \leqq p < \infty$ and $1 < p < 2$ as sufficient if the order of magnitude of the continuity modulus for the generalized λ-derivative of a function is to satisfy certain simple conditions. A derived result in Theorem 7 is best possible when applied to monotonic sets of coefficients of Fourier series. *E. L. Whitney* (Edmonton, Alta.)

Garsia, Adriano M. 4288

Existence of almost everywhere convergent rearrangements for Fourier series of L_2 functions.

Ann. of Math. (2) **79** (1964), 623–629.

The main result is that the Fourier series of every function in $L_2(-\pi, \pi)$ can be so rearranged as to converge almost everywhere. More generally it is shown that, if $f \in L_2(-\pi, \pi)$ is given, then, for almost all points of a certain probability space of permutations of the positive integers, the corresponding rearrangements of the Fourier series of f are convergent almost everywhere. The main tool used in the proof is a combinatorial lemma published by Spitzer [Trans. Amer. Math. Soc. **82** (1956), 323–339; MR **18**, 156]. {A note added in proof states that this lemma is due to H. F. Bohnenblust.} *P. B. Kennedy* (York)

Govil, Narendra K. 4289

A note on the Gibbs phenomenon for the (Λ) means of Fourier series.

Indian J. Math. **5** (1963), 37–40.

The author gives the following theorem. Let $\Lambda = (\lambda_{n,k})$ be a regular K-method of summation and $\lambda_{n,k} - \lambda_{n,k+1}$ be non-increasing, then the Λ means of the partial sums of the Fourier series must fail to exhibit the Gibbs phenomenon. His proof takes only several lines, because the Gibbs phenomenon does not occur in a K-method whose kernel $K_n(t)$ is bounded below for all n, t [B. Kuttner, J. London Math. Soc. **20** (1945), 136–139; MR **7**, 518]. *G. Sunouchi* (Sendai)

Muromskiĭ, A. A. 4290

On certain series in trigonometric functions. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 53–62.

Among the theorems proved we mention the following. Theorem 1: If the series $\sum_{k=1}^{\infty} (C_k/|\sin kx|^{\alpha})$, $\alpha > 1$, $C_k \geqq 0$, converges on a set of positive measure, then the series $\sum_{k=1}^{\infty} C_k^{4/(\alpha+3)+\delta}$ converges for arbitrary $\delta > 0$. Theorem 3: If the series $\sum_{k=1}^{\infty} a_k \cos k\pi x$ converges absolutely at an irrational point x with bounded quotients in the continued fraction expansion, then $\sum_{k=1}^{\infty} (|a_k|/k) < \infty$. Theorem 4: If the series $\sum_{k=1}^{\infty} a_k \cos 2\pi kx$ converges absolutely at an irrational algebraic point, then for every $\varepsilon > 0$ the series $\sum_{k=1}^{\infty} (|a_k|/k^{1+\varepsilon})$ converges. *J. E. Cigler* (Vienna)

Tandori, K. 4291

Beispiel der Fourierreihe einer quadratisch-integrierbaren Funktion, die in gewisser Anordnung ihrer Glieder überall divergiert.

Acta Math. Acad. Sci. Hungar. **15** (1964), 165–173.

By an extremely involved argument the author constructs the Fourier series of an L^2 function which has the property that some rearrangement of the series diverges everywhere. This generalizes an old result of Kolmogoroff [Kolmogoroff and Menchoff, Math. Z. **26** (1927), 432–441] who announced that there is an L^2 function whose Fourier series has a rearrangement which diverges almost everywhere. *R. A. Askey* (Madison, Wis.)

Takahashi, Shigeru 4292

Central limit theorem of trigonometric series.

Tôhoku Math. J. (2) **16** (1964), 1–14.

The author gives the limit distribution of the partial sum of a trigonometric series under certain conditions on a lacunary sequence of partial sums. In fact, his main theorem is the following: Let $S_N(t)$ be the Nth partial

sum of the series $\sum_1^\infty a_k \cos 2\pi k(t+\alpha_k)$ such that $B_N = (\frac{1}{2}\sum_1^N a_n^2)^{1/2} \uparrow \infty$, where a_k and α_k are real numbers. Let $\{n_k\}$ be a sequence of positive integers such that $n_{k+1}/n_k > q > 1$. Writing $R_1(t) = S_{n_1}(t)$, $R_k(t) = S_{n_k}(t) - S_{n_{k-1}}(t)$ $(k>1)$, and supposing that $B_{n_k}^2 - B_{n_{k-1}}^2 = o(B_{n_k}^2)$, $\sup_t |R_k(t)|^2 = O(B_{n_k}^2 - B_{n_{k-1}}^2)$ and

$$\lim_{k\to\infty} \int_0^1 \left| (1/B_{n_k}^2) \sum_{m=1}^{k} \{R_m^2(t) + 2R_m(t)R_{m+1}(t)\} - g(t) \right| dt = 0$$

for some function $g(t)$, we have that

$$\lim_{N\to\infty} \frac{1}{|E|} |\{t;\, t\in E, S_N(t)/B_N \leqq \omega\}| = \frac{1}{(2\pi)^{1/2}|E|} \int_E dt \int_{-\infty}^{\omega/g(t)^{1/2}} \exp(-u^2/2)\, du,$$

where $g(t)$ is seen to be essentially bounded and non-negative, E is any set of positive measure $|E|$, ω is any real number and $\omega/0$ is interpreted as $+\infty$ or $-\infty$ according as $\omega > 0$ or $\omega < 0$.

T. Kawata (Washington, D.C.)

Korovkin, P. P. 4293

An extremal problem and divergent functional series. (Russian. Azerbaijani summary)

Akad. Nauk Azerbaĭdžan. SSR Dokl. **19** (1963), *no.* 8, 3–8.

The paper contains misprints and undefined terms to such a degree that the reviewer could not understand it completely, and the review is therefore incomplete.

Let $\{f_k\}$ be a sequence of real-valued functions defined on a set E and Φ a functional such that for every series (1) $\sum_{k=0}^\infty c_k f_k(x)$ (c_k real numbers) we have $0 \leqq \Phi(\sum_0^\infty c_k f_k(x)) \leqq \infty$. Definitions:

$$(1) \in \bar{\Phi} \Leftrightarrow \Phi\left(\sum_0^\infty c_k f_k(x)\right) < \infty,$$

$(1) \in \Phi \Leftrightarrow$ there exists a sequence $n_1 < n_2 < n_3 < \cdots$ of natural numbers such that

$$\lim_{i\to\infty} \sup_{s>i} \Phi\left(\sum_{n_i+1}^{n_s} c_k f_k(x)\right) = 0.$$

Example: Let $f \in L_p(0, 2\pi)$, $p \geqq 1$ and f 2π-periodic with Fourier series

$$\sum_{i=0}^\infty c_i f_i(x) = \tfrac{1}{2}a_0 + \sum_{k=1}^\infty (a_k \cos kx + b_k \sin kx)$$

and $\Phi = \Phi_p$ with

$$\Phi_p\left(\tfrac{1}{2}a_0 + \sum_{k=1}^\infty (a_k \cos kx + b_k \sin kx)\right) = \int_{-\pi}^{\pi} |f(x)|^p\, dx.$$

Then $\bar{\Phi}_p = \bar{\bar{\Phi}}_p = L_p$ if $2 \geqq p > 1$, as is well known. Two of the eight theorems stated in the paper are the following: (1) If the system of functions $\{f_k\}$ is closed in the space $C[a, b]$ and orthonormal on $[a, b]$ and E is a set of type F_σ with measure zero, then there exists a series $\sum_{k=0}^\infty c_k f_k(x)$ with $\sum_{k=0}^\infty c_k^2 < \infty$ which diverges on E. (2) If there exists a Fourier series of a function $f \in L_p(0, 2\pi)$ $(1 < p \leqq 2)$ which diverges on a set of positive measure, then there also exists an everywhere divergent Fourier series of an $f \in L_p$. Various theorems are concerned with relations between $\bar{\Phi}$ and $\bar{\bar{\Phi}}$.

G. Goes (Lawrence, Kans.)

Ul'janov, P. L. 4294

Solved and unsolved problems in the theory of trigonometric and orthogonal series. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 1 (115), 3–69.

This survey article is divided into eleven paragraphs with the following headings (subtitles are stated in parentheses): Introduction. Fourier coefficients. Convergence (Convergence of orthogonal series. Convergence of trigonometric series. Coefficients of convergent orthogonal series. Bases. Series with respect to systems of the form $\{\varphi(nx)\}$. Series with respect to systems $\{e^{i\lambda_k x}\}$). Convergence to 0 and $+\infty$ (Convergence of trigonometric series to 0. Convergence of trigonometric series to $+\infty$. Convergence of orthogonal series to $+\infty$). Divergent series (Divergent trigonometric series. Divergent orthogonal series. Divergent trigonometric Fourier series). Summability (Summability of orthogonal series. Summability of trigonometric series. Summation and Lebesgue functions. Summability and convergence of subsequences). The problem of the representation of functions (Representation of functions. Correction of functions). Absolute convergence and absolute summability (Absolute convergence of orthogonal series. Absolute convergence of trigonometric series. Absolute convergence of Haar series. Absolute summability). Unconditional convergence (Criteria for unconditional convergence. Systems of unconditional convergence. On complete systems. Unconditional convergence of series with respect to concrete systems). Fourier series and integrals (The Denjoy integral. The A-integral). Main problems (Convergence. Representation of functions. Unconditional convergence. A-Fourier series). The bibliography consists of 180 titles. The paper also contains so far unpublished results and problems.

From the numerous problems stated only a few can be mentioned here. Still unsolved is the problem of Steinhaus: Does $s_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx) \geqq 0$ for all $x \in [0, 2\pi]$ imply that s_n is the partial sum of the Fourier series of a function $f \in L^1$? But Turán has shown that $s_n \geqq 0$ does not imply that s_∞ is the Fourier series of an $f \in L^2$, and M. Weiss has shown $\sup_n \|s_n\|_1 < \infty$ does not imply $f \in L^1$.

Another problem is the following: What conditions are fulfilled by an orthonormal system $\{\varphi_n(x)\}$ on $[a, b]$ if there exists a continuous function f over $[a, b]$ with Fourier coefficients $a_n = \int_a^b f(x)\varphi_n(x)\,dx$ such that $\sum |a_n|^{2-\varepsilon} = \infty$ for $\varepsilon > 0$? Two of the main problems stated are the following: Does there exist an $f \in L^p(0, 2\pi)$ $(1 < p \leqq \infty)$ whose Fourier series diverges everywhere? Does there exist a trigonometric series which converges to $+\infty$ on a set of positive measure? In the section about absolute convergence of trigonometric series it is remarked that the proof given by C.-T. Loo [Sci. Record (N.S.) 1 (1957), 363–368; MR **21** #2153] for the statement "$\sup_{0<h\leqq\delta} \|f(x+h) - f(x)\|_C = O(\log^{-2}(1/\delta))$ $(\delta \to 0+)$ with $f \in V(0, 2\pi)$ does not imply $\sum |a_n(f)| + |b_n(f)| < \infty$" contains an error, and it is an open question if the statement is correct.

G. Goes (Lawrence, Kans.)

Ul'janov, P. L. 4295

On Haar series. (Russian)

Mat. Sb. (*N.S.*) **63 (105)** (1964), 356–391.

This paper contains a detailed study of orthogonal expansions of functions in Haar series. Let $\chi_n^{(k)}(t)$ $(k = n = 0$ and $k = 1, 2, \cdots, 2^n;\ n = 0, 1, 2, \cdots)$ be the

orthonormal Haar system, and let $\chi_1(t)=\chi_0{}^0(t)$, $\chi_m(t)=\chi_n{}^{(k)}(t)$ ($m=2^n+k$, $1\leqq k\leqq 2^n$, $n=0,1,2,\cdots$), $a_m=a_m(f)=\int_0^1 f(t)\chi_m(t)\,dt$. Then $\sum_{m=1}^{\infty} a_m\chi_m(t)$ is the Haar series of the function $f(t)$. Denoting by $\omega_p(\delta,f)$ the pth integral modulus of continuity of $f(t)$ in $(0,1)$, the author estimates coefficients $a_m(f)$ for spaces of pth integrable functions ($1\leqq p\leqq\infty$), continuous functions, and functions of bounded variation. For instance, if $f\in L^p(0,1)$, $1\leqq p<\infty$, then

$$|a_m(f)|\leqq m^{1/p-1/2}\omega_p(1/m,f)\qquad(m>1),$$

$$\left\{\sum_{m=2^n+1}^{2^{n+1}}|a_m(f)|^p\right\}^{1/p}\leqq 8\cdot 2^{n(1/p-1/2)}\omega_p(1/2^n,f)\qquad(n>0).$$

A number of corollaries are given.

Next, the author deals with problems of convergence of Haar series. He shows that there exists a function of bounded variation whose Haar series diverges in a dense set in $(0,1)$. In the next section the author investigates absolute convergence of Haar series, proving that if $f(t)$ is of bounded variation $\overset{1}{\underset{0}{\mathrm{V}}} f$ in $(0,1)$, then

$$\sum_{m=1}^{\infty}|a_m(f)|\leqq\sup_{0\leqq t\leqq 1}|f(t)|+\frac{3}{2-\sqrt{2}}\overset{1}{\underset{0}{\mathrm{V}}} f.$$

Estimations of series $\sum_{m=1}^{\infty}|a_m(f)|^{\alpha}$, $\sum_{m=1}^{\infty}m^{\beta}|a_m(f)|$ and others are also given, and the case of $f(t)$ integrable in $(0,1)$ is considered. Finally, the author investigates unconditional convergence of Haar series, Haar series with coefficients a_m decreasing to 0, and the order of approximation of a function $f(t)$ by partial sums $S_m(t,f)$ of its Haar series. It is shown that if $f(t)$ is pth integrable in $(0,1)$ ($1\leqq p<\infty$), then $\|f(t)-S_m(t,f)\|_p\leqq 24\omega_p(1/m,f)$ ($m\geqq 1$) and that this estimation is the best one in the sense of order. *J. Musielak* (Poznań)

Leĭndler, L. [Leindler, L.] 4296

On unconditional convergence of trigonometric series. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 1 (115), 167–168.

Let $f\in L^2(0,2\pi)$ be a 2π-periodic function with Fourier series $\frac{1}{2}a_0+\sum(a_k\cos kx+b_k\sin kx)$. This series is called unconditionally convergent almost everywhere on $[0,2\pi]$ if it converges almost everywhere on $[0,2\pi]$ for every permutation of its terms. Here the set of points of divergence is dependent on the permutation. The author gives a new sufficient condition for the unconditional convergence almost everywhere of the Fourier series of f, namely,

$$\int_0^{2\pi}\frac{1}{t}\left\{\int_0^{2\pi}[f(x+t)-f(x-t)]^2\,dx\right\}^{1/2}dt<\infty.$$

G. Goes (Lawrence, Kans.)

Leindler, L. 4297

Über Approximation mit Orthogonalreihenmitteln unter strukturellen Bedingungen.

Acta Math. Acad. Sci. Hungar. **15** (1964), 57–62.

Let f be an L^2 function on (a,b), $f(x)\sim\sum a_n\varphi_n(x)$ be the Fourier expansion of f with respect to orthogonal functions $\varphi_n(x)$ which are orthogonal with respect to a positive measure $d\mu$ on (a,b). Then if E_n denotes the L^2 error of approximation of f by its partial sum, and if $\sum_{k=1}^{\infty}k^{2\alpha-1}E_k{}^2<\infty$ ($0<\alpha<1$), then the $(C,1)$ means σ_n of the expansion $f(x)\sim\sum c_n\varphi_n(x)$ satisfy $|\sigma_n(x)-f(x)|=o_x(n^{-\alpha})$ for almost all x in (a,b). If instead of the $(C,1)$ means the delayed means of de la Vallée Poussin are used, the same conclusion holds except that the restriction on α can be weakened to $0<\alpha<\infty$. Related results of this character have previously been obtained by Alexits and Králik [same Acta **11** (1960), 387–399; MR **24** #A2779]. Since de la Vallée Poussin introduced a number of different types of summability, the use of the term delayed means is preferable to the author's use of de la Vallée Poussin means. *R. A. Askey* (Madison, Wis.)

Bhatt, Shri Nivas 4298

An aspect of local property of $|\mathcal{N},p_n|$ summability of a Fourier series.

Indian J. Math. **5** (1963), 87–91.

The author proves the following theorem. If $\{p_n\}$ is a real positive sequence such that $\{p_n-p_{n-1}\}$ is bounded, non-increasing and $\sum P_n{}^{-1}$ is convergent, where $P_n=p_0+p_1+\cdots+p_n$, then the absolute Nörlund summability $|\mathcal{N},p_n|$ of $\frac{1}{2}a_0+\sum(a_n\cos nt+b_n\sin nt)$ depends only on the behaviour of the generating function $f\in L[-\pi,\pi]$ in the immediate neighborhood of the point $t=x$.

G. Goes (Lawrence, Kans.)

Singh, Tarkeshwar 4299

Nörlund summability of a sequence of Fourier coefficients.

Proc. Japan Acad. **39** (1963), 439–443.

Let $f(z)$ be a periodic function with period 2π and integrable in the Lebesgue sense over $(-\pi,\pi)$. Let the Fourier series of $f(x)$ be

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}A_n(x),$$

and let its conjugate series be

$$\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx)=\sum_{n=1}^{\infty}B_n(x).$$

We write $\psi(t)=f(x+t)-f(x-t)-l$.

In this paper the author proves the following. Theorem 1: If $(\mathcal{N},p_n)$ be a regular Nörlund method, defined by a real, non-negative, monotonic non-decreasing sequence of constants $\{p_n\}$ such that: $P_n=p_0+p_1+\cdots+p_n\to\infty$ and $\sum_{k=a}^{n}P_k/k\log k=O(P_n)$ as $n\to\infty$, where a is a fixed positive integer, and if

$$\int_0^t|\psi(u)|\,du=o\left[\frac{t}{\log(1/t)}\right],$$

as $t\to 0$, the sequence $\{nB_n(x)\}$ is summable $(\mathcal{N},p_n)\cdot C_1$ to the value l/π, where $(\mathcal{N},p_n)\cdot C_1$ is a method of summability obtained by superimposing $(\mathcal{N},p_n)$ on the Cesàro means of order one. Theorem 2: If $(\mathcal{N},p_n)$ is a regular Nörlund method defined by a real non-negative and non-increasing sequence such that $P_n\to\infty$ with n, and if

$$\int_0^t|\psi(u)|\,du=o\left[\frac{p(t^{-1})}{P(t^{-1})}\right]$$

as $t\to 0$, then the sequence $\{nB_n(x)\}$ is summable $(\mathcal{N},p_n)\cdot C_1$ to the value l/π, where $p(t^{-1})=p_\tau$, $P(t^{-1})=P_\tau$, and $\tau=[t^{-1}]$.

Theorem 1 includes a result of Varshney [Proc. Amer. Math. Soc. **10** (1959), 790–795; MR **22** #873] which corresponds to $p_n=1/(n+1)$. *R. Mohanty* (Cuttack)

Minakshisundaram, S. 4300

Summability of eigenfunction expansions.

Math. Z. **83** (1964), 321–335.

Given a bounded domain $D \subset E_n$, $n \geqq 2$, bdy $D = B$, such that $\Delta u + \lambda u = 0$ in D, $u(x) = 0$ on B, has eigenvalues $0 < \lambda_i$, $\lambda_i \uparrow \infty$ with a complete orthonormal set $\{\omega_i\}$ of eigenfunctions. Given, furthermore, $f \in L_2$ such that $f \sim \sum a_i\omega_i$, where $a_i = \int_D f\omega_i$. Then the typical means (λ, α) (λ denotes type λ_n) applied to $\sum a_i\omega_i$ is a local property for $\alpha = \frac{1}{2}(n-1)$ [Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Vol. 2, Clarendon, Oxford, 1958; MR **20** #1065].

Considerable preliminary work leads to the theorem: If $f \sim \sum a_i\omega_i$ and $\sum \lambda_i^k a_i^2 < \infty$, $0 \leq k \leq \frac{1}{2}(n-1)$, then $(\lambda, \frac{1}{2}(n-1)-k)$-summability of $\sum a_i\omega_i$ at an interior point of D depends on the behaviour of f in a neighbourhood of that point. The behaviour of f is judged by the convergence of an integral involving f and Bessel functions.

Under the same conditions $|\lambda, \alpha|$-summability is a local property if $\alpha > \frac{1}{2}(n+1)-k$. Three more theorems of this type conclude the paper.

J. Mayer-Kalkschmidt (Albuquerque, N.M.)

Bochner, S. 4301

Interpolation of general bounded and of almost periodic sequences by functions of stratified exponential type.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 164–168.

A holomorphic function $f(z)$ is said to be of stratified (exponential) type τ (at most) if for all z, $|f(x+iy)| \leq C_f e^{\tau|y|}$. The analytic almost periodic (a.a.p.) functions $f(z)$ occurring in the following are defined by $f(iz)$ being analytic almost periodic in the ordinary sense. It is proved that a bounded (two-sided) sequence of complex numbers $\{a_n\}$, $-\infty < n < \infty$, $|a_n| \leqq M$, can be interpolated by a function $A(z)$ of stratified type λ when $\lambda > \pi$. If the sequence is almost periodic (a.p.), $A(z)$ can be chosen a.a.p. of type $\lambda > \pi$. If a bounded sequence can be interpolated by a function $A(z)$ of stratified type π itself, then this $A(z)$ is uniquely determined except for an arbitrary additional term $c \sin \pi z$. If a function $A(z)$ is of stratified type π, and the sequence $a_n = A(n)$ is a.p., then $A(z)$ is a.a.p. If an a.p. sequence a_n with Fourier expansion $\sum_\lambda \gamma(\lambda)e^{i\lambda n}$ (normalized by $-\pi < \lambda \leqq \pi$) can be interpolated by a function of stratified type π, then there exists an a.a.p. function $f(z)$ with the "same" expansion $\sum_\lambda \gamma(\lambda)e^{i\lambda z}$, $-\pi < \lambda \leqq \pi$. Finally, an example is given of a.p. sequences which cannot be interpolated by a function of stratified type π. *E. Følner* (Copenhagen)

Topurija, S. B. 4302

On double lacunary Fourier series. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **33** (1964), 9–14.

Let $n_{k+1}/n_k \geqq A > 1$, and let $S_k(x)$ be the partial sums of the Fourier series of f. Two well-known theorems [Zygmund, *Trigonometric series*, 2nd ed., Vol. II, p. 164, Cambridge Univ. Press, New York, 1959; MR **21** #6498] state that if $f \in L^2$, then $S_{n_k}(x) \rightarrow f(x)$ almost everywhere and that if $f \in L$ and $a_n = b_n = 0$ except for $n = n_k$, then the Fourier series converges to $f(x)$ almost everywhere. These results were extended to double series by Fedulov [Učen. Zap. Kazan. Univ. **115** (1955), no. 14, 87–95; MR **18**, 303; Ukrain. Mat. Ž. **7** (1955), 433–442; MR **17**, 1075]. However, the author states that Fedulov's proofs are incorrect but that to give counterexamples to the theorems would be equivalent to solving the convergence problem for Fourier series of L^2. He gives a different definition of a lacunary double series and proves the analogous theorems with this definition. He calls a double sequence $\{m_i, n_k\}$ lacunary if

$$(1 - m_g/m_i)(1 - n_s/n_k) \geqq 1 - A^{-(i-g)(k-s)},$$

$A > 1$, $0 \leqq g \leqq i$, $0 \leqq s \leqq k$. Then if $f \in L^2$ and $\{m_i, n_k\}$ is lacunary, the rectangular partial sums $S_{m_i,n_k}(x, y) \rightarrow f(x, y)$ almost everywhere; if $f \log^+|f| \in L$ and the Fourier series of f is lacunary, then it converges almost everywhere to $f(x, y)$. The author also states that if $f \in L$ and its Fourier series is lacunary, then $\lim_{(m,n)\lambda} S_{m,n}(x, y) = f(x, y)$ almost everywhere.

R. P. Boas, Jr. (Evanston, Ill.)

Žižiašvili, L. V. 4303

Conjugate functions of two variables and double conjugate trigonometric series. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 521–523.

The author shows in seven theorems (without proofs) that theorems which are known to hold for 2π-periodic functions of one variable may not be true in the case of functions with two or more variables. Theorems: (1) There exists a non-negative 2π-periodic function $f(x, y)$ such that $f(x, y)[\log^+ f(x, y)]^\alpha \in L^1(R)$ ($R = [-\pi, \pi; -\pi, \pi]$) for every $\alpha \in [0, 1)$ and

$$\limsup_{(\varepsilon_m\eta_n)_\lambda \rightarrow 0} \left| \int_{\varepsilon_m}^{\pi} \int_{\eta_n}^{\pi} [f(x+s, y+t) - f(x-s, y+t) - f(x+s, y-t) + f(x-s, y-t)] \cot \tfrac{1}{2}s \cot \tfrac{1}{2}t \, ds dt \right| = +\infty$$

on a set of positive measure for arbitrary $\varepsilon_m \rightarrow 0$, $\eta_n \rightarrow 0$ and $\lambda \geqq 1$. In particular (the author claims), it follows from this that a result of H. Watanabe [Mem. Fac. Sci. Kyusyu Univ. Ser. A **12** (1958), 180–190; MR **22** #1789] is not correct. (2) There exists a non-negative 2π-periodic function $f(x, y) \in L^1(R)$ such that $\bar{f}(x, y) \in L^1(R)$; however, $f(x, y)[\log^+ f(x, y)]^\alpha \notin L^1(R)$ for every $\alpha \in (0, \beta]$, $\beta > 0$ (here $\bar{f}(x, y) = (1/4\pi^2)\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+s, y+t) \cot \frac{1}{2}s \cot \frac{1}{2}t \, ds dt$). (3) There exists a non-negative 2π-periodic function $f(x, y)$ such that $f(x, y) \log^+ f(x, y) \in L^1(R)$, $\bar{f}(x, y) \in L^1(R)$; however, $f(x, y)[\log^+ f(x, y)]^{1+\varepsilon} \notin L^1(R)$ for every $\varepsilon \in (0, \beta]$. (4) There exists an analytic function f of two variables given in the set $|z| < 1$, $|\eta| < 1$ and represented by a double integral of Cauchy type

$$f(z, \eta) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s, t) \frac{(\tau+z)(\sigma+\eta)}{(\tau-z)(\sigma-\eta)} ds dt$$

which, on a set of positive plane measure, does not have a finite limit if $(r, \rho) \rightarrow 1$ where $f(x, y)[\log^+|f(x, y)|]^\alpha \in L^1(R)$ for every $\alpha \in [0, 1)$, $z = \rho e^{ix}$, $\eta = re^{iy}$, $\tau = e^{is}$, $\sigma = e^{it}$. (5) There exists an even bounded 2π-periodic function $f(x, y)$ for which the functions $(1/x)\int_0^x \bar{f}_i(s, y) \, ds$ and $\int_x^\pi \bar{f}_i(s, y)/s \, ds$ ($i = 1, 2$) are not bounded. Here

$$\bar{f}_1(x, y) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+s, y) \cot \frac{s}{2} ds,$$

$$\bar{f}_2(x, y) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x, y+t) \cot \frac{t}{2} dt.$$

Other theorems are concerned with the behaviour of Cesàro means of different types for double trigonometric series and conjugate series. *G. Goes* (Lawrence, Kans.)

Gross, Leonard 4304

Harmonic analysis on Hilbert space.

Mem. Amer. Math. Soc. No. 46 (1963), ii+62 *pp.*

Let H be a real separable Hilbert space and ϕ a positive definite function on H. One of the questions the author considers concerns continuity conditions on ϕ so that Bochner's theorem is valid, that is, conditions in order that there exist a bounded non-negative measure $d\mu$, defined on the σ-ring generated by the open sets of H such that

$$(1) \qquad \phi(y) = \int_H \exp i(s|y)\, d\mu(s).$$

If ϕ is merely continuous in the strong topology of H, then it is easy to give a counter-example to Bochner's theorem. It turns out that ϕ has a representation (1) if and only if it is positive definite and continuous in the weakest topology T for which all Hilbert-Schmidt operators are continuous from the topology T to the strong topology of H. The author points out that this theorem has also been obtained by different methods by V. Sazonov [Teor. Verojatnost. i Primenen. **3** (1958), 201–205; MR **20** #4882].

The author also obtains an analogue of the Levy continuity theorem for characteristic functions. Using the notion of weak distribution it is possible to define a notion of normal distribution on a real Hilbert space, and hence any complex-valued uniformly continuous function f on H (in the topology T) gives rise to a random variable $f^\sim$ with respect to this normal distribution. The author shows that ϕ of the form (1) are uniformly continuous in the topology T, and hence the analogue of the Levy theorem reads as follows: Suppose μ_n is a sequence of probability measures on H with ϕ_n defined by (1). Let ϕ be uniformly continuous on H in the T topology with $\phi(0)=1$. If μ_n converges weakly to a measure whose characteristic function is ϕ, then $\phi_n \to \phi$ on H and $\phi_n^\sim \to \phi^\sim$ in probability. Conversely, if $\phi_n^\sim \to \phi^\sim$ in probability, then μ_n converges weakly to a probability measure μ with characteristic function ϕ.

The last part of this memoir is devoted to obtaining what the author calls inversion formulae. By this he means the possibility of obtaining the value for an integral $\int_H G(s)\, d\mu(s)$ in terms of G and the characteristic function ϕ of μ. Such an inversion theorem is given, together with a number of corollaries. *A. Devinatz* (St. Louis, Mo.)

INTEGRAL TRANSFORMS, OPERATIONAL CALCULUS

See also 4140, 4391.

Abdi, Wazir Hasan 4305

Certain inversion and representation formulae for q-Laplace transforms.

Math. Z. **83** (1964), 238–249.

The q-Laplace transform $(0<q<1)$ of $f(x)$ is defined by

$${}_qL_s f(x) = 1/(1-q) \mathop{S}_0^{s^{-1}} E_q(qsx) f(x)\, d(q,x),$$

where $E_q(x) = \sum_{j=0}^\infty q^j f(q^j x)$ and the "summation integral" is defined by

$$1/(1-q) \mathop{S}_0^{x} f(y)\, d(q,y) = x \sum_{j=1}^\infty q^j f(q^j x).$$

In a previous paper [Proc. Nat. Acad. Sci. India Sect. A **29** (1960), 389–408; MR **26** #2824a] the author derived a number of theorems analogous to those holding for the ordinary Laplace transform. The author now derives some inversion formulae for special cases. The work rests heavily on that of Hahn on Heine series [Math. Nachr. **2** (1949), 340–379; MR **11**, 720; ibid. **3** (1950), 257–294; MR **12**, 711]. *J. L. Griffith* (Kensington)

Widder, David V. 4306

The inversion of a transform related to the Laplace transform and to heat conduction.

J. Austral. Math. Soc. **4** (1964), 1–14.

Inversions of the linear integral transformation $f(t) = \int_0^\infty k(y,t)\phi(y)\, dy$ with kernel

$$k(x,t) = (4\pi t)^{-1/2} \exp[-x^2/(4t)]$$

are presented. An integral representation of solutions of the heat equation is used to establish conditions on ϕ such that

$$\phi(y) = \lim_{t\to 0+} 2 \sum_{n=0}^\infty y^{2n} f^{(n)}(t)/(2n)!.$$

A second inversion theorem is obtained with the aid of properties of the Laplace transformation. It represents ϕ as a limit as $n\to\infty$ of differential forms of order n that involve n, y and $f(t)$. Another inversion formula for the Laplace transformation follows from the first result. Some examples are given, as well as theorems for the corresponding Stieltjes integral transformation and a representation theorem. *R. V. Churchill* (Ann Arbor, Mich.)

Bhise, V. M. 4307

Certain rules and recurrence relations for Meijer-Laplace transform.

Proc. Nat. Acad. Sci. India Sect. A **32** (1962), 389–404.

This paper is a development of a previous paper by the same author [J. Vk. Univ. **3** (1959), no. 3, 57–63].

It concerns an integral transform involving G-functions. The work is essentially formal and many of the steps appear to require very heavy conditions on the functions concerned. The results are too complicated to reproduce or to check. *J. L. Griffith* (Kensington)

Jones, B. Frank, Jr. 4308

A class of singular integrals.

Amer. J. Math. **86** (1964), 441–462.

The singular integrals discussed in this paper generalize integral operators with kernels arising from parabolic differential equations with constant coefficients. Let $k(x,t)$ be defined on $E = R^n \times (0,\infty)$ by

$$k(x,t) = t^{-1}\{\phi(t)\}^{-n}\Omega(x/\phi(t)),$$

where $\phi(t)$ and $\Omega(x)$ satisfy appropriate conditions, one of the principal of them being that the integral of $\Omega(x)$ over

R^n is zero. Furthermore, given $\varepsilon > 0$, let $k_\varepsilon(x, t) = k(x, t)$ for $t \geqq \varepsilon$, and 0 for $t < \varepsilon$. Finally, given $f(x, t)$ on E, let

$$f_\varepsilon(x, t) = \int_E k_\varepsilon(x-\xi, t-\tau)\, d\xi d\tau.$$

The main result proved is that if $1 < p < \infty$ and $f \in L_p(E)$, then $\|f_\varepsilon\|_p \leqq A_p \|f\|_p$, and f_ε converges in $L_p(E)$ as $\varepsilon \downarrow 0$. The author points out that it would also be interesting to know whether f_ε converges pointwise almost everywhere.
A. C. Zaanen (Leiden)

Koizumi, Sumiyuki 4309
On the spectrum of function in the Weyl space.
J. Fac. Sci. Hokkaido Univ. Ser. I **17** (1963), 65–72.
The following notation is used. If K is integrable over $(-\infty, \infty)$, while f is bounded and measurable, $K * f$ is defined by $(K*f)(x) = \int_{-\infty}^{\infty} K(x-y)f(y)\, dy$. $\phi \sim 0$ means that

$$\limsup_{l\to\infty} \sup_{-\infty<x<\infty} \frac{1}{l}\int_{-\infty}^{\infty} |\phi(x)|^2\, dx = 0.$$

$\Lambda_*(f)$ is the set of real numbers λ such that $K * f \sim 0$ implies that $\int_{-\infty}^{\infty} e^{i\lambda y}K(y)\, dy = 0$.

Most of the paper deals with properties of functions f representable as Fourier-Stieltjes transforms

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{iux}\, d\sigma(u),$$

where σ is of bounded variation on $(-\infty, \infty)$. Theorem 3 shows that, for such an f,

$$\limsup_{l\to\infty} \sup_{-\infty<x<\infty} \frac{1}{l}\int_x^{x+l} |f(t)|^2\, dt = \frac{1}{2\pi}\sum |\sigma(\lambda+0)-\sigma(\lambda-0)|^2.$$

According to Theorem 4, $\Lambda_*(f)$ is the closure of the set $\{\lambda \,\|\, \sigma(\lambda+0)-\sigma(\lambda-0)| > 0\}$. Also, in Theorem 5, it is shown that $K * f \sim 0$ is equivalent to $\int_{-\infty}^{\infty} e^{i\lambda y}K(y)\, dy = 0$ for all λ in $\Lambda_*(f)$. Some results of a similar type are proved for functions f almost periodic in the sense of Weyl. The paper contains a considerable number of misprints. {For related results see H. Pollard [Duke Math. J. **20** (1953), 499–512; MR **15**, 215].}
H. Burkill (Sheffield)

Ramanujan, M. S. 4310
The moment problem in a certain function space of G. G. Lorentz.
Arch. Math. **15** (1964), 71–75.
Let $X(C)$ be the space of real-valued measurable functions f on $[0, 1]$ for which $\sup \int_0^1 |f(x)|c(x)\, dx = \|f\|$ is finite, where the supremum is taken over $c \in C$, a class of non-negative measurable functions on $[0, 1]$ with the properties: $1 \in C$, $c_1 \leqq c_2$ a.e. and $c_2 \in C$ imply $c_1 \in C$ and $\{\int c(x)\, dx : c \in C\}$ is bounded. The norm in $X(C)$ is rearrangement-invariant if $\|f\| = \|g\|$ whenever f and g are equimeasurable (for each a, $m\{x : f(x) \geqq a\} = m\{x : g(x) \geqq a\}$).

Theorem [G. G. Lorentz, *Bernstein polynomials*, Univ. Toronto Press, Toronto, Ont., 1953; MR **15**, 217]: Suppose the norm in $X(C)$ is rearrangement-invariant and that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $m(E) \leqq \delta$, $E \subset [0, 1]$, implies $\int_E f \leqq \varepsilon$ for all $f \in X(C)$ with $\|f\| \leqq 1$. A necessary and sufficient condition that $\{\mu_n\}$ be a moment sequence for $f \in X(C)$ with $\|f\| \leqq M$ is that $\|f_n\| \leqq M$, $n = 0, 1, 2, \cdots$, where $f_n(x) = (n+1)\binom{n}{s}\Delta^{n-s}\mu_s$, $s/(n+1) \leqq x < (s+1)/(n+1)$, $s = 0, 1, \cdots, n$.

The author proves the same theorem with $\{f_n\}$ replaced by $\{f_n^*\}$, where

$$f_n^*(x) = \frac{(k+1)(k+2)}{n+1}\binom{k}{n}\Delta^{k-n}\mu_{n+1},$$

$$(n+1)/(k+2) < x \leqq (n+1)/(k+1), \quad k = n, n+1, \cdots.$$

T. R. Jenkins (Los Altos, Calif.)

Sumner, D. B. 4311
A distribution function of Cantor-Vitali type.
Canad. Math. Bull. **7** (1964), 65–75.
The Cantor ternary set is, in this paper, replaced by a natural generalization. For a given positive integer N, the set is obtained by removing from $[0, 1]$ the intervals $I_r = ((2r-1)/(2N+1),\ 2r/(2N+1))$ $(r = 1, \cdots, N)$, by removing similar intervals from the remaining intervals and by continuing this process indefinitely. A continuous, but not absolutely continuous, distribution function $F(x)$ (which is 0 for $x < 0$ and 1 for $x > 1$) is defined by use of the representation of x in the scale $2N+1$; it is $r/(2N+1)$ in I_r and the definition is similar in other discarded intervals. The moments $M_n = \int_0^1 x^n F(x)\, dx$ and $\mu_n = \int_0^1 x^n\, dF(x)$ are shown to be calculable in virtue of three identities which are established. Finally, the author obtains a representation of the characteristic function $f(t) = \int_0^1 e^{itx}\, dF(x)$ in the form $\prod_1^\infty f_p(t)$, where each $f_p(t)$ is a simple sum of exponentials.
H. Burkill (Sheffield)

INTEGRAL EQUATIONS

See also 4139, 4890, 4902.

Az'belev, N. V.; Caljuk, Z. B. 4312
On the uniqueness of the solution of an integral equation. (Russian)
Dokl. Akad. Nauk SSSR **156** (1964), 239–242.
The authors give several existence theorems for unique solutions of certain systems of integral equations. These theorems are refinements of the existence and uniqueness theorems which one usually finds in treatises on integral equations.
H. P. Thielman (Alexandria, Va.)

Cooke, J. C. 4313
Some further triple integral equation solutions.
Proc. Edinburgh Math. Soc. (2) **13** (1962/63), 303–316.
A formal solution of a set of triple integral equations

$$\text{(B)}\quad \begin{aligned} &\int_0^\infty A(t)J_n(tx)\, dt = f(x) \quad (0 < x < a),\\ &\int_0^\infty t^{2\alpha}A(t)J_n(tx)\, dt = g(x) \quad (a < x < b),\\ &\int_0^\infty A(t)J_n(tx)\, dt = h(x) \quad (x > b) \end{aligned}$$

is given. This type was originally considered by Tranter [Proc. Glasgow Math. Assoc. **4** (1960), 200–203; MR **23** #A467], who reduced the problem to a pair of dual series

of Jacobi polynomials and obtained a complete solution of this problem only in the case $n=\frac{1}{2}$ and $\alpha=\frac{1}{2}$. At another place the author [Quart. J. Mech. Appl. Math. **16** (1963), 193–203; MR **27** #553] obtained a solution of (B) for the case when $f(x)=0$ and $h(x)=0$. Here a more general problem is formulated but the kernels of these equations are simpler. As in the previous paper the problem is reduced to the solution of Fredholm integral equations which, in some cases, may be solved by iteration. These results have been applied in the study of incompressible inviscid flow normal to an annular disc and to the flow due to the rotation of such a disc in a viscous fluid. An extension to a more complicated kernel is also given, but it is doubtful whether this result will be of any practical value. *K. N. Srivastava* (Bhopal)

Williams, W. E. 4314

Integral equation formulation of some three part boundary value problems.

Proc. Edinburgh Math. Soc. (2) **13** (1962/63), 317–323.

In a previous paper [Z. Angew. Math. Phys. **13** (1962), 133–152; MR **25** #414] the author gave a method whereby certain potential problems for circular disks can be shown to be governed by Fredholm integral equations of the second kind. This method is now extended to the general non-axisymmetric electrostatic problem for a circular annulus and it is found that this problem is governed by a pair of coupled Fredholm equations of the second kind, suitable for iteration when the ratio of the inner radius to the outer radius of the annulus is small. In the special case of axial symmetry these equations reduce to equations obtained previously by Gubenko and Mossakovskiĭ [Prikl. Mat. Meh. **24** (1960), 334–340; MR **22** #8804]. An application of the author's method is also made to boundary-value problems which can in some sense be regarded as perturbations on the electrostatic problem for the annulus. *W. D. Collins* (Manchester)

Sokolov, Ju. D. 4315

Sur l'application de la méthode des corrections fonctionnelles moyennes aux équations du type parabolique linéaires par rapport aux dérivées. (Russian. French summary)

Ukrain. Mat. Ž. **12** (1960), 181–195.

From the author's summary: "La méthode des corrections fonctionnelles moyennes exposée au paragraphe premier de ce travail se prête à la recherche des solutions approchées de l'équation intégrale du type mixte

$$(1)\quad u(x,t)=\varphi(x,t)+\int_0^t\int_a^b K(x,t;\xi,\tau)f[x,t;\xi,\tau;u(\xi,\tau)]\,d\xi d\tau \qquad (b-a=h>0).$$

Les §§ 2, 3 contiennent les raisonnements concernant les conditions de la convergence du procédé ainsi que l'estimation de l'erreur qu'on commet en s'arrêtant à $n$ième approximation. La même méthode suivie pour résoudre l'équation (1) s'étend immédiatement à l'équation intégrale plus générale

$$(1')\quad u(P,t)=\varphi(P,t)+\int_0^t\int_B K(P,t;Q,\tau)f[P,t;Q,\tau;u(Q,\tau)]\,d\omega_Q d\tau$$

où chacun des points P, Q reste dans un domaine B. On considère aux §§ 4, 5 un nombre d'exemples des applications de la méthode exposée aux équations intégrales de la forme (1) et aux équations différentielles du type parabolique linéaires par rapport aux dérivées

$$\frac{\partial u}{\partial t}-c^2\frac{\partial^2 u}{\partial x^2}=f(x,t,u)$$

avec les conditions

$$u(x,0)=\chi(x);\quad u(0,t)=\psi_1(t),\quad u(h,t)=\psi_2(t)."$$

Lučka, A. Ju. 4316

Approximate solution of Fredholm integral equations by the method of averaged functional corrections. (Russian. English summary)

Ukrain. Mat. Ž. **12** (1960), 32–45.

From the author's summary: "The author considers a linear integral equation

$$(1)\quad y(x)=f(x)+\lambda\int_a^b K(x,\xi)y(\xi)\,d\xi$$

under the assumption that $K(x,\xi)$ and $f(x)$ are complex-valued functions of real arguments belonging to $L^2(a,b)$. Equation (1) is solved by a method which is a generalization of the averaging method of Ju. D. Sokolov, i.e., successive approximations are constructed."

Nersesjan, A. B. 4317

On the theory of integral equations of Volterra type. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 1006–1009.

Consider Fredholm equations of the second type over n-dimensional space: $y(P)=\lambda\int K(P,Q)y(Q)\,dQ+f(P)$; the following discussion provides a significant generalization of the classical theory of Volterra equations even in the one-dimensional case and is clearly effective for arbitrary n. It is assumed that y and f are in $L_2(D)$ and K is in $L_2(D\times D)$. Define K to be an S-kernel, for $S\subset D\times D$, if all pairs (P,Q) such that $K(P,Q)\neq 0$ are in S. Define $S\subset D\times D$ to be a V-set if all S-kernels correspond to operators without finite eigenvalues. V-sets are characterized by the necessary and sufficient condition that the pair (P_1,P_k) is not in S if the pairs $(P_1,P_2),(P_2,P_3),\cdots,(P_{k-1},P_k)$ are in S. S and its complement in $D\times D$ are both V-sets if (a) either (P_1,P_2) or (P_2,P_1) is in S if $P_1\neq P_2$, and (b) if (P_1,P_2) and (P_2,P_3) are in S, then (P_1,P_3) is in S; such sets are maximal V-sets. If S is maximal, the set of operators corresponding to S-kernels is closed under composition. *D. C. Kleinecke* (Santa Barbara, Calif.)

Jones, D. S. 4318

On a certain singular integral equation. I.

J. Math. and Phys. **43** (1964), 27–33.

The author obtains explicit solutions of the singular integral equations

$$(*)\quad \int_0^\infty f(w)\{(w-v)^{-1}\exp[-i\alpha(w-v)]\pm(w+v)^{-1}\exp[-i\alpha(w+v)]\}\,dw=h(v),\qquad v>0,$$

where α is a real constant. These equations arise in diffraction theory. h is assumed to be such that

$$\int_0^\infty h(v)e^{-i\alpha v}(1+v)^{-1/2}\,dv$$

exists. Let

$$M(x,y) = [4xyH_1^{(2)}(\alpha x)J_0(\alpha y) - 4y^2H_0^{(2)}(\alpha x)J_1(\alpha y)]/(x^2-y^2).$$

Then, for the solution of (*) with the plus sign and positive α,

$$f(y) = \frac{i\alpha}{4\pi}\int_0^\infty h(v)M(v,y)\,dv.$$

A similar result holds for the minus sign and for negative α.

The results are interesting but presented in a rather mysterious fashion. It would be instructive to know their origin so that one might see if this represents a general method. *R. C. MacCamy* (Pittsburgh, Pa.)

Višik, M. I.; Èskin, G. I. 4319

Boundary-value problems for general singular equations in a bounded domain. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 24–27.

Let

$$K\varphi = K_\alpha\varphi + T\varphi = \int K_\alpha(x, x-y)\varphi(y)\,dy + \int T(x,y)\varphi(y)\,dy,$$

where the integral is over a bounded region $G\subset R^n$, $K_\alpha(x,z)$ and $T(x,z)$ are smooth in x but generalized functions in z, and where the Fourier transform $\tilde{K}_\alpha(x,\xi)$ of K with respect to z is homogeneous of order α in ξ and nonvanishing for $\xi\neq 0$ (generalized ellipticity). The solubility of the equation $K\varphi=F$ in certain spaces with norms of Sobolev type is discussed: conditions for normal solubility (existence of finite deficiency indices) are given, and the discussion is extended to the case when φ is required to satisfy certain boundary conditions.

J. L. B. Cooper (Cardiff)

Maroni, Pascal 4320

Sur l'équation de Chandrasekhar.

C. R. Acad. Sci. Paris **258** (1964), 2723–2726.

$\psi(s)$ is given and $H(s)$ is to be determined from the equation

$$(1)\qquad \frac{1}{H(s)} = 1 - s\int_0^1 \frac{\psi(u)H(u)}{u+s}\,du \quad (0 < s \leqq 1).$$

By a simple transformation, with $T(s)$ computed from $\psi(s)$, (1) can be transformed to

$$(2)\qquad T(s)H(s) - s\int_0^1 \frac{\psi(u)H(u)}{u-s}\,du = 1 \quad (0 < s < 1).$$

The homogeneous equation associated with (2), denoted by H.E.A., is obtained by writing zero for 1 on the right-hand side of (2). The author states that he has proved the following two theorems. Theorem 1: (i) Let $\psi(s)$ satisfy certain conditions, (ii) $\lim_{s\to 1-0} T(s)\leqq 0$; then H.E.A. possesses one non-zero solution (up to a constant factor) and (2) possesses an infinity of solutions dependent upon a parameter.

If (i) and $\lim_{s\to 1-0} T(s) > 0$, then H.E.A. has zero as its only solution and (2) possesses a unique solution.

Let

$$\sigma = \pi^{-1}\int_0^1 (u+s)^{-1} \operatorname*{arc\,tan}_{[0,\pi]} \{\pi u\psi(u)/T(u)\}\,du,$$

where $s\in C-[-1,0]$, $\psi_0=\int_0^1\psi(s)\,ds$, $\mu^{-2}=2\psi_0-1$, $\lambda^{-2}=1-2\psi_0$, $\nu^{-2}=2\int_0^1 s^2\psi(s)\,ds$. Theorem 2: If (i) $2\psi_0<1$, $\lim_{s\to 1-0} T(s)\leqq 0$, $T(w)=0$, then (1) possesses the two solutions given by $H(s)=\lambda(1+s)(w\pm s)^{-1}e^{-\sigma}$. If (ii) $2\psi_0<1$, $\lim_{s\to 1-0} T(s)>0$, then (1) has the unique solution $H(s)=\lambda e^{-\sigma}$. If (iii) $2\psi_0=1$, then (1) has the unique solution $H(s)=\nu(1+s)e^{-\sigma}$. If (iv) $2\psi_0>1$, $T(iw)=0$ $(w>0)$, then (1) possesses the two solutions $H(s)=\pm i\mu(1+s)\times(s\mp iw)^{-1}e^{-\sigma}$, where the upper signs (or the lower signs) go together. *C. Fox* (Montreal, Que.)

Anselone, P. M. (Editor) 4321

★Nonlinear integral equations.

Proceedings of an Advanced Seminar Conducted by the Mathematics Research Center, United States Army, at the University of Wisconsin, Madison, April 22–24, 1963.

The University of Wisconsin Press, Madison, Wis., 1964. xii+378 *pp.* \$6.50.

The papers of the seminar described in the heading will be reviewed individually.

Dolph, C. L.; Minty, G. J. 4322

On nonlinear integral equations of the Hammerstein type.

Nonlinear Integral Equations (Proc. Advanced Seminar Conducted by Math. Research Center, U.S. Army, Univ. Wisconsin, Madison, Wis., 1963), *pp.* 99–154. *Univ. Wisconsin Press, Madison, Wis.,* 1964.

This is a review of research done on the Hammerstein integral equation

$$y(s) = \int k(s,t)f\{t, y(t)\}\,dt.$$

There are two main types of conditions on k and f for existence and/or uniqueness of a solution. In conditions of the first type some kind of asymptotically linear behavior of f is required; one of these conditions, already due to Hammerstein himself, is that k be symmetric and positive definite, and $|f(t,u)|<A|u|+B$ with $A<\lambda_1$, the smallest eigenvalue of k. In conditions of the second type k and f are required to be monotone in some sense.

The review is divided into four sections. In the first section generalizations of conditions of the first type are discussed; symmetry of $k(s,t)$ is no longer required, and also the results go beyond the interval below the smallest eigenvalue of k. Many contributions to the use of topological methods in nonlinear functional analysis were made by E. H. Rothe (to whom the present exposition is dedicated), and some of these can be applied here. The second section contains remarks on some other approaches, and the motivation is mentioned for considering the right-hand member of the integral equation as an operator in an appropriate Orlicz space if f is of exponential type such as, e.g., $f(t,u)=\exp(u)$. The application of Orlicz spaces in the theory of the Hammerstein equation is then further worked out in the third section. The authors have

asked to correct their remark on p. 115, line 6, that the two Orlicz spaces introduced there are always a conjugate pair of reflexive Banach spaces. This is in general false. In the final section, devoted to monotonocity methods (conditions of the second type on k and f), a new method for proving the existence of a solution of Hammerstein's equation (in abstract form) in Hilbert space is presented.

The review ends with an extensive bibliography (in which E. H. Rothe, M. Krasnosel'skiĭ and Ya. Rutickiĭ figure prominently) and three appendices, one by I. I. Kolodner on contractive methods related to the material treated in the fourth section.

A. C. Zaanen (Pasadena, Calif.)

Guseĭnov, A. I.; Muhtarov, H. Š. 4323

Investigation of a class of non-linear singular integral equations with Cauchy kernel and vanishing on annuli. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 491–494.

The existence and uniqueness of the solution of the equation

$$u(x) = \lambda q(x) \int_a^b \frac{f[u(s)]}{s-x}\, ds$$

in the class $H^0_{M,\delta}$ is established. This class consists of the functions $u(x)$ which are defined on the interval $[a, b]$ and satisfy the conditions

$$|u(x)| \leqq Ml(x),$$

$$|u(x+\Delta x)-u(x)| \leqq M|\Delta x|^\delta,$$

where $M=\text{const}$, $l(x)=(x-a)^\delta(b-x)^\delta$, $0<\delta<1$, $q(x)=(x-a)^{\delta_1}(b-x)^{\delta_1}$, $0<\delta<\delta_1<1$.

H. P. Thielman (Alexandria, Va.)

Kurpel', N. S. 4324

Some approximate methods of solving non-linear equations in a coordinate Banach space. (Russian)

Ukrain. Mat. Ž. **16** (1964), 115–120.

For the nonlinear equation $u=Tu$ in a Banach space E, the method of Sokolov (as outlined by the author) consists in the judicious choice of a linear operator R with $R^2=R$ (projection operator), and the construction of successive approximations as solutions of (*) $u_{(n)}-RTu_{(n)}=Q_RTu_{(n-1)}$, where $Q_R=I-R$. The author discusses the application of this technique to the case $E=B_1+B_2+\cdots$ (a finite or infinite sequence of Banach spaces; here $\|u\|_E=\sup_i \|u^i\|_{B_i}$) with R chosen as $(u^1, u^2, \cdots, u^N, u^{N+1}, \cdots)\rightarrow(P_1u^1, \cdots, P_Nu^N, 0, 0, \cdots)$, where each P_i is a judiciously chosen projection operator in B_i. Under the assumption (not treated in this paper) that (*) are indeed successively solvable, the author exhibits a sufficient condition for $u_{(n)}$ to converge to a solution of $u=Tu$, essentially by finding a sufficient condition for the mapping implicit in (*) to be contractive. Other theorems concern the error of the approximation obtained by solving $u=RTu$, special theorems for Hilbert spaces B_i, and applications to (finite or infinite) systems of Hammerstein integral equations.

In several places, the author tacitly assumes Lipschitz constants to be less than unity. {Misprints: p. 116, line (−1): Q should be R; p. 117, line 19: v should be $u_{(n)}$; equation (3): $=$ should be $\leqq$.}

G. J. Minty (Ann Arbor, Mich.)

Nurekenov, T. 4325

A criterion for the complete continuity of the Urysohn integral operator. (Russian. Kazak summary)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 3, 13–16.

Suppose $(Ax)(t)=\int K[t, s, x(s)]\, ds$, where the integral is over a finite interval, defines a continuous operator from L_p to L_q. Suppose K is a non-decreasing function of t for all s and $x(s)$, $|K[t, s, u]|\leqq K_1[t, s, u]$, and $(Bx)(t)=\int K_1[t, s, x(s)]\, ds$ defines a completely continuous operator from L_p to L_q. Then A is also completely continuous. There are several obvious extensions and generalizations.

D. C. Kleinecke (Santa Barbara, Calif.)

FUNCTIONAL ANALYSIS

See also 4001, 4062, 4067, 4079, 4083, 4105, 4115, 4129, 4171, 4174, 4176, 4221, 4233, 4241, 4243, 4257, 4304, 4310, 4556, 4583, 4664, 4668.

Albrycht, J. 4326

Some remarks on the V^p space with mixed norm for finitely additive set functions.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 681–685.

The author constructs a normed linear space as a generalization of L_p-space for a finite system of numbers $p(p_1, \cdots, p_n)$, $1\leqq p_\nu\leqq +\infty$ $(\nu=1, 2, \cdots, n)$.

H. Nakano (Detroit, Mich.)

Bonsall, F. F. 4327

On the representation of points of a convex set.

J. London Math. Soc. **38** (1963), 332–334.

By using an elementary application of the Hahn-Banach theorem, the author proves a theorem which has as a corollary the important theorem of G. Choquet [C. R. Acad. Sci. Paris **243** (1956), 699–702; MR **18**, 219] concerning the representations of points of a compact metrizable convex subset X of a locally convex space E as centroids of unit measures with supports on the extreme points of X. Let X be a compact subset of E and let $h\in C_R(X)$. A point x in X is a point of strict convexity for h if there exist y, z in X and $t\in(0, 1)$ such that $x=ty+(1-t)z$ and $h(x)<th(y)+(1-t)h(z)$. The author proves the following theorem. Let $x_0\in X$ and $h\in C$. Then there exists a positive Radon measure μ on X with total mass 1 such that x_0 is the centroid of μ and the set of all points of strict convexity for h is a μ-null set.

R. G. Bartle (Urbana, Ill.)

Corson, Harry; Klee, Victor 4328

Topological classification of convex sets.

Proc. Sympos. Pure Math., Vol. VII, *pp.* 37–51. *Amer. Math. Soc., Providence, R.I.*, 1963.

Let E be an infinite-dimensional normed linear space and let C be a closed convex body containing the origin as an interior point. One of the main results of this note implies that C is either homeomorphic with (i) the product of an n-cell and a linear subspace of E with deficiency n, or (ii) a closed half-space of E. In case (ii) there is also a homeomorphism of C onto the unit cell of E which carries the boundary of C onto the unit sphere of E. This result reduces the problem of classifying closed convex bodies in E to the problem of classifying the unit cell and the closed subspaces of E with finite deficiency. Additional results:

Conditions are given which imply that E is homeomorphic with a positive cone in E, or that E is homeomorphic with all of its closed convex bodies. The authors conjecture that every (infinite-dimensional) normed linear space is homeomorphic with its unit cell. They reiterate the question of Fréchet and Banach: If $\{b_\alpha\}$ is a Schauder basis for a Banach space E, must E be homeomorphic with the positive cone $\{\sum_1^\infty t_i b_i : t_i \geq 0\}$?

R. G. Bartle (Urbana, Ill.)

de Branges, Louis; Rovnyak, James 4329

The existence of invariant subspaces.

Bull. Amer. Math. Soc. **70** (1964), 718–721.

If $\mathfrak{H}$ is a complex Hilbert space and T is a bounded linear operator on $\mathfrak{H}$, then a (closed) subspace $\mathfrak{M}$ of $\mathfrak{H}$ is said to be a proper *invariant subspace of* T if $T\mathfrak{M} \subset \mathfrak{M}$ and $\mathfrak{M}$ is neither the zero subspace nor $\mathfrak{H}$. The question of whether a proper invariant subspace must exist for each T (when $\mathfrak{H}$ has dimension greater than one) has been a long-standing and fundamental problem in operator theory that has been resolved only in certain special cases. The authors in this note announce the following theorem: Every bounded linear operator defined on a complex Hilbert space of dimension greater than one has a proper invariant subspace.

The present note consists of the statement of eight theorems along with several definitions and some expository material. As no proofs are given, most readers will probably find it more fruitful to wait for the complete account to appear. Despite this, it is possible to learn something of the authors' methods from this announcement. In particular, their first theorem is an improvement of a result of Rota [Rend. Circ. Mat. Palermo (2) **8** (1959), 182–184; MR **22** #2897]. Consider a countably infinite dimensional Hilbert space $\mathfrak{B}$, and let $\mathfrak{K}$ be the direct sum of a countable collection of copies of $\mathfrak{B}$ indexed by the set of positive integers. Recall that by a shift of infinite multiplicity U is meant the operator that acts on $\mathfrak{K}$ by "shifting" the indices of the vector components of a vector in $\mathfrak{K}$ by one. The authors' first theorem states that a contraction operator T defined on a separable Hilbert space $\mathfrak{H}$ for which the sequence $\{T^n\}$ of powers of T converges strongly to 0 is unitarily equivalent to the adjoint operator U^* restricted to some invariant subspace $\mathfrak{N}$ of U^*.

The authors are thus able to confine their attention to operators obtained by restricting U^* to one of its invariant subspaces. The space $\mathfrak{K}$ on which U acts is then realized as a Hilbert space of vector-valued analytic functions in which U is "multiplication by z". The invariant subspace $\mathfrak{N}$ to which U^* is restricted is the orthogonal complement of the range of an operator-valued analytic function $B(z)$, which acts on $\mathfrak{K}$ as a partial isometry. The authors propose to factor $B(z)=A(z)C(z)$ so that (1) $A(z)$ acts as a partial isometry on $\mathfrak{K}$ with range not equal to $\mathfrak{K}$, (2) $C(z)$ acts as a contraction on $\mathfrak{K}$, and (3) the range of $B(z)$ is a proper subset of the range of $A(z)$. The orthogonal complement of the range of $A(z)$ is then a proper invariant subspace of the operator obtained by restricting U^* to $\mathfrak{N}$. *R. G. Douglas* (Ann Arbor, Mich.)

Garling, D. J. H. 4330

A generalized form of inductive-limit topology for vector spaces.

Proc. London Math. Soc. (3) **14** (1964), 1–28.

The following generalization of the inductive limit of linear topological spaces is introduced. Let X be a linear space and let a system $\langle X_\alpha, \tau_\alpha, i_\alpha, S_\alpha\rangle_{\alpha\in A}$ be given in which X_α is a linear space with a locally convex topology τ_α, i_α is a one-to-one linear map: $X_\alpha \to X$, and S_α an absolutely convex subset of X_α. The generalized inductive limit topology τ under consideration is the strongest locally convex topology on X for which the restrictions j_α of i_α to S_α are all continuous. The system is subject to the following restrictions: (i) X is spanned algebraically by $\bigcup_{\alpha\in A} j_\alpha(S_\alpha)$; (ii) given $\alpha,\beta \in A$, there exists $\gamma\in A$ such that $j_\alpha(S_\alpha)+j_\beta(S_\beta)\subset j_\gamma(S_\gamma)$; (iii) if $j_\alpha(S_\alpha)\subset j_\beta(S_\beta)$, then the map $j_{\alpha,\beta}=j_\beta^{-1}\circ j_\alpha : S_\alpha\to S_\beta$ is continuous. A neighborhood basis for τ is formed by the family of all the sets $\operatorname{conv}\{\bigcup_{\alpha\in A} j_\alpha(S_\alpha\cap V_\alpha)\}$ as $\{V_\alpha\}_{\alpha\in A}$ ranges over all families of absolutely convex neighborhoods of 0 in $\langle X_\alpha, \tau_\alpha\rangle$ and conv { } denotes the absolute convex span of the set in { }. It follows that an absolutely convex set $V\subset X$ is a τ-neighborhood of 0 if and only if for each $\alpha\in A$ the set $j_\alpha^{-1}(V)$ is a neighborhood of 0 in S_α endowed with the topology induced by τ_α. Theorem 1: If $X_\alpha = X$, $\tau_\alpha=\tau_0$ for each $\alpha\in A$ and i_α is the identical map, if T_α is the closure of S_α in $\langle X_\alpha, \tau_\alpha\rangle$, then the topology is identical with the generalized inductive topology of $\langle X_\alpha, \tau_\alpha, i_\alpha, T_\alpha\rangle_{\alpha\in A}$. Theorem 2: Let $A=\{n\}$ be countable. If for each n the map $j_{n,n+1}$ is a homeomorphism, then τ induces the same topology on S_n as does τ_n. Moreover, supposing that $j_{n,n+1}(S_n)$ is closed in S_n for each n, then a set $B\subset X$ is τ-bounded if and only if, for certain n, $j_n(S_n)\supset B$ and $i_n^{-1}(B)$ is bounded in $\langle X_n, \tau_n\rangle$. Completeness conditions for τ are given as well as a discussion of the relations between τ and the linear (not necessarily convex) topologies which render all the maps j_α continuous.

A lot of examples follow. It is shown that each bornological topology and the topology of uniform convergence on compact subsets in a locally convex l.t. space may be obtained as a τ-topology. As principal example the author exhibits the two-norm spaces and their generalizations. Let X be a locally convex metrizable l.t. space with topology τ_1, let $\{B_n\}$ be an increasing sequence of absolutely convex, τ_1-closed and bounded sets such that (a) $\bigcup_{n=1}^\infty B_n = X$, (b) for each i and $a>0$ there exists a j such that $aB_i\subset B_j$. Then X is called a pseudo-two-norm space (when (b) is replaced by (b') $B_i = iB_1$, X is called a two-norm space). In these terms a sequence x_n is said to converge γ to x_0 ($x_n \xrightarrow{\gamma} x_0$) if $\{x_n\}\subset B_i$ for some i and $x_n\to x_0$ in the topology τ_1. There exists exactly one topology τ on X which is locally convex and such that (1) $x_n \xrightarrow{\gamma} 0$ if and only if $x_n\to 0$ in the topology τ; (2) each linear map from X to a linear topological space is (sequentially) γ-continuous if and only if it is τ-continuous. This topology may be obtained by the author's general procedure. Then follow definitions of dual pseudo-two-norm spaces and the second duals, and a duality theory is established. Weak γ-compactness, γ-reflexivity conditions are obtained, generalizing results from the theory of two-norm spaces.

A. Alexiewicz (Poznań)

Goldstine, H. H.; Horwitz, L. P. 4331

Hilbert space with non-associative scalars. I.

Math. Ann. **154** (1964), 1–27.

The authors consider a Hilbert space with Cayley numbers as coefficients. Restricting the coefficients to real numbers, an ordinary real-number Hilbert space is obtained, and

by this remark the whole apparatus of spectral resolutions, etc., may be carried over by the conventional theory. The Cayley coefficients then appear as a group of unitary transformations of the real Hilbert space, to the analysis of which various ideas of group representation theory are applied. *J. T. Schwartz* (New York)

Gordon, Hugh 4332
Relative uniform convergence.
Math. Ann. **153** (1964), 418–427.

Let E be a partially ordered linear space. A sequence $\{f_n\}$ in E converges relative uniformly to f if there exist a sequence $\{\varepsilon_n\}$ of real numbers tending to 0 and an element e of E such that $-\varepsilon_n e \leqq f_n - f \leqq \varepsilon_n e$. The author of the present paper shows that there is a strong connection between the relative uniform convergence and the convergence with respect to what this reviewer called order bound topology [Mem. Amer. Math. Soc. No. 24 (1957); MR **20** #1193]. Assume next that E is given a topology $\mathscr{T}$; then a sequence $\{f_n\}$ is said to converge to f locally for $\mathscr{T}$ if there exist a sequence $\{\beta_n\}$ of real numbers tending to 0 and a $\mathscr{T}$-bounded set B such that $f_n - f \in \beta_n B$. It is shown that if the positive cone of E is $\mathscr{T}$-sequentially complete, then the relative convergence and the local convergence for $\mathscr{T}$ are also closely related. When E is specialized to the order dual (or the double dual) of a vector lattice, the connection among the various modes of convergence introduced above takes an attractive form.
I. Namioka (Seattle, Wash.)

Kitajima, Kyosuke 4333
Convex functionals in a topological vector space.
J. Sci. Hiroshima Univ. Ser. A-I Math. **27** (1963), 167–172.

The author studies convex functionals and families of convex functionals defined on a domain in a linear topological vector space. Various conditions are given that a separately continuous functional f on a product of convex domains be continuous. The conditions all involve convexity assumptions on f, e.g., that $f(x, y)$ be a convex function of x for each fixed value of y.
T. W. Gamelin (Cambridge, Mass.)

Kreĭn, S. G.; Petunin, Ju. I. 4334
On the concept of minimal scale of spaces. (Russian)
Dokl. Akad. Nauk SSSR **154** (1964), 30–33.

This paper is related to and depends on earlier work of the authors; in particular, see S. G. Kreĭn [same Dokl. **130** (1960), 491–494; MR **22** #9860; ibid. **132** (1960), 510–513; MR **22** #12361] and the authors [ibid. **139** (1961), 1295–1298; MR **25** #5370]. If E_0 and E_1 are related Banach spaces, the authors introduce a family of norms in $E_1 \subseteq E_0$ by using functionals from E_0' and take completions to obtain a continuous normal scale, called the minimal scale, of Banach spaces connecting E_0 and E_1. This scale is majorized in norm by any "correct" scale joining these spaces. It is seen that if E_α and F_α are minimal scales and if A is a bounded linear operator from E_j into F_j, for $j=0, 1$, then A is a bounded linear operator from E_α into F_α and the norm $\|A\|_\alpha$ is a logarithmically convex function of α. The minimal scale connecting the spaces $L_1(0, 1)$ and $L_\infty(0, 1)$ is the space $M_\alpha{}^0$ of all measurable functions for which

$$\|x\|_\alpha = \sup \frac{\int_E |x(t)|\,dt}{(\text{mes } E)} < \infty,$$

$$\int_E |x(t)|\,dt = o(\text{mes } E)^\alpha,$$

as mes $E \to 0$. *R. G. Bartle* (Urbana, Ill.)

Lindenstrauss, Joram 4335
On projections with norm 1—an example.
Proc. Amer. Math. Soc. **15** (1964), 403–406.

The author gives an example of Banach spaces $Z \supset X$ with $\dim Z/X = 2$ satisfying: (1) There is no projection with norm 1 from Z onto X; (2) For every $\varepsilon > 0$ there is a projection with norm $< 1+\varepsilon$ from Z onto X; (3) For every Y with $Z \supset Y \supset X$ and $\dim Y/X = 1$ there is a projection with norm 1 from Y onto X.
T. W. Gamelin (Cambridge, Mass.)

Moreau, Jean-Jacques 4336
Théorèmes "inf-sup".
C. R. Acad. Sci. Paris **258** (1964), 2720–2722.

In this note the author relates the theory of dual functions on a topological vector space to the minimax theorems by proving the following theorem, together with some corollaries. Assume F a vector space, G a locally convex Hausdorff space (both over the reals) and a topology on the dual G' of G compatible with the duality $\langle\ ,\ \rangle$; let $f: F \times G \to \{\text{extended reals}\}$ be concave on each $F \times \{y\}$ and convex lower semicontinuous (or $+\infty$ or $-\infty$) on each $\{x\} \times G$. Write $g(x, y') = \sup_{y \in G} [\langle y, y'\rangle - f(x, y)]$, $\gamma(y') = \inf_{x \in F} g(x, y')$ for $y' \in G'$ and

$$\alpha = \inf_{y \in G} [\sup_{x \in F} f(x, y)], \quad \beta = \sup_{x \in F} [\inf_{y \in G} f(x, y)].$$

Then $\alpha = \beta$ implies that γ is lower semicontinuous at the origin, and conversely, provided that either $\alpha \neq +\infty$ or $\beta \neq -\infty$. *P. C. Deliyannis* (Chicago, Ill.)

Pták, Vlastimil 433
A combinatorial lemma on the existence of convex means and its application to weak compactness.
Proc. Sympos. Pure Math., Vol. VII, pp. 437–450. *Amer. Math. Soc., Providence, R.I.*, 1963.

The author uses an elementary (but not transparent) combinatorial lemma as the basis for a systematic theory of weak compactness. This lemma, which was first published by the author in an earlier paper [Czechoslovak Math. J. **9** (**84**) (1959), 629–630; MR **22** #890] has the form of a "double limit" condition, familiar in the study of weak compactness, and allows one to avoid the use of integration theory. Most of the results are too technical to be stated here, so we shall content ourselves with a few important samples. (I) Suppose that E is a locally convex space, then a subset A of E satisfies the double limit condition if there does not exist a neighborhood U of the origin and two sequences $(a_i) \subset A$, $(x_j') \subset U^0$ such that

$$\lim_i \lim_j \langle a_i, x_j' \rangle, \quad \lim_j \lim_i \langle a_i, x_j' \rangle$$

both exist and are not equal. The theorem is that if E is complete, A is bounded, and A satisfies the double limit condition, then A^{00} is weakly compact. This result contains both the famous theorem of W. F. Eberlein [Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 51–53; MR **9**, 42; errata, MR **10**, 855] and a theorem of Krein. (II) Let H be dense in the compact Hausdorff space T and let (x_n) be a bounded sequence in $C(T)$ which converges to 0 for each point of H. Then $\lim x_n(t)=0$ for each $t\in T$ if and only if $(x_n)\to 0$ in the weak topology of $C(T)$ if and only if for each $\varepsilon>0$ and each subsequence (x_{n_i}) there is a convex mean $\sum_1^p \lambda_i x_{n_i}$ such that $|\sum_1^p \lambda_i x_{n_i}|<\varepsilon$, if and only if for each sequence $(h_j)\subset H$ such that $\lim_j (x_n(h_j))$ exists for each n, then

$$\lim_n \lim_j x_n(h_j) = 0,$$

etc. (III) Conditions are given that a sequence (x_n) in a normed linear space E converges weakly to zero in terms of almost uniform convergence and a double limit condition. (IV) Weakly compact subsets of $C(T)$, T completely regular, are studied. *R. G. Bartle* (Urbana, Ill.)

Restrepo, Guillermo 4338

Differentiable norms in Banach spaces.

Bull. Amer. Math. Soc. **70** (1964), 413–414.

The norm in a Banach space X is said to be of class C^1 if it is Fréchet differentiable at each point of $X-\{0\}$ and if the derivative is continuous. The author announces the following (Theorem 1): A separable Banach space X admits an equivalent norm of class C^1 if and only if X^* is separable. This result was motivated by the problem (stated by S. Lang [*Introduction to differentiable manifolds*, p. 28, Interscience, New York, 1962; MR **27** #5192]) of whether any Banach space admits an equivalent norm which (or some function of which) is of class C^n for given n $(1\leqq n\leqq\infty)$.

The proof of Theorem 1 uses a characterization (Theorem 2) of C^1 norms from among smooth (=Gâteaux differentiable = weak differentiable) norms.

{Theorem 2 has also been proved by D. F. Cudia [Trans. Amer. Math. Soc. **110** (1964), 284–314, Corollaries 4.5 and 4.11; MR **29** #446] and the portions of it which are actually used in proving Theorem 1 were first proved by V. L. Šmuljan [Mat. Sb. (N.S.) **6** (**48**) (1939), 77–94; MR **1**, **242**; ibid. **9** (**51**) (1941), 545–561; MR **3**, 205].} Theorem 2 is applied as follows: V. Klee [Fund. Math. **49** (1960/61), 25–34; MR **23** #A3985] has shown that if X^* is separable, then X admits an equivalent norm with a number of useful properties, and the author's Theorem 2 shows that it is C^1. Conversely, in conjunction with a density theorem by E. Bishop and the reviewer [Proc. Sympos. Pure Math., Vol. VII, pp. 27–35, Amer. Math. Soc., Providence, R.I., 1963; MR **27** #4051], Theorem 2 shows that if X admits a C^1 norm, then X^* is separable. {This also follows immediately from the above density theorem together with two theorems of the reviewer [Proc. Amer. Math. Soc. **11** (1960), 976–983, Theorems 2.2 and 4.3 (iii); MR **23** #A501].}
R. R. Phelps (Seattle, Wash.)

R.-Salinas, Baltasar 4339

Generalization to modules of the Hahn-Banach theorem and its applications. (Spanish)

Collect. Math. **14** (1962), 105–151.

This paper contains results of considerable generality dealing with extensions of real-valued functions. Let A be a ring with unit and r a homomorphism of A onto the real numbers R. Let X be an A-module. A real-valued function f on X (or on a submodule of X) is said to be modular if $f(x+y)=f(x)+f(y)$ and $f(ax)=r(a)f(x)$ for all x, y in X, and $a\in A$. Suppose also that there is a subset $A^+\subset A$, closed under addition and multiplication and generating A, such that $r(a)>0$ for all $a\in A^+$. A function p on X to R is semimodular if $p(x+y)\leqq p(x)+p(y)$ and $p(ax)\leqq r(a)p(x)$ for all $a\in A^+$. The author proves a general extension theorem (Theorem 4) asserting that if f_0 and p_0 are given, with $f_0(x)\leqq p_0(x)$ for all $x\in \operatorname{Dom}(f_0)$, then there exist p and f with $p\leqq p_0$, f an extension of f_0, $f(y)=p(x+y)-p(x)=p(y)$ for all $x\in X$, $y\in\operatorname{Dom}(f)$, and such that if q is semimodular with $f_0\leqq q$, and $q\leqq p$, then $q=p$. Likewise, if g is modular and $g\leqq p$, then g is a contraction of f. Many of the more detailed results deal with circumstances under which $\operatorname{Dom}(f)=X$. This is the case, for example, if A is Abelian. More generally, a group G is said to be perfect if a certain commutability property holds, and is said to be accessible if the quotient groups in the normal series are all perfect. Any solvable group is accessible. Theorem 8 asserts that if A contains an accessible group G, and if G and the center of A^+ together generate A^+ in a simple way, then the domain of f in the extension theorem is all of X. The final section deals with applications of these results to the existence of rather general types of invariant measures and integrals.
R. C. Buck (Madison, Wis.)

Thorp, E. O. 4340

Internal points of convex sets.

J. London Math. Soc. **39** (1964), 159–160.

An internal point (also called a core point) of a convex subset of a topological linear space may fail to be an interior point. A sufficient condition for the two notions not to coincide is that the space has a neighborhood basis at zero whose cardinality is less than or equal to the dimension of the space. Examples: all infinite-dimensional metric linear spaces. *R. A. Raimi* (Rochester, N.Y.)

Veksler, A. I. 4341

Linear structures with a sufficient set of maximal l-ideals. (Russian)

Dokl. Akad. Nauk SSSR **150** (1963), 715–718.

The author continues his earlier work [see, e.g., Sibirsk. Mat. Ž. **3** (1962), 7–16; MR **27** #66] on K-lineals (linear lattices). The work itself is based on the theory expounded in the book of Vulih [*Introduction to the theory of partially ordered spaces* (Russian), Fizmatgiz, Moscow, 1961; MR **24** #A3494]. In the paper under review (a summary without proofs) the author studies l-ideals of a K-lineal X. A vector subspace I of X is called an l-ideal if $x\in I$, $y\in X$ and $|y|\leqq|x|$ imply $y\in I$. All order notions, including absolute values, are in the linear-lattice sense. His guiding principle is to develop for these ideals the best possible analogues of the elementary properties of ordinary ideals in linear algebras. For example, one of his theorems gives a necessary and sufficient condition in order that every l-ideal of X be contained in a maximal l-ideal of X. The main tool in the present work is the theorem, which can be found in the book of Vulih cited above, that every

K-lineal can be represented in a canonical manner as a space of continuous extended real-valued functions on a compact Hausdorff space Q. When X is thus represented, a maximal l-ideal turns out (roughly speaking) to consist of all functions vanishing at a given point of Q. Most of the theorems give properties of l-ideals in terms of the form they assume under the canonical representation of X.

J. Gil de Lamadrid (Minneapolis, Minn.)

Duren, P. L. 4342

On the spectrum of a Toeplitz operator.

Pacific J. Math. **14** (1964), 21–29.

A Toeplitz operator T is one which transforms a sequence $x=(x_0, x_1, x_2, \cdots)$ into a sequence y according to the formal law

$$(1)\qquad \sum_{k=0}^{\infty} c_{n-k}x_k = y_n, \quad n = 0, 1, 2, \cdots.$$

If the complex coefficients c_n satisfy the condition

$$(2)\qquad \sum_{n=-\infty}^{\infty} |c_n| < \infty,$$

then T carries each $l_p{}^+$ space $(1 \leqq p \leqq \infty)$ into itself, where $l_p{}^+$ is the Banach space of all complex sequences $x=(x_0, x_1, \cdots)$ for which the norm $\|x\|_p = \{\sum_{n=0}^{\infty} |x_n|^p\}^{1/p}$ is finite and $\|x\|_\infty = \sup |x_n|$.

Under the assumption (2), M. G. Kreĭn [Uspehi Mat. Nauk **13** (1958), no. 5 (83), 3–120; MR **21** #1507; Amer. Math. Soc. Transl. (2) **22** (1962), 163–288; MR **28** #1070] gave the detailed description on the spectrum of T as an operator in l_p.

In the present paper, the author picks up the special class of Toeplitz operators T (called multidiagonal Toeplitz operators) such that $c_n=0$, $n>l$, $n<-m$; $c_l\neq 0$, $c_{-m}\neq 0$, where l and m are certain non-negative integers, and gives a refinement of Kreĭn's result for such a special class.

S. Sakai (New Haven, Conn.)

Edwards, R. E. 4343

Endomorphisms of function-spaces which leave stable all translation-invariant manifolds.

Pacific J. Math. **14** (1964), 31–48.

Let X denote a locally compact Hausdorff group with left Haar measure. Let F denote one of the following complex topological vector spaces: (a) $L^1(X)$; (b) all bounded measures on X; (c) $L^\infty(X)$; (d) $C(X)$ for compact X; (e) $L^\infty(X)$ for compact X; (f) all continuous functions for Abelian X (with the topology of locally uniform convergence); (g) the space $\mathscr{D}$ of infinitely differentiable functions with compact support for $X=R^n$; (h) the dual of $\mathscr{D}$. Let T be an endomorphism of F such that $T(M)\subset M$ for each left [right] invariant closed subspace of F. For each of the above spaces F, the author finds conditions on T which imply that there is a measure or distribution μ on X such that $Tf=\mu * f$ [$f * \mu$] for all $f\in F$. He also concludes that T commutes with right [left] translations. Some related problems are also discussed.

K. A. Ross (Rochester, N.Y.)

Golovkin, K. K. 4344

On the non-existence of certain inequalities between functional norms. (Russian)

Trudy Mat. Inst. Steklov. **70** (1964), 5–25.

Let E_n be an n-dimensional euclidean space. Let Ω be a certain domain contained in E_n. By $M(\Omega)$ we denote the space of all finite, infinitely often differentiable functions $u(x_1, \cdots, x_n)$ determined on Ω.

In the paper definitions are given of certain constants connected with the norm N and, in the language of these constants, necessary and sufficient—or only necessary—conditions are given for an estimation of one norm by the other.

In § 1 the author defines a local dimension κ' with respect to the variables $x_1, \cdots, x_m$ such that

$$\frac{1}{\lambda^{\kappa'}} N(u(\lambda x_1, \cdots, \lambda x_m, x_{m+1}, \cdots, x_n)) \to C \neq 0 \quad \text{if } \lambda \to \infty.$$

A dimension of dilation with respect to the variables $x_1, \cdots, x_m$ is defined as a number κ'' such that

$$\frac{1}{\lambda^{\kappa''}} N(u(\lambda x_1, \cdots, \lambda x_m, x_{m+1}, \cdots, x_n)) \to C \neq 0 \quad \text{if } \lambda \to 0.$$

Theorem 1 gives necessary and sufficient conditions for estimation of one norm by the other in the language of the local dimension and the dimension of the dilation.

In § 2 the author defines the so-called differential order of a norm. In § 3 he gives for one variable the so-called logarithmic dimension of a norm. The definitions of these ideas are long and cannot be cited here. Some necessary conditions for an estimation of one norm by the other are given in the language of these ideas.

The paper contains many examples of countable differential orders and logarithmic dimensions for concrete norms so that concrete norms cannot be estimated in terms of certain other norms.

S. Rolewicz (Warsaw)

Guber, Siegfried 4345

Masstheoretische Kennzeichnung gewisser Funktionenkegel.

Arch. Math. **15** (1964), 58–70.

Let X be a Hausdorff space and let $\mathfrak{F}_1=(\sigma_i, x_i)_{i\in I}$ and $\mathfrak{F}_2=(\tau_j)_{j\in J}$ be families where σ_i, τ_j are Radon measures $\geqq 0$ supported by compact sets and where x_i are points in X such that $\int d\sigma_i \leqq 1$ and $\sigma_i(\{x_i\})=0$ $(i\in I)$. Let $\mathscr{C}(X)$ be the topological vector space of all continuous real-valued functions on X with the topology of uniform convergence on compact subsets. The set $\mathscr{K}$ of all functions $u\in\mathscr{C}(X)$ satisfying all the inequalities $\int u\, d\sigma_i \leqq u(x_i)$ $(i\in I)$ and $\int u\, d\tau_j \leqq 0$ $(j\in J)$ is then a closed convex cone in $\mathscr{C}(X)$ stable with respect to the operation (S): $(f, g)\to\inf(f, g+1)$. It is proved that, conversely, each closed convex cone $\mathscr{K}$ in $\mathscr{C}(X)$ which is stable with respect to (S) is associated to two families $\mathfrak{F}_1$ and $\mathfrak{F}_2$ in the above sense. This generalizes a theorem of G. Choquet and J. Deny [J. Math. Pures Appl. (9) **36** (1957), 179–189; MR **20** #1119] and two of the main results of the author's thesis [*Über die Struktur gewisser Funktionenkegel*, Verlag Mikrokopie GmbH, Munich, 1961; MR **26** #2871]. In this thesis the author treats by different methods the same problem for closed convex cones $\mathscr{K}$ satisfying in addition $\mathscr{K}\subset\mathscr{C}_+(X)$ or $\mathscr{K}\subset\mathscr{C}_-(X)$. It is proved that in both cases the stability with respect to (S) is equivalent to the stability with respect to the two operations $(f, g)\to\inf(f, g)$ and $(f, g)\to\inf(1, f)$. As an application of the main theorem a certain stability property of cones $\mathscr{K}$ is proved.

H. Bauer (Hamburg)

Itô, Takashi 4346
Remarks on completeness of continuous function lattice.
J. Fac. Sci. Hokkaido Univ. Ser. I **17** (1963), 149–151.
Let E be an arbitrary topological space. Theorem 1: The continuous function lattice $C(E)$ is conditionally σ-complete if and only if (a) there is a smallest open-closed set containing a given cozero-set and (b) any two disjoint cozero-sets are separated by open-closed sets. Theorem 2: $C(E)$ is conditionally complete if and only if (a), (b), and (c) the open-closed sets in E form a complete lattice.
L. Gillman (Rochester, N.Y.)

Kree, Paul 4347
Sur les multiplicateurs dans $\mathcal{F}$-L^p avec poids.
C. R. Acad. Sci. Paris **258** (1964), 1692–1695.
The author generalises inequalities of Littlewood and Paley [Proc. London Math. Soc. (2) **42** (1936), 52–89], Marcinkiewicz [Studia Math. **8** (1939), 78–91] and Mihlin [Dokl. Akad. Nauk SSSR **109** (1956), 701–703; MR **18**, 304] to cases in which the functions involved are vector- or operator-valued, and in which the measures involved, instead of being merely the Lebesgue measure on Euclidean space, involve a weight function. The following generalisation of the one-dimensional Paley-Littlewood inequality will suffice to give the flavor of the results stated. Theorem: Let μ be the measure on $R = R^1$ defined by

$$\mu(e) = \int_e (\beta_0 + |x|^2)^{\alpha p/2}\,dx,$$

where $\beta_0 = 0$ or 1, and where $-p^{-1} < \alpha < (p')^{-1}$, $p^{-1} + (p')^{-1} = 1$. If f is a vector-valued function with values in the Hilbert space H, such that $f \in L_p(\mu, H)$, let $\hat{f}$ denote its Fourier transform. Let $\hat{f}_n(k) = \hat{f}(k)$ if $2^{n-1} \le |k| < 2^n$, $\hat{f}_n(k) = 0$ otherwise, and let f_n be the function whose Fourier transform is $\hat{f}_n$. Then the norm

$$|f|_{p,\mu} = \left\{\int_R |f(x)|^p \mu\,(dx)\right\}^{1/p}$$

on $L_p(\mu, H)$ is equivalent to the norm

$$|f|^*_{p,\mu} = \left\{\int_R \left[\sum_{k=-\infty}^{+\infty} |f_k(x)|^2\right]^{p/2} \mu\,(dx)\right\}^{1/p}.$$

Proofs will appear in a subsequent publication.
J. T. Schwartz (New York)

Portnov, V. R. 4348
A theorem on the density of finite functions in the space $L^{(m)}_{p,a}(E_n)$. (Russian)
Dokl. Akad. Nauk SSSR **154** (1964), 530–533.
Let E_n be Euclidean n-space, and let $L^{(m)}_{p,a}(E_n)$ be the space of functions with the norm

$$\|f\| = \left\{\int_{E_n} a(x)\left[\sum_{|\alpha|=m} |D^\alpha f(x)|^2\right]^{p/2} dx\right\}^{1/p};$$

$a(x)$ satisfies certain conditions which are too complicated to be reproduced here.

The author proves that infinitely differentiable functions with compact supports are dense in $L^{(m)}_{p,a}(E_n)$. For $a(x) \equiv 1$ the theorem was proved by S. L. Sobolev [same Dokl. **149** (1963), 40–43; MR **26** #6758].
Z. Zieleźny (Wrocław)

Rota, Gian-Carlo 4349
Reynolds operators.
Proc. Sympos. Appl. Math., Vol. XVI, pp. 70–83. *Amer. Math. Soc., Providence, R.I.*, 1964.
Let $\mathcal{A}$ be an algebra over a field of characteristic not equal to 2. A Reynolds operator is a linear map of $\mathcal{A}$ into $\mathcal{A}$ which for any f and g in $\mathcal{A}$ satisfies

$$R(fg) = RfRg + R((f - Rf)(g - Rg)).$$

Using functional-analytic techniques, the author attacks the problem of characterizing Reynolds operators. Two major results are proved. Let $L_\infty(S, \Sigma, m)$ and $L_1(S, \Sigma, m)$ denote, respectively, bounded measurable and integrable functions on a σ-finite measure space. (1) Let R be a Reynolds operator defined in L_∞ and continuous with respect to the L_1-topology. Then R is an averaging operator, that is, R satisfies the identity $R(fRg) = (Rf)(Rg)$ for all f and g in L_∞ if and only if R has closed range. (2) Let R be a Reynolds operator in L_∞, $m(S) < \infty$, such that $R1 = 1$ and $\int |f|^2\,dm \ge \int |Rf|^2\,dm$ for all f in L_∞. Then R can be represented in the form $Rf = \int_0^\infty e^{-t} V^t Af\,dt$, where A is a unique positive conditional expectation operator and where V^t is a unique measure-preserving transformation defined for $t \ge 0$ on L_∞. An extensive bibliography on related papers is included.
G. Baxter (Minneapolis, Minn.)

Volkov, V. I. 4350
On a sequence of positive linear operators in the space of continuous functions. (Russian)
Kalinin. Gos. Ped. Inst. Učen. Zap. **29** (1963), 19–38.
Let $C^* = C^*_{[0,a]}$ be the set of all functions $f(x)$ which are defined on $[0, \infty)$, continuous in $[0, a]$ and continuous from the right at a. Furthermore, assume that $|f(x)| \le M(1 + x^2)$ on $(0, \infty)$ when the constant M depends on f. On C^* one defines a sequence of positive linear operators by the relations

$$(1) \qquad L_n(f; x) = \sum_{k=0}^{\infty} f(a_{nk})F_{nk}(x),$$

when the matrices a_{nk} and functions F_{nk} are subject to the following conditions: $a_{n,0} = 0$, $a_{nk} > 0$ for $k = 1, 2, \cdots$; $F_{nk}(x) \ge 0$ for $x \in [0, a]$ and (1) $\sum_{k=0}^\infty F_{nk}(x) = 1$, $x \in [0, a]$; (2) $a_{nk}F_{nk}(x) = xF_{p_n,k-1}(x)[1 + b_{nk}(x)]$, $x \in [0, a]$, $|b_{nk}(x)| \le b_n \to 0$ as $n \to \infty$; (3) $a_{nk} = a_{p_n,k-1} + d_{n,k-1}$, $|d_{n,k-1}| \le d_n \to 0$; (4) $p_n \to \infty$ as $n \to \infty$. The author proves that a sequence $L_n(f; x)$ tends to f, together with certain related results. Furthermore, for a uniformly continuous function on $[0, \infty)$ (1) uniformly tends to f and also $\sum_{k=0}^\infty f(t_{nk})F_{nk}(x)$ uniformly tends to f provided that $|a_{nk} - t_{nk}| \le g_n \to 0$ as $n \to \infty$. Operators of type (1) are generalisations of known operators used in this kind of problem [cf. V. A. Baskakov, Dokl. Akad. Nauk SSSR **113** (1957), 249–251; MR **20** #1153].
S. Kurepa (Zagreb)

Baouendi, Mohamed Salah 4351
Division des distributions dans $\mathcal{O}_M'$.
C. R. Acad. Sci. Paris **258** (1964), 1978–1980.
Cette note résume quelques résultats contenus dans la "thèse de 3ᵉ cycle" de l'auteur. Le problème de la divison des distributions tempérées a été résolu par Hörmander [Ark. Mat. **3** (1958), 555–568; MR **23** #A2044] et Łojasiewicz [Studia Math. **18** (1959), 87–136; MR **21** #5893] et généralisé à des systèmes par Malgrange [Séminaire

Schwartz (1959/60), Exp. 21–25, Fac. Sci. Paris, Paris, 1960]. D'une précision qu'il apporte à la division dans $\mathcal{S}'$, l'auteur déduit la solution du problème analogue dans l'espace $\mathcal{O}_M'$, dual de l'espace $\mathcal{O}_M$ des fonctions indéfiniment dérivables à croissance lente. Soit $P=(P_{ij})$, $1\leq i\leq p$, $1\leq j\leq q$, une matrice de polynômes à n variables réelles. Alors $T=(T_1,\cdots,T_p)\in\mathcal{O}_M'^p$ est de la forme $T=PS$ avec $S=(S_1,\cdots,S_q)\in\mathcal{O}_M'^q$ si et seulement si $QT=0$ pour toute matrice Q d'ordre $1\times p$ à coefficients polynômes qui vérifie $QP=0$. Dans sa thèse l'auteur montre que le résultat analogue est faux dans l'espace $\mathcal{O}_C'$.

J. Horváth (Nancy)

Beltrami, Edward J. 4352

Some alternative approaches to distributions.

SIAM Rev. **5** (1963), 351–357.

Es werden einige bekannte Definitionen für Distributionen angegeben und miteinander verglichen, und zwar die ursprüngliche Definition von L. Schwartz, die Definition mit Hilfe der Dualräume gewisser Hilberträume [vergl. P. D. Lax, *Theory of functions of a real variable*, New York Univ. Inst. Math. Sci., New York, 1959] und als Unterraum eines Raumes analytischer Funktionen [H. J. Bremermann und L. Durand, III, J. Mathematical Phys. **2** (1961), 240–258; MR **25** #5380]. Obwohl die Resultate fast durchweg bekannt sind, ist die Zusammenstellung nützlich.

D. Laugwitz (Darmstadt)

Miller, J. B. 4353

Hilbert spaces of generalized functions extending L^2. II.

J. Austral. Math. Soc. **3** (1963), 267–281.

The present note continues the discussion, begun in the first part [same J. **1** (1959/60), 281–298; MR **23** #A2047], of classes of spaces $\mathfrak{H}_{-\lambda}$ which are extensions of $L^2(0,\infty)$ and whose elements ('sequence-functions') exhibit some of the properties of distributions. In this paper the analogy is pursued between these sequence-functions and other types of generalized functions.

At first the author studies their local behavior and obtains the following Theorem 1: If $F\in\mathfrak{H}_{-\lambda}$ is numerical on an open subset E of $(0,\infty)$ (i.e., F is equivalent to a function of $L^2(E)$ without belonging to $L^2(0,\infty)$) and if $\lambda>\frac{1}{2}$, then all Cauchy sequences for F which converge in $L^2(E)$ have the same limit.

Then (in §§ 3, 4 and 5) the existence of ordinary and convolution-type products is discussed: Introducing (in § 3) $\|\phi\|_0=\{\int_0^\infty|\phi(t)|^2t^{-1}\,dt\}^{1/2}$ and $\mathcal{P}_0$ as the space of all measurable functions ϕ for which $\|\phi\|_0<+\infty$, and finally, $\mathcal{P}_\lambda$ as the space of all (measurable) functions ϕ for which a function $\phi^{(\lambda)}$ exists with the properties:

$$\phi(t)=\Gamma(\lambda)\int_t^\infty(\tau-t)^{\lambda-1}\phi^{(\lambda)}(\tau)\,d\tau,$$

$$\psi\in\mathcal{P}_0,\quad \psi(t)=t^\lambda\phi^{(\lambda)}(t),$$

and $\mathcal{P}_{[\lambda]}$ in a similar way, the author proves Theorem 2: Let $F\in\mathfrak{H}_{-l}$ (l being a positive integer). Then $F\phi$ exists as a member of $\mathfrak{H}_{-(l+1)}$ if $\phi\in\mathcal{P}_0\cap\mathcal{P}_l$, or even as a member of $\mathfrak{H}_l$ if $\phi\in\mathcal{P}_l$ and ϕ is bounded, or if $\phi\in\mathcal{P}_{[l]}$.

Now (in § 4) the author considers a product of the form

$$(f\cdot g)(t)=\int_0^\infty f(t\tau)\frac{1}{\tau}g\left(\frac{1}{\tau}\right)d\tau,$$

where the integral is understood to be absolutely convergent for almost all positive t. (Clearly $f\cdot g$ exists if both f and g belong to $L^2(0,\infty)$.) In order to be able to discuss the existence of this resulting product in $\mathfrak{H}_{-\lambda}$, the author introduces new spaces $\mathfrak{B}_{-\lambda}$, derived from the norm

$$|f|_0=\int_0^\infty|f(t)|t^{-1/2}\,dt$$

in the same way as the $\mathcal{P}$-spaces are derived from the norm $\|\cdot\|_0$. Then the following Theorem 3 holds: Let $\lambda\geq0$, $\alpha\geq0$. If $F\in\mathfrak{H}_{-\lambda}$ and $G\in\mathfrak{B}_{-\alpha}$, then $F\cdot G$ exists as an element of $\mathfrak{H}_{-(\lambda+\alpha)}$. If instead $F\in\mathfrak{B}_{-\lambda}$ and $G\in\mathfrak{B}_{-\alpha}$, then $F\cdot G$ exists as an element of $\mathfrak{B}_{-(\lambda+\alpha)}$. In § 5 a theory like that of § 4 is sketched for the (convolution) product

$$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Turning to the definition of derivatives and integrals of sequence-functions, the author shows that expressions of the form $t^\alpha F^{(\alpha)}(t)$ and $t^{-\alpha}F^{(-\alpha)}(t)$ arise more naturally than $F^{(\alpha)}(t)$ and $F^{(-\alpha)}(t)$; therefore he first gives conditions for the existence of these 'affixed' derivatives and integrals (in § 6) and succeeds in detaching the factors $t^{\pm\alpha}$ by using the ordinary product of § 3 to define $t^{\mp\alpha}\cdot t^{\pm\alpha}F^{(\pm\alpha)}(t)$. The method expounded here defines sequence-functions which are interpretable as derivatives and integrals only over intervals excluding neighborhoods of 0 and ∞. For this the results on the local behavior of sequence-functions obtained in the beginning of the paper are used.

H. Pachale (Berlin)

Guelfand, I. M. [Gel'fand, I. M.]; Chilov, G. E. [Šilov, G. E.] 4354

★Les distributions. Tome 2: Espaces fondamentaux.

Traduit par Serge Vasilach. Collection Universitaire de Mathématiques, XV.

Dunod, Paris, 1964. xv+258 *pp.* 42 *F.*

This is a translation into French of the classical work of the authors [*Generalized functions*, Part 2, *Spaces of fundamental and generalized functions* (Russian), Fizmatgiz, Moscow, 1958; MR **21** #5142a]. The book has also been translated into German [*Verallgemeinerte Funktionen* (*Distributionen*), II, *Lineare topologische Räume, Räume von Grundfunktionen und verallgemeinerten Funktionen*, VEB Deutscher Verlag der Wiss., Berlin, 1962; MR **26** #6765].

Schwartz, Laurent 4355

Convergence de distributions dont les dérivées convergent.

Studies in mathematical analysis and related topics, pp. 364–372. *Stanford Univ. Press, Stanford, Calif.,* 1962.

Sia Ω un aperto di R^n e $\mathcal{A}(\Omega)$ uno spazio di distribuzioni su Ω con $\mathcal{E}(\Omega)\subset\mathcal{A}(\Omega)\subset\mathcal{D}'(\Omega)$ (iniezioni continue) e verificante le seguenti due condizioni: (1) $\mathcal{A}(\Omega)$ è di tipo locale, cioè: (α) $T\in\mathcal{A}(\Omega)\Leftrightarrow\forall\alpha\in\mathcal{D}(\Omega)\ni\alpha T\in\mathcal{A}(\Omega)$, ($\beta$) $T\to0$ in $\mathcal{A}(\Omega)\Leftrightarrow\forall\alpha\in\mathcal{D}(\Omega)\ni\alpha T\to0$ in $\mathcal{A}(\Omega)$, (2) Per ogni misura μ su R^n, a supporto contenuto nella sfera di centro l'origine è di raggio ρ e per ogni distribuzione T di $\mathcal{A}(\Omega)$, la distribuzione $T*\mu$ appartiene ad $\mathcal{A}(\Omega_0)$

($\Omega_0 = \{x | x \in R^n$, distanza di x dalla frontiera di $\Omega > \rho\}$); inoltre per ρ e μ fissati, $T \to T * \mu$ è continua da $\mathscr{A}(\Omega)$ in $\mathscr{A}(\Omega_0)$ ($\mathscr{A}(\Omega_0)$ è lo spazio delle $T \in \mathscr{D}'(\Omega_0)$ tale che $\alpha T \in \mathscr{A}(\Omega)$ per ogni $\alpha \in \mathscr{D}(\Omega_0)$, munito della topologia tale che $T \to \alpha T$ sia continua). Diciamo d^m l'applicazione $T \to \{D^pT\}_{|p|=m}$, che a ogni T fa corrispondere il sistema di tutte le sue derivate di ordine m; e indichiamo con $\mathscr{A}^m(\Omega)$ lo spazio delle T appartenenti ad $\mathscr{A}(\Omega)$ insieme a tutte le sue derivate d'ordine $\leq m$, munito della topologia la meno fine per cui ogni derivazione D^p, $|p| \leq m$, è continua da $\mathscr{A}^m(\Omega)$ in $\mathscr{A}(\Omega)$. Il risultato principale del lavoro è allora il seguente notevole teorema: l'applicazione d^m è un omomorfismo di $\mathscr{A}^m(\Omega)$ in $\{\mathscr{A}(\Omega)\}^M$ (M = numero delle derivate d'ordine m).

Ne vengono dedotte diverse interessanti applicazioni supposto Ω connesso: (a) Sia $\mathscr{F}$ un filtro su $\mathscr{A}^m(\Omega)$ tale che la sua immagine attraverso d^m converga a zero in $\{\mathscr{A}(\Omega)\}^M$ e $\mathscr{F}$ converga a zero in una topologia di spazio vettoriale su $\mathscr{A}^m(\Omega)$, meno fine di quella di $\mathscr{A}^m(\Omega)$, non necessariamente separata ma inducente su $\mathscr{P}$ (spazio vettoriale dei polinomi di grado $\leq m-1$) una topologia separata; allora $\mathscr{F}$ converge a zero nella topologia iniziale di $\mathscr{A}^m(\Omega)$. (b) Sia $\mathscr{B}$ un sottospazio vettoriale di $\mathscr{A}^m(\Omega)$: condizione necessaria e sufficiente perchè la convergenza di d^mT a zero in $\{\mathscr{A}(\Omega)\}^M$, $T \in \mathscr{B}$, equivalga alla convergenza a zero di T in $\mathscr{A}^m(\Omega)$ è che $\mathscr{B} \cap \mathscr{P} = \{0\}$, oppure che esista un sistema di forme lineari e continue su $\mathscr{A}^m(\Omega)$, nulle su $\mathscr{B}$, ma non tutte nulle su alcun polinomio $\neq 0$ di $\mathscr{P}$. Tra i corollari del teorema (b) particolarmente interessante è il seguente: sia K un insieme chiuso di Ω; condizione necessaria e sufficiente perchè, se $f \in H^1_{loc}(\Omega)$ ed è supporto contenuto in $\Omega - K$, la convergenza di f in $H^1_{loc}(\Omega)$ comporti la convergenza di f in $H^1_{loc}(\Omega)$, è che K sia di capacità positiva. *E. Magenes* (Pavia)

Slowikowski, W. 4356

The concept of inductive families of (F)-spaces in connection with solvability of linear equations.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 517–520.

Let X be the inductive limit of a sequence of convex spaces X_n and X_0 a subspace of X; if f is a linear functional defined on X_0 and continuous on each $X_0 \cap X_n$, we may ask under what conditions f may be extended to a linear functional on X. Also, if X_1 and X_2 are two LF spaces and A is a linear continuous mapping of X_1 into X, when does A' map X' onto X_1'. In the more general setting of inductive families (introduced here) these questions are answered. The conditions are too complicated to be reproduced here. No proofs are given. *V. Pták* (Prague)

Slowikowski, W. 4357

Linear operators in spaces of distributional type.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 521–524.

Let X_1 and X be two inductive families of convex spaces and A a continuous linear mapping of X_1 into X. Conditions (too complicated to be reproduced here) are given for A' to map X' onto X_1'. Further specialization to spaces of infinitely differentiable functions is given. No proofs are given. *V. Pták* (Prague)

Bessmertnyh, G. A. 4358

The behavior of corrections in some methods of approximate calculation of bounds of the spectrum of a self-adjoint operator.

Dokl. Akad. Nauk SSSR **136** (1961), 1265–1268 (*Russian*); *translated as Soviet Math. Dokl.* **2** (1961), 173–176.

The author studies the iteration processes given by M. A. Krasnosel'skiĭ [Uspehi Mat. Nauk **11** (1956), no. 3 (69), 151–158; MR **18**, 676] and V. N. Kostarčuk and B. P. Pugačev [Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. No. 2 (1956), 25–30; MR **18**, 713]. For instance, let A be a self-adjoint operator, $m < M$ the boundaries of its spectrum $\sigma(A)$; suppose that m is isolated in $\sigma(A)$, and that if m_1 denotes the least point in $\sigma(A) - \{m\}$, then $m > \frac{1}{2}(M - m_1)$. Define

$$x_{k+1} = x_k - \frac{1}{\gamma_k}\Delta_k, \qquad \Delta_k = Ax_k - \mu_k x_k,$$

$$\mu_k = \frac{(Ax_k, x_k)}{(x_k, x_k)}, \qquad \gamma_k = \frac{(A\Delta_k, \Delta_k)}{(\Delta_k, \Delta_k)}, \qquad k = 0, 1, 2, \cdots.$$

Under the above conditions, if x_0 is conveniently chosen, then x_k, $\Delta_k \neq 0$ and

$$\frac{(Ax_k, x_k)}{(x_k, x_k)} \to m, \qquad \frac{(A\Delta_k, \Delta_k)}{(\Delta_k, \Delta_k)} \to m_1.$$

C. Foiaş (Bucharest)

Birman, M. Š. 4359

Existence conditions for wave operators. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 883–906.

If A and B are self-adjoint operators in a Hilbert space, and if $P(A)$ and $P(B)$ are the orthogonal projections on their absolutely continuous subspaces, respectively, then the wave operators $W_\pm(A, B) = \lim_{t\to\pm\infty} e^{iAt}e^{-iBt}P(B)$, $W_\pm(B, A) = \lim_{t\to\pm\infty} e^{iBt}e^{-iAt}P(A)$ (strong limits) are of interest in the quantum theory of scattering and also in the spectral theory of operators inasmuch as they are used to prove the unitary equivalence of the absolutely continuous parts of A and B. Conditions for existence of the wave operators have been investigated for abstract operators by M. Rosenblum [Pacific J. Math. **7** (1957), 997–1010; MR **19**, 756], T. Kato [Proc. Japan Acad. **33** (1957), 260–264; MR **19**, 1068], and S. T. Kuroda [J. Math. Soc. Japan **11** (1959), 246–262; MR **22** #8345; ibid. **12** (1960), 243–257; MR **25** #4360]. Because of limited applicability of their theorems to differential operators in higher dimensions and to problems of perturbation of the boundary and boundary conditions, additional investigations concerning more general conditions for existence of the wave operators have been carried out recently by the author [Dokl. Akad. Nauk SSSR **143** (1962), 506–509; MR **26** #6823] and by the author and M. Kreĭn [ibid. **144** (1962), 475–478; MR **25** #2447]. In this article the author gives detailed proofs of further results in this direction reported in an earlier paper [ibid. **147** (1962), 1008–1009]. His basic theorem is as follows. Let φ be a regular function in some open set of the complex plane containing the spectra of A and B, and let φ have certain additional properties connected with its behavior on the spectra of A and B. Suppose that U, V, $\tilde{U}$, $\tilde{V}$ are the Cayley transforms, respectively, of A, B, $\tilde{A} = \varphi(A)$, $\tilde{B} = \varphi(B)$. If $(\tilde{B} - iE)^{-1} - (\tilde{A} - iE)^{-1}$ is a nuclear

operator, then the wave operators $W_\pm(B, A)$, $W_\pm(V, U) = \lim_{n\to\pm\infty} V^n U^{-n} P(U)$ exist, and

$$(*)\quad W_\pm(B, A) = W_\pm(\hat{B}, \hat{A}) = W_\pm(\hat{V}, \hat{U}) = W_\pm(V, U).$$

Analogous assertions are true for $W_\pm(A, B)$ and $W_\pm(U, V)$. The important step of the proof is to verify the first equation of (*) for weak limits. Since it is known from a theorem of the author and Kreĭn [loc. cit.] that $W_\pm(\hat{B}, \hat{A})$ and $W_\pm(\hat{V}, \hat{U})$ exist as strong limits and are equal, the first and second equations of (*) therefore follow for strong limits. The remaining equation follows by introducing two operators which are functions of A and B, respectively, and using what has already been proved. Theorems different from but analogous to the above theorem have also been obtained by T. Kato [Office of Naval Res. Dept. Math. Univ. California Tech. Rep. No. 11 (1963)].

R. C. Gilbert (Madison, Wis.)

Borisovič, Ju. G. 4360

An application of the concept of rotation of a vector field. (Russian)

Dokl. Akad. Nauk SSSR **153** (1963), 12–15.

The author announces several Leray-Schauder type theorems about the fixed points of completely continuous operators on linear topological spaces. Among them is the following generalization of a theorem of F. E. Browder [Duke Math. J. **26** (1959), 291–303; MR **21** #4368]: Let F be a completely continuous operator defined on $\bar{U}$, a closed convex subset of E, a linear locally convex topological space; let D be a convex closed subset of U, where $F^j D \subset U$, $j = 1, 2, \cdots, m-1$, $F^m R \subset D$ (F^j denotes the jth iterate of F, R is the domain of F^m, $R \subset \bar{U}$); let the convex closure of every compact set be compact in E, let the rotation of the field $x - Fx$ be unity; consequently, there is a fixed point of the operator F in D. {Related papers by the author are in Dokl. Akad. Nauk SSSR **131** (1960), 230–233 [MR **23** #A3446]; ibid. **136** (1961), 1269–1272 [MR **22** #8179].}

W. T. Kyner (Los Angeles, Calif.)

Fan, Dyk-Tin' 4361

On functionals which are concentrated on a surface. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 1 (115), 183–186.

Soient P une fonction réelle indéfiniment différentiable dans R^n telle que grad $P \neq 0$ sur la surface S définie par l'équation $P(x) = 0$, $\mathscr{D}$ l'espace des fonctions réelles indéfiniment différentiables à support compact définies sur R^n et $f = \sum_{k=1}^m b_k \delta^{(k)}(P)$ une fonctionnelle sur $\mathscr{D}$ telle que les b_k sont indéfiniment différentiables dans R^n. On prouve que f peut s'écrire d'une seule manière sous la forme $f = \sum_{k=1}^m c_k \delta^{(k)}(P)$ où les fonctions c_k sont indéfiniment différentiables dans R^n et constantes sur les normales à S dans un certain voisinage de S.

N. Dinculeanu (Bucharest)

Halperin, Israel 4362

The spectral theorem.

Amer. Math. Monthly **71** (1964), 408–410.

This paper gives a slight modification of von Neumann's original proof of the spectral theorem for bounded Hermitian operators. *F. H. Brownell* (Seattle, Wash.)

Kalisch, G. K. 4363

Direct proofs of spectral representation theorems.

J. Math. Anal. Appl. **8** (1964), 351–363.

An expository account is given of known results on integral representations of bounded operators in Hilbert space with emphasis on normal operators. Proofs depend on analytic function theory but avoid the general use of characteristic operator functions. The only non-normal operators A considered have the properties: (i) $A + A^*$ has one-dimensional range; (ii) the spectrum of A consists of zero alone. If A has no non-zero kernel, it is unitarily equivalent to a constant multiple of the operator $\varphi(t) \to \int_0^t \varphi(s)\,ds$ in $L^2(0, 1)$. *L. de Branges* (Lafayette, Ind.)

Inoue, Sakuji 4364

A note on the functional-representations of normal operators in Hilbert spaces.

Proc. Japan Acad. **39** (1963), 647–650.

The author seeks to construct a class of bounded normal operators in Hilbert space having prescribed spectral properties. The last sentence of his Theorem A contradicts the known fact that if A is a self-adjoint operator in l^2 with matrix (α_{ij}) and if $0 \neq \sum_{i,j} |\alpha_{ij}|^2 < \infty$, then A has a non-zero eigenvalue. *D. A. Edwards* (Oxford)

Inoue, Sakuji 4365

Some applications of the functional-representations of normal operators in Hilbert spaces. VII, VIII, IX.

Proc. Japan Acad. **39** (1963), 338–341; *ibid.* **39** (1963), 455–460; *ibid.* **39** (1963), 566–568.

The author here continues an investigation of the spectral theory of normal operators in Hilbert space by means of his functional representation for such operators [for the first six papers see same Proc. **37** (1961), 566–570; MR **27** #4077a; ibid. **37** (1961), 614–618; MR **27** #4077b; ibid. **38** (1962), 263–268; MR **27** #4077c; ibid. **38** (1962), 452–456; MR **27** #4077d; ibid. **38** (1962), 641–645; MR **27** #4077e; ibid. **38** (1962), 646–650; MR **27** #4077f; ibid. **38** (1962), 706–710; MR **27** #4077g; ibid. **39** (1963), 109–113; MR **27** #4077h]. All three papers use complex function theory to study, for given ζ, the zeros of the author's functions $R(\lambda) - \zeta$ and $S(\lambda) - \zeta$. The results are too complicated for summary here, and are concerned entirely with intrinsic problems of the theory. Some corrections to earlier papers are offered. *D. A. Edwards* (Oxford)

Kolomý, Josef 4366

On the generalization of Wiarda's method of solution of non-linear functional equations. (Russian summary)

Czechoslovak Math. J. **13** (88) (1963), 159–165.

The following theorem is a special case of a theorem proved by the author by reduction to a contraction mapping. Let F be a mapping of Hilbert space H into H which has a continuous Gâteaux derivative F' which is positive definite. If θ satisfies $0 < \theta < [1 + \sup_{y \in H} \|F'(y)\|]^{-1}$ and if $y_0 \in H$, then the iteration

$$y_{n+1} = (1-\theta)y_n - \theta F(y_n) + \theta f, \qquad n \geqq 0,$$

converges to the unique solution of the equation $y + F(y) = f$ lying in a neighborhood of y_1.

R. G. Bartle (Urbana, Ill.)

Naĭmark, M. A. 4367

On commuting unitary operators in spaces with indefinite metric.

Acta Sci. Math. (*Szeged*) **24** (1963), 177–189.

The author considers a Hilbert space H with the usual inner product $[x, y]$, and also an indefinite inner product (x, y) which, for some complete orthonormal set e_α in H, is defined by $(x, y) = \sum_1^k \xi_\alpha\bar\eta_\alpha - \sum_{k+1}^\infty \xi_\alpha\bar\eta_\alpha$, where $\xi_\alpha = [x, e_\alpha]$, $\eta_\alpha = [y, e_\alpha]$. Here k is a fixed positive integer less than the dimension of H. H with the structure $(\cdot, \cdot)$ is denoted by Π_k [see I. S. Iohvidov and M. G. Kreĭn, Trudy Moskov. Mat. Obšč. **8** (1959), 413–496; MR **21** #6543]. A linear operator U in Π_k is called unitary if it is onto and if $(Ux, Uy) = (x, y)$ for all x, y. There exists for such a U a k-dimensional non-negative subspace which is invariant under U [L. S. Pontrjagin, Izv. Akad. Nauk SSSR Ser. Mat. **8** (1944), 243–280; MR **6**, 273]. The author generalizes this result: If $\mathcal{U}$ is a set of commuting unitary operators U in Π_k, there exists a k-dimensional non-negative subspace invariant with respect to all $U \in \mathcal{U}$. The proof proceeds through six lemmas. An extension to hermitian operators ($(Ax, y) = (x, Ay)$) is immediate. A further theorem is that if $X_0, \cdots, X_m$ are sets of bounded operators in Π_k and $H_0, \cdots, H_{m-1}$ are hermitian; if $X_0 \supset \cdots \supset X_m$ and X_ν is generated by H_ν and $X_{\nu+1}$; if $H_\nu A - AH_\nu \in X_{\nu+1}$ for every $A \in X_{\nu+1}$; if X_m is commutative and $A \in X_m$ implies $A^* \in X_m$; then there exists a non-negative k-dimensional subspace in Π_k which is invariant with respect to all operators from X_0; there exists a vector $X_0 \neq 0$ which is a common eigenvector of all operators of X_0. This is an infinite-dimensional generalization of the Lie theorem for solvable Lie algebras.

E. R. Lorch (New York)

Bognár, János 4368

Non-negativity properties of operators in spaces with indefinite metric.

Ann. Acad. Sci. Fenn. Ser. A I No. 336/10 (1963). 9 *pp*.

Let H_k be a Hilbert space with an indefinite inner product in the sense of L. S. Pontrjagin [Izv. Akad. Nauk SSSR Ser. Mat. **8** (1944), 243–280; MR **6**, 273], I. S. Iohvidov and M. G. Kreĭn [Trudy Moskov. Mat. Obšč. **5** (1956), 367–432; MR **18**, 320]. For a bounded self-adjoint operator A on H_k, the author considers 19 non-negativity conditions which are closely related to the condition $(Ax, x) \geq 0$ for all $x \in H_k$. In case $k = 0$ (i.e., the ordinary Hilbert space), these conditions are all equivalent. The paper deals with the logical implications between these conditions in the case $k > 0$. The various implication relations are given in a diagram, but their proofs are to be found in another paper by the author [Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 201–212]. Several examples are given to show certain non-implications.

Ky Fan (Evanston, Ill.)

Pełczyński, A. 4369

On weakly compact polynomial operators on B-spaces with Dunford-Pettis property.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 371–378.

Let X and Y be B-spaces and consider a polynomial operator $P = \sum_1^k P_j$ mapping X into Y. It is proved that P is (weakly) compact [i.e., it maps bounded sets into (weakly) sequentially compact sets] if and only if its homogeneous terms P_j are (weakly) compact. Following A. Grothendieck [Canad. J. Math. **5** (1953), 129–173; MR **15**, 438] a B-space X is said to have the Dunford-Pettis property [respectively, polynomial D.-P. property] if for every B-space Y and every weakly compact linear operator [polynomial operator] $T: X \to Y$, then T maps weak Cauchy sequences into norm Cauchy sequences. It is conjectured that X has the polynomial D.-P. property if and only if X has the D.-P. property. It is proved that if X has the D.-P. property, then every homogeneous polynomial maps weak Cauchy sequences into weak Cauchy sequences.

R. G. Bartle (Urbana, Ill.)

Pełczyński, A. 4370

A theorem of Dunford-Pettis type for polynomial operators.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 379–386.

This note continues the work reviewed in the preceding paper [#4369]. It is proved that a weakly compact multilinear operator on the product of abstract L-spaces or abstract M-spaces to an arbitrary Banach space maps weak Cauchy sequences into strong Cauchy sequences. This is done by replacing the multilinear operator by a linear operator on a tensor product space. Hence, abstract L- and M-spaces have the polynomial Dunford-Pettis property.

R. G. Bartle (Urbana, Ill.)

Pfluger, Albert 4371

Verallgemeinerte Poisson-Stieltjes'sche Integraldarstellung und kontraktive Operatoren.

Ann. Acad. Sci. Fenn. Ser. A I No. 336/13 (1963), 14 *pp*.

The author gives a new proof of the spectral representation for unitary operators and generalizes it to give a spectral representation for contraction operators and for maximal symmetric operators. An analytic function F from the unit disk into the space of bounded linear operators on a Hilbert space is said to have positive real part if for each λ the operator $F(\lambda) + F^*(\lambda)$ is positive (here * denotes the adjoint). The author shows that for such an F there is a monotone mapping Φ of $[0, 2\pi]$ into the subspace of self-adjoint operators such that for $|z| < 1$ we have

$$F(z) = \int_0^{2\pi} (e^{i\theta} + z)(e^{i\theta} - z)^{-1} \, d\Phi(\theta) + i \operatorname{Im} F(0).$$

This generalization of the Poisson-Stieltjes integral formula is then used to give the representations of contraction and unitary operators.

H. L. Royden (Stanford, Calif.)

Sankaran, S. 4372

Stochastic convergence for operators.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 97–102.

Let Γ be a probability gauge space, in the sense of I. E. Segal [Ann. of Math. (2) **57** (1953), 401–457; MR **14**, 991]. The author introduces stochastic convergence of measurable operators over Γ and proves that $A_n \to A$ stochastically if and only if every subsequence of $\{A_n\}$ has a subsequence approaching A nearly everywhere. However, the author is apparently unaware of the fundamental paper of W. F. Stinespring [Trans. Amer. Math. Soc. **90** (1959), 15–56; MR **21** #1547] in which convergence

in measure for gauge spaces is defined. For probability gauge spaces, stochastic convergence and convergence in measure coincide. The present author's theorem is then just Theorems 3.1 and 5.1 in the Stinespring paper.

R. J. Blattner (Los Angeles, Calif.)

Slugin, S. N. 4373

The method of steepest descent in a Hilbert module over a finite-dimensional complex K-space. (Russian)

Dokl. Akad. Nauk SSSR **152** (1963), 834–837.

In earlier work [same Dokl. **139** (1961), 1059–1062; MR **26** #2849; ibid. **147** (1962), 306–309; MR **27** #4061] the author has defined the notion of a Banach module X over a complete vector lattice V, the "norm" of an element of X being itself an element of V. This generalizes the notion of a Banach space, and what the author calls a Hilbert module is a generalization of Hilbert space. In this paper he takes V to be the space of complex m-tuples (with coordinate-wise partial ordering) and X to be a Hilbert module over V. He considers the problem of solving the equation $Ux=f$ for a bounded, positive definite operator U from X into itself, where the notions "bounded" and "positive definite" are defined in terms of the vector-valued "scalar product" on X. An algorithm is constructed for generating approximations to the solution of the above equation, which is analogous to the algorithm in the method of steepest descent. Estimates for the rate of convergence are given, under various additional hypotheses on U.

R. R. Phelps (Seattle, Wash.)

Balachandran, V. K. 4374

The Weyl group of an L^*-algebra.

Math. Ann. **154** (1964), 157–165.

This brief paper contains eight theorems, eight propositions and eleven lemmas which extend the properties of the Killing-Weyl group familiar in the case of semi-simple Lie algebras to the L^*-algebras defined by J. R. Schue [Trans. Amer. Math. Soc. **95** (1960), 69–80; MR **22** #8352; ibid. **98** (1961), 334–349; MR **24** #A3242]. As stated by the author, the principal results are as follows: "(i) Every element of W is of finite order (Theorem 3). (ii) The centre of W is trivial (Theorem 5). (iii) The commutator W' of W in the algebra B (of all bounded operators on H) consists of only the scalar operators (Theorem 4). (iv) The Weyl group W of a semi-simple L^*-algebra L is a direct product of the Weyl groups W_j of its simple components (Theorem 6). There are two natural topologies available for W, namely, the uniform (operator) topology and the strong topology. It is shown that W is discrete in the former and non-discrete in the latter."

A. J. Coleman (Kingston, Ont.)

Banaschewski, Bernhard 4375

Maximal monoid subalgebras in group algebras.

Arch. Math. **15** (1964), 6–9.

Let G be a discrete group, $A(G)$ the group algebra of G. For M a submonoid of G (subsemigroup containing the unit element of G), $A(M)$ denotes the subalgebra of $A(G)$ generated by M, i.e., the set of elements of $A(G)$ of form $\sum_{s\in G}\alpha(s)s$, with $\alpha(s)$ complex, vanishing off M, and the series absolutely convergent. In 1955, Wermer [Proc. Amer. Math. Soc. **6** (1955), 692–694; MR **17**, 512] studied the maximality of $A(M)$ in $A(G)$, when G is totally ordered abelian and M the monoid of all non-negative elements. The author shows that the monoid subalgebras considered by Wermer are essentially the only ones which are maximal closed. Call a monoid "pure" if none of its non-unit elements has an inverse. The author proves that if M is a submonoid of the infinite abelian group G, then $A(M)$ is a maximal closed subalgebra of $A(G)$ if and only if M is a pure maximal submonoid of G. The infinite abelian group G contains pure maximal submonoids if and only if G is isomorphic to a subgroup of the additive group of real numbers, if and only if G is torsion-free and has cardinality not exceeding that of the continuum. If G is a finite abelian group of more than two elements, then no monoid subalgebra of $A(G)$ is maximal. The author's results carry over to non-abelian groups if one considers only "normal" monoids ($sMs^{-1}\subset M$ for every s in M).

A. Browder (Berkeley, Calif.)

Birtel, F. T. 4376

Singly-generated Liouville F-algebras.

Michigan Math. J. **11** (1964), 89–94.

An F-algebra A is the dense inverse limit of a sequence $A_1, A_2, \cdots$ of Banach algebras. For an element g of A let $\sigma(g, n)$ be the spectrum of g relative to A_n. Suppose that A has a single generator (in the topological sense) g for which the intersection of the frontiers of the $\sigma(g, n)$ is empty. Suppose A has a unit, and that only the scalars have uniformly bounded spectra. Then A is topologically isomorphic to the algebra of entire functions. The same result is also obtained if the condition on the frontiers is replaced by the assumption that there are no topological zero-divisors, or by the assumption that the algebra is an algebra of continuous functions on **C**.

R. Arens (Los Angeles, Calif.)

Broise, Michel 4377a

Sur les opérateurs unitaires et les fonctions de type positif.

C. R. Acad. Sci. Paris **258** (1964), 2471–2473.

Broise, Michel 4377b

Sur les représentations unitaires des groupes abéliens.

C. R. Acad. Sci. Paris **258** (1964), 3157–3160.

In the first paper, the author shows the following theorem. Let $\{A_t\}_{t\in R}$ be a family of unitary operators depending on the real parameter t. If for every $x\in H$, the function $X(t)=(A_tx, x)$ is continuous and positive definite, then the family $\{A_t\}_{t\in R}$ forms a one-parameter group.

In the second paper, the author generalizes the above theorem as follows. Theorem: Let G be an abelian group, $\{A_g\}_{g\in G}$ a family of unitary operators on a Hilbert space such that for every $x\in H$, the function $X(g)=(A_gx, x)$ is positive definite; then the family $\{A_g\}_{g\in G}$ is a representation of G (namely, $A_{g_1g_2}=A_{g_1}A_{g_2}$ for $g_1, g_2\in G$).

It is an open question whether the assumption of the commutativity of G can be dropped or not.

S. Sakai (New Haven, Conn.)

Choda, Hisashi; Echigo, Marie 4378
A new algebraical property of certain von Neumann algebras.
Proc. Japan Acad. **39** (1963), 651–655.
The authors show that, in the proof of the existence of two non-isomorphic and non-hyperfinite factors of type II_1 due to J. Schwartz [Comm. Pure Appl. Math. **16** (1963), 19–26; MR **26** #6812], Schwartz's property P may be replaced by the following: a von Neumann algebra $\mathfrak{A}$ has property Q if there is an amenable group of unitary operators in $\mathfrak{A}$ which generates $\mathfrak{A}$ (say, in the σ-weak topology). Unlike property P, property Q is a priori algebraic. The methods of proof are all essentially from Schwartz's paper. *R. J. Blattner* (Los Angeles, Calif.)

Deckard, Don; Pearcy, Carl 4379
On algebraic closure in function algebras.
Proc. Amer. Math. Soc. **15** (1964), 259–263.
Let X be a topological space, $C(X)$ the algebra of all continuous complex-valued functions on X. The authors ask when is $C(X)$ algebraically closed in the sense that every polynomial equation with coefficients in $C(X)$ and leading coefficient 1 has a solution in $C(X)$. They find that this is so: (a) if X is totally disconnected compact Hausdorff; (b) if X is any interval on the real line; (c) if X is any linearly ordered, order-complete space. Result (a) improves a previous result of the authors [same Proc. **14** (1963), 322–328; MR **26** #5438] (X Stonian); (c) evidently contains (b). Neither (a) nor (c) implies the other. The question remains open: For which compact Hausdorff X is $C(X)$ algebraically closed? *A. Browder* (Berkeley, Calif.)

Pearcy, Carl 4380
On certain von Neumann algebras which are generated by partial isometries.
Proc. Amer. Math. Soc. **15** (1964), 393–395.
In the present paper, the author shows the following theorem: There exists a partial isometry P [respectively, Q] which generates a von Neumann algebra of type II_1 [II_∞].

By combining Lemma 1 in this paper and the result of Suzuki and Saito [Tôhoku Math. J. (2) **15** (1963), 277–280; MR **27** #4094], the above theorem can be extended to the following theorem: There exists a partial isometry P [respectively, Q and R] which generates a von Neumann algebra of type II_1 [II_∞ and III]. {In general, problems of this nature can be always reduced to the construction of von Neumann algebras of type II, if we use the deep theorem of Glimm [Ann. of Math. (2) **73** (1961), 572–612; MR **23** #A2066]; in fact, let P be a partial isometry which generates a von Neumann algebra M of type II and let $\mathfrak{A}$ be a C^*-subalgebra of M generated by P; then $\mathfrak{A}$ is not of type I in the sense of Glimm, so that it has a type III-*-representation π; then the von Neumann algebra generated $\pi(P)$ is of type III and $\pi(P)$ is a partial isometry.} *S. Sakai* (New Haven, Conn.)

Pearcy, Carl 4381
Entire functions on infinite von Neumann algebras of type I.
Michigan Math. J. **11** (1964), 1–7.
The author considers algebras $\mathscr{A}=\mathfrak{Z}\otimes B$ of weakly closed abelian algebras $\mathfrak{Z}$ on Hilbert space H with the algebra B of all bounded operators on H (and, more generally, direct sums of uncountably many such algebras). Generalising a result of A. Brown [same J. **10** (1963), 91–96; MR **26** #6815] he shows that if an entire function f has the property $F(\mathscr{A})=\mathscr{A}$, then $f(\mathscr{A})=\mathscr{A}$ for every Banach algebra. Some questions concerning the representation of $\mathfrak{A}$ as a set of operator-valued functions on the "spectrum" or "maximal ideal space" of $\mathfrak{Z}$ are also studied. *J. T. Schwartz* (New York)

Dixmier, Jacques 4382
Utilisation des facteurs hyperfinis dans la théorie des C^*-algèbres.
C. R. Acad. Sci. Paris **258** (1964), 4184–4187.
For separable type I C^*-algebras, the direct integral decomposition of a *-representation into irreducible ones is essentially unique, so that we can take irreducible *-representations as building blocks in the representation theory. On the other hand, the situation for algebras not of type I is quite bad, as has been shown by various pathological phenomena.

In the present paper, the author shows that four pathological phenomena (too many to state here), each of which tells us the difficulty of the reducibility problem in the representation theory of algebras not of type I, occur always in every separable C^*-algebra not of type I. These phenomena are given as applications of the following theorem obtained by help of Glimm's result [Ann. of Math. (2) **73** (1961), 572–612; MR **23** #A2066]. Theorem 1: Let A be a separable C^*-algebra not of type I. (i) There is a *-representation π of A in a separable Hilbert space such that $\pi(A)'$ is hyper-finite. (ii) If A is primitive NGCR, there is a faithful π as above. *S. Sakai* (New Haven, Conn.)

Glicksberg, I. 4383
Some uncomplemented function algebras.
Trans. Amer. Math. Soc. **111** (1964), 121–137.
Let X be a locally compact Hausdorff space, and A a closed subalgebra of $C_0(X)$, the sup-normed algebra of all continuous complex-valued functions on X vanishing at infinity. The author raises the question: When does A admit a (closed) complementary subspace, i.e., when does there exist a bounded projection of $C_0(X)$ onto A? He obtains partial results, notably: If X is a compact group or locally compact abelian group, and A is translation-invariant, then A admits a complement if and only if A is self-adjoint. This generalizes a theorem of Rudin (obtained also by Arens and Curtis) to the effect that the familiar "disk algebra" on the unit circle admits no complement. Rudin's method yielded also another proof of D. J. Newman's result that the Hardy class H^1 is uncomplemented in L^1 of the circle; the author obtains a generalization of this fact as well.

The general question (X not a group) remains open. The author does prove that if X is the Šilov boundary for the closed subalgebra A of $C_0(X)$, B a proper closed A-module without common zeroes on X, then any projection of $C_0(X)$ onto B must have norm strictly greater than 2. *A. Browder* (Berkeley, Calif.)

Hewitt, Edwin; Kakutani, Shizuo 4384
Some multiplicative linear functionals on $\mathbf{M}(G)$.
Ann. of Math. (2) **79** (1964), 489–505.
The authors continue their study of multiplicative linear functionals on measure algebras [Illinois J. Math. **4** (1960), 553–574; MR **23** #A527; see also Simon, ibid. **5** (1961), 398–408; MR **24** #A1635]. Theorem (3.1): Let H be a locally compact abelian group with character group Γ. Let ϕ be a continuous mapping of a compact abelian group G into H. Let μ be normalized Haar measure on G; μ and ϕ induce a measure ρ on H. Suppose that there is a sequence $(\psi_n)_{n=1}^{\infty}$ in Γ such that

$$\lim_{n\to\infty}\int_G \chi\cdot(\psi_n\circ\phi)\,d\mu = 0$$

for every character χ of G different from the identity, and $\lim_{n\to\infty}\int_G(\psi_n\circ\phi)\,d\mu=\gamma$ for some complex number γ where $|\gamma|\leqq 1$. Then there is a multiplicative linear functional M on the measure algebra $\mathbf{M}(H)$ such that $M(\sigma)=\gamma\cdot\sigma(H) = \lim_{n\to\infty}\int_H \psi_n\,d\sigma$ for all measures σ absolutely continuous with respect to ρ. Moreover, regarding Γ as embedded in the structure space of $\mathbf{M}(H)$, M belongs to $\bigcap_{k=1}^{\infty}\{\psi_k,\psi_{k+1},\cdots\}^-$.
By means of intricate calculations, the authors construct measures ρ and functionals M as in (3.1) in the following cases: (4.3) H is the real line R and the support of ρ is a very thin set; (4.4) $H=R$ and the support of ρ is the Cantor set; (4.8) H is the group of a-adic integers or numbers. In each case, G turns out to be a countably infinite product of finite cyclic groups. The principal result of Šreĭder [Mat. Sb. (N.S.) **29 (71)** (1951), 419–426; MR **13**, 755] is also a consequence of (3.1).
K. A. Ross (Rochester, N.Y.)

Sherbert, Donald R. 4385
The structure of ideals and point derivations in Banach algebras of Lipschitz functions.
Trans. Amer. Math. Soc. **111** (1964), 240–272.
The object mainly studied here is the class $\mathrm{Lip}(X,d)$ of all bounded Lipschitz functions on a space X with a metric d. With the least possible Lipschitz constant for f as the norm of f, this is a Banach algebra under ordinary multiplication. The structure of the ideals $J(K)$ generated by the functions vanishing in a neighborhood of K is described when X is compact; they are the intersections of closed primary ideals. The point-derivations of $\mathrm{Lip}(X,d)$ are also investigated.
R. Arens (Los Angeles, Calif.)

Varopoulos, Nicholas Th. 4386
Sur les mesures de Radon d'un groupe localement compact abélien.
C. R. Acad. Sci. Paris **258** (1964), 3805–3808.
Let G be a locally compact Abelian group with character group X. A Θ-measure $\mu\in M^+(G)$ is a measure such that all convolution powers $\varepsilon_e,\mu,\mu^2,\cdots$ are mutually singular and also $\hat\mu\in\mathfrak{C}_0(X)$. The main result is that $\mathbf{P}_{n=1}^{\infty}Z(p_n)$ has a Θ-measure if $p_1,p_2,\cdots$ are prime numbers. This implies that $\{\mu\in M(\mathbf{P}_{n=1}^{\infty}Z(p_n))\colon \hat\mu\in\mathfrak{C}_0\}$ is asymmetric and that the only functions operating on the transforms of this class are entire. *Edwin Hewitt* (Seattle, Wash.)

Varopoulos, Nicolas Th. 4387
[Varopoulos, Nicholas Th.]
Continuité des formes linéaires positives sur une algèbre de Banach avec involution.
C. R. Acad. Sci. Paris **258** (1964), 1121–1124.
Let R be a commutative Banach algebra with involution $x\to x^*$ which is an isometry. In the note under review, some sufficient conditions for a positive linear functional f (i.e., a linear functional f such that $f(xx^*)\geqq 0$ for each x in R) to be continuous are established. The main theorem states that if R^3 (the smallest linear subspace of R generated by the elements of the form xyz, where $x,y,z\in R$) is a closed subspace of R of a finite codimension, each positive linear functional on R is continuous. The hypothesis of the main theorem is satisfied if R possesses an approximate identity, for in this case $R=R^2$ by a theorem of P. Cohen [Duke Math. J. **26** (1959), 199–205; MR **21** #3729].
I. Namioka (Seattle, Wash.)

Babbitt, Donald 4388
The Wiener integral and perturbation theory of the Schrödinger operator.
Bull. Amer. Math. Soc. **70** (1964), 254–259.
Let $V(x)$ be a real-valued function on E^m. Under certain conditions on V, a semigroup $\{T_t^V\}$ of bounded self-adjoint operators on $L^2(E^m)$ can be defined by

$$(T_t^V\psi)(x) = E\left(\exp\left[-\int_0^t V(x(t))\,dt\right]\psi(x(t))\,|\,x(0)=x\right),$$

where E denotes the Wiener integral (expectation) over the Wiener process which begins at x at time 0. The infinitesimal generator of the semigroup $\{T_t^V\}$ is seen to be an extension of the differential operator $\Delta-V$ restricted to $C_0^{\infty}(E^m)$. The dependence of $\{T_t^V\}$ on V is considered in detail, which leads to useful results on the perturbation theory of the operator $\Delta-V$ when V is changed. Details of the proofs are not given. *T. Kato* (Berkeley, Calif.)

Daleckiĭ, Ju. L. 4389
Functional integrals associated with operator evolution equations. (Russian)
Uspehi Mat. Nauk **17** (1962), *no.* **5** (107), 3–115.
Quasi-measures and functional integrals, related to transition functions in the same manner as the Wiener integral to the transition probabilities of the Brownian process are constructed. Applications to the Cauchy initial-value problem are discussed. This is generalized to a construction of functional integrals $\int\Phi[x(t)]\,d\mu$, where $x(t)$ is from $(0,l)$ to a vector space B, Φ a functional on B, $\mu(q,\Gamma)$ is the measure of a parallelepiped $Q(q,\Gamma)$ in function space where for a partition $q=(t_0,t_1,\cdots,t_n)$ of $(0,l)$ and a choice $\Gamma=(\gamma_1,\cdots,\gamma_n)$ of sets $\gamma_k\subset B$, $Q(q,\Gamma)$ is the set of $x(t)$ such that $x(t_k)\in\gamma_k$, and μ is vector-valued so that the product $\Phi\mu$ is meaningful. The integral is considered as a generalized element in a Hilbert space. μ can be defined in terms of the transition function $U(t,\tau)$, where $U(t,\tau)\varphi$ is the solution of $d\psi/dt=-A(t)\psi$ with $\psi(0)=\varphi$. The integrals constructed are used to solve initial-value problems for the equation $d\psi/dt=[-A(t)+B(t)]\psi$, where $B(t)$ is self-adjoint and A and B satisfy various conditions: in one case, of boundedness of operators of the form $BA^{-\gamma}$, $0\leqq\gamma\leqq 1$, in another of a

type which makes the equation $d\psi/dt = iA\psi$ analogous to a hyperbolic equation in the fashion in which subsets of the spectrum of B are displaced, in a third case having properties which make the equation a generalization of ordinary parabolic equations. *J. L. B. Cooper* (Cardiff)

Gasymov, M. G. 4390

Applications of an inequality for the sum of the differences of the eigenvalues of two self-adjoint operators. (Russian. Azerbaijani summary)

Akad. Nauk Azerbaĭdžan. SSR Dokl. **20** (1964), *no.* 1, 3–8.

A study of the spectra of self-adjoint differential operators is based on the following inequality. Let A and C be self-adjoint operators in the same space, which are bounded below, have discrete spectra with finite multiplicities, and have the same domain. Let $\lambda_1 \leqq \lambda_2 \leqq \cdots$ and $\mu_1 \leqq \mu_2 \leqq \cdots$ be the eigenvalues of A and C, respectively, and let $\varphi_1, \varphi_2, \cdots$ and $\psi_1, \psi_2, \cdots$ be corresponding orthonormal sets of eigenvectors. Then

$$\sum_1^N (B\psi_n, \psi_n) \leqq \sum_1^N (\mu_n - \lambda_n) \leqq \sum_1^N (B\varphi_n, \varphi_n),$$

where $B = C - A$. (Several variants and generalizations are also given.) The following result is a typical application. Let $A = d^2/dx^2 - q(x)$ and $B = d^2/dx^2 - p(x) - q(x)$ in $L^2(0, \pi)$, where $p(x)$ and $q(x)$ are real-valued and continuous. A necessary and sufficient condition that $\sum_{\lambda_n < \lambda} (\mu_n - \lambda_n)$ and $\sum_{\mu_n < \lambda} (\mu_n - \lambda_n)$ are $o(\sqrt{\lambda})$ as $\lambda \to +\infty$ is that $\int_0^\pi p(x)\,dx = 0$. In this case the Gel'fand-Levitan formula, $\sum (\mu_n - \lambda_n) = \frac{1}{4}[p(0) + p(\pi)]$, holds. Similar results are stated for self-adjoint, second-order, partial differential operators in a region and for differential operators perturbed by integral operators. These results should be compared with those of C. J. A. Halberg, Jr. and V. A. Kramer [Duke Math. J. **27** (1960), 607–617; MR **22** #5894].

L. de Branges (Lafayette, Ind.)

Hirschman, I. I., Jr. 4391

Extreme eigen values of Toeplitz forms associated with Jacobi polynomials.

Pacific J. Math. **14** (1964), 107–161.

Let $\alpha, \beta > 1$ be fixed and let $P_n^{(\alpha,\beta)}(x)$ denote the Jacobi polynomials. If $w(x) = (1-x)^\alpha(1+x)^\beta$, then

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) w(x)\,dx = \delta(n, m) h_n$$

for suitable constants h_n. Let $t(x)$ be a real function in $L^1(w)$ and call C_n the $(n+1) \times (n+1)$ matrix whose (j, k)-entry is

$$(h_j h_k)^{-1/2} \int_{-1}^{1} P_j^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(x) t(x) w(x)\,dx.$$

Then C_n is the generalized Toeplitz matrix associated with t. If the eigenvalues of C_n be written as $\lambda_{n,1} \geqq \lambda_{n,2} \geqq \cdots \geqq \lambda_{n,n+1}$, the author has in previous work [J. Analyse Math. **12** (1964), 187–242] investigated the asymptotic behavior of $\lambda_{n,k}$ (for fixed k) as $n \to \infty$ whenever t has an absolute maximum at a single point x_0 and is regularly behaved near x_0; it was assumed that t was continuously differentiable near x_0 and that $t''(x_0) < 0$ if $x_0 \neq \pm 1$ and that $t'(1) > 0$ if $x_0 = 1$. In the present pap[er] these conditions are relaxed so that any order of conta[ct] with the maximum is allowed. Thus in the case $x_0 = 1$ is assumed that $t(1) - t(x) \sim (1-x)^\omega L(1-x)$ near $x =$ where $\omega > 0$ and L is slowly oscillating, and it is prove[d] that $\lambda_{n,k} = t(1) - 2^{-\omega} L(n^{-2}) n^{-2\omega}[\mu_k + o(1)]$, where $0 < \mu_1$ $\mu_2 \leqq \cdots$ are the eigenvalues of an operator dependin[g] only upon ω and α. *H. Widom* (Ithaca, N.Y[.])

Jeanquartier, Pierre 439[2]

Distributions homogènes et invariantes, opérateu[rs] différentiels associés.

C. R. Acad. Sci. Paris **258** (1964), 2963–2965.

Author's summary: "Soit $\mathscr{H}_\nu$ l'espace des distributio[ns] $f(x, t)$ $(x \in R^n, t \in R)$ invariantes par le groupe des tran[s]formations linéaires de R^{n+1} qui conservent t et u[ne] forme quadratique $u = u(x)$ de signature quelconque [et] homogènes au sens suivant: $f(\lambda x, \lambda^a t) = \lambda^\nu f(x, t)$, po[ur] tout $\lambda > 0$, a étant un nombre rationnel > 0. On donn[e] une représentation paramétrique de $\mathscr{H}_\nu$ à l'aide du du[al] d'un espace de fonctions d'une variable réelle et l'o[n] considère les opérateurs différentiels à coefficients co[n]stants qui conservent $\mathscr{H} = \bigcup \mathscr{H}_\nu$."

J. Elliott (New Yor[k])

Kisyński, J. 439[3]

Sur les opérateurs de Green des problèmes de Cauch[y] abstraits.

Studia Math. **23** (1963/64), 285–328.

Let X be a Banach space, and consider an abstrac[t] Cauchy problem in X: (1) $dx(t)/dt = A(t)x(t)$ for $t \in [0, T$ $x(0) = x_0$. Here $A(t)$ is a closed linear operator wit[h] domain $D(A(t))$ dense in X and range $R(A(t))$ in X. Th[e] author assumes essentially the same hypothesis a[s] T. Kato [J. Math. Soc. Japan **5** (1953), 208–234; MR 1[5] 437]: (i) $D(A(t))$ is independent of t so that $Y = D(A(t))$ i[s] dense in X. (ii) Each $A(t)$ is the infinitesimal generat[or] of a contraction semi-group of class (C_0) so that th[e] resolvent $(I - n^{-1}A(t))^{-1}$ exists with the estimat[e] $\|(I - n^{-1}A(t))^{-1}\| \leqq 1$ for $n = 1, 2, \cdots$. (iii) For ever[y] $y \in Y$, $A(t)y$ is weakly continuously differentiable wit[h] respect to t. To solve (1), consider an approximate Cauch[y] problem: (2) $dx_n(t)/dt = A_n(t)x_n(t)$ for $t \in [0, T]$, $x(0) = x_0$ where $A_n(t) = A(t)(I - n^{-1}A(t))^{-1}$ [cf. the reviewer, Pro[c.] Internat. Congr. Mathematicians (Amsterdam, 1954 Vol. 1, pp. 405–420, Noordhoff, Groningen, 1957; MR 2[0] #1920]. Since $A_n(t)$ is a bounded linear operator on X int[o] X which is strongly continuous in t, we may construct fundamental solution $G_n(t, s)$, $0 \leqq s, t \leqq T$, of (2) as follows $G_n(s, s) = I$, $G_n(t, r)G_n(r, s) = G_n(t, s)$ for $0 \leqq r \leqq s \leqq t \leqq T$ $\partial G_n(t, s)/\partial t = A_n(t)G_n(t, s)$ and $\partial G_n(t, s)/\partial s = -G_n(t, s)A_n(s)$. Since $A(t)$ is closed, Y becomes a Banach space wit[h] respect to the norm $|||y|||$ which is equivalent to $\|y\| +$ $\|A(t)y\|$. It is proved that, for $y \in Y$,

$$\|G_n(t, s)y - G_m(t, s)y\| \leqq C|||y||| + \|G_{n,k}(t, s)y - G_{m,k}(t, s)y\|,$$

where C is a constant and

$$G_{n,k}(t, s) = \exp\left((t-s)A_n\left(\frac{i-1}{k}T\right)\right)$$

for $((i-1)/k)T \leqq s$, $t \leqq (i/k)T$ $(i = 1, 2, \cdots, k)$. By th[e] semi-group, $s\text{-}\lim_{n\to\infty} G_{n,k}(t, s)y$ exists for fixed k, and so by the above inequality, $x(t) = s\text{-}\lim_{n\to\infty} G_n(t, s)x_0$ exist[s]

for $x_0 \in Y$. By introducing the operator $H_n(t, s) = (I-A(t))G_n(t, s)(I-A(s))^{-1}$ which satisfies the equation

$$H_n(t, s) = G_n(t, s) - \int_s^t G_n(t, r)\frac{dA(r)}{dr}(I-A(r))^{-1}H_n(r, s)\,dr,$$

it is proved that $x(t)$ satisfies (1). *K. Yosida* (Tokyo)

Ladohin, V. I. 4394

Evaluation of continual integrals of the functionals

$$\Phi\left[\int_0^T \alpha_1(\tau)\,dx(\tau); \cdots; \int_0^T \alpha_m(\tau)\,dx(\tau)\right].$$

(Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 1 (115), 155–159.

Let M be the space of step-functions continuous on the right $t \rightarrow x(t)$ defined for $0 \leq t < +\infty$ with $x(0)=0$; and let us consider the functional

$$\Phi\left[\int_0^T \alpha_1(\tau)\,dx(\tau); \cdots; \int_0^T \alpha_m(\tau)\,dx(\tau)\right]$$

with $\alpha_k(\tau)$ real-valued and continuous functions with $\alpha_k(0)=0$, for $k=1, \cdots, m$. Ju. L. Daleckiĭ has studied [see #4389 above; English transl., Russian Math. Surveys **17** (1962), no. 5, 1–107] some special classes of functionals associated with certain differential evolution equations for which there exist continuous integrals with respect to certain quasi-measures. In this paper sufficient conditions are given for the existence of continuous integrals connected with parabolic and hyperbolic equations.

G. Geymonat (Pavia)

Lions, J. L. [Lions, Jacques-Louis] 4395

Remarques sur les espaces d'interpolation et les problèmes aux limites.

Les Équations aux Dérivées Partielles (Paris, 1962), *pp.* 75–86. *Éditions du Centre National de la Recherche Scientifique, Paris,* 1963.

Expository paper describing the topics connected with the "interpolation space". The contents are: (i) Two modes of construction of the spaces of interpolation [cf., e.g., the author, C. R. Acad. Sci. Paris **250** (1960), 2104–2106; MR **22** #4940]; (ii) Espaces de traces et semi-groupes [cf., e.g., the author, Math. Scand. **9** (1961), 147–177; MR **28** #2429]; (iii) Fractional power of operators and complex spaces (cf., e.g., the paper referred to in (i)); (iv) Spaces of interpolation and elliptic boundary-value problems [c.f., e.g., T. Kato, Nagoya Math. J. **19** (1961), 93–125; MR **26** #631]; (v) Spaces of interpolation and operational differential equations [cf., e.g., the author, *Equations différentielles opérationnelles et problèmes aux limites*, Springer-Verlag, Berlin, 1961; MR **27** #3935].

K. Yosida (Tokyo)

Malgrange, Bernard 4396

Sur la propagation de la régularité des solutions des équations à coefficients constants.

Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **3 (51)** (1959), 433–440.

Let $D \neq 0$ be a linear differential operator on R^n with constant coefficients. Theorem 1: Let $\Omega \subset R^n$ be open and convex, $p \geqq n+2$ be a continuous real-valued function on Ω. If φ is a distribution on Ω having compact support, such that $D\varphi$ is $[p(x)]$ continuously differentiable in some neighborhood of x for every $x \in \Omega$, then φ is $[p(x)]-n-2$ continuously differentiable in some neighborhood of x for every $x \in \Omega$ (where $[\lambda]$ denotes the largest integer smaller than $\lambda \in R$). The author remarks that a similar result holds if "continuously differentiable" is replaced by "L^2 differentiable". The above "regularity theorem" is used to prove the following result due to Ehrenpreis. Theorem 2: Let $\Omega \subset R^n$ be open and convex. For every distribution g on Ω, there is some distribution f on Ω such that $Df=g$. Theorem 1 is also used to establish the following results about "propagation of regularity". Proposition 1: Let $\Omega \subset R^n$ be open and φ a distribution on Ω which is indefinitely differentiable outside $L \subset \Omega$ compact. If $K \subset L$ is compact and convex such that $D\varphi$ is indefinitely differentiable outside K, then φ has the same property. Theorem 3: Let $0 \in E$ be an extreme point of $E \subset R^n$, E closed and convex. There is a basis $\mathfrak{B}$ of open neighborhoods of 0 in R^n such that every $\Omega \in \mathfrak{B}$ has the following property: If a distribution f on Ω satisfies $Df=0$ on Ω and is indefinitely differentiable on $\Omega - E$, then f is indefinitely differentiable on Ω ($\mathfrak{B}$ is independent of D). An example due to Zerner shows that Theorem 3 may fail to hold simultaneously for all D if 0 is a boundary point which is not extremal for E. The author remarks that Theorem 3 overlaps with results obtained independently and about the same time by F. John [Comm. Pure Appl. Math. **13** (1960), 551–585; MR **24** #A317].

L. Nachbin (Paris)

Nelson, Edward 4397

Feynman integrals and the Schrödinger equation.

J. Mathematical Phys. 5 (1964), 332–343.

This paper is concerned with the group or semigroup $\{U^t_{m,V}\}$ of contraction operators generated by the l-dimensional Schrödinger operator $i[(1/2m)\Delta - V(x)]$, where m is the mass of the particle and V is a real function which may have very strong singularities not heretofore considered. The idea is to construct $U^t_{m,V}$ by analytic continuation starting from pure imaginary m with $\operatorname{Im} m > 0$. The only assumption on V is that V is continuous on the complement of a closed set F of capacity 0. If m is pure imaginary with $\operatorname{Im} m > 0$, $U^t_{m,V}$ can be defined directly by the Wiener integral

$$U^t_{m,V}\psi(x) = \int_\Omega \exp\left[-i\int_0^t V(\omega(s))\,ds\right]\psi(\omega(t))\,Pr_x(d\omega)$$

for any $\psi \in L^2(R^l)$, where Ω is the set of all trajectories $\omega(t)$ and Pr_x is the Wiener measure on Ω. Then it is shown that (1) $U^t_{m,V} = \text{strong}\lim_{n\to\infty}(K_m^{t/n}M_V^{t/n})^n$, where $K_m^t = \exp(it\Delta/2m)$ and $M_V^t = \exp(-itV)$. This shows that $\{U^t_{m,V}\}_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(R^l)$, and the associated infinitesimal generator is seen to be an extension of $i[(1/2m)\Delta - V]$ restricted to C_0^2 (the set of functions of class C^2 with compact supports in $R^l - F$). Now K_m^t is holomorphic in m in the upper half-plane. Hence $(K_m^{t/n}M_V^{t/n})^n$ has the same property, and, since it is uniformly bounded, the limit (1) exists and defines $U^t_{m,V}$ as a holomorphic function of m in this half-plane. Finally, $U^t_{m,V}$ is defined for real m as the boundary value of this holomorphic function except for a

set N of m with measure 0 (the Fatou-Privaloff theorem). An interesting application of the result is given to the potential $V = -1/r^2$ in R^3. It turns out that if m is real and $> \frac{1}{8}$, $\{U^t_{m,V}\}$ is a semigroup but not a group, the infinitesimal generator being a dissipative (but not symmetric) extension of $i[(1/2m)\Delta - V]$ restricted to $C_0^2(R^3 - \{0\})$. Physically this is interpreted to correspond to collision of the particle with the center of force. There are expository appendices devoted to Wiener integrals and to semigroup theory, which are very useful to non-specialists. *T. Kato* (Berkeley, Calif.)

Nelson, Edward 4398

L'équation de Schrödinger et les intégrales de Feynman.

Les Équations aux Dérivées Partielles (Paris, 1962), pp. 151–158. Éditions du Centre National de la Recherche Scientifique, Paris, 1963.

An expository article on the solution of the Schrödinger equation by means of the Wiener integral and analytic continuation. Details are given in the preceding paper [#4397]. *T. Kato* (Berkeley, Calif.)

Odhnoff, Jan 4399

Un exemple de non-unicité d'une équation différentielle opérationnelle.

C. R. Acad. Sci. Paris **258** (1964), 1689–1691.

Let $\mathfrak{H}$ be a separable Hilbert space. The author produces a closed operator A with dense domain and a regular $\mathfrak{H}$-valued function $u(\cdot)$ such that $u'(t) + Au(t) = 0$ for $t \in [0, 2]$ and such that $u(0) = u(2) = 0$ but $u(t) \neq 0$ for $t \in (0, 2)$. The operators A and A^* have the same domain. {The last seven lines on p. 1690 repeat the lines immediately above and should be deleted.} *R. J. Blattner* (Los Angeles, Calif.)

Olariu, V. 4400

Problèmes aux limites normaux.

Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **4 (52)** (1961), *no.* 1–2, 69–75 [1963].

The author makes some remarks about expansion theorems for differential operators which are normal, and applies these remarks to a mixed problem in partial differential equations. *R. R. D. Kemp* (Kingston, Ont.)

Palamodov, V. P. 4401

Fourier transforms of infinitely differentiable functions of rapid growth. (Russian)

Trudy Moskov. Mat. Obšč. **11** (1962), 309–350.

A given function space $\mathscr{E}_1$ of type $\mathscr{E}$ is interpreted as the space of operators of multiplication by functions of $\mathscr{E}_1$ on a corresponding space S_1 of type S of infinitely differentiable and rapidly decreasing functions. The Fourier image $\tilde{S}_1$ of S_1 also consists of infinitely differentiable and rapidly decreasing functions and the Fourier image of $\mathscr{E}_1$ is then a space of convolutions on $\tilde{S}_1$. Such a Fourier image is characterised by the author as the dual of a space of type $\mathscr{E}$ explicitly described in terms of $\mathscr{E}_1$. This generalises theorems of Paley-Wiener-Schwartz [L. Schwartz, *Théorie des distributions*, Tome I, Actualités Sci. Indust., No. 1091, Hermann, Paris, 1950; MR **12**, 31; Tome II, Actualités Sci. Indust., No. 1122, Hermann, Paris, 1951; MR **12**, 833], Gel'fand and Šilov [*Generalized functions* (Russian), Part 2, Chap. 3, Fizmatgiz, Moscow, 1958; MR **21** #5142a], and others. The advance over previous work lies in the fact that the author obtains information concerning both the smoothness and the rate of growth of the Fourier transforms of increasing functions.

Let Φ, Ψ be locally convex linear topological spaces of functions or functionals. Let $\mathscr{L}(\Phi, \Psi)$ denote the space of continuous linear maps from Φ into Ψ with the topology of uniform convergence on the bounded sets of Φ. Let $\mathscr{S}$ be the Schwartz space of rapidly decreasing functions. The space of multiplications $M(\Phi, \Psi)$ is defined as the closure in $\mathscr{L}(\Phi, \Psi)$ of the operators of multiplication by those functions $\sigma \in \mathscr{S}$ such that $\varphi \in \Phi$ implies $\sigma\varphi \in \Psi$. The space of convolutions $C(\Phi, \Psi)$ is defined using convolution instead of multiplication. Put $M(\Phi) = M(\Phi, \Phi)$ and $C(\Phi) = C(\Phi, \Phi)$.

With the usual multidimensional notation, put $e_{\alpha,A}(x) = \exp((\alpha/e)|x/A|^{1/\alpha})$ if $\alpha > 0$; $e_{0,A}(x) = 1$ if $|x| \leq A$, $= \infty$ if $|x| > A$. Let $S^{\beta,B}_{\alpha,A}$, $\mathscr{E}^{\beta,B}_{\alpha,A}$ be the Banach spaces with norms

$$\|\varphi\|_A{}^B = \sup_{x,q} \left| e_{\alpha,A}(x) \frac{D^q\varphi(x)}{B^q q^q} \right|,$$

$$\|\varphi\|_{-A}{}^B = \sup_{x,q} \left| e_{\alpha,A}(x)^{-1} \frac{D^q\varphi(x)}{B^q q^q} \right|,$$

respectively. The spaces of type $\mathscr{E}$ and of type S are constructed by taking inductive and projective limits; for example,

$$\check{\mathscr{E}}^{\alpha-}_{\beta+} = \lim_{A\to 0} \mathrm{pr} \lim_{B\to 0} \mathrm{ind}\, \mathscr{E}^{\alpha,A}_{\beta,B},$$

$$\hat{\mathscr{E}}^{\beta-}_{\alpha+} = \lim_{A\to 0} \mathrm{ind} \lim_{B\to 0} \mathrm{pr}\, \mathscr{E}^{\beta,B}_{\alpha,A},$$

$$S^{\alpha-}_{\beta-} = \lim_{B\to 0,\, A\to 0} \mathrm{pr}\ S^{\beta,B}_{\alpha,A}.$$

The principal results consist of thirty-six isomorphisms concerning $M(\mathscr{E})$, $M(S)$, $C(\mathscr{E})$, $C(S)$, $\tilde{\mathscr{E}}$, $\tilde{S}$, where $\mathscr{E}$, S denote generically a space of type $\mathscr{E}$ and a space of type S. Thus $\widetilde{\hat{\mathscr{E}}^{\beta+}_{\alpha+}} \approx \mathscr{E}^{\alpha-*}_{\beta-}$, the asterisk signifying strong dual. These results are achieved by an interesting process of approximation at the level of the intervening spaces $S^{\alpha,A}_{\beta,B}$, $\mathscr{E}^{\alpha,A}_{\beta,B}$. *E. J. Akutowicz* (Bologna)

Palamodov, V. P. 4402

Well-posed solvability conditions in the large for a certain class of equations with constant coefficients. (Russian)

Sibirsk. Mat. Ž. **4** (1963), 1137–1149.

The arguments in this paper depend upon notions developed in the paper of the preceding review [#4401]. Consider a partial differential operator with constant coefficients $p(D)$, where $p(s) = p(s_1, s_2, \cdots, s_n)$ is a polynomial without real zeros. It then follows that there is a domain $T = \{s \in C^n : \exists \sigma \in R^n, |s - \sigma| \leq c|\sigma + i|^\gamma\}$ in which $p(s)$ remains different from zero. The author derives an explicit formula for the least upper bound of numbers γ for which this holds.

The main part of the paper is devoted to determining a class of right-hand sides w of the equation (*) $p(D)u = w$ such that in some other class there exists a unique solution u. Such classes are called classes of correctness. It

turns out that the cases $\gamma \geqq 0$ and $\gamma < 0$ are quite distinct as far as the existence of maximal classes of correctness is concerned. In the case $\gamma < 0$ it is observed that if the right-hand side grows no faster than a power of $|x|$ and has sufficiently high smoothness, then a bona fide solution u (i.e., sufficiently differentiable solution) of (*) exists. Conversely, if a bona fide solution u of (*) exists when w grows no faster than a power of $|x|$, then w must have a degree of smoothness which is the higher, the greater is its growth at infinity.

Assume $\gamma < 0$. Put $H_{(k)} = \{\varphi(x) : |x+i|^k \varphi(x) \in L_2\}$. The following are then classes of correctness: (1) $w \in \hat{\mathscr{E}}^{\beta\,-}_{\alpha\,+}$ ($\alpha > 1$, $\alpha + \beta\gamma \geqq 1$) and $u \in \hat{\mathscr{E}}^{\beta\,-}_{\alpha\,+}$; (2) $D^j w \in H_{(-k)}$, $|j| \leqq -\gamma(k+m)$, $k > 0$, and $u \in H_{(-k)}$; (3) $D^j w \in H_{(k)}$, $|j| \leqq -\gamma(k+m)$, $k > 0$, and $u \in H_{(k)}$; (4) $w \in S^{\beta\,-}_{\alpha\,-}$ ($\alpha > 1$, $\alpha + \beta\gamma \geqq 1$) and $u \in S^{\beta\,-}_{\alpha\,+}$. The number m is the degree of p.

Assume $\gamma \geqq 0$. Put $E_{\alpha,A} = \{\varphi(x) : \exp(|x/A|^{1/\alpha})\varphi(x) \in L_2\}$. The following are then classes of correctness: (1) $w \in E^*_{\alpha,A}$, $\alpha \geqq 1$, and $u \in E^*_{\alpha,A_1}$ for some A, A_1; (2) $w \in H_{(-k)}$ and $u \in H_{(-k)}$; (3) $w \in H_{(k)}$ and $u \in H_{(k)}$; (4) $w \in E_{\alpha,A}$, $\alpha \geqq 1$, and $u \in E_{\alpha,A_1}$ for some A, A_1.

E. J. Akutowicz (Bologna)

Gol'denštein, L. S. 4403

Criteria for one-sided invertibility of functions of several isometric operators and their applications. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 28–31.

The author studies methods for solving certain classes of singular integral equations using results from the theory of commutative Banach algebras. Some criteria for deciding whether an element of a certain Banach algebra is invertible are given. As an application a necessary and sufficient condition for the existence of a unique solution to a multidimensional Wiener-Hopf type equation is given.

J. E. Cigler (Vienna)

CALCULUS OF VARIATIONS

See also 4466, 4972, 4973.

Troickiĭ, V. A. 4404

The Mayer-Bolza problem of the calculus of variations and the theory of optimum systems.

Prikl. Mat. Meh. **25** (1961), 668–679 (*Russian*); *translated as J. Appl. Math. Mech.* **25** (1962), 994–1010.

This paper seeks first-order necessary conditions for a Mayer problem with differential and "finite" side-conditions in a state vector x and a control vector u, apparently with respective restriction to classes of piecewise smooth and piecewise continuous functions. Multipliers that tacitly assume normality are used but no multiplier rule applicable to such admissable functions is proved or cited. The author concludes for the case of differential constraints that are linear with constant coefficients, and consistent with various authors including Pontryagin et al. [*The mathematical theory of optimal processes*, Chapter 3, Interscience, New York, 1962] and Leitmann [*Optimization techniques*, Chapter 5, pp. 171–204, Academic Press, New York, 1962; MR **27** #3467], that if the control u is bounded, then only boundary values can occur.

G. M. Ewing (Norman, Okla.)

GEOMETRY

See also 4439, 4443, 4449, 4450, 4451.

Jaglom, I. M. [Яглом, И. М.] 4405

★Complex numbers and their application in geometry [Комплексные числа и их применение в геометрии].

Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1963. 192 *pp.* 0.31 *r.*

The introductory sections of the book's three chapters are accessible to many high school students. Chapter I contains sections on the complex field, quadratic fields and quaternions. Chapter II is devoted to geometric interpretations of complex numbers and elements of quadratic fields and the role of complex numbers in Lobachevskian geometry. Chapter III contains material on linear fractional transformations and two sections on their role in Lobachevskian geometry.

The author never introduces an idea, the cross-ratio for example, without using it to prove a succession of geometric theorems. Some of these theorems are exceedingly intricate. Figure 15 involves nine circles, eleven line segments, and faintly printed letters and has a most jumbled appearance.

N. D. Kazarinoff (Ann Arbor, Mich.)

Pol'skiĭ, N. I. [Польский, Н. И.] 4406

★On various geometries [О различных геометриях].

Second edition, revised.

Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1962. 100 *pp.* 0.15 *r.*

This book is written for those who have taken a high school course in Euclidean geometry, and its goal is to introduce its readers to the idea of geometry as an axiomatic system and to show them models of many such systems. The connections between non-Euclidean geometry and relativity are discussed.

N. D. Kazarinoff (Ann Arbor, Mich.)

Brehmer, S. 4407

Zur freien Beweglichkeit in der Ebene.

Acta Math. Acad. Sci. Hungar. **15** (1964), 47–52.

Let V be a two-dimensional vector space over an ordered commutative field K. One-dimensional subspaces of V are called lines through O. The author considers a group $\mathfrak{G}$ of linear transformations of V with the following properties: (I) $\mathfrak{G}$ is transitive on the lines through O, (II) $P\mathfrak{G} \cap l$ is finite for any $P \in V$ and any line l through O. ($P\mathfrak{G}$ denotes the $\mathfrak{G}$-orbit containing P.) The main result is the following: If the subgroup $\mathfrak{G}^+$ of linear transformations with positive determinant in $\mathfrak{G}$ is abelian, then either (i) $\mathfrak{G}$ contains all reflections on lines of V and $\mathfrak{G}^+$ consists of the rotations of V (i.e., the transformations of determinant 1); or (ii) $\mathfrak{G}$ is simply transitive on the lines through O, so that the subgroup fixing one line through O consists of dilatations. In case (i), K is pythagorean (i.e., $1+x^2$ is a square for any $x \in K$), and in case (ii), $\mathfrak{G}$ is abelian. This theorem is used to generalize results of Baer [Trans. Amer. Math. Soc. **68** (1950), 439–460; MR **12**, 9] and Lenz [Arch. Math. **8** (1957), 477–480; MR **20** #5446] on the group of motions of a plane over an ordered field.

P. Dembowski (Frankfurt a.M.)

Trigg, Charles W. 4408

A model of a tetrahedron.

Math. Student **31** (1963), 15–16 (1964).

Methods are given for the construction of a model of a regular tetrahedron out of a strip of four edge-joined equilateral triangles. If the strip is composed of four edge-joined congruent acute triangles, the tetrahedron will be isosceles. *S. R. Mandan* (Kharagpur)

Skopec, Z. A.; Asekritov, U. M. 4409

Mapping of space onto the plane by means of a cubical circle. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1963**, *no.* 5 (36), 113–116.

Kubische Kreise sind kubische Normkurven des R_3, die auf Kreiszylindern liegen. Ist ein solcher Zylinder Z_2 durch den Kreis k_2 in der Ebene E_2 bestimmt, so gibt es ∞^1 kubische Kreise auf Z_2 durch einen vorgegebenen Punkt A auf k_2 und mit der durch $M \subset k_2$ gehenden Erzeugenden von Z_2 als Fernasymptoten. Einer dieser kubischen Kreise C_3 wird zur Abbildung des R_3 auf E_2 benutzt, womit sich vorher schon Brauner beschäftigt hatte [vgl. H. Brauner, Monatsh. Math. **59** (1955), 258–273; MR **17**, 895; ibid. **60** (1956), 231–248; MR **18**, 334]. Günstig ist hierfür der Fall, wo M und A auf k_2 diametral liegen. Dann ist nämlich die Spur der Bisekante durch die Punkte $P_1'Q' \subset C_3$ der Fusspunkt des von A auf PQ gefällten Lotes [Punkte des R_3 und ihre senkrechten Projektionen in E_2 sollen stets durch Striche unterschieden werden]. Die Spur der Ebene $P'Q'R'$ ist dann die Simsongerade von A bezüglich des Dreiecks PQR. Durch den nicht auf C_3 gelegenen Punkt P' gibt es die Bisekante g' mit den Treffpunkten U_1', V' auf C_3. Auf der Geraden $g = UV$ liegt dann P_1 und das Linienelement $(P_1 g) \subset E_2$ kann als Bild von P' aufgefasst werden. Die Verfasser geben jetzt Konstruktionen an für die Spur der Verbindungsgeraden zweier Punkte des R_3 an deren zugeordnete Linienelemente in E_2 bekannt sind, auch dann wenn die Geraden der Elemente k_2 nicht reell schneiden. Ebenen im R_3 sind Tripel von Punkten auf k_2 zugeordnet, wovon 2 Punkte imaginär sein können. Allen üblichen Inzidenzaufgaben im R_3 entsprechen Konstruktionen in E_2, wie an einigen Beispielen gezeigt wird. *W. Burau* (Hamburg)

Balakrishnan, R. 4410

On a representation of linear complexes by conics in a plane.

Math. Student **31** (1963), 41–45 (1964).

Given a conic-locus Γ and a conic-envelope Σ, there exists, in general, a conic-locus S such that Σ is the ϕ-conic of Γ and S. If, in particular, Σ is triangularly inscribed in Γ, then S is inpolar to Γ.

Representing points on Γ in terms of a parameter t, any triad of points on Γ is given by the roots of a cubic equation in t. The four coefficients, in order, of this cubic equation, not all being zeros, can be taken to represent the homogeneous coordinates of a definite point in a 3-space $\{3\}$. Thus, a pencil of triads inscribed in Γ, determined by two fixed triads, is then represented by a straight line in $\{3\}$. Now, the triads of the pencil are all apolar to a conic S' which is inpolar to Γ, so that the inpolar conics of Γ correspond to the lines of $\{3\}$ in this representation. Considering conics S' inpolar to Γ and outpolar to another conic Σ, lines in $\{3\}$ that correspond to S' will all belong to a linear complex.

Special properties of linear complexes that correspond to certain relations between the inpolar conics of Γ are considered. *S. R. Mandan* (Kharagpur)

Dubikajtis, L. 4411

Une extension de la notion d'ordre linéaire à celle d'ordre de dimension n.

Ann. Polon. Math. **14** (1963/64), 211–238.

In einem angeordneten projektiven Raum der Dimension n wird für $1 \leqq k \leqq n$ eine $(k+3)$-äre Relation "zyklisch liegend" in der Menge der $(k-1)$-dimensionalen linearen Unterräume rekursiv folgendermaßen definiert: Die Punkte a_1, a_2, a_3, a_4 liegen genau dann zyklisch, wenn sie kollinear sind und das Paar (a_1, a_3) das Paar (a_2, a_4) trennt; die k-dimensionalen Unterräume $a_1, \cdots, a_{k+4}$ liegen zyklisch, wenn alle $a_{ij} = a_i \cap a_j$ $(i \neq j)$ die Dimension $k-1$ haben und für jedes i die a_{ij} mit $j \neq i$ zyklisch liegen. Diese Relation sowie die durch Dualisieren in einem k-dimensionalen Unterraum daraus entstehende $(k+3)$-äre Relation in der Menge der Punkte werden näher untersucht. Zum Schluß stellt der Verfasser das Problem, bei gegebenem k mit $1 < k \leqq n$ Axiome für die zuletzt genannte Relation anzugeben, aus denen sich die üblichen Anordnungsaxiome gewinnen lassen. *G. Pickert* (Giessen)

Edge, W. L. 4412

Fundamental figures, in four and six dimensions, over $GF(2)$.

Proc. Cambridge Philos. Soc. **60** (1964), 183–195.

In projective space $[n]$ of n dimensions any $n+2$ points, no $n+1$ of which lie in a space $[n-1]$, can be taken as basis (vertices of the simplex of reference) and unit point. Over $GF(2)$, if $\{A_i\}$, $i = 0, \cdots, n+1$, is such a set, the relation of linear dependence among them is $\sum A_i = 0$.

The points A_i determine the $\binom{n+2}{2}$ others $A_i + A_j$ (say ij); these lie in a space $[n-1]$ and form a symmetrical system of $n+2$ interlocking polygons, each consisting of the $n+1$ points having one index fixed. The points ij determine a null polarity in the $[n-1]$ in which the polar of ij is the $[n-2]$ containing the $\binom{n}{2}$ points rs $(r, s \neq i, j)$.

The paper discusses the Richmond-type configurations consisting of the points $\{ij\}$ and the Kummer-type configurations consisting of the 2^n points not in the hyperplane spanned by the points ij. *T. G. Room* (Sydney)

Heyting, A. 4413

★Axiomatic projective geometry.

Bibliotheca Mathematica, Vol. V.

Interscience Publishers [John Wiley & Sons, Inc.], New York; P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1963. ix + 148 *pp.* $6.00.

Dieses aus Vorlesungen entstandene Lehrbuch gibt eine vorzügliche Einführung in die neueren Untersuchungen über die Grundlagen der projektiven Geometrie, bei denen vor allem die Übersetzung geometrischer Inzidenzaussagen

in algebraische Aussagen über einen Koordinatenbereich eine Rolle spielen. Um nicht zu viel an algebraischen Vorkenntnissen voraussetzen oder das Buch zu umfangreich machen zu müssen, zieht der Verfasser jedoch oft rein geometrische Methoden den algebraischen vor. Das erste Kapitel führt äußerst sorgfältig in die Grundbegriffe der Axiomatik ein und behandelt die in den folgenden Kapiteln benötigten Tatsachen aus Algebra und Analytischer Geometrie. Vier Kapitel sind der ebenen Geometrie gewidmet. Die zu den trivialen Inzidenzaxiomen hinzutretenden weiteren Annahmen werden nicht wie vielfach üblich als "Sätze" bezeichnet sondern als "Aussagen" ("propositions"). Die Desarguessche Aussage D_{11} mit ihren Ausartungsfällen D_{10}, D_9 sowie ihren durch Festlassen von Zentrum und Achse entstehenden Spezialisierungen werden ausführlich behandelt, ferner die Aussage vom vollständigen Viereck, die Pappos-Aussage und ihre Konsequenzen und Zusammenhänge untereinander. Jedes Punktquadrupel O, W, X, Y, bei dem keine drei Punkte kollinear sind, führt zu einem Ternärkörper, dessen Elemente die nicht auf XY liegenden Punkte von OW sind, und dieser liefert eine Koordinatendarstellung. Die Bedeutung von D_{11}, D_{10}, D_9 und der Pappos-Aussage für den Ternärkörper wird untersucht. Besonders anzumerken ist, daß sich dabei die Inversbedingungen der Multiplikation bereits aus D_9 direkt ergeben. Wohl zum ersten Mal in einem Lehrbuch findet man die geometrische Anordnung nicht nur durch die Trennungsrelation (zwischen den Punkten jeder Geraden) sondern auch durch Anordnungseigenschaften des Ternärkörpers dargestellt. Zwei Kapitel behandeln die dreidimensionale Geometrie unter Benutzung der Grundbegriffe Punkt, Gerade, Ebene, Inzidenz. Ein rein geometrischer Beweis zeigt die Äquivalenz der Pappos-Aussage mit der 16-Punkte-Aussage: Existieren 15 der Schnittpunkte $a_i \cap b_k$ $(i, k=0, 1, 2, 3)$ der 8 Geraden a_i, b_k, so auch der 16-te. Die Einbettbarkeit einer Desarguesschen Ebene folgt mittels der Koordinatendarstellung. Das gut ausgestattete Buch ist klar und sorgfältig geschrieben. Trotz des reichen Inhalts bei geringem Umfang dürfte es leicht zu lesen sein.

G. Pickert (Giessen)

Bruck, R. H.; Bose, R. C. 4414

The construction of translation planes from projective spaces.

J. Algebra **1** (1964), 85–102.

Nach André [Math. Z. **60** (1954), 156–186, § 8; MR **16**, 64] lassen sich genau diejenigen Translationsebenen, für die ein Koordinatenquasikörper endliche Dimension über seinem Kern hat, dadurch beschreiben, daß man für alle $t \geqq 2$ in einem $(2t-1)$-dimensionalen projektiven Raum Σ alle Mengen S von $(t-1)$-dimensionalen projektiven Unterräumen angibt, bei denen jeder Punkt von Σ in genau einem Element von S liegt. Hierfür wird ein neuer Beweis gegeben. Darauf folgt eine nähere Untersuchung der Mengen S im Falle $t=2$ bei endlichem Raum Σ. Enthält hierbei eine solche Geradenmenge S eine Regelschar R, so entsteht bei Ersetzen von R durch die konjugierte Regelschar wieder eine Geradenmenge, die eine Translationsebene liefert. Dieser Übergang von einer Translationsebene zu einer anderen wird als Sonderfall eines von Ostrom [Trans. Amer. Math. Soc. **111** (1964), 1–18; MR **28** #2472] für affine Ebenen beschriebenen Konstruktionsverfahrens erkannt. *G. Pickert* (Giessen)

Wagner, Richard 4415

Projektivitäten auf Quadriken.

Math. Z. **83** (1964), 336–344.

Les transformations d'une quadrique en une autre, qui peuvent se prolonger en une collinéation des espaces ambiants ont été caractérisées dans le cas des quadriques non dégénérées par la propriété que les sections planes de la quadrique initiale se représentent par les sections planes de la seconde. Extension au cas général où l'auteur démontre que la transformation biunivoque entre les quadriques f et f' se laisse prolonger pour un espace de dimension $n \geqq 3$ en une collinéation des espaces ambiants si 4 points coplanaires de f ont pour images 4 points coplanaires de f' et si f' contient au moins deux points non conjugués (dans la bilinéarité liée à l'équation de la quadrique). *B. d'Orgeval* (Dijon)

Haruki, Hiroshi 4416

On a characteristic property of confocal conic sections.

Proc. Japan Acad. **39** (1963), 564–565.

An analytic function f maps a coordinate rectangle into a curvilinear rectangle with vertices C_1, C_2, C_3, C_4 numbered in cyclic fashion. Suppose that for all such curvilinear rectangles, $|C_1-\gamma|+|C_3-\gamma|=|C_2-\gamma|+|C_4-\gamma|$, where γ is some fixed complex number. Then the function f maps lines parallel to the coordinate axes into two families of conics with foci at γ.

John W. Green (Los Angeles, Calif.)

Abbasov, N. T. 4417

Spinor representations of motions of quasi-non-Euclidean spaces. (Russian)

Trudy Sem. Vektor. Tenzor. Anal. **11** (1961), 241–252.

The quasi-non-euclidean space ${}^{k,l}R_n{}^m$ is defined in a projective space P_n by an m-dimensional absolute plane, an absolute cone, and by an absolute quadric (where k and l determine the signature of the latter). Such a space occupies an intermediate place between euclidean and non-euclidean spaces. The author establishes a one-to-one correspondence between the points x of the ${}^{k,l}R_n{}^m$ and the expressions $\sum_{i=1}^n x^i e_i$, the e_i being the basis elements of an algebra of rank 2^n, for which $e_ie_j=-e_je_i$ $(i\neq j)$, respectively, $e_i{}^2=\pm 1$ or 0. An arbitrary motion of the space can be written in the form $x'=\alpha x\alpha^{-1}$, where α is formed by certain alternations from the e_i, and from the terms determining the motion of the space and the absolute configurations. The author comes to the spinor representation through the group of alternations. The author uses (or generalizes) the notions and notations of the book of B. A. Rozenfel'd [*Non-Euclidean geometries* (Russian), GITTL, Moscow, 1955; MR **17**, 293]. The results are not summarized in theorems.

L. Tamássy (Debrecen)

Escher, M. C. 4418

★The graphic work of M. C. Escher.

Oldbourne Press, London, 1961. 61 *pp.* 21*s.*

This unusual book is a translation, with important additions, of the author's *Grafiek en Tekeningen* [Koninklijke Uitgeverij van de Erven J.J. Tijl N.V., Zwolle, 1960]. It is a wonderful blend of art, mathematics, and whimsical philosophy. All the 44 full-page plates are

fascinating, several of them achieve real beauty, and one's enjoyment of them is greatly enhanced by the accompanying text. In the course of his Introduction, the author remarks: "Although I lack any training or knowledge in the exact sciences, I often feel closer to mathematicians than to my fellow-artists." The twelve small drawings that precede Plate I were all made long before the mathematical technique involved was explained by Heesch and Kienzle [*Flächenschluss*, Springer, Berlin, 1963; MR **27** #6185]. Plates 10, 11, 12 are the most deeply mathematical, as they are conformal representations of symmetrical patterns in the hyperbolic plane. In Plates 10 and 12, one possible fundamental region for the symmetry group is a triangle with angles $\pi/6$, $\pi/6$, $\pi/2$. In the former, the group is generated by a half-turn (or digonal rotation) about the midpoint of the hypotenuse, along with reflections in the other two sides; it is thus somewhat analogous to the Euclidean group [the reviewer and Moser, *Generators and relations for discrete groups*, Fig. 4.5k, Springer, Berlin, 1957; MR **19**, 527; and 1964]. In Plate 12 ("Heaven and Hell", a particularly beautiful woodcut), using the same right-angled isosceles hyperbolic triangle, the group is generated by a reflection in the hypotenuse and a quarter-turn (or tetragonal rotation) about the opposite vertex, i.e., the group (somewhat analogous to **p4g**) is $[4^+, 6]$ in the notation of the reviewer and Moser [loc. cit. (4.44)].

Plate 11, a woodcut printed in five blocks, might well have been entitled "The miraculous draught of fishes". It can be appreciated mathematically on two levels. If the colors are disregarded, the fundamental region is a fish or, equally well, a rhombus with angles $\pi/4$, $2\pi/3$, $\pi/4$, $2\pi/3$; and the group, generated by trigonal rotations about the two obtuse-angled vertices, is (3, 3, 4) [the reviewer and Moser, loc. cit. (6.41)]. The centers of tetragonal rotation, where the right fins of four fishes come together, are the vertices of the regular tessellation {3, 8}: each vertex is surrounded by eight equilateral triangles. A reader who is familiar with automorphic functions may be disturbed by the fact that the white arcs running along the backs of the fishes are visibly not orthogonal to the peripheral circle (absolute). Far from being a mistake, this is a striking instance of the author's flair for geometry. In the hyperbolic plane, such an arc is not a straight line but one branch of an "equidistant curve" (hypercycle); that is why it is possible for three of the arcs to form an equilateral "triangle" whose angles are $\pi/3$.

The second level of appreciation of Plate 11 arises when we distinguish the four colors, thus obtaining, instead of (3, 3, 4), a subgroup of index 12 generated by half-turns about the midpoints of the five sides of a certain pentagon with angles $\pi/8$, $\pi/4$, $\pi/4$, $\pi/4$, $\pi/8$ [the reviewer, *Introduction to geometry*, p. 299, Ex. 5, Wiley, New York, 1961; MR **23** #A1251]. Since this pentagon is one half of a regular octagon with angles $\pi/4$, the group generated by five half-turns can be derived from the fundamental group for the surface of genus 2, in the form

$$abcda^{-1}b^{-1}c^{-1}d^{-1} = 1,$$

by adjoining a half-turn which transforms each of the generating translations a, b, c, d into its inverse [P. Bergau and J. Mennicke, Math. Z. **74** (1960), 414–435; p. 421; MR **27** #1960]. This octagon, into which the surface of genus 2 could be unfolded, is a face of one of the six regular tessellations {8, 8} that have together the same vertices as the above-mentioned {3, 8}. Each tetragonal center determines a pair of colors (for instance, yellow-green in the middle of the whole picture, where two yellow fins meet two green fins). In this way, the six pairs of the four colors serve to distinguish the six {8, 8}'s, as in Figure 7 of the reviewer's paper [Proc. Roy. Soc. London Ser. A **278** (1964), 147–167; MR **29** #507], where the letters A, B, C, D, E, F correspond to the color-pairs yellow-blue, yellow-red, yellow-green, red-green, green-blue, blue-red. Thus the author could have painted a finite version of the same pattern in the form of a solid figure-of-eight covered with 24 fishes, 6 of each color. Although he has not yet carried out this suggestion, it is well within his powers, for Plate 13 is a photograph of a beechwood ball on which twelve fishes have been carved in low relief. *H. S. M. Coxeter* (Toronto, Ont.)

Shirokov, P. A. [Širokov, P. A.] 4419

★*A sketch of the fundamentals of Lobachevskian geometry.*

Prepared for publication by I. N. Bronshtein. Translated from the first Russian edition by Leo F. Boron with the assistance of Ward D. Bouwsma.

P. Noordhoff Ltd., Groningen, 1964. 88 *pp.* *Dfl.* 11.75.

Lobachevskian geometry is discussed without carrying out an analysis of axioms by making use of the simplest theorems of elementary geometry that do not depend on Euclid's fifth postulate. The chapters are: (I) Euclid's and Lobachevsky's parallel postulates and their relation to the problems of the sum of the angles of a triangle and of the existence of similar figures; (II) Parallel lines and their simplest properties; (III) Fundamental properties of intersecting, divergent and parallel lines; (IV) Pencils of lines and simplest curves in the Lobachevskian plane; (V) Lobachevskian stereometry; (VI) Derivation of Lobachevskian trigonometry; (VII) Survey of the further construction of Lobachevskian geometry; (VIII) Interpretations.

This fine translation [see Russian original, GITTL, Moscow, 1955; MR **17**, 655] is recommended to high school teachers and college students.

N. D. Kazarinoff (Ann Arbor, Mich.)

G.-Rodeja F., E. 4420

Representations over quadrics. (Spanish)

Rev. Mat. Hisp.-Amer. (4) **23** (1963), 84–92.

In this paper the author gives a method for the direct representation over a quadric of (1) a cubic surface with 27 real lines; (2) a ruled cubic surface; (3) a surface of order 4 with double twisted cubic curve.

M. Piazzolla-Beloch (Ferrara)

Fejes Tóth, László 4421

What is "discrete geometry"? (Hungarian)

Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **13** (1963), 229–238.

In dieser Arbeit werden Begriff und Gegenstand der diskreten Geometrie erörtert. Die Entwicklung der diskreten Geometrie wird von der altägyptischen Ornamentik ausgehend im Rahmen der Gesamtentwicklung der Mathematik dargestellt. Dabei wird bis an die modernsten Probleme der diskreten Geometrie herangegangen, und

es werden die wichtigsten Resultate derselben, von denen viele dem Verfasser gehören, in klarer und übersichtlicher Form dargestellt. *A. Rapcsák* (Debrecen)

Craig, Robert T. 4422

Extension of finite projective planes. I. Uniform Hjelmslev planes.

Canad. J. Math. **16** (1964), 261–266.

Es wird bewiesen, daß jede endliche projektive Ebene π' der Ordnung n als Restklassenebene (nach der Nachbar-Relation) einer uniformen Hjelmslev-Ebene π (der Ordnung n^2N, mit $N=n^2+n+1$) aufgefaßt werden kann. Zu diesem Zweck geht der Verfasser von $n+1$ verschiedenen Einteilungen der Menge der natürlichen Zahlen $\leqq n^2$ in n Teilmengen zu je n Elementen aus, bei denen jedes Paar verschiedener natürlicher Zahlen $\leqq n^2$ in genau einer Teilmenge bei genau einer der $n+1$ Einteilungen vorkommt. Daß die Existenz solcher Einteilungen dasselbe bedeutet wie die Existenz einer projektiven Ebene der Ordnung n, beweist der Verfasser auf einem Umweg über orthogonale lateinische Quadrate, obwohl man direkt sehen kann, daß $n+1$ Einteilungen mit den genannten Eigenschaften gerade die $n+1$ Parallelenscharen einer affinen Ebene mit den Punkten $1, 2, \cdots, n^2$ bilden. Zu der k-ten von solchen $n+1$ Einteilungen wird die $(n^2\times n^2)$-Matrix M_k gebildet, bei der an der Stelle (i, j) eine Eins steht, wenn i, j zur gleichen Teilmenge (der k-ten Einteilung) gehören, andernfalls eine Null. Die Inzidenzmatrix von π' läßt sich als Summe von $n+1$ Permutationsmatrizen schreiben. Ersetzt man nun in der k-ten von diesen jede Null durch die $(n^2\times n^2)$-Null-Matrix und jede Eins durch M_k, so ist die Summe der entstehenden $(n^2N\times n^2N)$-Matrizen die Inzidenzmatrix einer uniformen Hjelmslev-Ebene π, die eine zu π' isomorphe Restklassenebene besitzt.

G. Pickert (Giessen)

Karzel, Helmut 4423

Ebene Inzidenzgruppen.

Arch. Math. **15** (1964), 10–17.

Eine projektive Ebene zusammen mit einer auf der Menge ihrer Punkte erklärten Gruppenverknüpfung heißt ebene Inzidenzgruppe, wenn die Linkstranslationen der Gruppe Projektivitäten der Ebene sind. Falls auch die Rechtstranslationen Projektivitäten sein sollen, muß die Gruppe kommutativ sein. Ist die ebene Inzidenzgruppe desarguessch (als projektive Ebene), so läßt sich der zur Koordinatendarstellung der Ebene verwandte dreidimensionale Vektorraum durch Einführung einer Multiplikation derart zu einem Fastkörper machen, daß die Zuordnung der Punkte zu den sie darstellenden Vektoren ($\neq 0$) ein Homomorphismus der multiplikativen Gruppe des Fastkörpers auf die Inzidenzgruppe ist. Bei kommutativer Inzidenzgruppe ist dieser Fastkörper kommutativ und damit ein Körper. *G. Pickert* (Giessen)

Lingenberg, Rolf 4424

Über die Gruppe einer projektiven Dualität.

Math. Z. **83** (1964), 367–380.

A one-to-one incidence-preserving mapping δ of a projective plane which maps points onto lines and lines onto points is called a projective duality if the induced mapping between the lines through P and the points on $P\delta$ is a projectivity. The author considers the group $B(\delta)=\{\alpha|\alpha^{-1}\delta\alpha=\delta,\ \alpha$ a central collineation$\}$ and determines its structure when δ^2 has at least two fixed points. Let K be the underlying field and $F(\delta^2)$ the set of all points and lines fixed by δ^2. If δ^2 fixes at least two points, there are four possibilities: (1) All points and lines belong to $F(\delta^2)$; then $B(\delta)\cong O_3^+(K,f)$, the proper orthogonal group of a form f of rank 3, if Fano's axiom is valid. If Fano's axiom is not valid, then $B(\delta)$ depends upon the structure of the set of self-conjugate points (i.e., points incident with their images under δ). If there is a line all of whose points are self-conjugate, then $B(\delta)\cong SL_2(K)$. If δ has but one self-conjugate point, $B(\delta)\cong K^+$, the additive group of K. If there are no self-conjugate points, $B(\delta)$ consists of the identity alone. (2) $F(\delta^2)$ consists of a line together with its points and a point together with all the lines through this point. There are two possibilities: if δ is a perspective duality (i.e., there is a point Z and a line a not incident with Z such that $Z\delta=a$ and $Zl\Rightarrow l\cdot a$), then $B(\delta)\cong SL_2(K)$ and char $K\neq 2$; otherwise $B(\delta)\cong O_2(K,F)$ for a field K of characteristic different from 2 and a symmetric bilinear form F of rank 2. (3) $F(\delta^2)$ consists of three non-collinear points together with the lines joining them. Then $B(\delta)\cong K^\times$, the multiplicative group of K. (4) $F(\delta^2)$ consists of two points, the line joining them and one further line through one of these points. Then $B(\delta)=N(\delta)+\omega N(\delta)$, whereby the subgroup $N(\delta)$ has index 2 in $B(\delta)$ and $N(\delta)\cong K^+$; furthermore, ω may be taken as an involutory homology. The characteristic of K is different from 2.

W. Jonsson (Kingston, Ont.)

Roth, Richard 4425

Collineation groups of finite projective planes.

Math. Z. **83** (1964), 409–421.

Let π be a finite projective plane of order n, G a collineation group of π, and π_0 the system of points and lines left fixed by G. The author proves several results on π_0. First, he is concerned with the question as to when π_0 has equally many points and lines. The reviewer has shown [same Z. **69** (1958), 59–89; MR **20** #255] that this is the case if G is a p-group with $p\nmid n$. The author shows that if G is solvable, if $(|G|, n)=1$, and if π has property S (which means that every subplane which is the system of fixed elements of a collineation has order dividing n), then π_0 also has equally many points and lines. The author proves that π has property S if π is (C, h)-transitive for at least one incident point-line pair C, h and states that the Hughes planes also have property S. No finite planes without property S are known. Second, the author investigates the possible order m of π_0 in case π_0 is a (non-degenerate) subplane of π. It is well known [Bruck, Trans. Amer. Math. Soc. **78** (1955), 464–481; MR **16**, 1081] that $m^2=n$ or $m^2+m\leqq n$ for arbitrary subplanes. The author shows that in the special situation where π_0 is the set of fixed elements of G the second alternative can be improved to $m^2+m\leqq n-d$ with $d\geqq 2$, regardless of property S. For the proof of this the tactical decompositions of the reviewer [loc. cit.] are employed successfully. It is also shown that if G is of prime order p, then $d=2$ implies $p=3$ and $d=3$ implies $p=3$ or $p=7$. *P. Dembowski* (Frankfurt a. M.)

Swift, J. D. 4426

Chains and graphs of Ostrom planes.

Pacific J. Math. **14** (1964), 353–362.

Let R be a finite planar ternary ring [in the sense of Hall, Trans. Amer. Math. Soc. **54** (1943), 229–277; MR **5**, 72] which, with the usual definition of addition and multiplication, is a two-dimensional vector space over a subfield K. Let $\{1, t\}$ be a K-basis, and denote by α the permutation

$$(x_1t+x_2,\ y_1t+y_2) \to (x_1t+y_1,\ x_2t+y_2)$$

of the points of the affine plane coordinatized by R. The author defines two transformations O_T and O_S yielding planes which are in general not isomorphic to the original: O_T means leaving all lines with slope in K unchanged and replacing all other lines by their images under α; and O_S is essentially a dual operation. The resulting planes have, under appropriate choice of coordinates, ternary rings which are also vector spaces over K. If the original plane is desarguesian, i.e., if R is a quadratic field extension of K, then O_T gives the Hall plane, O_S its dual, O_S and O_T commute, and the result of $O_SO_T = O_TO_S$ is a self-dual plane. If the order is 9, this self-dual plane is the Hughes plane [Canad. J. Math. **9** (1957), 378–388; MR **19**, 444] but for higher orders new planes arise. This last result is not proved; the author refers to a recently published paper by Ostrom [Trans. Amer. Math. Soc. **111** (1964), 1–18; MR **28** #2472].

P. Dembowski (Frankfurt a. M.)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 4327.

Lyusternik, L. A. [Ljusternik, L. A.] 4427

★Convex figures and polyhedra.

Translated from the Russian by T. Jefferson Smith. *Dover Publications, Inc., New York*, 1963. x + 173 pp. $1.50.

This translation might be termed a popular variant of A. D. Alexandrov's monograph [*Convex polyhedra* (Russian), GITTL, Moscow, 1950; MR **12**, 732]. The book may also be characterized as one much in the spirit of *Convex figures* (Russian), by I. M. Jaglom and V. G. Boltjanskiĭ [GITTL, Moscow, 1951; MR **14**, 197]. However, it covers a set of complementary topics.

The book is an exposition of a variety of important, beautiful, and reasonably elementary theorems. There are no problems; many non-elementary theorems are stated without proof, for example, S. P. Olavyanshchikov's result that if all the plane sections of a convex body Q dividing its volume in a given ratio $r \neq 1$ are centrally symmetric, then Q is an ellipsoid. Two of the main classes of theorems proved are theorems involving lattice points and convex figures and theorems characterizing polyhedra.

The introductory material consists of sections on lines and planes of support, cones, the relation of these objects to convex figures (with an excursion into theorems on ovals of constant width), and central symmetry and parallel displacement. There follow proofs of Minkowski's theorem on maximal centrally symmetric bodies in an integral lattice, Cauchy's theorem that two convex polyhedra with corresponding congruent and similarly situated faces are either congruent or symmetric (Pogorelov's generalization to isometric convex surfaces is stated), and theorems on realizations of polyhedra.

Next is a chapter on linear systems of convex bodies including the Brunn-Minkowski inequality and some of its consequences. A. D. Alexandrov wrote the next chapter, which contains an elementary proof of his theorem that a convex polyhedron is determined by the areas and directions of its faces. The last chapter gives precise definitions and generalizations of certain concepts used earlier in the book, and some related theorems, such as the Bol-Brouwer fixed-point theorem, are proved.

More important to the value of the book than the choice of topics included is the author's careful, well-motivated, clear exposition. It is full of vitality and truly communicates the spirit of mathematics. This is a book for all mathematicians to enjoy, a book for bright young students especially.

N. D. Kazarinoff (Ann Arbor, Mich.)

Firey, William J. 4428

Some applications of means of convex bodies.

Pacific J. Math. **14** (1964), 53–60.

Jeder reellen positiv-definiten $n \times n$-Matrix A wird im R_n das Ellipsoid $E(A)$ zugeordnet: $(x, Ax) = 1$. Zu diesem Ellipsoid gehört die Distanzfunktion $F(x) = \sqrt{(x, Ax)}$ und die Stützfunktion $H(x) = \sqrt{(x, A^{-1}x)}$. Der Verfasser wendet eigene Resultate über konvexe Körper [dasselbe J. **11** (1961), 1263–1266; MR **25** #3427; Math. Scand. **10** (1962), 17–24; MR **25** #4416; Canad. J. Math. **13** (1961), 444–453] hierauf an, um Aussagen über Matrizen zu erhalten. $A, B, \cdots$ bezeichnen durchweg quadratische reelle positiv-definite Matrizen. $A \geqq B$ wird als Abkürzung dafür verwendet, daß $A - B$ positiv semidefinit ist. Es wird bewiesen $(1-\vartheta)A_0 + \vartheta A_1 \geqq \{(1-\vartheta)A_0^{-1} + \vartheta A_1^{-1}\}^{-1}$, wobei Gleichheit nur in den trivialen Fällen $A_0 = A_1$ oder $\vartheta = 0$, 1 gilt. Es werden sodann Analoga der Minkowskischen Determinantenungleichung bewiesen, in denen die Determinante von A ersetzt wird durch das Produkt der k größten oder der k kleinsten Eigenwerte. Die Gültigkeit des Gleichheitszeichens wird diskutiert. Schließlich werden noch Anwendungen auf das Löwner-Ellipsoid eines konvexen Körpers gegeben, welches bekanntlich das Ellipsoid mit festem Mittelpunkt (hier 0) ist, welches den konvexen Körper enthält. Hier wird gezeigt, daß dieses Ellipsoid auch die Minimumeigenschaft besitzt, wenn man das Volumen durch die k-ten Quermaßintegrale ersetzt. Entsprechendes gilt für das maximale unter den im konvexen Körper enthaltenen Ellipsoiden. Für das Volumen hat man, daß Eindeutigkeit der beiden Extremalellipsoide auch bei freiem Mittelpunkt gilt. Diese Frage ist für die Quermaßintegrale offen.

D. Laugwitz (Darmstadt)

Sobczyk, Andrew 4429

Convex polygons.

Proc. Amer. Math. Soc. **15** (1964), 438–446.

A planar convex body K is said to be of standard type (with respect to A and B) provided there exist chords A and B of K such that K is contained in the parallelogram with sides parallel and equal to A and B, respectively. The principal result of the paper is: (*) Every centrally symmetric convex polygon is of standard type, even if it is required that A and B be parallel to sides, or else to diagonals, of the polygon. From (*) it easily follows that each convex polygon is of standard type; this, in turn, implies: (**) Every planar convex body is of standard

type. The proof of (*) is by induction on the number of sides of the polygon and by duality. {Reviewer's remarks: A proof of (**) can be given by considering the quadrilateral of greatest area contained in K; its diagonals may serve as A and B; a simple additional argument establishes (*). For strictly convex and smooth K, (**) is known; see, e.g., S. K. Stein [Math. Z. **68** (1957), 282–283; MR **19**, 877]. For centrally symmetric convex K, the n-dimensional analogue of (**) is well known; see, e.g., M. M. Day [Trans. Amer. Math. Soc. **62** (1947), 315–319; MR **9**, 246]; H. Lenz [Arch. Math. 8 (1957), 209–211; MR **19**, 977]; A. E. Taylor [Bull. Amer. Math. Soc. **53** (1947), 614–616; MR **8**, 588; errata, MR **8**, 709]; from this it is easy to derive for general K the n-dimensional analogue of (**), conjectured in the paper under review.}

B. Grünbaum (Jerusalem)

Asplund, Edgar 4430

A k-extreme point is the limit of k-exposed points.

Israel J. Math. **1** (1963), 161–162.

Let C be a compact convex set in n-space. A point of C is k-extreme if it is not the centroid of a non-degenerate $(k+1)$-simplex. A point of C is k-exposed if it is contained in a closed half-space K such that $K \cap C$ is at most k-dimensional. Generalizing a result of S. Straszewicz [Fund. Math. **24** (1935), 139–143], which corresponds to $k=0$, the author proves that, for $k \geq 0$, the closure of the set of k-exposed points contains the set of k-extreme points.

B. Grünbaum (Jerusalem)

Ehrhart, E. 4431

Sur le nombre de points à coordonnées entières d'une région convexe plane ou spatiale.

Enseignement Math. (2) **10** (1964), 138–146.

The author obtains a number of inequalities on the number j of points with integral coordinates contained in a closed convex body in two or three dimensions. Typical results are the following: In two dimensions, let S be the area of a convex body and l its perimeter; then $j \leq S + \frac{1}{2}l + 1$, the equality being obtained only for rectangles with sides parallel to the coordinate axes. In three dimensions, let V be the volume of a convex body, h its width in the x-direction, s the maximum area and l the maximum perimeter of plane sections perpendicular to the x-axis. Then $j \leq V + s + (h+1)(\frac{1}{2}l + 1)$. The following conjecture is made: Let a, b, c, be the widths in the x, y, and z directions of a convex body of volume V and area S. Then $j \leq V + \frac{1}{2}S + a + b + c + 1$.

John W. Green (Los Angeles, Calif.)

Eggleston, H. G. 4432

Minimal universal covers in E^n.

Israel J. Math. **1** (1963), 149–155.

A compact convex subset K of Euclidean n-space E^n is a "universal cover" if for every set $X \subset E^n$ of diameter 1 there is a subset of K congruent to X. A universal cover K is minimal provided no proper subset of K is a universal cover. In the plane, any minimal cover has diameter less than 3 [the reviewer, Proc. Sympos. Pure Math., Vol. VII, pp. 271–284, Amer. Math. Soc., Providence, R.I., 1963; MR **27** #4134]. The author solves a problem of V. Klee by proving that for $n \geq 3$, there exist minimal universal covers of arbitrarily great diameter. The highly ingenious proof uses a number of interesting lemmas; the following two deserve special mention: (1) Every planar set of constant width 1 contains a semicircle of diameter 1. (This generalizes a partial result obtained by A. S. Besicovitch [ibid., pp. 15–18; MR **27** #2912].) (2) The union of a circular disc and of a Reuleaux triangle, both of unit diameter, so placed that two of the vertices of the triangle are contained in the disc, is a minimal universal cover.

B. Grünbaum (Jerusalem)

Groemer, Helmut 4433

Über die Zerlegung eines konvexen Körpers in homothetische konvexe Körper.

Math. Ann. **154** (1964), 88–102.

A finite family $\{K_1, \cdots, K_m\}$, $m \geqq 2$, of n-dimensional compact convex sets is a "partition" of an n-dimensional compact convex set K provided $K = \bigcup K_i$ and each point of int K_i belongs to no K_j, $j \neq i$. The author establishes a number of results on the nature of sets K for which there exist partitions $\{K_i\}$ such that the sets K_i satisfy certain additional requirements. Typical examples are: K has a partition $\{K_i\}$ such that the sets K_i are mutually positively homothetic [translates of each other] if and only if K is a truncated cone [cylinder]. Then all K_i are also truncated cones [cylinders]. ("Truncated cone" means the convex hull of two positively homothetic $(n-1)$-dimensional convex sets contained in parallel hyperplanes.)

B. Grünbaum (Jerusalem)

Croft, H. T. 4434

A net to hold a sphere.

J. London Math. Soc. **39** (1964), 1–4.

Von A. S. Besicovitch [Math. Gaz. **41** (1957), 106–107; MR **19**, 58] stammt der folgende Satz: Ein aus einem unausdehnbaren Faden geknüpftes Netz, das eine Einheitskugel κ so umschließt, daß die Kugel aus ihm nicht ausschlüpfen kann, hat mindestens die Länge 3π. Diese Konstante ist dabei die bestmögliche; ersetzt man sie nämlich durch eine größere $3\pi + \varepsilon$ mit $\varepsilon > 0$, so existiert stets ein (aus sechs Großkreisbogen bestehendes) Netz kleinerer Länge mit den geforderten Eigenschaften.

Für diesen Satz wird hier ein neuer Beweis gegeben. Dieser arbeitet mit einem einfachen, schon von H. Steinhaus [Colloq. Math. **3** (1954), 1–13; MR **16**, 121] und L. A. Santaló [Duke Math. J. **9** (1942), 707–722; MR **4**, 252] benutzten Maß für Mengen C von Großkreisen c der Einheitskugel κ, das als die Hälfte des zweidimensionalen Lebesgueschen Maßes jener Punktmenge C^* erklärt wird, welche auf κ von den beiden Polen c_1^*, c_2^* der Kreise c gebildet wird.

In ähnlicher Weise hat inzwischen auch S. K. Stein [Math. Gaz. **47** (1963), 44] den Beweis des Satzes von Besicovitch geführt.

Etwas allgemeiner ist die folgende Frage: wie lang muß ein um die Einheitskugel κ gelegtes Netz sein, wenn die Kugel ihm auch dann nicht entschlüpfen soll, falls k der Seiten des Netzes (Großkreisbogen) reißen? Es wird gezeigt, daß dann die Länge $> K\pi$ sein muß mit $K = 2k+3$ und daß diese Zahl wieder die bestmögliche ist, d.h. daß für jedes $\varepsilon > 0$ die Netzlänge $K\pi + \varepsilon$ hinreicht.

K. Strubecker (Karlsruhe)

Fejes Tóth, L.; Heppes, A. 4435
Über stabile Körpersysteme.
Compositio Math. **15**, 119–126 (1963).
A body [domain] is the closure of a bounded region in euclidean 3-space [plane]. A packing P is a set of such bodies [domains] without common interior points. If the only possible euclidean motions of the elements of P are rigid motions of the whole packing P, P is called stable. P is "relatively stable" if each element of P can be moved only if all the others are. (Relative) translation stability is defined in an analogous manner. The following are some of the authors' results: If P consists of (finitely many) convex domains, then P is not (relatively) translation stable in any direction. However, there are stable P's of finitely many convex bodies [cf. de Bruijn, Nieuw Arch. Wisk. (3) **2** (1954), 67]. *P. Scherk* (Toronto, Ont.)

Few, L. 4436
Multiple packing of spheres.
J. London Math. Soc. **39** (1964), 51–54.
Let S be a set of points in n-dimensional space. Let spheres of radius r be centred at the points of S. If no point of space is inside more than k spheres (k a positive integer), the density of this k-fold packing is

$$\delta_k(S) = \limsup_{t\to\infty} r^n J_n N_t(S)(2t)^{-n},$$

where J_n is the volume of the unit sphere and $N_t(S)$ is the number of points of S inside a cube of side $2t$, centred at the origin O. $\delta_k = \overline{\text{bd}}_S\, \delta_k(S)$ is the density of the closest packing. The following inequalities are established:

$$\delta_1\{2k/(k+1)\}^{n/2} \leqq \delta_k \leqq (1+n^{-1})\{(n+1)^k-1\}/\{(k+1)/k\}^{n/2}.$$

For $k=1$ (the classical packing problem) the upper bound is weaker than Blichfeldt's [Math. Ann. **101** (1929), 605–608] and Rogers' [Proc. London Math. Soc. (3) **8** (1958), 609–620; MR **21** #847]; for $k=2$, the upper bound is significantly better than the author's earlier inequality [J. London Math. Soc. **28** (1953), 297–304; MR **14**, 1115]. The proof of the upper bound depends upon the following interesting lemma: Let T be a set of points in n-space, inside a sphere of radius $\{(k+1)/k\}^{1/2}$, which provides a k-fold packing for spheres of radius 1. Then T contains at most $(1+n^{-1})\{(n+1)^k-1\}$ points.
W. Moser (Winnipeg, Man.)

DIFFERENTIAL GEOMETRY

See also 4059, 4065, 4141, 4265, 4266, 4360, 4417, 4912.

Raševskiĭ, P. K. [Рашевский, П. К.] 4437
★Riemannian geometry and tensor analysis [Риманова геометрия и тензорный анализ].
Second edition.
Izdat. "Nauka", Moscow, 1964. 664 *pp.* 2.10 *r.*
This second edition differs from the first edition [GITTL, Moscow, 1953; MR **16**, 1051] only in the correction of misprints and in the revision of three sections on spinors (§§ 57–59). A German translation of the first edition has appeared [VEB Deutscher Verlag der Wiss., Berlin, 1959; MR **21** #2258].

El'cov, A. I. 4438
On the squarability in the Lebesgue sense of a type of ruled surface. (Russian)
Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 40–45.
On rappelle que l'aire au sens de Lebesgue d'une surface S est la limite inférieure exacte des limites inférieures de toutes les suites d'aires de polyèdres inscrits dans S. L'auteur donne des conditions nécessaires et suffisantes pour que la portion de surface S définie en axes rectangulaires par

$$X = vf_1(u), \quad Y = vf_2(u), \quad Z = vf_3(u)+\varphi_3(u),$$
$$0 \leqq v \leqq 1,\ \alpha \leqq u \leqq \beta,$$

ait une aire au sens de Lebesgue. *M. Decuyper* (Lille)

Fabricius-Bjerre, Fr. 4439
On the double tangents of plane closed curves.
Math. Scand. **11** (1962), 113–116.
Given a closed differentiable curve c in the affine plane. A double tangent of c is called exterior [interior] if it does not separate [if it separates] small neighbourhoods on c of the points of contact. Assume that the inflection points and double points and the exterior and interior double tangents of c are all simple singularities and that their numbers $2i$, d, t, s are finite. Then $t-s=d+i$. We quote the following corollary: Let ν denote the rotation number of c, i.e., the degree of its tangential map. If $\nu=0$, then $d\geqq 1$ [H. Whitney, Compositio Math. **4** (1937), 276–284] and $2i\geqq 2$; hence $t\geqq 2$. If $\nu>0$, then $d\geqq \nu-1$ [H. Whitney, loc. cit.]; hence $t\geqq \nu-1$ for $i=0$ and $t\geqq\nu$ for $i>0$.
P. Scherk (Toronto, Ont.)

Gołąb, S.; Moszner, Z. 4440
Sur le contact des courbes dans les espaces métriques généraux.
Colloq. Math. **10** (1963), 305–311.
Let $E=\{p, q, \cdots\}$ be a metric space with the distance function ρ and containing two Jordan arcs $C_i: p_i=p_i(\tau)$, $0\leqq\tau\leqq 1$, with $p_i(0)=p_0$ $(i=1, 2)$. For any q, the point $q'=p_1(\tau')\subset C_1$ is called the projection of q onto C_1 if $\rho(q, q')\leqq\rho(q, p_1(\tau))$ for all τ, strict inequality holding for $0\leqq\tau<\tau'$. The arc C_2 is called tangent to C_1 at p_0 if $\lim_{p_2\to p_0}\rho(p_2, p_2')/\rho(p_0, p_2)=0$. Tangency is reflexive and transitive. Even in the euclidean plane it is not necessarily symmetric. However, the authors prove: Let C_1 be rectifiable. For each p_1 let $l(p_0, p_1)$ denote the length of the subarc of C_1 from p_0 to p_1. Assume

$$\lim_{p_1\to p_0} l(p_0, p_1)/\rho(p_0, p_1) = 1.$$

Then C_1 is tangent to C_2 at p_0 if C_2 is tangent to C_1 at that point. *P. Scherk* (Toronto, Ont.)

Klotz, Tilla 4441
Further geometric consequences of conformal structure.
Trans. Amer. Math. Soc. **112** (1964), 67–78.
The author continues her investigation of various conformal structures defined in terms of the coefficients of the first and second fundamental forms on a surface immersed in E^3. In earlier papers [Michigan Math. J. **9** (1962), 129–136; MR **26** #1453; Trans. Amer. Math. Soc. **108** (1963),

38–53; MR **27** #1882] she used the conformal structures defined by the first and second fundamental forms themselves. The results in the present paper parallel closely her earlier results, but use in addition a third conformal structure which is defined on surfaces of negative curvature. *R. Osserman* (Stanford, Calif.)

Müller-Pfeiffer, E. 4442

Über Kurven, die gewissen ihrer Evolutoiden direkt ähnlich sind.

Math. Nachr. **27** (1964), 229–252.

Generalizing a problem of Euler, the author considers arcs that are similar to some arc (defined by a constant shift of the parameter) of their nth evolutoid for a prescribed sequence of inclinations of the tangent at each successive evolutoid. For the definitions, see, e.g., the reviewer's *Differential geometry*, problem 3-2.8 [McGraw-Hill, New York, 1963; MR **27** #6194]. The resulting functional differential equation

$$(*)\qquad f(u+1) = L(f),$$

where $L(f)$ is an nth-order linear differential operator with constant coefficients, is treated both analytically and kinematically. The existence theorem for equation (*) with prescribed initial condition $f(u)=f_0(u)$, $-1\leq u<0$, shows that an arbitrary arc of sufficient differentiability can always be imbedded in an arc so that it is similar to some part of the nth evolutoid with arbitrarily prescribed angles of inclination.

H. W. Guggenheimer (Minneapolis, Minn.)

Wunderlich, W. 4443

Böschungslinien, die mit ihren Planevoluten zusammenfallen.

J. Reine Angew. Math. **214/215** (1964), 31–42.

Die Planevolute c_1 einer (weder sphärischen noch ebenen) Raumkurve c ist die Einhüllende der Normalebenen von c; die Punkte von c_1 sind die Mitten der Schmiegkugeln von c und die Tangenten von c_1 sind die Krümmungsachsen von c. Die Schmiegebenen in entsprechenden Punkten von c und c_1 sind zueinander normal. Die Fernkurven u und u_1 der Tangentenflächen von c und c_1 sind daher zueinander bezüglich des absoluten Kegelschnittes i polar.

Soll c mit c_1 zusammenfallen, also die Schmiegkugelmitte des Punktes P von c wieder ein Punkt P_1 von $c_1=c$ sein, so muß die Fernkurve u der Tangentenfläche von c bezüglich des absoluten Kegelschnittes i autopolar $(u=u_1)$ sein. Die einfachste Annahme ist dabei die, daß u ein Fernkreis der Öffnung $\pi/4$ ist, etwa die Fernkurve des Kegels $x^2+y^2-z^2=0$. Die mit ihrer Planevolute c_1 zusammenfallende Raumkurve c ist dann eine zur z-Richtung gehörende Böschungslinie mit dem Steigwinkel $\pi/4$. Schreibt man die Schar der Schmiegebenen σ von c in der Form

$$(*)\qquad \sigma \equiv x\sin\varphi - y\cos\varphi - h(\varphi) = 0,$$

so ist c dann und nur dann mit seiner Planevolute c_1 identisch, wenn $h(\varphi)$ der "Hystero-Differentialgleichung"

$$(**)\quad h(\varphi)+2h''(\varphi) = h(\varphi+\alpha) \quad \text{mit } \alpha \equiv \pi \pmod{2\pi}$$

genügt, deren Elementarlösungen $h=e^{r\varphi}$ lauten; die Konstante r genügt dabei noch der Bedingung

$$(***)\qquad 1+2r^2 = e^{\alpha r} \quad \text{mit } \alpha = (2j-1)\pi.$$

Deren einzige reelle Lösung $r=0$ ergibt für c nur einen festen Punkt. Wirkliche Kurven c erhält man aus den konjugiert komplexen Lösungspaaren $r=(q\pm in)$ von (***), indem man aus ihnen reelle Lösungen von (**) kombiniert. Man findet für $h(\varphi)$ die normierte Darstellung $h(\varphi)=e^{q\varphi}\sin n\varphi$.

Die Ebenenschar (*) kann (auch bei beliebigen q und n) kinematisch aus der Ebene $\varphi=0$ durch eine gedämpfte harmonische Umschwungbewegung erzeugt werden; sie hüllt dabei eine Umschwungtorse Γ ein. Es interessieren die Gratlinie c von Γ, die Planevolute c_1 und die Spurkurve c_0 von Γ auf der Grundebene π $(z=0)$, die eine ebene Filarevolvente von c ist. Die Spurkurve c_0 von Γ erweist sich als eine Spiraloide [der Verfasser, Simon Stevin **36** (1962/63), 57–71; MR **27** #620]; der Normalriß der Böschungslinie c auf die Grundebene π ist eine zu c_0 ähnliche Spiraloide. Die Planevolute c_1 ist zu c ähnlich und fällt (als ganzes) mit c zusammen, wenn $r=(q\pm in)$ der Gleiching (***) genügt. Die Böschungslinie c gestattet eine diskontinuierliche Gruppe von Ähnlichkeiten und gehört einer abzählbar unendlichen Menge von Spiralflächen an, die von rational-algebraischen Kurven erzeugt werden. In den Fällen $n=0$ und $n=\pm 1$ sowie $q=0$ ist die Bedingung (***) nicht erfüllbar.

Das eingeschlagene Verfahren kann auch zur Bestimmung der Böschungslinien c angewandt werden, die sich mit ihren höheren Planevoluten $c_{2\nu+1}$ ungerader Ordnung decken. Es gibt auch Planevoluten $c_{2\nu}$ gerader Ordnung, die sich mit Böschungslinien c decken; der Böschungswinkel ω kann dann $\neq\pi/4$ sein. In Sonderfällen können dann c und $c_{2\nu}$ sogar punktweise übereinstimmen.

K. Strubecker (Karlsruhe)

Havel, Václav 4444

Über die begleitenden Normaldreibeine der Fläche $\mathscr{A}^2_{0,3}$. I. (Russian summary)

Czechoslovak Math. J. **13** (**88**) (1963), 327–334.

L'auteur utilise la méthode de l'immersion fictive dans un espace affine, méthode exposée par A. Švec [Časopis Pěst. Mat. **86** (1961), 425–432; MR **24** #A3600]; il construit un repère normal pour une surface dont les lignes asymptotiques sont distinctes; c'est un trièdre dont les axes sont les tangentes asymptotiques et la normale affine (droite de Demoulin); puis il définit les transformations asymptotiques normales et introduit un invariant pour les transformations unimodulaires: la courbure affine.

M. Decuyper (Lille)

Kamenskiĭ, N. P. 4445

On the coincidence of metric and affine normals on a hypersurface. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1958**, *no.* 3 (4), 107–110.

A euclidean $(n+1)$-space E_{n+1} is at the same time an affine space. At all points on a hypersurface V_n of E_{n+1} one can form the metric normals (with respect to E_{n+1}) and the affine normals. The author shows that these two coincide if and only if the covariant derivative of Δ/g vanishes on V_n (Δ being the determinant of the metric

tensor of E_{n+1}, and g being analogously defined on the V_n induced by the E_{n+1}). In connection with these, certain other questions are investigated, and examples given. *L. Tamássy* (Debrecen)

Lopšic, A. M. 4446
Families of lines of force in a dimensionless affine space. (Russian)
Trudy Sem. Vektor. Tenzor. Anal. **12** (1963), 175–201.
Continuing research initiated by E. Kasner and Ya. S. Dubnov, the author investigates the field strength corresponding to given lines of force (the problem inverse to that of dynamics). The space is supposed to be a dimensionless affine space, and the lines of force to be plane curves. The following is shown. (a) If the lines of force are of zero affine curvature (parabolas), then the field strength consists of parallel vectors

$$a(M) = a(M_0)[1+\varphi\overline{M_0M}]^{-3/2},$$

where M is an arbitrary point and M_0 is a fixed point of the space, a is the vector of field strength and φ denotes a scalar-valued linear function of a vector argument with the condition $\varphi a(M_0)=0$. (b) If the lines of force are individually of constant affine curvature, then the field strength consists likewise of parallel vectors or of radial vectors (vectors going through a point). In both cases (a) and (b) the concrete form of the vector of field strength is also determined.
Finally, the solutions of some differential equations of the simplest kind in dimensionless affine space are investigated. *L. Tamássy* (Debrecen)

Magazinnikov, L. I. 4447
Centro-affine theory of ruled surfaces. (Russian)
Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 101–110.
The problem is to construct the simplest centro-affinely invariant frame of reference in the case of ruled surfaces. This frame renders it possible to specify the complete classification of the different classes of ruled surfaces. The author uses results proved by O. Mayer [Ann. Sci. Univ. Jassy **21** (1934), 1–77]. In particular, Mayer's centro-affine arc length ds can be expressed by a simple relation (Theorem 1). Between ds and the invariant α of I. Popa [C. R. Acad. Sci. Paris **198** (1934), 2051–2053] the relation

$$ds^* = \alpha^{1/3}ds, \quad ds^* = ((\bar{x}', \bar{x}'', \bar{x}''')/(\bar{x}, \bar{x}', \bar{x}''))^{1/3}$$

holds along a curve $\bar{x}=\bar{x}(u)$ (Theorem 2). Two other theorems are interpretations of other invariants of the theory of Mayer. *M. Pinl* (Moscow, Idaho)

Merza, József 4448
A new characterization of the affine geodesic curvature of surface curves. (Hungarian)
Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **13** (1963), 119–124.
In dieser Arbeit untersucht der Verfasser die affine geodätische Krümmung der Kurven auf einer in den dreidimensionalen affinen Raum eingebetteten Fläche. Es wird gezeigt, dass die affine geodätische Krümmung einer Flächenkurve mit dem Masse der Abweichung von einer passend gewählten berührenden autoparallelen Kurve identisch ist. *A. Rapcsák* (Debrecen)

Murgescu, V. 4449
Sur l'introduction d'une métrique dans le plan affin. (Romanian. Russian and French summaries)
Gaz. Mat. Fiz. Ser. A **14** (67) (1962), 581–587.
Author's summary: "D'abord on établit le théorème suivant, qui représente une propriété caractéristique, de nature affine, pour les coniques à centre. Soit, dans un plan, un point fixe O et une courbe (Γ), régulière, qui ne passe pas par O. Soit encore deux rayons quelconques (OM_1) et (OM_2) qui touchent la courbe (Γ) dans les points M_1, respectif M_2, et encore N_1, N_2 les projections des points M_2 et M_1 sur ces rayons, faites selon les directions tangentes à (Γ) dans les points M_1, respectif M_2. Les coniques non dégénérées, à centre O, sont les courbes (Γ) qui jouissent de la propriété $ON_1/OM_1=ON_2/OM_2$. Ce résultat représente une nouvelle démonstration du fait que l'indicatrice de Finsler doit être une conique à centre pour que la mesure de l'angle soit indépendante de l'ordre des directions qui le composent. Par conséquent, on montre comment on peut définir dans le plan affin les mesures usuelles pour les angles et longueurs par des rapports simples, bien-entendu, en utilisant une conique comme indicatrice."

Malahovskiĭ, V. S. 4450
Congruences of parabolas in equi-affine geometry. (Russian)
Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 76–86.
Etude de congruences de paraboles en se limitant aux congruences dites P, pour lesquelles une au moins des surfaces focales: (1) n'est pas enveloppe d'une famille de plans de paraboles; (2) n'est pas développable; (3) sur elle les lignes focales ne sont pas des asymptotiques. Définition d'un repère canonique; propriétés géométriques. *B. d'Orgeval* (Dijon)

Malahovskiĭ, V. S. 4451
The canonical frame of a congruence of central curves of second order in an equi-affine geometry. (Russian)
Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 87–92.
Etude des congruences de coniques à centre possédant deux surfaces focales, non dégénérées, dont les points focaux correspondants n'appartiennent pas à un même diamètre d'une conique (congruences A). Définition d'un repère canonique; propriétés géométriques. *B. d'Orgeval* (Dijon)

Vasenin, V. V. 4452
On the equi-affine theory of triples of non-holonomic surfaces. (Russian)
Sibirsk. Mat. Ž. **5** (1964), 22–33.
The author considers special triples B_i^j ($j \leq i$; $i,j=0,1,2,3$) of non-holonomic surfaces; i denotes the number of affine normals incident with tangent planes of non-holonomic surfaces of the triple and j denotes the number of these affine normals which coincide with the intersecting lines of tangent planes of non-holonomic surfaces of the triple. The geometric construction of all types of triples B_i^j (with the exception of B_3^2) is found. *A. Urban* (Prague)

Ščerbakov, R. N. 4453

Line complexes whose metric and affine frames coincide. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 93–100.

The canonical moving frame of a line complex in E^3 (defined by Finikov) coincides, except for the lengths of the vectors, with the canonical equi-affine frame defined by the author only for complexes of constant metric curvature. The equi-affine transforms of such complexes are affine-symmetric. The author also constructs a semi-canonical equi-affine frame. An analogous semi-canonical metric frame is defined, and those complexes are determined which contain non-holonomic congruences where corresponding semi-canonical metric frames coincide with the affine. These complexes are also affine-symmetric.

H. Busemann (Los Angeles, Calif.)

Gol'dberg, V. V. 4454

A mapping of L sequences of an N-dimensional space onto a fixed hyperplane. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 39–53.

This paper is a continuation of the author's paper [Mat. Sb. (N.S.) **58** (**100**) (1962), 749–784; MR **27** #662], where special types of Laplace sequences of congruences and conjugate nets have been introduced. The present paper is devoted to the connections existing between such a sequence in P_m and its intersection with a fixed subspace $P_{m-1} \subset P_m$.

A. Švec (Prague)

Romanovič, V. A. 4455

On a class of pairs of congruences. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 24–28.

The author deals with pairs of parabolic congruences A_1 generated by arbitrary pairs of corresponding ruled surfaces. For the totality of all pairs A_1 the following properties have been established: (1) developable surfaces within the pairs A_1 are corresponding surfaces; (2) the pairs of congruences A_1 turn out to be pairs of congruences discussed by S. E. Karapjatjan; (3) on corresponding rays of the pairs of congruences A_1 a focus can be shown such that the corresponding focal planes pass through these foci. Differentiating the Pfaffian system belonging to A_1 the author gets the following results: (1) there are pairs A_1 depending on two arbitrary functions of two variables; an arbitrary congruence can be transformed into a congruence A_1; (2) the corresponding transformation depends on three arbitrary functions of one variable; (3) there are pairs of congruences which are projectively developable congruences; they are determined together with all their deformations apart from an arbitrary function of two variables. Projectively developable pairs of congruences, which can be deformed by a bending of first order into pairs A_1, turn out to be pairs A_1.

M. Pinl (Moscow, Idaho)

Ržehina, N. F. 4456

Theory of curves in an $(n-1)$-dimensional projective space. (Russian)

Trudy Sem. Vektor. Tenzor. Anal. **11** (1961), 153–164.

It is very convenient to start from the geometry of curves in a centro-affine space E_n for a construction of the theory of curves in a projective space P_{n-1}. V. V. Vagner [same Trudy **6** (1948), 257–364; MR **14**, 1124] has done it for $n=3$ and L. V. Gol'dštein [ibid. **9** (1952), 288–308; MR **14**, 687] for $n=4$. The author constructs in the same way a theory of curves in an $(n-1)$-dimensional projective space for arbitrary n. Some results are more general than the results obtained earlier (1932) by V. Hlavatý in a different way.

W. Wrona (Warsaw)

Šepelenko, L. M. 4457

Projective bending of a two-parameter family of $(p-1)$-planes in $(2p-1)$-dimensional projective space. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 29–38.

The author studies, in projective space P_{2p-1}, the projective bending of first and second order (in the sense of Fubini and Cartan) of a two-parameter family W_2 of $(p-1)$-dimensional planes. He also investigates the foci and focal directions of the planes of the W_2. The author proves that a W_2 admits projective bendings of first order preserving foci and focal directions and containing p^2-p arbitrary functions of two variables. A necessary and sufficient condition is given for a W_2 and a W_2' to admit projective bendings of first order on each other. An arbitrary W_2 does not admit a projective bending of second order. It turns out that there exists a W_2 admitting projective bendings of second order with $2p-3$ arbitrary functions in two variables. Finally those W_2 having an enveloping surface are investigated.

L. Tamássy (Debrecen)

Stanilov, G. 4458

Kanonisches Bezugssystem der Regelscharen der zweiachsigen Geometrie. (Russian summary)

C. R. Acad. Bulgare Sci. **16** (1963), 473–476.

Die zweiachsige Geometrie im reellen dreidimensionalen projektiven Raum stützt sich auf die Gruppe jener Kollineationen, welche zwei windschiefen Geraden als Ganzes in sich überführen. Unter Anwendung alternierender Differentialformen wird ein kanonisches Bezugssystem eines Strahlkomplexes im Rahmen dieser Geometrie festgelegt. Die Grundpunkte des Fundamentaltetraeders werden dabei unter Ausnützung von Eigenschaften der im Komplex enthaltenen Kurven möglichst zweckmäßig gewählt.

H. Brauner (Stuttgart)

Švec, Alois 4459

Sur la géométrie différentielle des réseaux conjugués dans E_n. (Russian summary)

Czechoslovak Math. J. **13** (**88**) (1963), 539–550.

Après avoir rappelé les formules fondamentales qui définissent le déplacement du repère mobile (orthogonal) attaché au point courant d'une surface quelconque (A) de E_n, l'auteur se place dans le cas où (A) admet un réseau conjugué, et adapte ces formules à un choix particulier du repère, consistant à prendre les vecteurs I_1 et I_2 de celui-ci situés dans le plan tangent en A suivant les bissectrices des angles formés par les tangentes aux courbes du réseau conjugué, les courbes coordonnées u, v admettant en chaque point I_1, I_2 pour vecteurs unitaires tangents. Il

peut ainsi (étant donné un domaine de paramètres (u, v) et deux fonctions indépendantes $F_\rho(x_1, x_2, x_3, u, v)$, $\rho = 1, 2$) montrer qu'il existe, dans E_n, des surfaces (A) dépendant de $2n$ fonctions d'une variable, douées d'un réseau conjugué, telles que les tangentes aux courbes u, v soient les axes des angles des tangentes conjuguées, et que, si 2φ est l'angle de ces dernières tangentes et si $\overline{AA}_1$, $\overline{AA}_{-1}$ désignent les distances du point A à ses deux transformés de Laplace, on ait $F_\rho(2\varphi, \overline{AA}_1, \overline{AA}_{-1}, u, v) = 0$. Ayant égard à la notion de surfaces pseudodéveloppables d'une congruence [l'auteur, Czechoslovak Math. J. **10 (85)** (1960), 523-550; MR **26** #2974], et à celle qui s'y rattache de réseaux pseudoconjugués d'une surface, l'auteur, après quelques remarques générales sur l'existence de ces derniers réseaux, montre l'existence de tels réseaux sur les surfaces (π) douées d'un réseau conjugué. Il établit qu'il existe sur ces surfaces un réseau pseudoconjugué orthogonal différent du réseau conjugué, et termine par quelques considérations relatives aux familles de lignes asymptotiques de (π). *P. Vincensini* (Marseille)

Vaĭntrub, M. R. 4460

Cyclic and spherical manifolds in conformal geometry. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 164–168.

The Frenet formulas for a curve in K_2 and for a curve or a surface in K_3 are presented, K_i being the conformal i-space. *A. Švec* (Prague)

Pergamenščikov, M. B. 4461

Inclusion of H-pairs in a Möbius configuration. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 13–23.

The present paper deals with a Möbius configuration formed by four different congruences C, C_1, C_1', C_2, where the pairs (C, C_1), (C, C_1'), (C_1, C_2), (C_1', C_2) are T-pairs and the pair (C, C_1) is a P-pair [the author, same Trudy **160** (1962), 39–44; MR **27** #5178]. There exists such a Möbius configuration in which (C, C_1) and (C_2, C_1') are H-pairs [R. N. Ščerbakov, Mat. Sb. (N.S.) **46 (88)** (1958), 159–194; MR **21** #881]. *A. Urban* (Prague)

Auslander, Louis; MacKenzie, Robert E. 4462

★Introduction to differentiable manifolds.

McGraw-Hill Book Co., Inc., New York-Toronto-London, 1963. ix + 219 *pp*. $9.95.

This book is intended to introduce differentiable manifold theory to readers with some undergraduate background in mathematics and it is, to the reviewer's knowledge, the first book which treats the subject at the introductory level. Indeed, in the preface of this book the authors explain clearly their method of exposition: "The book begins with a leisurely introduction to the general concept of a differentiable manifold. The path that we have chosen to this goal leads through a careful re-examination of the differentiable structure of euclidean space. Once the reader has understood euclidean space as a differentiable manifold, we define the general object and study its elementary properties. We have accompanied our abstract discussion of differentiable manifolds with a liberal dose of some of those historical examples which originally motivated their definition. Once these foundations have been laid, we have selected topics (see the Table of Contents for exact details) which illustrate further the historical background for differentiable manifolds and the directions in which the concepts have been applied. In particular, we have tried to sample the various techniques that have been found useful in handling differentiable manifolds. In a few places where the technicalities become oppressively intricate or repetitious, the arguments have been left incomplete, and the reader is urged to consult the references at the end of the book for the missing parts. Above all, we hope that the reader will be encouraged by this introduction to pursue the references well beyond the material which is presented here."

The following Table of Contents, list of chapters, will give a general idea of the scope of the book: (1) Euclidean, affine, and differentiable structure on R^n; (2) Differentiable manifolds; (3) Projective spaces and projective algebraic varieties; (4) The tangent bundle of a differentiable manifold; (5) Submanifolds and Riemann metrics; (6) The Whitney imbedding theorem; (7) Lie groups and their one-parameter subgroups; (8) Integral manifolds and Lie subgroups; (9) Fibre bundles; (10) Multilinear algebra.

The book is well organized and written with much consideration for the educational effect. The presentation is clear and unhurried. In each chapter basic notions are introduced after illustrating and discussing simple cases of the objects, and almost all sections end with a set of exercises which facilitate the understanding of the subject. It is very regrettable that the book contains nevertheless the following careless mistakes. On page 88, lines 11–14, the statement which pretends the uniqueness of submanifold structure (if it exists) on a subset of a manifold is false, and indeed the lemniscate in the plane has two different submanifold structures which are both diffeomorphic to the real line; this example is given by N. Iwahori [*Theory of Lie groups* (Japanese), Iwanami-Kôza Gendai-Ôyô-Sûgaku, Iwanami, Tokyo, 1957]. As a result, Exercise 5 on page 89 cannot be solved unless the mapping $\mu: N \to M'$ is continuous. On page 118, the last statement of § 7-1, asserting that a subgroup of a Lie group which is also a submanifold is a Lie group and on which Theorem 7-1 on page 120 depends essentially, is not easy to prove at all, and Exercise 5 on page 119 requiring its proof would be too heavy for the readers. On page 174, Theorem 9-6 is not true if one does not assume that the isotropy subgroup at a point of S be a closed (Lie) subgroup. For Theorem 9-7 on page 175, it would be very desirable to note here that the assertion always holds as a consequence of the separability assumption on G and M. On page 203, the last statement and Exercise 1 of § 10-6 should be put in § 10-8.

Apart from these points, the authors' way of treatment confines evidently the material to rather basic manifold theory, and no mention is made of advanced theories such as infinitesimal connection theory. After reading this book, the reader may quickly approach advanced theories by means of the many publications listed in the references. {Perhaps one should now add to these references the following recent books: S. Lang, *Introduction to differentiable manifolds* [Interscience, New York, 1962; MR **27** #5192]; S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. I [Interscience, New York, 1963; MR **27** #2945].} *S. Murakami* (Osaka)

Auslander, Louis 4463

The structure of complete locally affine manifolds.

Topology **3** (1964), *suppl.* 1, 131–139.

Generalizing Bieberbach's theorems on complete flat Riemannian manifolds, the author proves the following theorems on complete locally affine manifolds M. (1) The radical of the fundamental group Γ of M is of finite index in Γ provided that Γ is finitely generated. (2) If M is compact, it is finitely covered by a compact solvmanifold. (3) If compact M_1 and M_2 have isomorphic fundamental group, then M_1 and M_2 are homeomorphic.

S. Kobayashi (Berkeley, Calif.)

Kneser, Hellmuth 4464

Abzählbarkeit und geblätterte Mannigfaltigkeiten.

Arch. Math. **13** (1962), 508–511.

Der Verfasser definiert über einer n-dimensionalen Mannigfaltigkeit M den Begriff des Atlasses, welcher zuerst von Ch. Ehresmann und G. Reeb eingeführt wurde. Es seien O_λ ($\lambda \in \Lambda$) eine offene Menge von M und φ_λ ein Homeomorphismus zwischen O_λ und einer offenen Teilmenge von R^n. Gehört die Abbildung

$$\varphi_\mu \varphi_\lambda^{-1} : \varphi_\lambda(O_\lambda \cap O_\mu) \to \varphi_\mu(O_\lambda \cap O_\mu)$$

zu einer bestimmten Kategorie Γ, dann ist der Atlas ein Γ-Atlas. Ist derselbe vollständig, so ist er eine Γ-Struktur. Die Struktur einer k-Blätterung ($0<k<n$) einer n-dimensionalen Mannigfaltigkeit M besteht aus denjenigen Abbildungen von R^n auf sich, bei welchen die Koordinaten $(y_{k+1}, y_{k+2}, \cdots, y_n)$ des Bildpunktes $(y_1, y_2, \cdots, y_n)$ im kleinen nur von den Koordinaten $(x_{k+1}, x_{k+2}, \cdots, x_n)$ des Urbildes $(x_1, x_2, \cdots, x_n)$ abhängen. Diese Blätterung gibt Anlass zu einer zweiten, gegenüber der Topologie T von M verfeinerten Topologie T', die mit Hilfe derjenigen Ebene von R^n eingeführt wird, für welche die Koordinaten $x_{k+1}, \cdots, x_n$ konstant sind. Ist O offen in M für die Topologie T, so ist (OTT') die in O induzierte, und (MTT') die ursprüngliche Blätterung. Der Verfasser zeigt: Ist (MTT') eine k-geblätterte M-Mannigfaltigkeit und O ein T-offener Teil von M, und hat M eine abzählbare Offen-Basis, so enthält ein Blatt von (MTT') höchstens abzählbar viele Blätter von (OTT'). Daraus ergibt sich: Hat eine k-geblätterte n-Mannigfaltigkeit (MTT') eine abzählbare Offen-Basis, so hat auch jedes Blatt eine solche.

A. Rapcsák (Debrecen)

Kneser, Martin 4465

Beispiel einer dimensionserhöhenden analytischen Abbildung zwischen überabzählbaren Mannigfaltigkeiten.

Arch. Math. **11** (1960), 280–281.

Der Verfasser konstruiert eine reell-analytische, zusammenhängende, zweidimensionale Mannigfaltigkeit, die sich eindeutig und analytisch auf eine reell-analytische dreidimensionale Mannigfaltigkeit abbilden lässt. Die Konstruktion wird dadurch möglich, dass die Mannigfaltigkeiten das zweite Hausdorffsche Abzählbarkeitsaxiom nicht erfüllen.

A. Rapcsák (Debrecen)

Černyĭ, D. E. 4466

Geometry of the Lagrange variational problem with a double integral in X_4. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 153–165.

Let $L(x, B)$ be defined in $A^4 \times V_2^4$, where A^4 is the 4-dimensional affine space, V_2^4 is the space of all bivectors B with $L(x, B) > 0$ for $B \neq 0$. The paper derives from the Lagrange problem of minimizing $\int L(x, B)$ on the oriented integral surfaces of a system of differential equations $f_i(x, B) = 0$ ($i = 1, 2, 3$), where B is a simple bivector tangent to the surface. It is assumed that L and the f_i satisfy the conditions which make the integral independent of the parametrization of the surface. $f_i = 0$ determines for each x a curve on the Grassmann cone $\mathfrak{G}_2^4 \subset V_2^4$ of simple bivectors. Consequently, first the affine geometry of curves on $\mathfrak{G}_2^4$ is studied by means of a properly defined affine connection. Some algebraic comitants of the bivectors are studied. The surface $f_i = 0$ may be considered as fibered with A^4 as base space and the intersections with the Grassmann cones as fibers. A general theory of these objects is developed. The principal aim and results concern conditions under which such a problem of Lagrange can be differentiably transformed into another one. The analytic details are much too involved to be indicated here.

H. Busemann (Los Angeles, Calif.)

Szybiak, A. 4467

Covariant derivative of geometric objects of the first class.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 687–690.

We consider an n-dimensional manifold L_n with a linear connexion $\Gamma_{\mu\lambda}{}^\kappa$. Let $\omega = \{\omega^K\}$, $K = 1, 2, \cdots, M$, be a geometric object of the first class and let its transformation rule be $\omega^{K'} = \varphi^{K'}(\omega, A)$, where the A's are elements $A_\lambda{}^{\kappa'} = \partial\xi^{\kappa'}/\partial\xi^\lambda$ of the Jacobian of the coordinate transformation $\xi^{\kappa'} = \xi^{\kappa'}(\xi)$. We denote by $\varphi^{K'}|_{\kappa'}{}^\lambda$ the partial derivative $\partial\varphi^{K'}/\partial A_\lambda{}^{\kappa'}$ and by $\hat\varphi^K|_\kappa{}^\lambda$ the values of $\varphi^{K'}|_{\kappa'}{}^\lambda$ when we put $A_\lambda{}^{\kappa'} = \delta_\lambda{}^\kappa$ in $\varphi^{K'}|_{\kappa'}{}^\lambda$.

The author puts

$$\nabla_\mu \omega^K = \partial_\mu \omega^K + \hat\varphi^K|_\kappa{}^\lambda \Gamma_{\mu\lambda}{}^\kappa$$

and proves the following theorems. (1) To every geometric object ω of the first class corresponds a covariant derivative, viz., $\{\nabla_\mu \omega^K\}$ if ω is a linear object and $\{\omega^K, \nabla_\mu \omega^H\}$ if ω is a non-linear object. (2) If, using Theorem (1), we construct the covariant derivatives of the objects ω and η, then the linear displacement of η is consistent with that of ω. In particular, all the comitants of tensors will be displaced consistently with the classical displacements of tensors.

K. Yano (Tokyo)

Szybiak, A. 4468

Covariant derivative of the geometric objects.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 751–755.

The author generalises the results of the paper reviewed above [#4467] to the covariant derivatives of the more general geometric objects using the so-called generalized connexion object.

K. Yano (Tokyo)

Mutō, Yosio 4469

On some (1, 1) tensor field and connections.

Sci. Rep. Yokohama Nat. Univ. Sect. I No. 10 (1963), 1–11.

A tensor field φ of type 1/1 (once covariant and once contravariant) whose minimal polynomial has no multiple

roots, has diagonal Jordan form. It is called numerical if coordinate systems covering the manifold can be found so that φ has constant components. Conditions necessary for this to happen are Trace φ = const and $N(\varphi) = 0$, where $N(\varphi)$ is the torsion tensor of type 1/2 of φ ("Nijenhuis tensor").

Equivalently, these conditions may be replaced by the existence of a connection such that $\nabla\varphi = 0$ ("φ-connection") which is symmetric. Recipes are given for the construction of φ-connections in rather explicit form.

The bibliography fails to mention the work of A. G. Walker [Proc. Sympos. Pure Math., Vol. III, pp. 94–100, Amer. Math. Soc., Providence, R.I., 1961; MR **23** #A1314], which contains useful hints for notation and overlaps considerably with the present work, since $\varphi = \sum \lambda_a a_a$, where the a_a are the projection operators on the eigenspaces of φ. *A. Nijenhuis* (Amsterdam)

Širokov, A. P. 4470

On symmetric spaces defined by algebras. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1963**, *no.* 6 (37), 159–171.

In previous papers [same Izv. **1961**, no. 1 (20), 163–170; MR **25** #5476; Kazan. Gos. Univ. Učen. Zap. **123** (1963), no. 1] the author introduced the notion of a connection without torsion defined by a given algebra. If $\{\underset{1}{e}, \underset{2}{e}, \cdots, \underset{n}{e}\}$ denotes the base of such an algebra and γ^k_{ij} are its structure constants, then this connection has the form $\Gamma^q_{rk} = p^m\gamma^j_{ms}\gamma^s_{kr}$, p^m being functions such that for every integer $0 < r \leq n$ and $0 < k \leq n$, $P(\underset{r}{e}\,\underset{k}{e} - \underset{k}{e}\,\underset{r}{e}) = 0$, where $P = p^m\underset{m}{e}$. In this paper the author studies the symmetric spaces defined by some special algebras. *W. Wrona* (Warsaw)

Jakubowicz, A. 4471

Über die Metrisierbarkeit der affin-zusammenhängenden Räume.

Tensor (*N.S.*) **14** (1963), 132–137.

We consider an n-dimensional manifold A_n with a symmetric affine connexion $\Gamma_{ji}{}^h$. If there exists a symmetric covariant tensor g_{ji} such that $\partial_k g_{ji} - \Gamma_{kj}{}^a g_{ai} - \Gamma_{ki}{}^a g_{ja} = 0$ and $\mathrm{Det}(g_{ji}) \neq 0$, the A_n is said to be metrisable. A necessary condition that the above equation admits a solution is that

$$(*) \qquad K^a_{ikj} g_{ai} + K^a_{iki} g_{ja} = 0,$$

where K^a_{ikj} is the curvature tensor of $\Gamma_{ji}{}^h$. For a solution g_{ji} of (*), we put p = rank of (g_{ji}), and for all the solutions of (*) we put $q = \mathrm{Max}\ p$.

The author calls q the algebraic rank of the object Γ. S. Gołąb [Tensor (N.S.) **9** (1959), 1–7; MR **21** #4449] studied the classification of A_2 following the rank q. The author studies similar problems for the case of A_3. *K. Yano* (Tokyo)

Nakagawa, Hisao 4472

On differentiable manifolds with certain almost contact structures.

Sci. Rep. Tokyo Kyoiku Daigaku Sect. A 8, 144–163 (1964).

It is well known that an almost contact structure in a $(2n+1)$-dimensional differentiable manifold M with local coordinates x^h is defined by a tensor $\varphi_i{}^h$ and vectors ξ^h, η_i satisfying

$$\varphi_j{}^i\varphi_i{}^h = -\delta_j^h + \eta_j\xi^h, \quad \varphi_i{}^h\xi^i = 0, \quad \varphi_i{}^h\eta_h = 0, \quad \xi^h\eta_h = 1$$

and an almost contact metric structure by these and a Riemannian metric g_{ji} satisfying

$$\varphi_j{}^c\varphi_i{}^b g_{cb} = g_{ji} - \eta_j\eta_i, \qquad g_{ji}\xi^i = \eta_j.$$

The product manifold $M \times R$ admits an almost complex structure

$$F = \begin{pmatrix} \varphi_i{}^h & \xi^h \\ -\eta_i & 0 \end{pmatrix},$$

and the Nijenhuis tensor of $M \times R$ gives four tensors $N_{ji}{}^h$, $N_i{}^h$, N_{ji} and N_i of the manifold M with an almost contact structure.

In § 1 of the present paper, the author first proves Theorem 1.1: A necessary and sufficient condition that $N_{ji}{}^h$ of M vanishes identically is that

$$\varphi_j{}^a\nabla_a\varphi_i{}^h + \varphi_i{}^a\nabla_j\varphi_a{}^h - (\nabla_j\xi^h)\eta_i = 0.$$

Then in § 2, the author defines various special spaces with an almost contact metric structure. If the tensor field $\varphi_i{}^h$ satisfies $\nabla_h\varphi_i{}^h = 0$ and the vector field ξ^h satisfies $\nabla_h\xi^h = 0$, then the M is said to be of type A. (The M corresponds to the manifold studied by Apte [C. R. Acad. Sci. Paris **238** (1954), 1091–1093; MR **15**, 649]. If the tensor field φ_{ji} satisfies $\nabla_j\varphi_{ih} + \nabla_i\varphi_{hj} + \nabla_h\varphi_{ji} = 0$ and the vector field η_i satisfies $\nabla_j\eta_i - \nabla_i\eta_j = 0$, then the M is said to be of type AK. (The M corresponds to an almost Kähler manifold.) If $\varphi_i{}^h$ and η_i satisfy $\nabla_j\varphi_i{}^h + \nabla_i\varphi_j{}^h = 0$ and $\nabla_j\eta_i + \nabla_i\eta_j = 0$, then the M is said to be of type AT. (The M corresponds to an almost Tachibana manifold.) If $\varphi_i{}^h$ and η_i satisfy $\nabla_j\varphi_i{}^h = 0$ and $\nabla_j\eta_i = 0$, then the M is said to be of type K. (The M corresponds to a Kähler manifold.) Then the author proves the following theorems. Theorem 2.1: The M of type AT is of type A. Theorem 2.2: If, in M of type AT, either $N_i{}^h$ or N_{ji} vanishes identically, then $\nabla_j\eta_i$ vanishes too. Theorem 2.3: A necessary and sufficient condition that M of type AT is of type K is that $N_{ji}{}^h$ vanishes identically. Theorem 2.4: An M of type AK is of type A. Theorem 2.5: If, in M of type AK, $N_i{}^h$ vanishes identically, then $\nabla_j\eta_i$ vanishes too.

In § 3, the author derives some identities and inequalities and proves Theorem 3.1: If, in an M, the vector field ξ^h satisfies $\nabla_h\xi^h = 0$, then we have

$$\nabla_j\nabla_i(\xi^j\xi^i) = (\nabla_j\xi^i)(\nabla_i\xi^j) + R_{ji}\xi^j\xi^i.$$

Theorem 3.2: In an M of type A, we have

$$R - R^* = (\nabla_j\varphi_{ih})(\nabla^i\varphi^{hj}) - (\nabla_j\eta_i)(\nabla^i\xi^j),$$

where $R^* = R_{ji}{}^*g^{ji}$, and $R_{ji}{}^* = \frac{1}{2}\varphi^{ts}\varphi_j{}^r T_{tshi}$. These theorems yield interesting corollaries.

The author discusses, in § 4, the curvature tensors of some special M, and proves Theorem 4.1: There does not exist an Einstein space M of type AT such that the curvature scalar R is negative. There does not exist an Einstein space M of type AK such that the curvature scalar R is positive. Theorem 4.2: On a conformally flat M of type A, the identity

$$\frac{2}{2n-1}[(n-1)R + R_{ji}\xi^j\xi^i] = (\nabla_j\varphi_{ih})(\nabla^i\varphi^{hj}) - (\nabla_j\eta_i)(\nabla^i\xi^j)$$

is valid. Theorem 4.3: Let M be of constant holomorphic curvature k. If M is of type AT, then $R \geqq n(n+2)k \geqq 0$. If an M of type AT is not of type K, then $R > n(n+2)k \geqq 0$.

In § 5, the author studies infinitesimal transformations in M and proves Theorem 5.1: If a vector field v in M satisfies $\pounds(v)\varphi_i{}^h = 0$ and $\pounds(v)\xi^h = 0$, then $\pounds(v)\eta_j = 0$, where $\pounds(v)$ denotes the Lie differentiation with respect to v. If it satisfies $\pounds(v)\varphi_i{}^h = 0$ and $\pounds(v)\eta_i = 0$, then $\pounds(v)\xi^h = 0$. Theorem 5.2: If, in an M such that $\nabla_i\xi^i = 0$ and $N_i = 0$, a conformal Killing vector v satisfies one of the two conditions $\pounds(v)\xi^h = -\lambda\xi^h$ and $\pounds(v)\eta_j = \lambda\eta_j$, where λ is a scalar, then it is homothetic. Theorem 5.3: If, in an M such that $\nabla_i\xi^i = 0$ and $N_i = 0$, a projective Killing vector v satisfies the condition $\pounds(v)\xi^h = -\lambda\xi^h$ and $\pounds(v)\eta_j = \lambda\eta_j$, then it is an affine transformation and λ is a constant.

In the last chapter, the author derives some integral formulas and obtains a necessary and sufficient condition that a contravariant vector field v satisfies $\pounds(v)\varphi_i{}^h = 0$ under certain conditions. *K. Yano* (Tokyo)

Šulikovskiĭ, V. I. 4473

Differential-topological characterization of the family of nets with equal Chebyshev vectors and a common apolar net. (Russian)

Trudy Sem. Vektor. Tenzor. Anal. **11** (1961), 141–151.

The author investigates the one-parameter family of nets with equal Chebyshev vectors and a common apolar net in a two-dimensional space of affine connection without torsion. Every pair of nets of this family has determined differential-topological characteristics. The author studies the characteristics and classification of the families of nets considered. *W. Wrona* (Warsaw)

Satô, Saburô 4474

On a projective connection and Riemannian metric.

Tensor (*N.S.*) **14** (1963), 1–5.

The author considers an n-dimensional differentiable manifold M_n and attaches at each point of M_n a projective tangent space P_n. The author then assumes that there is given in each P_n a non-degenerate quadric Q, $a_{\mu\lambda}X^\mu X^\lambda = 0$, and that, x^λ being the coordinates of the point of contact, we have $\det(a_{\mu\lambda}) = \pm 1$, $a_{\mu\lambda}x^\mu x^\lambda = k^2$, k being a positive scalar.

The author obtains the conditions characterizing the parameters of projective connexion for the constant normalizing function k for the constant quadric Q and for constant k and Q, respectively.

The author also shows that the constant k and the $a_{\mu\lambda}$ characterize a space of non-Euclidean connexion without torsion and, moreover, if the projective curvature is zero, then the space becomes a Riemannian space of constant curvature. *K. Yano* (Tokyo)

Stoka, Marius I. 4475

Sur les correspondances entre des espaces projectifs à connexion projective linéaire. (Russian summary)

Czechoslovak Math. J. **13** (**88**) (1963), 622–631.

Etant donné une correspondance entre deux espaces projectifs le rapporteur a montré [Boll. Un. Mat. Ital. (3) **12** (1957), 489–506; MR **19**, 1075] que l'on peut associer à la correspondance un tenseur $\Pi_{jk}{}^i$ qui satisfait pour $n=2$ aux conditions $\Pi_{11}{}^1 + \Pi_{21}{}^2 = 0$, $\Pi_{12}{}^1 + \Pi_{22}{}^2 = 0$. Le tenseur est défini donc dans le cas des plans projectifs par les quatres composantes $\Pi_{22}{}^1$, $\Pi_{12}{}^1$, $\Pi_{21}{}^2$, $\Pi_{11}{}^2$ et on montre que ces composantes satisfont à un système de trois équations aux dérivées partielles du second ordre et inversement.

Le rapporteur a considéré aussi le cas où le tenseur $\Pi_{jk}{}^i$ est constant et a trouvé que les correspondances appartiennent à quatre cas différents. L'auteur considère les correspondances à tenseur $\Pi_{jk}{}^i$ linéaire et montre en utilisant des transformations linéaires de variables, que l'on peut distinguer 28 cas différents dont deux sont de la troisième espèce, six de la seconde espèce et 20 de la première espèce. La notion d'espèce étant celle introduite par Borůvka [1926]. Dans les deux cas de la troisième espèce, on peut s'arranger de façon que $\Pi_{22}{}^1$ soit égal à x ou à y, les autre composantes étant nulles. *G. Vrănceanu* (Bucharest)

Datta, D. K. 4476

Some theorems on symmetric recurrent tensors of the second order.

Tensor (*N.S.*) **15** (1964), 61–65.

Consider an n-dimensional Riemannian space with metric tensor g_{ji}. A symmetric tensor a_{ji} is said to be recurrent if it satisfies an equation of the form $\nabla_k a_{ji} = \lambda_k a_{ji}$. If $\lambda_k = 0$, it is said to be covariantly constant. If there exists a scalar f such that $\nabla_k(fa_{ji}) = 0$, it is said to be semi-covariantly constant. The author proves the following theorems. (1) If a_{ji} is a recurrent tensor and if the elementary divisors of $|a_{ji} - \rho g_{ji}| = 0$ are simple, then the congruences determined by its principal vectors are normal and the congruences corresponding to a simple root of $|a_{ji} - \rho g_{ji}| = 0$ form a field of parallel vectors. (2) If g_{ji} is a recurrent tensor and the corresponding equation $|a_{ji} - \rho g_{ji}| = 0$ has simple elementary divisors, then the roots of this equation are proportional to one another. (3) A recurrent tensor is semi-covariantly constant if and only if the recurrence vector is the gradient of a scalar. (4) If a recurrent tensor is semi-covariantly constant, the function f is proportional to the principal invariants of the tensor provided that elementary divisors of $|a_{ji} - \rho g_{ji}| = 0$ are all simple. *K. Yano* (Tokyo)

Grabiel, Federico 4477

Algebra of set tensors.

Tensor (*N.S.*) **14** (1963), 53–59.

From the author's introduction: "In Riemann spaces there exist no conservation relations in which the quantities involved possess tensorial character, and this is due to the unavailability of a law of addition for tensors attached to different points of the space. In an earlier paper [Tensor (N.S.) **10** (1960), 1–20; MR **22** #5004] the author developed an algebra and analysis of tensors attached to sets in the space of concern, and that theory was applied [Rev. Un. Mat. Argentina **17** (1955), 69–71; MR **18**, 669] to the study of physical measurements. The set tensors entering as terms of an addition or as factors of a product in the paper cited first, however, were tensors defined over a common set S. In this paper those operations are going to be extended to tensors defined over arbitrary sets S and R of the space or region of space of concern. Then in succeeding papers we shall present a theory of integration of set tensors, and the conservation theorems that follow from it." *K. Yano* (Tokyo)

Gupta, Bandana 4478

On projective-symmetric spaces.

J. Austral. Math. Soc. **4** (1964), 113–121.

An n-dimensional Riemannian space V_n with metric tensor g_{ji} for which the covariant derivative of Weyl's projective curvature tensor vanishes is called a projective-symmetric space and is denoted by P_n. The author proves the following series of theorems. (1) Every V_2 is a P_2. The scalar curvature of a P_n $(n>2)$ is a constant but that of a P_2 is, in general, not so. A P_2 is of constant scalar curvature if and only if it is symmetric in the sense of Cartan. (2) A decomposable P_n is symmetric in the sense of Cartan. (3) Every P_3 is a conformally flat symmetric space. (4) A conformally flat P_n $(n \geqq 4)$ is symmetric in the sense of Cartan. If, further, the rank of the matrix

$$C_{ji} = -\frac{R_{ji}}{n-2}+\frac{Rg_{ji}}{2(n-1)(n-2)}$$

is n, then the P_n is a symmetric space of the first kind, where R_{ji} is the Ricci tensor and R the curvature scalar [V. Hlavatý, Rend. Circ. Mat. Palermo (2) **9** (1960), 125–146; MR **25** #3469]. (5) A non-flat P_n $(n \geqq 4)$ cannot be a recurrent space. (6) A necessary and sufficient condition that a P_n $(n \geqq 4)$ be a Ricci recurrent space specified by a non-zero vector λ_m is that

$$\nabla_l R_{kjih} = \lambda_l(R_{kjih}-W_{kjih}),$$

where R_{kjih} is the curvature tensor and W_{kjih} the projective curvature tensor. (7) In a Ricci-recurrent P_n $(n \geqq 4)$ the rank of the Ricci tensor is 1 and the vector of recurrence is a null vector and the gradient of a scalar. (8) In a non-flat P_n $(n>2)$ there cannot exist a field of concurrent directions. A field of concurrent directions is defined by a vector v^h satisfying $\nabla_j v_i = g_{ji}$. {It seems to the reviewer that the author is not aware of the results of S. Sasaki and M. Goto [Trans. Amer. Math. Soc. **80** (1955), 148–158; MR **17**, 659].} *K. Yano* (Tokyo)

Hsu, Chen-Jung 4479

Remarks on certain almost product spaces.

Pacific J. Math. **14** (1964), 163–176.

An almost product metric space is a C^∞-manifold with a non-trivial tensor field F_i^j and a riemannian metric g_{ij} satisfying $F_i^j F_j^k = \delta_i^k$, $f_{ij} F_h^i F_k^j = g_{hk}$. Properties of various classes of such spaces corresponding to certain classes of almost Hermitian spaces are given, and relations between spaces of these classes established. A number of theorems analogous to known theorems are proved.

A. G. Walker (Liverpool)

Moór, A. 4480

Über konforme und projektive Vernäderung der Krümmung in Punkträumen.

Acta Math. Acad. Sci. Hungar. **15** (1964), 67–75.

Consider a manifold with a Riemannian metric tensor g_{ji} and denote by Γ_{ji}^h the Christoffel symbols formed with g_{ji}. The author calls V^* a manifold with the linear connexion

$$\overset{*}{\Gamma}{}_{ji}^h = \Gamma_{ji}^h + T_{ji}{}^h,$$

where $T_{ji}{}^h$ is a tensor symmetric with respect to j and i. The author calls V^* a space of scalar curvature tensor of the first, second and third kinds when the curvature tensor of the V^* is one of the forms:

$$R^*_{kjih} = R^*(h_{kh}h_{ji}-h_{jh}h_{ki});$$
$$R^*_{kjih} = R^*(h_{kh}g_{ji}-h_{jh}g_{ki})$$
or
$$R^*_{kjih} = R^*(g_{kh}h_{ji}-g_{jh}h_{ki});$$
$$R^*_{kjih} = R^*(g_{kh}g_{ji}-g_{jh}g_{ki}),$$

respectively, where h_{ji} is a tensor determined by g_{ji} and $T_{ji}{}^h$ and R^* is a scalar.

Now, if we effect a conformal change $\tilde g_{ji} = e^{2f} g_{ji}$ to the metric tensor, we have a manifold with linear connexion

$$\tilde\Gamma_{ji}^h = \Gamma_{ji}^h + f_j\delta_i^h + f_i\delta_j^h - g_{ji}f^h \qquad (f_i = \partial_i f, f^h = f_i g^{ih}).$$

The author calls this space $\tilde V$. Effecting a projective change to Γ_{ji}^h, we get a space with linear connexion

$$\hat\Gamma_{ji}^h = \Gamma_{ji}^h + p_j\delta_i^h + p_i\delta_j^h,$$

where p_i is a covariant vector. The author calls this space $\hat V$. Finally, effecting a conformal change to g_{ji} and a projective change to Γ_{ji}^h, we get a space with linear connexion

$$\overset{+}{\Gamma}{}_{ji}^h = \Gamma_{ji}^h + f_j\delta_i^h + f_i\delta_j^h - g_{ji}f^h + p_j\delta_i^h + p_i\delta_j^h.$$

The author calls this space $\overset{+}{V}$.

In the paper under review the author studies the conditions under which the spaces $\tilde V$, $\hat V$ and $\overset{+}{V}$ are spaces of scalar curvature tensor of the first, the second or the third kind. *K. Yano* (Tokyo)

Takasu, Tsurusaburo 4481

The relativity theory in the Einstein space under the extended Lorentz transformation group.

Proc. Japan Acad. **39** (1963), 620–625.

Tret'jakov, V. D. 4482

A conformal interpretation of symmetric, conformally Euclidean spaces. (Russian)

Comment. Math. Univ. Carolinae **4** (1963), 65–73.

It is known that the line geometry of a three-dimensional space of constant curvature appears to be a geometry of symmetric conformally Euclidean four-dimensional space of nul signature. A. P. Širokov has shown [Kazan. Gos. Univ. Učen. Zap. **116** (1956), no. 1, 15–19] that such a geometry can be realized as an intrinsic geometry of a B-hyperquadric normalized by means of the absolute involution. On the basis of these results, the author gives a conformal interpretation of symmetric, conformally Euclidean spaces. *W. Wrona* (Warsaw)

Katsurada, Yoshie 4483

On a certain property of closed hypersurfaces in an Einstein space.

Comment. Math. Helv. **38** (1964), 165–171.

In a previous paper [Ann. Mat. Pura Appl. (4) **57** (1962), 283–293; MR **26** #2989], the author generalized the classical theorems of Liebmann and Süss to a hypersurface of a Riemannian space of constant curvature. In the

present paper, she establishes a further generalization to the case of an Einstein space. The main theorem is as follows. Let R^{m+1} be an Einstein space and V^m a closed orientable hypersurface with constant mean curvature. If there exists an infinitesimal conformal transformation ξ in R^{m+1} such that the inner product of ξ with the unit normal vector n to V^m does not change its sign (and is not $\equiv 0$), then every point of V^m is umbilic. Note that if V^m is convex in a euclidean space R^{m+1}, then the vector field ξ with components $\xi^i = x^i$, where $x^1, \cdots, x^{m+1}$ is a rectangular coordinate system in R^{m+1} with origin in the interior of V^m, is an infinitesimal conformal transformation with the required property.

The proof is based on some integral formulas of Minkowski type, one of which was given in the previous paper and is proved here by a different method.

K. Nomizu (Providence, R.I.)

Raghunathan, M. S. 4484

Deformations of linear connections and Riemannian manifolds.

J. Math. Mech. **13** (1964), 97–123.

The author develops a theory of deformations of linear connections and Riemannian metrics on a differentiable manifold which is analogous to the theory for complex structures [K. Kodaira and D. C. Spencer, Ann. of Math. (2) **67** (1958), 328–466; MR **22** #3009]. The results are most definitive for a deformation of regular connections or metrics. A linear connection or Riemannian metric is said to be regular if the vector space of germs of Killing vector fields has a constant dimension. A locally homogeneous or analytic connection (or metric) is regular.

For a deformation of regular linear connections on a manifold with finitely presentable fundamental group, the author proves upper semi-continuity of the dimension of the 0-dimensional and 1-dimensional cohomology groups with coefficients in the sheaf of germs of Killing vector fields. It follows, in particular, that every deformation family of regular linear connections on M with a finite fundamental group is locally trivial. As special cases, the deformations on compact Riemannian manifolds with constant curvature or with positive or negative definite Ricci curvature are studied [cf. E. Calabi, Proc. Sympos. Pure Math., Vol. III, pp. 155–188, Amer. Math. Soc., Providence, R.I., 1961; MR **24** #A3612].

Finally, using the theory of harmonic forms with coefficients in a local system, the author gives a variation of Weil's results on discrete subgroups of semi-simple Lie groups [I, Ann. of Math. (2) **72** (1960), 369–384; MR **25** #1241; II, ibid. (2) **75** (1962), 578–602; MR **25** #1242]. This portion of the paper is similar to part of the work of Matsushima and Murakami [ibid. (2) **78** (1963), 365–416; MR **27** #2997].

K. Nomizu (Providence, R.I.)

Tanno, Shûkichi 4485

Some transformations on manifolds with almost contact and contact metric structures. II.

Tôhoku Math. J. (2) **15** (1963), 322–331.

Using the same notations as in Part I [same J. (2) **15** (1963), 140–147; MR **27** #703], the group Φ of diffeomorphisms leaving ϕ invariant is a Lie group. If M is complete, an infinitesimal transformation of Φ can be extended completely. Under $\mu \in \Phi$, the metric g is transformed by $\mu^* g = \alpha g + \alpha(\alpha-1)\eta \otimes \eta$, α a positive constant. The transformation formula of the scalar curvature and its applications are obtained.

Y. Tashiro (Okayama)

Tsukamoto, Yôtarô 4486

A proof of Berger's theorem.

Mem. Fac. Sci. Kyushu Univ. Ser. A **17** (1963), 168–175.

The theorem referred to in the title is the following. Let M be a compact, simply connected Riemannian manifold of even dimension n. If the sectional curvature K of M satisfies the inequalities $\frac{1}{4} \leqq K \leqq 1$, then M is either homeomorphic with an n-sphere or isometric with a symmetric space of rank 1. This was proved by Berger [C. R. Acad. Sci. Paris **250** (1960), 442–444; MR **22** #1865] who used, among other results, the triangle theorem of Toponogov [Dokl. Akad. Nauk SSSR **120** (1958), 719–721; MR **20** #6139]. The purpose of the present paper is to prove Berger's theorem without using the triangle theorem. The principal idea of this proof is already contained in the author's paper [Mem. Fac. Sci. Kyushu Univ. Ser. A **15** (1961/62), 90–96; MR **25** #2561].

W. Klingenberg (Mainz)

Wong, Yuen-Fat 4487

Similarity transformations of hypersurfaces.

Proc. Amer. Math. Soc. **15** (1964), 286–287.

Let M and M' be two n-dimensional closed orientable Riemannian manifolds of class C^3 imbedded in $(n+1)$-dimensional Euclidean space. We denote by g_{ji} the first fundamental tensor, by R_{ji} the Ricci tensor and by R the curvature scalar $g^{ji}R_{ji}$ of M and by g_{ji}', R_{ji}' and R' the corresponding quantities of M', respectively.

The purpose of the present note is to prove the following theorem. Given M, M' with positive R, R', respectively, and a diffeomorphism $h: M \to M'$ which preserves Rg_{ji}. Then h is a similarity. For the proof the author utilises the Hopf-Bochner principle: In a compact Riemannian manifold with positive definite metric if a function f satisfies $g^{ji}\nabla_j\nabla_i f \geqq 0$ everywhere, then f is a constant.

K. Yano (Tokyo)

Wu, H. 4488

On the de Rham decomposition theorem.

Illinois J. Math. **8** (1964), 291–311.

The decomposition theorem for a simply connected complete Riemannian manifold according to the reducibility of the holonomy group [de Rham, Comment. Math. Helv. **26** (1952), 328–344; MR **14**, 584; for a different proof, see Kobayashi and the reviewer, *Foundations of differential geometry*, Vol. I, Interscience, New York, 1963; MR **27** #2945] is now extended to the case of an indefinite metric. To be more precise, let M be a manifold with a non-degenerate metric. The holonomy group Φ of M at a point m is said to be nondegenerately reducible if it leaves invariant a proper subspace $M_m{}^1$ of the tangent space M_m on which the restriction of the metric is also non-degenerate. In this case, M_m is the direct sum of $M_m{}^1$ and its orthogonal complement $M_m{}^2$, which is invariant by Φ and on which the restriction of the metric is also non-degenerate. The main result is the following. Let M be a simply connected complete manifold with an indefinite metric and suppose that the holonomy group Φ

at m is non-degenerately reducible: $M_m = M_m^1 \oplus M_m^2$ as before. Then M is isometric to the direct product of the maximal integral manifolds M^1 and M^2 of the parallel distributions T_1 and T_2 obtained from M_m^1 and M_m^2, respectively.

The first step of the proof is, of course, to establish a local decomposition and it is here that the assumption of non-degenerate reducibility is essential. An example, attributed to Holzsager, is given to illustrate that mere reducibility is not enough.

The construction of a global decomposition is based on a generalization by Hicks [Illinois J. Math. **3** (1959), 242-254; MR **21** #6597] of a result of Ambrose on parallel translation of curvature [Ann. of Math. (2) **64** (1956), 337-363; MR **21** #1627]. As the author observes, this part of the proof is valid for an affine connection which is not necessarily metric; once a local product structure is established, simple connectivity and completeness together will give a global decomposition, as worked out in a different manner by Kashiwabara [Tôhoku Math. J. (2) **8** (1956), 13-28; MR **18**, 332]; see also the remarks on the de Rham theorem by R. Hermann [Trans. Amer. Math. Soc. **108** (1963), 170-183; MR **27** #1905].

The author lists a number of known results based on the de Rham decomposition theorem which are now valid in suitable form for a manifold with an indefinite metric, thanks to his main theorem. *K. Nomizu* (Providence, R.I.)

Bonan, Edmond 4489

Structures presque quaternioniennes.

C. R. Acad. Sci. Paris **258** (1964), 792-795.

The author studies various infinitesimal properties of almost quaternionic manifolds (i.e., manifolds of dimension $4n$ which admit two tensor fields $\mathscr{I}$ and $\mathscr{J}$ of type (1, 1) such that $\mathscr{I}^2 = \mathscr{J}^2 = -$ iden, and $\mathscr{I}\mathscr{J} + \mathscr{J}\mathscr{I} = 0$).

S. Kobayashi (Berkeley, Calif.)

Bonan, Edmond 4490

Connexions presque quaternioniennes.

C. R. Acad. Sci. Paris **258** (1964), 1696-1699.

La note fait suite à une note antérieure [#4489 ci-dessus]. On démontre d'abord le théorème: Toute connexion linéaire réelle sur une variété presque quaternionienne V_{4n} induit sur cette variété une connexion presque quaternionienne. Pour que la connexion linéaire soit naturellement associée à la connexion presque quaternionienne qu'elle induit, il faut et il suffit que les différentielles absolues des tenseurs $\mathscr{I}$ et $\mathscr{J}$ dans la connexion linéaire soient nulles.

Etant données deux structures presque quaternioniennes $\mathscr{I}$, $\mathscr{J}$ ($\mathscr{K} = \mathscr{I}\mathscr{J}$) et $\mathscr{U}$, $\mathscr{V}$ ($\mathscr{W} = \mathscr{U}\mathscr{V}$) determinées respectivement par les sous-espaces vectoriels S_x^H et Σ_x^H de T_x^H, les conditions suivantes sont equivalentes: (a) $\mathscr{U} = a\mathscr{I} + b\mathscr{J} + c\mathscr{K}$; $\mathscr{V} = a'\mathscr{I} + b'\mathscr{J} + c'\mathscr{K}$; $a, b, c, a', b', c' \in R$; (b) Il existe un autre système de biconjugaisons de H tel que $\mathscr{U}$ et $\mathscr{V}$ soient définis par S_x^H; (c) Il existe un champ $\mathscr{L}$ d'opérateurs linéaires appliquant T_x sur lui-même tel que Σ_x^H soit l'image de S_x^H par l'extension linéaire de $\mathscr{L}$ à T_x^H avec $\mathscr{L} = \alpha\mathscr{E} + \beta\mathscr{I} + \gamma\mathscr{J} + \delta\mathscr{K}$ ($\alpha, \beta, \gamma, \delta \in R$), $\mathscr{E}$ étant l'identité de T_x^H.

Dans ces conditions, nous dirons que les structures S_x^H et Σ_x^H sont fortement équivalentes.

L'auteur démontre ensuite le théorème: Si deux structures presque quaternionienne sont fortement équivalentes,

$$(\mathscr{I}, \mathscr{I}) + (\mathscr{J}, \mathscr{J}) + (\mathscr{K}, \mathscr{K}) = (\mathscr{U}, \mathscr{U}) + (\mathscr{V}, \mathscr{V}) + (\mathscr{W}, \mathscr{W}),$$

où $(\mathscr{I}, \mathscr{I})$ est le tenseur de structure de $\mathscr{I}$.

K. Yano (Tokyo)

Bonan, Edmond 4491

Structures presque hermitiennes quaternioniennes.

C. R. Acad. Sci. Paris **258** (1964), 1988–1991.

La Note fait suite à deux Notes antérieures [#4489 et #4490 ci-dessus]. L'auteur définit d'abord la structure presque hermitienne quaternionienne et démontre ensuite deux théorèmes suivants. Théorème: Si $\mathscr{K}$ est une structure presque complexe de V_{2m} on peut toujours construire, à partir d'une métrique arbitraire, une autre métrique riemannienne orthoéchangeable avec F et G et, par suite, une structure presque hermitienne quaternionienne les admettant comme deux formes fondamentales. Il en resulte $m = 2n$. Théorème: Il y a identité entre l'ensemble des variétés admettant une structure presque quaternionienne et celles qui admettent une structure presque symplectique complexe.

Dans la dernière partie l'auteur étudie les connexions presque hermitiennes quaternioniennes. *K. Yano* (Tokyo)

Bouzon, Jean 4492

Structures presque-cohermitiennes.

C. R. Acad. Sci. Paris **258** (1964), 412–415.

Continuing two previous notes [same C. R. **255** (1962), 822-824; MR **25** #5474a; ibid. **255** (1962), 1061–1063; MR **25** #5474b], the author studies almost co-hermitian connections, co-kählerian connections, their holonomy groups and reducibility, the Laplacian operator, and the Betti numbers of compact co-kählerian manifolds.

K. Nomizu (Providence, R.I.)

Nagai, Tamao; Kôjyô, Hidemaro 4493

On some properties of hypersurfaces with certain contact structures.

J. Fac. Sci. Hokkaido Univ. Ser. I **17** (1963), 160–167.

Let us consider an almost Hermitian manifold, that is, a manifold with an almost complex structure $F_\lambda{}^\kappa$ and a Riemannian metric $g_{\mu\lambda}$ satisfying $F_\mu{}^\gamma F_\lambda{}^\beta g_{\gamma\beta} = g_{\mu\lambda}$, and denote by ∇_μ the covariant differentiation with respect to $g_{\mu\lambda}$. The manifold is called a Kähler manifold if $\nabla_\mu F_{\lambda\kappa} = 0$, an almost Kähler manifold if $\nabla_\mu F_{\lambda\kappa} + \nabla_\lambda F_{\kappa\mu} + \nabla_\kappa F_{\mu\lambda} = 0$ and an almost Tachibana manifold if $\nabla_\mu F_{\lambda\kappa} + \nabla_\lambda F_{\mu\kappa} = 0$.

Consider an orientable hypersurface $x^\kappa = x^\kappa(y^a)$ in an almost Hermitian manifold, put $B_a{}^\kappa = \partial_a x^\kappa$ ($\partial_a = \partial/y^a$) and denote by N^κ the normal to the hypersurface. If we put

$$\varphi_i{}^h = B_i{}^\lambda B^h{}_\kappa F_\lambda{}^\kappa, \quad \xi^h = -N^\lambda B^h{}_\kappa F_\lambda{}^\kappa, \quad \eta_i = B_i{}^\lambda N_\kappa F_\lambda{}^\kappa,$$

we have

$$\varphi_j{}^i\varphi_i{}^h = -\delta_j{}^h + \eta_j\xi^h, \; \varphi_i{}^h\xi^i = 0, \; \varphi_i{}^h\eta_h = 0, \; \xi^h\eta_h = 1,$$

and

$$\varphi_j{}^c\varphi_i{}^b g_{cb} = g_{ji} - \eta_j\eta_i, \; g_{ji}\xi^i = \eta_j.$$

We say that $\varphi_i{}^h$, ξ^h, η_i, g_{ji} satisfying these equations

define an almost contact metric structure on the hypersurface. If an almost contact structure satisfies

$$n_{ji}{}^h = \varphi_j{}^a(\partial_a\varphi_i{}^h - \partial_i\varphi_a{}^h) - \varphi_i{}^a(\partial_a\varphi_j{}^h - \partial_j\varphi_a{}^h) - (\partial_j\xi^h)\eta_i + (\partial_i\xi^h)\eta_j = 0,$$

the structure is said to be normal.

Now, the authors prove the following theorems. (1) If a hypersurface of an almost Tachibana space has the normal contact structure, the second fundamental tensor has the form

$$(*) \qquad h_{ji} = -g_{ji} + \phi\eta_j\eta_i.$$

If the second fundamental tensor of a hypersurface has the form (*), the authors call it an η-umbilical hypersurface. If the Ricci tensor of the hypersurface has the form $R_{ji} = \lambda g_{ji} + \mu\eta_j\eta_i$, the authors call the space an η-Einstein space. (2) An η-umbilical hypersurface in a space of constant curvature is not a space of constant curvature. (3) An η-umbilical hypersurface in a space of constant curvature is an η-Einstein space. (4) If a hypersurface of an almost Tachibana space with constant curvature has a normal contact metric structure, then the second fundamental tensor has the form (*), where ϕ is a constant, and the hypersurface has constant mean curvature. (5) In order that an induced contact metric structure of a hypersurface in a Kähler space is normal, it is necessary and sufficient that the second fundamental tensor h_{ji} satisfies

$$h_i{}^h + h_b{}^a\varphi_i{}^b\varphi_a{}^h - h_a{}^h\xi^a\eta_i = 0.$$

(6) In order that an induced almost contact metric structure of a hypersurface in a Kähler space is normal contact metric structure, it is necessary and sufficient that the second fundamental tensor has the form (*).

K. Yano (Tokyo)

Moalla, M. Fatma 4494

Espaces de Finsler complets.

C. R. Acad. Sci. Paris **258** (1964), 2251–2254.

The theorem of Hopf-Rinow, generalised by de Rham, states that for a complete riemannian manifold the following properties are equivalent: (a) either every geodesic can be continued indefinitely in either direction or else it is closed; (b) every Cauchy sequence is convergent; (c) every bounded subset is relatively compact.

In this paper it is shown that the Hopf-Rinow theorem applies to Cartan-Finsler spaces for which the geodesics coincide with the extremals of the length integral. The proof follows closely that given by de Rham [*Variétés différentiables*, pp. 133–134, Actualités Sci. Indust., No. 1222, Hermann, Paris, 1955; MR **16**, 957; see also Comment. Math. Helv. **26** (1952), 328–344; MR **14**, 584].

T. J. Willmore (Liverpool)

Kropina, V. K. 4495

Projective two-dimensional Finsler spaces with special metric. (Russian)

Trudy Sem. Vektor. Tenzor. Anal. **11** (1961), 277–292.

Following the investigations of L. Berwald the author shows that: (a) A two-dimensional Finsler space F_2 with a fundamental function $L(x; X) = a_{\alpha\beta}(x)X^\alpha X^\beta / b_\mu(x)X^\mu$ is flat projective if and only if L in a suitable coordinate system has the form

$$L = \frac{1}{X}\left(\frac{\partial\varphi}{\partial x^1}X^2 + \frac{\partial\varphi}{\partial x^2}XY + Y^2\right)$$

($\varphi(x^1, x^2)$ being an arbitrary function); (b) It is a Minkowski space M_2 if and only if φ is linear in x, or

$$\varphi = [1/(k_1 - x^2)][(x^1)^2 + k_2x^1 + k_3] + k_4$$

(k_i being constants); (c) Similarly, it has constant scalar curvature only if it is an M_2; (d) If an F_2 with $L = [a_{\alpha\beta\gamma}(x)X^\alpha X^\beta X^\gamma]^{1/3}$ is flat projective, then it is an M_2.

L. Tamássy (Debrecen)

Soós, Gyula 4496

On the theory of fiber spaces of Finsler type. (Hungarian)

Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **13** (1963), 17–64.

Der Verfasser begründet die Finslersche Geometrie mit Hilfe eines prinzipalen fibrierten Raumes über einer differenzierbaren Mannigfaltigkeit. Dies geschieht so, dass auf einem Tangentenbündel V über einer differenzierbaren Mannigfaltigkeit M derjenige fibrierte Raum in Betracht gezogen wird, welcher dadurch entsteht, dass jedem Punkt von R als Fibrum die Gesamtheit der n-Kanten zugeordnet wird. Die Parallelverschiebung wird global definiert. Es wird gezeigt, dass diejenigen differenzierbaren Automorphismen des Raumes, welche den linearen Zusammenhang bewahren, eine Liesche Gruppe bilden, und dass dieser Gruppe gegenüber die Parallelverschiebung invariant ist.

Es werden noch für ein beliebiges Vektorfeld X eine kovariante Derivation die eine solche vierter Art genannt wird, und ein sogenannter Endomorphismus eingeführt. Für den letzteren wird auch die geometrische Bedeutung angegeben. Endlich wird gezeigt, dass sich jeder fibrierte Finslersche Raum Q mit einem linearen Zusammenhang versehen lässt, falls er nur dem zweiten Abzählbarkeitsaxiom genügt.

A. Rapcsák (Debrecen)

Kirk, W. A. 4497

On curvature of a metric space at a point.

Pacific J. Math. **14** (1964), 195–198.

Let U be a neighborhood of the point p in a metric space such that any two points of U can be connected by a segment and no neighborhood of p is linear. If, for a given $\varepsilon > 0$, any four points sufficiently close to p, three of which lie on a segment, are isometric to a quadruple in the simply-connected space S_k of constant curvature k with $|k - k'(p)| < \varepsilon$, then $k'(p)$ is the curvature (Wald) at p. Consider also points A, B, C in some S_k isometric to three points a, b, c in U and points $a_t, b_t, A_t{}^k, B_t{}^k$ on segments from c to a, c to b, C to A, C to B dividing these segments in the ratio $t : 1$ $(0 < t < 1)$, i.e., $cb_t : cb = t$, etc. The space has at p the curvature $R(p)$ (Rinow) if for a, b, c sufficiently close to p, both $a_tb_t \leqq A_t{}^kB_t{}^k$ with $k \leqq R(p) + \varepsilon$ and $a_tb_t \geqq A_t{}^kB_t{}^k$ with $k \geqq R(p) - \varepsilon$.

It is shown that if one of these curvatures exists, then so does the other and they are equal.

H. Busemann (Los Angeles, Calif.)

Nass, Josef 4498

Über die ebenen ersten Fundamentalformen gemeiner Flächen.

Math. Nachr. **26** (1963), 149–160.

Ordinary surfaces in Euclidean E_3 are defined by a vector-valued function $X(w)=(X^1(w), X^2(w), X^3(w))$, where $w=(u,v)$ is a point in an interval J of E_2 and the functions are of class C^2. In this paper the functions $X^i(w)$ are replaced by two-dimensional distributions, and the resulting object is called a "generalized surface".

Since a distribution can be expressed as an equivalence class of sequences of smooth functions, a generalized surface is an equivalence class of sequences of smooth ordinary surfaces. From this the author defines the first fundamental form of a generalized surface, and finds that it has the usual properties of the first fundamental form of an ordinary surface.

C. B. Allendoerfer (Seattle, Wash.)

Tandai, Kwoichi 4499

On general connections in an areal space. II. General connections on the tangent m-frame bundle.

Tensor (*N.S.*) **14** (1963), 26–46.

From the author's introduction: "The present paper is a continuation of the author's preceding paper [Tensor (N.S.) **13** (1963), 277–291; MR **27** #4189]. As was mentioned in this paper, the present paper is devoted to the investigations of the theory of general connexions on the tangent m-frame bundle $F^{(m)}(M_n)$ of a differentiable manifold M_n of n dimensions with an areal space structure. All the notations used in the paper cited above are preserved in the present paper. In § 1, we explain the group reduction of the principal bundle $P(F^{(m)}(M_n))$ associated with the tangent bundle $T(F^{(m)}(M_n))$ of $F^{(m)}(M_n)$. On making use of the group reduction in § 1, we introduce the canonical connexion of the principal bundle $\rho: F^{(m)}(M_n) \to G^{(m)}(M_n)$ [$\rho': F^{(m)}(M_n) \to H^{(m)}(M_n)$] in § 2. We shall consider the general theory of general connexions on $F^{(m)}(M_n)$ in § 3, and two important general connexions Γ and Γ' are defined by means of the fundamental function F of an areal space over M_n in § 4. In § 5, curvature tensor field and Ricci identities for Γ [Γ'] are obtained. It must be noticed that Ricci identities do not hold for an arbitrary tensor field, but only for an $\tilde{\omega}$-basic [$\tilde{\omega}'$-basic] tensor field. In § 6, we shall introduce a general connexion $\tilde{\gamma}$ [$\tilde{\gamma}'$] complementary to Γ [Γ'] by means of the canonical connexion $\tilde{\omega}$ [$\tilde{\omega}'$] defined in § 4, and define an affine connexion C [C'] as the sum of Γ [Γ'] and $\tilde{\gamma}$ [$\tilde{\gamma}'$]. There are many unsolved problems arising from the theory developed in the present paper. Further and more profound investigations are expected in the future."

K. Yano (Tokyo)

GENERAL TOPOLOGY

See also 4058, 4066, 4346, 4544.

Baum, John D. 4500

★Elements of point set topology.

Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964. x+150 *pp.* $7.95.

This book is an introduction to general topology; as such, it goes from the basic concepts up to the level of the metrization theorems and the more elementary facts about connectedness. Being designed for undergraduates at small colleges, it spells out the subject in great detail. When there are two essentially equivalent alternatives, both are often mentioned. While the author carries out his program with great care, the reviewer cannot help but feel that the result is too soft, leaving the good student little chance to supply part of the theory himself. Perhaps the book will be most useful in conjunction with the concise treatment in a more advanced text.

D. W. Kahn (New York)

Banaschewski, Bernhard 4501

Extensions of topological spaces.

Canad. Math. Bull. **7** (1964), 1–22.

An extension of a topological space E is a pair (E', φ), where E' is a topological space and φ is a homeomorphism of E onto a dense subset of E'. In the author's words, "This paper is neither intended to present substantial new results nor to provide an encyclopaedic survey, but, rather, to give a unified account ...". The author discusses such topics as the properties of $\varphi(E)$ as a subset of E', the properties of E' as a topological space, the relations on the set of all (E', φ) for a given E, and the various systematic methods of defining an (E', φ) for each E in a fixed collection of spaces.

H. H. Corson (Seattle, Wash.)

Cohen, Henry B. 4502

The k-extremally disconnected spaces as projectives.

Canad. J. Math. **16** (1964), 253–260.

Let k be an infinite cardinal. A space is called extremally disconnected [k-extremally disconnected] if it is the Stone space for a complete [k-complete] Boolean algebra. A subset of a topological space is k-open if it is the union of k or fewer sets each of which is the complement of a zero set. A k-space is one in which every k-open set has a k-open complement. A map (continuous) $f: X \to Y$ is a k-map if, for each k-open subset U of X, there is a k-open subset V of $f(X)$ such that $f(\text{cl}(U)) = \text{cl}(V)$. Theorem: Every k-space X is the image under a minimal k-map f of a k-extremally disconnected space M; if M' is a k-extremally disconnected space and $f': M' \to X$ is a minimal k-map, there is a homeomorphism $h: M \to M'$ such that $f'h = f$. Corollary: In the category of k-spaces and k-maps, a space is projective if and only if it is k-extremally disconnected. This is a refinement of a result of A. M. Gleason [Illinois J. Math. **2** (1958), 482–489; MR **22** #12509].

R. W. Bagley (Coral Gables, Fla.)

Comfort, W. W. 4503

An example in density character.

Arch. Math. **14** (1963), 422–423.

The author gives a simple example of a non-separable Hausdorff space whose Stone-Čech compactification is separable (i.e., contains a countable dense subset). It is of interest that the spaces involved are also groups.

F. Burton Jones (Riverside, Calif.)

Engelking, R.; Pełczyński, A. 4504
Remarks on dyadic spaces.
Colloq. Math. **11** (1963), 55–63.

A dyadic space is one which is the continuous image of a Cartesian product of two point spaces. (Space means Hausdorff space.) Some necessary conditions are given for a space to be a dyadic space, for instance, there is no infinite, extremally disconnected dyadic space; and some sufficient conditions are given, for instance, every closed G_δ in a dyadic space is a dyadic space itself.

H. H. Corson (Seattle, Wash.)

Engelking, R. 4505
Quelques remarques concernant les opérations sur les fonctions semi-continues dans les espaces topologiques.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 719–726.

Let $F: Y \to 2^X$ be a function from a topological space Y to a family 2^X of all closed subsets (including the empty one) of a topological space X. Introducing in 2^X a topological base as finite intersections of sets $2^G \cap 2^X$ or $2^X - 2^H$, where $G \subset X$ is open and $H \subset X$ is closed, one defines upper semicontinuity (u.s.c.) and lower semicontinuity (l.s.c.) of F at a point $y \in Y$. Operations like multiplication or cartesian product applied to values of different functions F allow one to define in an obvious way new such functions. Some of them preserve u.s.c. or l.s.c. at each point where all defining functions are, respectively, u.s.c. or l.s.c. On the other hand, preserving of u.s.c. or l.s.c. by some of these operations implies such properties of X like compactness, regularity, etc.

R. Duda (Cambridge, England)

Flachsmeyer, Jürgen 4506
Topologische Projektivräume.
Math. Nachr. **26** (1963), 57–66.

Author's introduction: "In der homologischen Algebra weiss man um die Wichtigkeit des Begriffes des projektiven Moduls. Für gewisse Kategorien topologischer Räume ist eine entsprechende Begriffsbildung inhaltsreich, was erstmal A. M. Gleason [Illinois J. Math. **2** (1958), 482–489; MR **22** #12509] von der Kategorie der kompakten topologischen Räume und stetigen Abbildungen sowie von der Kategorie der lokal kompakten Räume und eigentlichen Abbildungen gezeigt hat. Nachstehende Ausführungen sollen die unnatürlichen Beschränkungen auf die genannten Raumkategorien aufheben."

S.-T. Hu (Los Angeles, Calif.)

Kuratovski, K. [Kuratowski, K.] 4507
Characterization of regular lattices by means of the exponential topology. (Russian)
Dokl. Akad. Nauk SSSR **155** (1964), 751–752.

This paper deals with the lattice-theoretical description of relations between a topological space X and the space 2^X of all closed subsets of X. For a distributive lattice L the notions of regularity and normality can be introduced (which, for the lattice 2^X, are equivalent to regularity and normality of the space X), and a topology in L can be defined (which is the usual one for $L = 2^X$). Connections between regularity, normality, separation properties of the topology and semi-continuity of the lattice-theoretical operations for a distributive lattice L are studied in an earlier paper of the author [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 9–16; MR **29** #593]. In the present paper these results are completed. The following theorem is proved: A distributive lattice L which has the Wallman disjunction property is regular if and only if the set $F = \{(x, y) : x \subset y\}$ is closed in $L \times L$.

R. Engelking (Warsaw)

Papert, Seymour 4508
An abstract theory of topological subspaces.
Proc. Cambridge Philos. Soc. **60** (1964), 197–203.

The open subsets of a topological space X form a complete Brouwerian, i.e., distributive and pseudo-complemented lattice $L(X)$. Many topological properties of X can be formulated as properties of $L(X)$, although, in general, X is not determined by $L(X)$. In this paper, the author develops sections of general topology as part of the theory of complete Brouwerian lattices.

S.-T. Hu (Los Angeles, Calif.)

Kent, D. 4509
Convergence functions and their related topologies.
Fund. Math. **54** (1964), 125–133.

Let S be a set and q be a (partial) order-preserving function from the set $F(S)$ of all filters on S to the set $P(S)$ of all subsets of S. The function q is a convergence function if for each $x \in S$, $x \in q(\mathscr{F}_x)$, where $\mathscr{F}_x$ is the ultrafilter determined by $\{x\}$. The set $C(S)$ of all convergence functions on S is partially ordered by $q_1 \leqq q_2$ if for each $\mathscr{F} \in F(S)$, $q_1(\mathscr{F}) \supset q_2(\mathscr{F})$, and the result is a complete lattice.

If $\mathscr{T}(S)$ is the set of all topologies on S, then there is a natural inclusion $\mathscr{T}(S) \to C(S)$, because given a topology we can define $q(\mathscr{F})$ to be the set of all x to which $\mathscr{F}$ converges in the given topology. A linkage function is a retraction of $C(S)$ to $\mathscr{T}(S)$. It is a linkage homomorphism if it also preserves partial order.

This paper studies three linkage homomorphisms and one linkage (non-homomorphic) function, and relations among them. The three homomorphisms agree on any convergence function q satisfying: $x \in q(\mathscr{F}_1)$ and $x \in q(\mathscr{F}_2)$ imply $x \in q(\mathscr{F}_1 \cap \mathscr{F}_2)$. An example is given to show that this is not true for all convergence functions.

M. L. Curtis (Tallahassee, Fla.)

Oeconomidis, Nicolas 4510
Sur la convergence relative d'une suite d'ensembles et de leurs applications sur les suites de fonctions.
Ann. Sci. École Norm. Sup. (3) **80** (1963), 107–133.

Soit $G = \prod A_n$ le produit cartésien d'une suite des ensembles non-vides situés dans un espace L^* (éventuellement dans un espace métrique). Nous dirons que $f \in G$ est convergente si la suite $f(n)$ est convergente; posons dans ce cas $\lim f(n) = \hat{f}$ et $G^{\to}(A_n) = [f : f \in G,\ f$ convergente$]$, et $\hat{G}(A_n) = [\hat{f} : f \in G^{\to}(A_n)]$. Alors, on dit que la suite A_n converge vers A, relativement à H, et l'on écrit $(H) \lim A_n = A$, si: (1) $H \subset G^{\to}(A_n)$, (2) $\bigcup_{f \in H} f(\phi) = \bigcup_{n \in \phi} A_n$ (ϕ est l'ensemble des nombres naturels, positifs), et (3) $A = [\hat{f} : f \in H]$. Cette notion de convergence est invariante sous transformations continues.

L'auteur considère ensuite les conditions nécessaires et

suffisantes, pour que la limite relative d'une suite d'ensembles dans un espace métrique, soit un ensemble borné, compact, fermé ou ouvert. Dans la seconde partie, applications de cette notion aux suites des fonctions, définies dans un espace métrique et dont les valeurs appartiennent aussi à un espace métrique.
V. S. Krishnan (Madras)

Oeconomidis, Nicolas 4511
Sur la convergence pseudo-continue d'une suite de fonctions.
C. R. Acad. Sci. Paris **258** (1964), 3154–3156.
Etant donné une suite des fonctions F_n, définies sur un ensemble E, situé dans un espace métrique X, et dont les valeurs appartiennent à un espace métrique Y, la condition nécessaire et suffisante pour que cette suite converge à une fonction F continue au point $x_0 \in E$ est que cette suite converge pseudo-continûment à F au point x_0. [Voir aussi #4510 ci-dessus.] *V. S. Krishnan* (Madras)

Reichaw-Reichbach, M. 4512
On compactification of metric spaces.
Israel J. Math. **1** (1963), 61–74.
Throughout, all spaces considered are separable metric. An absolute G_δ space X is of the "first kind" if there exists a compact (metric) space in which X is the intersection of a sequence of open sets G_i with $\dim(\mathrm{Fr}(G_i)) < \dim X$; X is of the "second kind" otherwise. Let n be a given positive integer. After summarising some known results about compactifications, the author answers some questions of Knaster and Lelek by constructing examples as follows. (In each case the space constructed is a σ-compact absolute G_δ, and is the union of a sequence of cells.) (1) An n-dimensional space of the second kind which contains no set of the form $N \times Z$, where $\dim Z = n$ and N is the space of irrationals. (2) An n-dimensional space X, locally compact except at one point, such that, for every (metrisable) compactification X^* of X, $\dim(X^* - X) \geq 1$. (3) An infinite-dimensional space Y, of the first kind, such that, for every compactification Y^* of Y, $\dim(Y^* - Y) = \infty$.
A. H. Stone (Rochester, N.Y.)

Lodato, Michael W. 4513
On topologically induced generalized proximity relations.
Proc. Amer. Math. Soc. **15** (1964), 417–422.
A P_2-relation δ (a generalization of proximity) is defined on the power set of a set X. A cluster π from (X, δ) is a collection of subsets of X satisfying the conditions: (1) $A\delta B$ for all $A, B \in \pi$. (2) If $A \cup B \in \pi$, then $A \in \pi$ or $B \in \pi$. (3) If $A\delta B$ for every $A \in \pi$, then $B \in \pi$. Theorem A: If X is a set and δ is a binary relation on the power set of X, then the following are equivalent: (I) There is a T_1-space Y and a mapping $f: X \to Y$ such that $f(X)$ is regularly dense in Y and $A\delta B$ in X if and only if $\mathrm{cl}(f(A))$ meets $\mathrm{cl}(f(B))$. (II) δ is a P_2-relation satisfying: (α) if $A\delta B$ in X, then there is a cluster from (X, δ) which contains A and B. (A subset X of Y is regularly dense in Y if $p \in \mathscr{U}$, open in Y, implies the existence of $E \subset X$ such that $p \in \bar{E} \subset \mathscr{U}$, the closure being taken in Y.) A symmetric P_1-relation is a P_2-relation which satisfies the condition: $x\delta y$ implies $x = y$ for $x, y \in X$. Theorem B: Given X and δ as in Theorem A, the following are equivalent: (I) There is a T_1-space Y in which X can be topologically imbedded as a regularly dense subset so that $A\delta B$ if and only if $\bar{A}$ meets $\bar{B}$ in Y. (II) δ is a symmetric P_1-relation satisfying condition (α).
R. W. Bagley (Coral Gables, Fla.)

Pervin, William J. 4514
Equinormal proximity spaces.
Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 152–154.
The author proves that a normal proximity space (X, δ) is equinormal (pairs of disjoint closed sets are δ-distant) if and only if every real-valued continuous function on X is equicontinuous. Here the term equicontinuous (applied to a single function) means that pairs of near sets are carried onto near sets. *R. W. Bagley* (Coral Gables, Fla.)

Mrówka, S. G.; Pervin, W. J. 4515
On uniform connectedness.
Proc. Amer. Math. Soc. **15** (1964), 446–449.
Connectedness of topological spaces can be defined in terms of continuous functions to a (non-degenerate) discrete space or to the real line. The authors consider similar properties in uniform spaces and proximity spaces by using uniformly continuous functions or equicontinuous functions and what they call uniform connectedness, which is more general than topological connectedness but, as may be suspected, depends on uniformity rather than topology. Theorem 1 lists several equivalent conditions along this line. A neat result is Theorem 2: Let (X, δ) be a Lindelöf proximity space. If every real-valued equicontinuous function on X has the Darboux property (i.e., has a connected image), then X is connected (in the topological sense). A counter-example is given to show that the hypothesis of the Lindelöf property here is essential. *K. W. Kwun* (Tallahassee, Fla.)

Mrówka, S. 4516
An elementary proof of Katětov's theorem concerning Q-spaces.
Michigan Math. J. **11** (1964), 61–63.
The theorem in question states that if X is paracompact and every closed discrete subspace of X is a Q-space (i.e., real-compact), then X is also a Q-space; it is a particular case of Shirota's result that if X admits a complete uniformity and every closed discrete subspace is real-compact, then X is real-compact. {Inasmuch as the author's proof refers nontrivially to Kelley's Theorem 5.28 [*General topology*, Van Nostrand, Toronto, Ont., 1955; MR **16**, 1136], it seems to the reviewer to be no more elementary than the proof of Shirota's theorem itself given by the reviewer and Jerison [*Rings of continuous functions*, Van Nostrand, Princeton, N.J., 1960; MR **22** #6994].} *L. Gillman* (Rochester, N.Y.)

Pasynkov, B. 4517
Universal spaces for certain classes of spaces. (Russian)
Dokl. Akad. Nauk SSSR **153** (1963), 1009–1012.
The author generalizes the following four classes of spaces: (i) metrizable spaces, (ii) strongly metrizable

spaces (i.e., spaces having a σ-star-finite basis), (iii) paracompact spaces and (iv) strongly paracompact spaces (i.e., spaces whose open coverings admit star-finite refinements). For example, the generalization in the case (ii) yields the class of (completely regular) spaces X having a τ-star-finite basis, i.e., a basis which is the union of τ copies of star-finite normal coverings ω_α such that, given any $x \in X$ and any neighborhood $U(x)$, there is an $\alpha = \alpha(x, U(x))$ for which $\mathrm{St}(x, \omega_\alpha) \subseteq U(x)$ (here a covering ω is said to be normal provided there is a refinement ω' such that every $U \in \omega$ contains a $U' \in \omega'$ which is functionally separated from the complement of U). In the cases (i), (iii) and (iv) the author introduces spaces having a τ-locally finite basis, τ-paracompact spaces and τ-strongly paracompact spaces respectively.

A long list of theorems is announced (without proofs). The following are two sample theorems. Theorem 9: Every space X having a τ-star-finite basis and being of weight κ and (covering) dimension $\dim X \leqq n$ is imbeddable in the product $F^n \times B(\kappa, \tau)$ of a bicompactum F^n of weight κ and dimension $\dim F^n = n$ and of the product $B(\kappa, \tau)$ of τ copies of the space N_κ consisting of κ isolated points. Theorem 15(a): Every mapping $f: X \to R$ of an n-dimensional completely regular space X onto a strongly paracompact metric space R admits a factorization $f = hg$, $g: X \to S$, $h: S \to R$ ($g(X)$ dense on S), where S is a metric strongly paracompact space whose weight equals that of R and whose dimension $\dim S = n$.

S. Mardešić (Zagreb)

Westwick, R. 4518

On uniform sets in a complete separable metric space.

Fund. Math. **54** (1964), 1–6.

From the author's introduction: "Let R denote a complete separable metric space. . . . Let T denote a group of homeomorphisms of R onto itself. For each set A contained in R, we let $S(A)$ denote the collection of all those subsets B of R for which a sequence $\tau_1, \tau_2, \cdots$ of elements of T exists with $B \subset \bigcup_{i=1}^{\infty} \tau_i(A)$. We call a subset A of R uniform (with respect to the group T) if, for each open set U for which $U \cap A \neq \emptyset$, we have $A \in S(U \cap A)$. The purpose of this paper is to prove Theorem 4: Let A be a closed uniform subset of R, and let $\mathfrak{A}$ be an uncountable collection of closed subsets of A, such that $A \in S(B)$ for each $B \in \mathfrak{A}$. If either (i) T is an abelian group, or (ii) R is the real line and T the group of isometries of R, then there are at least two sets B_1 and B_2 of $\mathfrak{A}$ such that $A \in S(B_1 \cap B_2)$.

"This theorem reminds one of the theorem: If $\mathfrak{A}$ is an uncountable collection of Lebesgue measurable subsets of the real line, each with positive Lebesgue measure, then at least two of the sets have an intersection with positive Lebesgue measure. The similarity is even closer when we notice that the real line can be covered, to within a set of measure zero, by a countable number of isometric copies of any set with positive Lebesgue measure."

P. R. Halmos (Ann Arbor, Mich.)

Kimura, Nobuo 4519

On the inductive dimension of product spaces.

Proc. Japan Acad. **39** (1963), 641–646.

The following theorem (and several corollaries thereto) is shown. If Y is a metric space and if $X \times Y$ is totally normal and countably paracompact, then $\mathrm{Ind}(X \times Y) \leqq \mathrm{Ind}\, X + \mathrm{Ind}\, Y$. Where X and Y are not both empty sets, "Ind" denotes the large inductive dimension (defined in terms of closed sets) and "totally normal" is used in Dowker's sense [Quart. J. Math. Oxford Ser. (2) **4** (1953), 267–281; MR **16**, 157], i.e., a space is totally normal if it is normal and each open subset has a locally finite covering by open sets each of which is an F_σ set of the space.

Haskell Cohen (Baton Rouge, La.)

Bing, R. H.; Borsuk, K. 4520

A 3-dimensional absolute retract which does not contain any disk.

Fund. Math. **54** (1964), 159–175.

The authors construct a 3-dimensional absolute retract which does not contain any homeomorph of a 2-dimensional disc. It is the image of a map f defined on a 3-dimensional disc such that f has only countably many non-degenerate inverses, each of them being an arc suitably approximating a member of a carefully constructed family of Antoine's necklaces in the 3-disc.

E. Dyer (Chicago, Ill.)

Trohimčuk, Ju. Ju. 4521

Continuous mappings of domains in Euclidean space. (Russian. English summary)

Ukrain. Mat. Ž. **16** (1964), 196–211.

In an open set in R^n ($n \geqq 3$) let f be a continuous open mapping, locally homeomorphic except at a zero-dimensional set of points; then f is a local homeomorphism. The proof involves several preliminary results on light mappings.

J. R. Isbell (Princeton, N.J.)

Taĭmanov, A. D. 4522

Open pre-images of spheres. (Russian. Kazak summary)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 3, 8–12.

The author shows that each compact space that can be mapped onto an n-cell ($n \geqq 1$) [an n-sphere ($n \geqq 2$)] with an open local homeomorphism is the union of a finite collection of disjoint continua homeomorphic to the n-cell [n-sphere]. Other results concerning open, exactly $(k, 1)$ maps and open local homeomorphisms are given. The author defines a map f to be semi-monotone if and only if there is an integer l greater than 1 such that $f^{-1}(y)$ has exactly l components for each point y of the range. Then there is no semi-monotone open map of an n-cell ($n \geqq 2$) onto a cell of higher dimension. As the author points out, this is related to well-known results of L. Keldyš [e.g., Mat. Sb. (N.S.) **43** (**85**) (1957), 187–226; Dokl. Akad. Nauk SSSR **114** (1957), 472–475; MR **19**, 972] concerning open monotone dimension-raising maps.

R. Bennett (Knoxville, Tenn.)

Dickman, R. F.; Rubin, L. R.; Swingle, P. M. 4523

Characterization of n-spheres by an excluded middle membrane principle.

Michigan Math. J. **11** (1964), 53–59.

The authors present a new characterization of the 1-sphere, namely, "Let X be a nondegenerate, compact, perfectly

separable Hausdorff continuum. Then X is a 1-sphere if and only if for every pair $x, y \in X$ with $x \neq y$, there exist irreducible continua S_1 and S_2 from x to y, such that $S_1 \neq S_2$, $X = S_1 + S_2$, and if S_3 is a continuum irreducible from x to y, then either $S_3 = S_1$ or $S_3 = S_2$." They also succeed in giving a considerably more complicated characterization of the n-sphere which in some sense generalizes the above. *R. H. Rosen* (Princeton, N.J.)

Zieschang, Heiner 4524

Über einfache Kurvensysteme auf einer Vollbrezel vom Geschlecht 2.

Abh. Math. Sem. Univ. Hamburg **26** (1963/64), 237–247.

Let V be a solid torus of genus p. A simple system of curves on V is a collection of mutually disjoint simple closed curves which lie on the boundary of V. The author investigates the question of when two such systems are equivalent under a global homeomorphism. He studies the problem in a polyhedral setting and uses elements of the edge path group—from a fixed triangulation of V—associated with the curves of such a system. Sufficient conditions are given concerning when a fixed automorphism of the group relating the edge path classes associated with two such systems may be covered by a homeomorphism of V carrying one of the systems onto the other. Subsequently [same Abh. **27** (1964/65), 13–31; MR **28** #5105] the author has stated that there is a gap in one of his key lemmas on p. 239 and he reproves this as Theorem 7 of his new paper. D. R. McMillan, Jr. has obtained similar results [Proc. Amer. Math. Soc. **14** (1963), 386–390; MR **27** #770].

R. H. Rosen (Princeton, N.J.)

Gillman, David S. 4525

Note concerning a wild sphere of Bing.

Duke Math. J. **31** (1964), 247–254.

In his paper [same J. **28** (1961), 1–15; MR **23** #A630] R. H. Bing produced a 2-sphere S in E^3 such that every arc on S is tame, yet S itself is wild. Bing observed that one component of $E^3 - S$ is not simply connected, and raised the question: Is a 2-sphere in E^3 tame if each of its arcs is tame and each of its complementary domains is simply connected? The author answers this question in the negative. *L. Neuwirth* (Princeton, N.J.)

Greathouse, Charles 4526

Locally flat strings.

Bull. Amer. Math. Soc. **70** (1964), 415–418.

Let us call a topological $(n-1)$-sphere in S^n and a topological $(n-1)$-space in E^n flat if there is in each case a global homeomorphism which carries the $(n-1)$-sphere onto an equator and the $(n-1)$-space onto a hyperplane. The well-known annulus conjecture states that the closed region in S^n between two flat $(n-1)$-spheres is topologically $S^{n-1} \times I$. The author dubs the following the "slab conjecture": The closed region in E^n bounded by two closed disjoint locally flat $(n-1)$-spaces is topologically $E^{n-1} \times I$. He then shows that the slab conjecture implies the annulus conjecture. Since in the form given, as the author points out, the slab conjecture is false in dimension three, the reviewer proposes that a more natural form for the slab conjecture would be that the closed region in E^n bounded by two disjoint flat $(n-1)$-spaces is topologically $E^{n-1} \times I$. Stated this way, it would seem reasonable that the two conjectures are equivalent.

R. H. Rosen (Princeton, N.J.)

Kwun, Kyung Whan 4527

Uniqueness of the open cone neighborhood.

Proc. Amer. Math. Soc. **15** (1964), 476–479.

The author elaborates on an idea due to M. Brown [same Proc. **12** (1961), 812–814; MR **23** #A4129] and comes up with the following theorem: "Let U and V be any two open cone neighborhoods of a point x in a locally compact Hausdorff space X. Then there is a homeomorphism of V onto U which leaves a neighborhood of x pointwise fixed." This neat generalization implies several scattered results in the theory of manifolds. *R. H. Rosen* (Princeton, N.J.)

McMillan, D. R., Jr. 4528

A criterion for cellularity in a manifold.

Ann. of Math. (2) **79** (1964), 327–337.

A set X in a manifold M^n is cellular if it is the intersection of a nested sequence of n-cells, each lying in the interior of its predecessor. The principal result of the present paper is that if M^n is piecewise linear and $n \geqq 5$, then a compact set is cellular if and only if M/X is a homotopy manifold. If X is an absolute retract, then M^n/X will be a homotopy manifold if and only if for each open set U containing X there is an open set V such that $X \subset V \subset U$ and each loop in $V - X$ is nullhomotopic in $U - X$ ($n \geqq 3$).

The hypotheses are sufficient to obtain a nested sequence $\{M_i\}$ of manifolds with boundary such that each M_{i+1} injects trivially into Int M_i. Then the engulfing lemma is applied (after obtaining a complex R of codimension $\geqq 3$ in M_{i+1}) to get a cell in M_i. The trick is to move $M_{i+1} - X$ off the 2-skeleton of M_{i+1}, then find the cell and then move back so that the cell will contain X.

The principal theorem applies to the case $n = 3$ provided some neighborhood of X can be imbedded in E^3. It is first shown that X has small neighborhoods which are cubes with handles. Then the hypotheses of the theorem are used to show that the handles may be cut to yield a 3-cell.

Next some applications are given. If a complementary domain W of an S^{n-1} in S^n ($n \neq 4$) is uniformly locally simply connected, then it is homeomorphic to E^n. A general theorem about subsets of cellular sets is proved, having as a corollary that a subarc of a cellular arc in a piecewise linear manifold M is cellular. Finally, a general theorem (generalizing the result of the reviewer and the author [Michigan Math. J. **9** (1962), 299–302; MR **27** #1925]) about cellularity in products is proved. This has the important corollary that if X is a compact absolute retract in $E^n \times 0 \subset E^n \times E^1$, then X is cellular in $E^n \times E^1$ (any n). *M. L. Curtis* (Tallahassee, Fla.)

Quintas, Louis V. 4529

One-dimensional cell complexes with homeotopy group equal to zero.

Canad. J. Math. **16** (1964), 353–357.

Let K denote a connected finite 1-dimensional cell complex, $G(K)$ its group of homeomorphisms, and $D(K)$ the group of homeomorphisms of K which are isotopic to the identity. The group $\mathfrak{H}(K) = G(K)/D(K)$ is a topological invariant of K, called the homeotopy group of K

[McCarty, Trans. Amer. Math. Soc. **106** (1963), 293–304; MR **26** #3062]. The purpose of the paper is to study 1-dimensional cell complexes with homeotopy group 0, that is, for which every homeomorphism is isotopic to the identity.

Let $a_0(K)$ and $a_1(K)$ denote the number of vertices and edges, respectively, in K, and let $N(K)=a_1(K)-a_0(K)+1$ denote the nullity of K. The author obtains the following theorem: If $\mathfrak{H}(K)=0$, then $a_0(K)\geqq 7$, $a_1(K)\geqq 10$ and $N(K)\geqq 2$. Furthermore, there exist linear graphs K without vertices of degree 2 such that $\mathfrak{H}(K)=0$ and (i) K has a_0 vertices for all $a_0\geqq 7$; (ii) K has a_1 edges for all $a_1\geqq 10$; and (iii) K has nullity N for all $N\geqq 2$. Conditions (i), (ii), and (iii) are not necessarily satisfied simultaneously. *H. R. Gluck* (Cambridge, Mass.)

Sonneborn, L. M. 4530

Level sets on spheres.

Pacific J. Math. **13** (1963), 297–303.

Let m denote the usual n-dimensional measure on the n-sphere, S^n, normalized so that $m(S^n)=1$. If A is a measurable subset of S^n, then the author calls A too big if $m(A)>\frac{1}{2}$, and an open or closed subset of S^n cuts up S^n if no component of its complement is too big. His main result is Theorem 2: Let $F:S^n\times I\to E^1$ be continuous, $n>1$, and define $f_t:S^n\to E^1$ for each t, $0\leqq t\leqq 1$, by $f_t(x)=F(x,t)$ for each $x\in S^n$. Then for each t, $0\leqq t\leqq 1$, there exists a unique real number k_t such that $f_t^{-1}(k_t)$ contains a closed, connected subset which cuts up S^n. This subset contains a pair of antipodal points of S^n. Further, k_t is a continuous function of t on $0\leqq t\leqq 1$. The stated purpose of the paper, to prove that for any map $f:S^n\to E^1$ there is a connected set in S^n containing two antipodal points and on which f is constant, can be easily accomplished using some simple homology theory. However, Theorem 2 is much stronger, and does not appear to have a statement in homology theory.

G. R. Livesay (Ithaca, N.Y.)

Davis, Allen S. 4531

Fixpoint theorem for contractions of a well-chained topological space.

Proc. Amer. Math. Soc. **14** (1963), 981–985.

An analogue of the contraction mapping theorem is proved, somewhat more general than the following. Let X be a space with a complete uniform structure, let A be a map from X to X, and suppose there exist positive integers n, r, s, with $r<s$, and a base $\mathscr{V}$ of entourages such that, for each $V\in\mathscr{V}$, $V^s\subseteq A^{-n}V^rA^n$. Then A has a unique fixed point, provided X is well-chained (that is, for each $V\in\mathscr{V}$, $\bigcup\{V^k|k=1,2,\cdots\}=X\times X$); and this proviso cannot be omitted. {In several places "=" should apparently be "$\subseteq$", e.g., in the statement $AA^{-1}=\Delta$.} *A. H. Stone* (Rochester, N.Y.)

Whittlesey, E. F. 4532

Fixed points and antipodal points.

Amer. Math. Monthly **70** (1963), 807–821.

This expository paper presents a concise survey, with proofs, of many of the elementary results on fixed points and vector fields for spheres and real projective spaces. Elementary homology theory is assumed.

E. Dyer (Chicago, Ill.)

ALGEBRAIC TOPOLOGY

See also 4526, 4529, 4550.

Erdős, P. 4533

On the structure of linear graphs.

Israel J. Math. **1** (1963), 156–160.

Let $G(n;m)$ denote a simple graph with n vertices and m edges; c_i is a positive absolute constant. It is shown that every $G(n;[n^2/4]+1)$ contains a subgraph consisting of a complete u_n-by-u_n bipartite graph to which an extra edge has been added, where $u_n=[c_1\log n]$. Such a graph also contains a circuit of length h, if $3\leqq h<c_2n$, and for $t>t_0$, every $G(n;[tn^{3/2}])$ contains a circuit of length $2h$, if $2\leqq h<c_3t^2$. *J. W. Moon* (London)

Halin, R. 4534

Bemerkungen über ebene Graphen.

Math. Ann. **153** (1964), 38–46.

This paper contains some theorems on planar graphs intended for use in a following paper on the four-colour problem [see #4535 below]. The graphs considered have no loops or multiple edges. The notation used is as follows. $G'\subseteq G$ means that G' is a subgraph of G, $G\succ G'$ means that G is "homomorphic" to G', that is, can be transformed into an isomorph of G' by deleting some edges and contracting others into single points, and $G\supseteq U(G')$ means that G contains an isomorph of a subdivision of G'. The symbol $(a_1,a_2,\cdots,a_n)$, where the a_i are positive integers, denotes a graph G in which there are n disjoint classes of vertices, a_i vertices in the ith class, which has all possible joins between members of distinct classes, and which has no join between vertices of the same class. If $a_1=a_2=\cdots=a_n=1$, we have the complete graph $S(n)$. Just two non-isomorphic graphs can be obtained by adjoining two new edges to the Thomsen graph (3, 3). One of these is (1, 2, 3), and the other is denoted by L.

Special interest attaches to the planar triangulations. Some of these are obtainable from sets of tetrahedra by identifying pairs of triangular faces. The author shows that each of the others contains a subdivision of the octahedron (2, 2, 2).

Let a graph G be constructed from a planar triangulation D by joining two vertices a and b, non-adjacent in D, by a new edge. It is shown that either a and b belong to a $D_6=(1,1,1,2)\subseteq D$ or G contains a $U(1,1,2,2)$, and that in the first case G contains a subdivision of a graph X_8 consisting of an $S(4)$ and an $S(5)$ whose intersection is a common triangle.

The author uses these results in a new proof of Kuratowski's Theorem. *W. T. Tutte* (Waterloo, Ont.)

Halin, R. 4535

Über einen Satz von K. Wagner zum Vierfarbenproblem.

Math. Ann. **153** (1964), 47–62.

The author uses the notation of the paper reviewed above [#4534]. In addition, a graph is called "prime" if it cannot be represented as the union of two graphs H and K whose intersection is a complete graph properly contained in each of H and K. It is known that each graph has a unique decomposition into prime graphs [K. Wagner and the author, Math. Ann. **147** (1962), 126–142; MR **26** #756].

Given a graph A, the author considers the maximal members of the class of all graphs not homomorphic to A. The prime graphs into which these can be decomposed constitute the "homomorphism-basis" of A.

The author determines the homomorphism-bases of $(1, 2, 3)$ and L. In the case of L the basis consists of the prime planar triangulations, the complete graph $S(5)$, and a graph W consisting of an octagon with its four long diagonals.

Hadwiger has conjectured that if a graph G has chromatic number $\phi(G) \geqq n$, then $G \succ S(n)$. K. Wagner has given a formal proof that, for $n=5$, this assertion is equivalent to the four-colour conjecture. The present author uses his results on homomorphism-bases to obtain more detailed results of the same general character. He shows for example that the four-colour conjecture is equivalent to the following proposition: $(\phi(G) \geqq 5) \rightarrow (G \geqq U(S(5))$ or $G \succ (1, 2, 3))$.

The author concludes with a discussion of the general Hadwiger conjecture. He shows that for every positive integer n there is an integer $f_n \geqq n$ such that $(\phi(G) \geqq f_n) \rightarrow (G \succ S(n))$. *W. T. Tutte* (Waterloo, Ont.)

Hoffman, A. J. 4536

On the line graph of the complete bipartite graph.

Ann. Math. Statist. **35** (1964), 883–885.

Except when $m=n=4$, the line graph of the complete m-by-n bipartite graph is characterized by the following properties: (1) It has mn vertices, each of degree $m+n-2$; (2) Any two non-adjacent vertices are mutually adjacent to exactly two vertices; (3) Of the $\frac{1}{2}mn(m+n-2)$ pairs of adjacent vertices, $n\binom{m}{2}$ pairs are mutually adjacent to exactly $m-2$ vertices while the remaining $m\binom{n}{2}$ pairs are mutually adjacent to exactly $n-2$ vertices. This was proved by Shrikhande [same Ann. **30** (1959), 781–798; MR **22** #1048] when $m=n$, and by the reviewer [ibid. **34** (1963), 664–667; MR **26** #5559] when $m > n$ and $(m, n) \neq (5, 4)$ or $(4, 3)$. In the present paper a proof is supplied for the remaining two cases. *J. W. Moon* (London)

Jung, H. A. 4537

Maximal-Γ-prime Graphen.

Math. Ann. **153** (1964), 210–226.

A factor of a graph G, finite or infinite, is a subgraph of G including all the vertices. Let $V(G)$ be the set of vertices of G, and Γ a mapping of $V(G)$ into the class of non-negative integers. The author defines a Γ-factor of G as a factor G' in which the valency of each vertex v is $\Gamma(v)$. If G has no Γ-factor it is called Γ-prime.

G is said to be "Γ-vollständig", abbreviated as Γ_v, between two vertices x and x' if the number of edges joining them is at least $\mathrm{Min}(\Gamma(x), \Gamma(x'))$ when $x \neq x'$, and at least $2[\Gamma(x)/2]$ when $x = x'$. It is called maximal-Γ-prime if it is Γ-prime but ceases to be so when a new edge is added joining any two vertices, not necessarily distinct, between which G is not Γ_v. The author discusses the problem of characterizing maximal-Γ-prime graphs.

Let E_v be the set of all vertices x such that G is Γ_v between x and each $x' \in V(G)$. Let E_{uv} be the set of all $x \in V(G) - E_v$ such that G is not Γ_v between x and any $x' \in V(G) - E_v$. The author solves his problem for the case in which E_{uv} is finite. He then obtains necessary and sufficient conditions for G to be maximal-Γ-prime. These conditions involve the properties of E_v, E_{uv}, and the residual components of $E_v \cup E_{uv}$ in G. The author thus extends the earlier theories of the reviewer [J. London Math. Soc. **22** (1947), 107–111; MR **9**, 297], H. B. Belck [J. Reine Angew. Math. **188** (1950), 228–252; MR **12**, 730], and K. Wagner [Math. Ann. **141** (1960), 49–67; MR **22** #6728].

The author concludes by obtaining a condition for an infinite Γ-prime graph G to be a factor of some maximal-Γ-prime graph, and showing that this condition is satisfied by all locally finite graphs. *W. T. Tutte* (Waterloo, Ont.)

Kelly, Paul 4538

On some mappings related to graphs.

Pacific J. Math. **14** (1964), 191–194.

The author defines the order of a graph as the number of vertices, and the join measure as the number of edges. Loops and multiple joins are excluded. Two subgraphs are said to have maximal intersection if each has only one vertex not belonging to the other.

The author shows that two nth-order graphs, where $n \geqq 3$, are isomorphic if and only if there is a one-to-one correspondence between their subgraphs of some order h, where $1 < h < n-1$, such that corresponding subgraphs have equal join measure and the correspondence preserves maximal intersections. *W. T. Tutte* (Waterloo, Ont.)

Harary, F.; Kodama, Y. 4539

On the genus of an n-connected graph.

Fund. Math. **54** (1964), 7–13.

The authors define the connectivity of a graph G as the smallest number of vertices whose removal results either in a disconnected graph or a graph with one vertex and no edges. G is said to be n-connected if the connectivity is not less than n. An n-component of G is a maximal n-connected subgraph. The genus $\gamma(G)$ of G is the smallest integer m such that G can be embedded in the orientable surface of genus m.

The authors consider the case of a graph G which is n-connected and is the union of two distinct $(n+1)$-components B and C. They show that $B \cap C$ has exactly n vertices, and that

$$\gamma(G) \leqq \gamma(B) + \gamma(C) + n - 1.$$

A further theorem asserts that if $\gamma(G)$ cannot be increased by joining two vertices of $B \cap C$ by a new edge, then the equality holds in the above formula.

W. T. Tutte (Waterloo, Ont.)

Moon, J.; Moser, L. 4540

On Hamiltonian bipartite graphs.

Israel J. Math. **1** (1963), 163–165.

Authors' summary: "Various sufficient conditions for the existence of Hamiltonian circuits in ordinary graphs are known. In this paper the analogous results for bipartite graphs are obtained." *W. T. Tutte* (Waterloo, Ont.)

Nash-Williams, C. St. J. A. 4541

Decomposition of finite graphs into forests.

J. London Math. Soc. **39** (1964), 12.

This is an addendum to an earlier paper [same J. **36** (1961), 445–450; MR **24** #A3087]. For every set X of vertices in a finite graph G the number $\Delta_G(X)$ is defined as $k(|X|-1)-e_X$, where k is a fixed positive integer and e_X is the number of edges with both ends in X. The author shows that G can be decomposed into k forests if and only if $\Delta_G(X) \geqq 0$ for each non-null X.

W. T. Tutte (Waterloo, Ont.)

Trahtenbrot, B. A. 4542

An estimate of the weight of a finite tree. (Russian)
Sibirsk. Mat. Ž. **5** (1964), 186–191.

Die Arbeit beschäftigt sich mit solchen endlichen Automaten, die ein beliebiges Wort in ein Wort derselben Länge transformieren. Die Anzahl der Buchstaben des Eintrittsalphabetes sowie des Ausgangsalphabetes ist endlich aber nicht notwendig dieselbe. Die Tätigkeit dieses Automaten kann man ähnlich wie in dem zitierten Buche [N. E. Kobrinskiĭ und B. A. Trahtenbrot, *Einführung in die Theorie der endlichen Automaten* (Russian), Fizmatgiz, Moscow, 1962; MR **26** #4866] mittels eines Baumes v beschreiben. Die minimale Anzahl der Zustände des endlichen Automaten, der diese Transformation realisiert, ist gleich dem Gewichte $k(v)$ des Baumes v, d.h. sie ist gleich der Anzahl der Knotenpunkte der Basis (im Sinne des zitierten Buches) des Baumes v.

In der Arbeit werden zwei Sätze bewiesen, die eine asymptotische Beschätzung der Anzahl $k(v)$ für die Länge des Wortes grenzlos wachsende ermöglichen.

A. Kotzig (Bratislava)

Youngs, J. W. T. 4543

Irreducible graphs.
Bull. Amer. Math. Soc. **70** (1964), 404–405.

"A graph G is n-irreducible if it cannot be imbedded in an orientable 2-manifold M of genus $(n-1)$, but, for any arc a in G, the graph G_a, obtained by removing a from G, can be imbedded in M."

The author shows how to construct a separable n-irreducible graph of any given genus $n>1$.

W. T. Tutte (Waterloo, Ont.)

Bing, R. H.; Klee, V. L. 4544

Every simple closed curve in E^3 is unknotted in E^4.
J. London Math. Soc. **39** (1964), 86–94.

The authors give an elementary proof that any topologically imbedded 1-complex in $R^n \times 0$ is tame in $R^n \times R^1$ $(n \geqq 3)$. Since a tame S^1 in R^4 is known to be unknotted, this yields as a corollary the title theorem. This corollary also follows from results of the reviewer and Lashof [Proc. Amer. Math. Soc. **13** (1962), 934–937] and Homma [Yokohama Math. J. **10** (1962), 5–10; MR **27** #4236], but the general theorem about 1-complexes does not seem to.

The key lemma shows that if M is an n-manifold with boundary B and $A_1, \cdots, A_r$ are mutually exclusive arcs in $M \times [0, 1]$, each A_i intersecting each horizontal slice $M \times t$ in exactly one interior point, then there is a homeomorphism of $M \times [0, 1]$ which is fixed on the bottom and sides and sends each A_i into a vertical arc.

M. L. Curtis (Tallahassee, Fla.)

Hammer, Gerald 4545

Ein Verfahren zur Bestimmung von Begleitknoten.
Math. Z. **81** (1963), 395–413.

This paper continues the work of Haken [Acta Math. **105** (1961), 245–375; MR **25** #4519a; Math. Z. **76** (1961), 427–467; MR **25** #4519c] and of Schubert [ibid. **76** (1961), 116–148; MR **25** #4519b]. The work concerns decision procedures for problems in three-dimensional manifolds.

In the present paper a procedure is given by which one can construct from a given knot in S^3 all its companion knots (Begleitknoten). Companion knots were first defined by Schubert [Acta Math. **90** (1953), 131–286; MR **17**, 291]. A by-product of the work on companion knots by the author is that, given a 3-manifold M, there is a procedure for constructing a 2-dimensional disc $\mathfrak{c}$ in M such that $\dot{\mathfrak{c}} = \mathfrak{c} \cap \dot{M}$ and $\dot{\mathfrak{c}}$ does not bound a disc in $\dot{M}$.

The author's methods are, for the most part, those of Haken and Schubert, to whom reference has been made above.

D. B. A. Epstein (Cambridge, England)

Tsuchida, Kisuke 4546

Operation and co-operation in the general categories.
Tôhoku Math. J. (2) **16** (1964), 15–33.

In this paper, the author defines and discusses operation and co-operation in the framework of general categories corresponding to those established by Eckmann and Hilton.

S.-T. Hu (Los Angeles, Calif.)

Milnor, J. 4547

On the Betti numbers of real varieties.
Proc. Amer. Math. Soc. **15** (1964), 275–280.

The author presents estimates on the sum of the Betti numbers of an algebraic set in terms of the degrees of the polynomials defining the set. Thus he shows that if V is defined in R^m by polynomial equations $f_i = 0$, each polynomial being of degree $\leqq k$, then the sum of the Betti numbers of V is $\leqq k(2k-1)^{m-1}$. If V is defined by polynomial inequalities: $f_1 \geqq 0, \cdots, f_p \geqq 0$, then it is shown that the sum of the Betti numbers of V is bounded by $\frac{1}{2}(2+d)(1+d)^{m-1}$, where $d = \deg f_1 + \cdots + \deg f_p$. The author also treats the case of complex and projective varieties.

The proofs are based on the following construction. If V is given by $f_i = 0$, $i = 1, \cdots, p$, the author considers the set $K(\varepsilon, \delta)$ defined by $f_1{}^2 + \cdots + f_p{}^2 + \varepsilon^2(x_1{}^2 + \cdots + x_n{}^2) \leq \delta^2$ and its "boundary" $\partial K(\varepsilon, \delta)$ defined by the equality sign in the above equation. By Sard's lemma, $\partial K(\varepsilon, \delta)$ will be a non-singular hypersurface for nearly all ε and δ. By Alexander, the rank $H^*(K)$ is therefore $\frac{1}{2}$ rank $H^*(\partial K)$. In this way the author uses an estimate on the sum of the Betti numbers of a non-singular hypersurface (which he obtains by means of the Morse theory) to estimate the rank of the restriction homomorphism $H^*(V) \to H^*(P)$ for any compact polyhedron in V.

R. Bott (Cambridge, Mass.)

Nakagawa, Ryosuke 4548

On Fuks' homotopy duality.
Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **8**, 93–98 (1964).

In this paper, the author indicates that Fuks's duality

can be considered as a functorial duality in the sense of Eilenberg and MacLane and proves that such a functorial duality is unique in a certain sense.

S.-T. Hu (Los Angeles, Calif.)

Brown, E. H., Jr.; Peterson, F. P. 4549

Whitehead products and cohomology operations.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 116–120.

If $n+1$ is not a power of 2, then Sq^{n+1} is decomposable in the Steenrod algebra, say as $\sum_i a_i' a_i$. On classes u of dimension n we have $Sq^{n+1}u=0$, yielding a relation $\sum_i a_i'(a_i u)=0$. This relation gives rise to a non-stable secondary operation Φ defined on a subgroup of $H^n(X; Z_2)$. It is shown that Φ is not primitive; indeed,

$$\Phi(x+y) = \Phi(x)+\Phi(y)+xy.$$

It follows that Φ is non-zero in the complex $S^n \cup_f e^{2n}$, where f represents the Whitehead product $[\iota, \iota]$.

As a corollary, there are spaces with comparatively few homotopy groups which have non-zero Whitehead squares. The case $n+1=2^r$, $r>3$ is discussed more briefly.

J. F. Adams (Manchester)

TOPOLOGY AND GEOMETRY OF DIFFERENTIABLE MANIFOLDS

See also 4060, 4148, 4245, 4465, 4484, 4547, 4557.

Mazur, Barry 4550

Differential topology from the point of view of simple homotopy theory.

Inst. Hautes Études Sci. Publ. Math. No. 15 (1963), 93 *pp.*

Smale's point of view in Morse theory is to develop a compact manifold M^n by successive stages $\{M_i\}$ with M_{i+1} obtained from M_i by attaching a thickened cell or handle $D^{k_i}\times D^{n-k_i}$ to M_i by a map $\varphi_i: S^{k_i-1}\times D^{n-k_i}\to\partial M^i$. Smale's success in higher-dimensional simply connected manifolds lies in the possibilities of rearranging the order in which the handles are attached and finally being able to cancel out handles which serve no useful purpose from the point of view of the algebraic-topological data on the manifold.

In this work, the author studies compact manifolds which are not necessarily simply connected and their developments $\{M_i\}$ as above. Primarily, the tool advanced is the study of a related space X with a "cell decomposition" $\{X_i\}$ with X_{i+1} obtained from X_i by attaching D^{k_i} along a map $\bar\varphi_i: \partial D^{k_i}\to X_i$. The relation being that X is obtained from M^n by "unthickening". Specifically, one requires maps $\pi_i: M_i\to X_i$ which inductively (through the previous π_{i-1}) have the attaching map φ_i of $D^{k_i}\times D^{n-k_i}$ restricted to $\partial D^{k_i}\times\{0\}$ correspond to the attaching map $\bar\varphi_i$ of D^{k_i}. The basic theme is promoted by showing that the resulting map $\pi: M\to X$ is a simple homotopy equivalence in the sense of J. H. C. Whitehead. The author makes the association of M^n to X functorial by introducing equivalence relations among manifolds with handle-body decompositions on the one hand and spaces with cell decompositions on the other. An equivalence class of the latter is called a cell filtration.

Viewing matters in reverse, the author defines the sets $N^n(X)$ which unthicken to X (in the equivalence relation previously defined for manifolds). Various theorems illustrating that simple homotopy type is critical are then deduced. In particular, if X and Y are simply homotopy equivalent, then $N^n(X)\cong N^n(Y)$ if $n\geq 2\max(\dim X, \dim Y)+1$. The inequalities can be improved to $n\geq \max(\dim X, \dim Y)+3$ if one ignores trivial attachments. It is also shown that a cell filtration X is an unthickening if X has the same simple homotopy type of a manifold M^n with $n\geq\dim X+3$. Proper counter-examples are usually provided for all inequalities. A relation between $N^n(X)$ and $\widetilde{KO}(X)$ is established which is one-to-one in the stable range, which can be viewed by (non-functorially) imbedding X in M^n and restricting the tangent bundle.

The previously announced theorems of the author on the diffeomorphism of the cell bundle of ξ over $M_1{}^n$ and of $f^*(\xi)$ over $M_2{}^n$, where $f: M_1{}^n\to M_2{}^n$ is a simple homotopy equivalence and $\dim\xi$ is large, are demonstrated and various corollaries deduced.

The author views the elements of $N^n(X)$ as "neighborhoods" of X and provocatively develops this theme. Unfortunately, the work is difficult to read, with proofs often only indicated. Much worse, the definitions are so immediately geared to making the machinery work as to be rendered clumsy from an intrinsic point of view.

The author has, since the publication of this work, accomplished much more significant research in this direction and has managed to eliminate the above-mentioned difficulties, and one awaits its publication.

T. Stewart (Notre Dame, Ind.)

Lima, Elon L. 4551

On the local triviality of the restriction map for embeddings.

Comment. Math. Helv. **38** (1964), 163–164.

The author gives a short and elegant proof of a theorem of the reviewer [same Comment. **34** (1960), 305–312; MR **23** #A666] and J. Cerf [Bull. Soc. Math. France **89** (1961), 227–380; MR **25** #3543], namely, given a C^∞ manifold M, a C^∞ manifold W, and a compact C^∞ submanifold V of W, let $\mathscr{E}^r(W, M)$ and $\mathscr{E}^r(V, M)$ denote the spaces of C^∞ embeddings of W and V into M with the C^r topology $(1\leq r\leq\infty)$. Then the restriction map $j:\mathscr{E}^r(W, M)\to\mathscr{E}^r(V, M)$ (i.e., $j(f)=f|V$) is a locally trivial fibration.

R. S. Palais (Waltham, Mass.)

Munkres, James 4552

Obstructions to extending diffeomorphisms.

Proc. Amer. Math. Soc. **15** (1964), 297–299.

The following theorem is proved. Let M and N be C^∞ manifolds such that the boundary of M is the disjoint union of M_0 and M_1 and, similarly, Bd $N=N_0\cup N_1$. Let $f: M_0\to N_0$ be a diffeomorphism which can be extended to an isomorphism between C^∞ triangulations of M and N. Then the obstructions to extending f to a diffeomorphism between M and N lie in the homology groups $\mathfrak{H}_m(M, M_1; \Gamma^{n-m})$, based on infinite chains with possibly twisted coefficients, and the extension is possible when the obstructions vanish. As a corollary, the author reproves the Thom theorem [Proc. Internat. Congress Math., 1958, pp. 248–255, Cambridge Univ. Press, New York, 1960; MR **22** #12536] which states that the obvious

differentiable structure on $M \times I$ is determined up to diffeomorphism by its combinatorial structure and the differentiable structure on $M \times 0$.

H. B. Shutrick (Manchester)

Milnor, J. 4553a

Topological manifolds and smooth manifolds.

Proc. Internat. Congr. Mathematicians (Stockholm, 1962), *pp.* 132–138. *Inst. Mittag-Leffler, Djursholm,* 1963.

Milnor, J. 4553b

Microbundles. I.

Topology **3** (1964), *suppl.* 1, 53–80.

In this paper the author proves the results stated in his lecture to the International Congress of Mathematicians, 1962 [#4553a]. We refer to this lecture as [L].

The notion of a "microbundle" (§§ 1, 2) is essentially obtained from that of a vector bundle by (i) restricting attention to a neighbourhood of the zero cross-section, and (ii) abandoning all conditions of "linearity", so that one uses only topological conditions. This notion is introduced so that one may assign (§ 2) to each topological manifold a tangent microbundle, analogous to the tangent bundle of a smooth manifold. Suppose given a smooth manifold M; then on the one hand we can first take its tangent vector bundle ξ, and then take the microbundle underlying ξ; on the other hand we can first take the topological manifold $|M|$ underlying M, and then take the tangent microbundle of $|M|$. It is proved that the two results agree (Theorem 2.2 = Theorem 1 of [L]).

Much of the theory of vector bundles carries over to microbundles. Thus, we have induced microbundles (§ 3) and Whitney sums (§ 4). It is stated in § 3 that if two maps are homotopic, then the corresponding induced microbundles are isomorphic (Theorem 3.1 = Theorem 3 of [L]). The proof, which requires some work, is given in § 6.

One can introduce a factor k_{Top} based on microbundles, analogous to the factor $k_O = \tilde{K}$ based on vector bundles, and one has a natural transformation $k_O \to k_{\mathrm{Top}}$ (§ 4; Corollary 4.3 = Theorem 4 of [L]). For this purpose the key result (Theorem 4.1) is that any microbundle x over a finite-dimensional simplicial complex B admits an "inverse" y such that the Whitney sum $x+y$ is trivial. The proof, which requires some work, is given in §§ 4, 7.

Next, one tries to introduce normal microbundles. If we have topological manifolds $M \subset N$, then in general M does not have a normal microbundle. However, M has a normal microbundle in $N \times R^q$ for sufficiently large q (Theorem 5.8). The proof takes some work (§ 5). A normal bundle n is related in the usual way to the two tangent bundles: $t_M + n \cong t_N$.

This leads to results on the smoothing problem. Theorem 5.12, which corresponds to Theorem 2 of [L], is sharpened to give Theorem 5.13: Let ξ be a vector bundle over the topological manifold M; then some product $M \times R^q$ can be smoothed so as to have tangent bundle isomorphic to ξ plus a trivial bundle if and only if the homomorphism $k_O(M) \to k_{\mathrm{Top}}(M)$ carries the class of ξ to the class of the tangent microbundle t_M.

It is therefore interesting to examine the difference between $k_O(X)$ and $k_{\mathrm{Top}}(X)$. The fact is that the natural transformation $k_O(X) \to k_{\mathrm{Top}}(X)$ need be neither mono nor epi (Lemmas 9.1 and 9.4 = Theorem 5 of [L]). The key result (Theorem 8.1) states that the image in $k_{\mathrm{Top}}(S^{4n})$ of the generator in $k_O(S^{4n})$ is divisible by a certain integer (whose definition involves the Bernoulli numbers). Heavy hammers are needed for the proof (§ 8); the author uses the methods of Milnor and Kervaire ["Groups of homotopy spheres", II, in preparation] and Wall [Ann. of Math. (2) **75** (1962), 163–189; MR **26** #3071]; he also quotes results from Hirsch, from Smale and from the reviewer.

Finally, the following results are proved in § 9 by exploiting the fact that k_O and k_{Top} are different. (i) The tangent vector bundle of a manifold is not a topological invariant (Theorem 9.2 = Corollary 1 of [L]). (ii) The Pontryagin classes of an open manifold are not topological invariants (Corollary 9.3). (iii) There exists a topological manifold M such that no product $M \times M'$ can be smoothed (Theorem 9.5 = Corollary 2 of [L]).

This paper is clearly required reading for any worker in the field.

J. F. Adams (Manchester)

Hermann, Robert 4554

A Poisson kernel for certain homogeneous spaces.

Proc. Amer. Math. Soc. **12** (1961), 892–899.

Démonstration et généralisation de résultats énoncés par D. Lowdenslager [Ann. of Math. (2) **67** (1958), 467–484; MR **21** #2836]. On donne en particulier une construction du noyau de Poisson pour des espaces homogènes G/K, où G est un groupe de Lie non compact, K un sous-groupe compact connexe.

P. Lelong (Paris)

Griffiths, Phillip A. 4555

Deformations of holomorphic mappings.

Illinois J. Math. **8** (1964), 139–151.

We are in the category of complex manifolds and holomorphic mappings, and M is a compact manifold. A deformation of a map $f\colon M \to M'$ is a map $g\colon M \times V \to M'$, where the parameter space V has a preferred point v_0 such that $f = g(\ , v_0)$. Theorem 1 points out that g determines and is determined by a deformation of the submanifold $X = i(M)$ of $Y = M' \times M$, where $i = (f, 1)$. A deformation of a compact submanifold X of Y is a submanifold W of $X \times Y$, codim $W = \dim Y - \dim X$, meeting $V \times \{v_0\}$ in X and having the right transversality conditions. Thus the normal bundle $\mathbf{N}_W(X)$ is a sub-bundle of $\mathbf{N}_{Y \times V}(X) \cong \mathbf{N}_Y(X) \oplus \mathbf{T}_{v_0}(V)$ transversal to the first factor, and so is the graph of a linear mapping $\rho\colon \mathbf{T}_{v_0}(V) \to H^0(X, \mathfrak{N}_Y(X))$, where $\mathfrak{N}$ means the sheaf of germs of local holomorphic sections of $\mathbf{N}$. A theorem of Kodaira [*A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds*, Mimeographed Notes, Princeton Univ., Princeton, N.J., 1962] states that any such ρ comes from a deformation if $H^1(X, \mathfrak{N}_Y(X)) = 0$.

In the present context, $\mathbf{N}_Y(X) = \mathbf{N}_{M' \times M}(X) \cong f^{-1}(\mathbf{T}(M'))$ and so, Theorem 2, a cohomology condition for deformations of f is that $H^1(M, f^{-1}(\mathfrak{T}(M'))) = 0$. If $\mathfrak{T}_f$ $(= f(\mathfrak{T}(M')))$ and $\mathfrak{N}$ are the coherent sheaves of holomorphic tangents and normals, respectively, to $f(M)$, then there is a short exact sequence $0 \to f^{-1}(\mathfrak{T}_f) \xrightarrow{i} f^{-1}(\mathfrak{T}(M')) \to f^{-1}(\mathfrak{N}) \to 0$ and the cohomology sequence gives $H^0(M, f^{-1}(\mathfrak{T}(M'))) \cong \operatorname{Im} i_* \oplus \operatorname{Ker} \delta$. The obstructions to infinitesimal deformations parametrised by $\operatorname{Im} i_*$ occur in

$$H^1(M, \mathcal{K}er(\mathfrak{T}(M) \to f^{-1}(\mathfrak{T}_f)))$$

and those parametrised by Ker δ occur in $H^1(f(M), \mathfrak{N})$, when $f(M)$ is non-singular in M'. The author is thus able to weaken the cohomology condition, and he gives a geometrical interpretation of the above splitting.

H. B. Shutrick (Manchester)

PROBABILITY

See also 3910, 3953, 4076, 4083, 4084, 4253, 4388, 4610, 4634, 4650, 4944, 4945, 4956, 4957, 4959, 4971, 4975.

Gangolli, Ramesh 4556

Wide-sense stationary sequences of distributions on Hilbert space and the factorization of operator valued functions.

J. Math. Mech. **12** (1963), 893–910.

Let W' be a function on the circle $C: z = e^{i\theta}$ taking values in the space of bounded positive definite self-adjoint operators on a q-dimensional Hilbert space $\mathfrak{H}$ to $\mathfrak{H}$. For the case $q = \infty$, the author obtains necessary and sufficient conditions for the factorization $W' = \Phi^*\Phi$, where Φ is an "outer" (or "optimal") function having only non-negative frequencies in its Fourier series and * indicates the adjoint. He thereby extends a result for $q < \infty$ established, among others, by N. Wiener and the reviewer [Acta Math. **98** (1957), 111–150; MR **20** #4323].

When $q < \infty$, W' is simply a $q \times q$ matrix-valued function, and thus $\mathfrak{H}$ is simply the Cartesian product $\mathfrak{C}^q$, where $\mathfrak{C}$ is the complex number field. The factorization is accomplished by considering the q-variate weakly stationary stochastic process (S.P.) $(f_n)_{-\infty}^{\infty}$, having spectral density W'. Then $f_n \in \mathfrak{H}'^q$, where $\mathfrak{H}'$ is a Hilbert space. Products Af_n, where A is an arbitrary $q \times q$ matrix, and Gram matrices $\langle f_m, f_n \rangle$ enter essentially into the discussion. This approach fails when $q = \infty$ since such products will not always yield vectors in $\mathfrak{H}'^q$ and Gram matrices will cease to have a finite trace.

To circumvent this difficulty the author redefines an S.P. as a bisequence $(F_n)_{-\infty}^{\infty}$ of bounded linear operators from $\mathfrak{H}$ to $\mathfrak{H}'$, unfortunately calling such operators "distributions" and denoting the space they form by $\mathfrak{D}$. (For $q < \infty$ this definition is equivalent to the classical one: just associate with $f_n = (f_n^1, \cdots, f_n^q) \in H'^q$ the operator F_n defined by $F_n(x) = \sum_1^q f_n^i x_i$.) The "Gramian" of F, G in $\mathfrak{D}$ is defined by $\langle F, G \rangle = F^*G$, an operator from $\mathfrak{H}$ to $\mathfrak{H}$. F is "orthogonal" to G if $\langle F, G \rangle = 0$. F is "normalized" if $\langle F, F \rangle = I$. A "linear manifold" $\mathfrak{M}$ of $\mathfrak{D}$ is such that $F, G \in \mathfrak{M}$ implies that $FA + GB \in \mathfrak{M}$, for all bounded linear operators on $\mathfrak{H}$ to $\mathfrak{H}$. A "subspace" of $\mathfrak{D}$ is a linear manifold which is closed under the Banach-norm topology of $\mathfrak{D}$.

In § 2 the author shows that the structure so defined in $\mathfrak{D}$ closely resembles that introduced in $\mathfrak{H}'^q$ by Wiener and the reviewer. This enables him to carry over many of the definitions, results and proofs of q-variate prediction theory in the time domain to the case $q = \infty$ (§ 3). Thus, the S.P. $(F_n)_{-\infty}^{\infty}$ is "weakly stationary" if the Gramian $\langle F_m, F_n \rangle$ depends only on $m - n$. The "nth innovation" G_n is defined as the difference between F_n and the "projection" of F_n on the subspace $\mathfrak{M}_{n-1}$ spanned by the F_k for $k \leq n-1$; $S = \langle G_0, G_0 \rangle$ is the "prediction error operator". The S.P. has "full-rank" if S has a bounded inverse; it has "nearly full-rank" if S is merely one-one. Every non-deterministic S.P. admits a Wold decomposition into a deterministic process and a one-sided moving-average of innovations (3.3).

The extension of the spectral theory (§ 5) is much more complicated. One observes first that, associated with our S.P. $(F_n)_{-\infty}^{\infty}$ is a unitary operator $U = \int_0^{2\pi} e^{-i\theta}\,dE$ on $\mathfrak{H}'$ onto $\mathfrak{H}'$ such that $U \cdot F_n = F_{n+1}$. The function W such that $W(\theta) = F_0{}^*E(\theta)F_0$ is called the "spectral distribution" of the S.P. Unfortunately, its derivative W' turns out to be an unbounded operator-valued function. To cope with this situation the author introduces "subprocesses" $(F_nP)_{-\infty}^{\infty}$, where P is a projection on $\mathfrak{H}$ onto a finite-dimensional subspace (§ 4). The spectral distribution of the subprocess is PWP. Let $\lambda_-(S_P)$ be the smallest eigenvalue of the prediction error operator S_P of this subprocess. The author shows that the original process $(F_n)_{-\infty}^{\infty}$ has full-rank if and only if $\lambda_-(S_P) \geq c > 0$, uniformly for all such P. A purely spectral criterion for full-rank takes the form (6.2)

$$\int_0^{2\pi} \log \Delta(\alpha(P), PW'(e^{i\theta})P)\,d\theta > c > -\infty,$$

where $\Delta(\ ,\ )$ is an asymmetric geometric mean of the eigenvalues of $PW'P$, the definition of which is too technical to state here. The same condition is shown to be necessary and sufficient for the factorization described at the outset of this review (6.7.3). The author concludes by pointing out a large number of open questions in the subject.

P. Masani (Bloomington, Ind.)

Gangolli, Ramesh 4557

Isotropic infinitely divisible measures on symmetric spaces.

Acta Math. **111** (1964), 213–246.

The author extends some theorems on infinitely divisible probability measures on the reals R to the case of a symmetric space. A probability measure μ is said to be infinitely divisible if for each positive integer n there is a probability measure ν such that $\mu = \nu^n$ (the n-fold convolution). Some of the classical theorems generalized are the following: (1) If μ is infinitely divisible, then $\hat{\mu}(\lambda) \neq 0$ for each real number λ, where $\hat{\mu}$ is the characteristic function of μ. (2) μ is infinitely divisible if and only if $\hat{\mu}(\lambda) = \lim \exp(-\psi_j(\lambda))$, where $\psi_j(\lambda) = \int_R (1 - \varphi_\lambda(x))\,dL_j(x)$, with $\varphi_\lambda(x)$ a character of the additive group of R and L_j a finite measure on R. (3) μ is infinitely divisible if and only if the Lévy-Khintchine formula holds. (4) (Khintchine's theorem) μ is infinitely divisible if and only if it is a generalized limit. (For more detail about (3) and (4) see, for instance, Gnedenko and Kolmogorov [*Limit distributions for sums of independent random variables*, Addison-Wesley, Cambridge, Mass., 1954; MR **16**, 52].)

The author considers G a non-compact semi-simple Lie group with a finite center and K a maximal compact subgroup of G. An isotropic measure on G is a finite regular non-negative Borel measure which is invariant under both left and right translation by K. Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the Lie algebras of G and K respectively. Then there is a subspace $\mathfrak{p}_0$ of $\mathfrak{g}_0$ such that $\mathfrak{g}_0 = \mathfrak{p}_0 + \mathfrak{k}_0$ is a Cartan decomposition of $\mathfrak{g}_0$. Let $\mathfrak{h}_{\mathfrak{p}_0}$ be a maximal abelian subalgebra of $\mathfrak{p}_0$. Let E_R be the space of real-valued linear functionals on $\mathfrak{h}_{\mathfrak{p}_0}$. To each λ in E_R there is associated a positive definite elementary spherical function $\varphi_\lambda(x)$, $x \in G$. The author proves the following for isotropic

probability measures: (1) (Lemma 5.1) If μ is infinitely divisible, then $\hat{\mu}(\lambda) \neq 0$ for $\lambda \in E_R$, where $\hat{\mu}(\lambda) = \int_G \varphi_\lambda(x)\, d\mu(x)$. (2) (Corollary 5.4) μ is infinitely divisible if and only if $\hat{\mu}(\lambda) = \lim \exp(-\psi_j(\lambda))$, where $\psi_j(\lambda) = \int_G (1-\varphi_\lambda(x))\, dL_j(x)$ and L_j is an isotropic measure on G. (3) (Theorem 6.2) (Lévy-Khintchine formula) μ is infinitely divisible if and only if

$$\hat{\mu}(\lambda) = \exp\left[P_D(\lambda) - \int_{|x|>0} [1-\varphi_\lambda(x)]\, dL(x)\right],$$

where L is a spherical measure satisfying

$$\int |x|^2/(1-|x|^2)\, dL(x) < \infty$$

and $P_D(\lambda)$ is the eigenvalue corresponding to the eigenfunction φ_λ of a second-order elliptic differential operator $D \in \mathbf{D}(G/K)$ which annihilates constants. (4) (Theorem 7.2) (Khintchine's theorem) μ is infinitely divisible if and only if it is a generalized limit.

In Section 8 the author outlines the case where G/K is a compact symmetric space. In this case (1) does not necessarily hold and (2) has the following form: If $\hat{\mu}(n) \neq 0$ for all n then μ is infinitely divisible if and only if $\hat{\mu}(n) = \exp(P_D(n) - \int_{|x|>0} [1-\varphi_n(x)]\, dL(x))$, where $P_D(n)$ is the eigenvalue corresponding to φ_n of an elliptic second-order differential operator $D \in \mathbf{D}(G/K)$ and L is a spherical measure on $\{x : |x| > 0\}$ such that $\int [1 - \operatorname{Re} \varphi_1(x)]\, dL(x) < \infty$.

F. Hahn (New Haven, Conn.)

Stackelberg, Olaf P. 4558

On the law of the iterated logarithm. I, II.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 48–55, 56–67.

Denote by $f_i(x)$ the ith Rademacher function. The author proves that for almost all x

$$(1) \qquad \limsup_{N=\infty} \frac{\left|\sum_{i=1}^{N} (1-((i-1)/N))f_i(x)\right|}{\sqrt{(\frac{2}{3}N \log\log N)}} = 1.$$

The author in fact proves a more general theorem, which is too long to state here, from which he deduces (1).

P. Erdős (Hamilton, Ont.)

Chartres, B. A. 4559

A geometrical proof of a theorem due to D. Slepian.

SIAM Rev. **5** (1963), 335–341.

The theorem referred to in the title [D. Slepian, Bell System Tech. J. **41** (1962), 463–501; MR **24** #A3017] is that if x and y are two Gaussian vector random variables with respective covariance matrices $R = (r_{ij})$, $Q = (q_{ij})$ and with $r_{ii} = q_{ii}$, $r_{ij} \geqq q_{ij}$, $i, j = 1, 2, \cdots, n$, then $P(x \in A) \geqq P(y \in A)$, where A is a set in E^n of the form $\{x_i > a_i, i = 1, 2, \cdots, n\}$.

The author's geometrical proof is longer and less concise than Slepian's and applies only if R and Q are positive definite, but he is able to show that if for some i, j we have $r_{ij} > q_{ij}$, then $P(x \in A) > P(y \in A)$.

D. A. Darling (Ann Arbor, Mich.)

Usiskin, Zalman 4560

Max-min probabilities in the voting paradox.

Ann. Math. Statist. **35** (1964), 857–862.

Consider n real-valued random variables $X_1, X_2, \cdots, X_n$. This paper contains results on the problem of maximizing

$$\min[P(X_1 > X_2), \cdots, P(X_{n-1} > X_n), P(X_n > X_1)].$$

With no restrictions imposed on $X_1, \cdots, X_n$, the author shows that the max-min probability is $(n-1)/n$ and is easily achieved. Under the assumption of independence, the max-min probability, $b(n)$, is the value of a certain continued fraction. The function $b(n)$ is monotone increasing and converges to $\frac{3}{4}$ as n increases indefinitely.

This paper generalizes some results in the voting paradox problem. *D. S. Adorno* (Wilton, Conn.)

Engelberg, Ora 4561

Exact and limiting distributions of the number of lead positions in "unconditional" ballot problems.

J. Appl. Probability **1** (1964), 168–172.

In an election return with two candidates A and B, if α_r is the number of votes for A in the first r counted, β_r the similar number for B, then r is a strict lead position for A if $\alpha_r > \beta_r$, a weak lead position for A if $\alpha_r \geqq \beta_r$. In a paper in course of publication, the author has given the probability distributions of the number of both strict and weak lead positions, given that the total vote is a for A, b for B; these results are also given in a paper by the reviewer [Ann. Math. Statist. **35** (1964), 369–379; MR **28** #5008]. Here she uses these results to find the corresponding distributions when the total vote (for both A and B) is fixed, assuming, of course, equiprobable arrangements of the vote count; put in another way, these are the distributions by number of lead positions for symmetric random walks on the line with a fixed number of steps. Writing $W_n(x)$ for the probability generating function with total vote n in the weak lead case and

$$f_n(x) = \frac{1}{2}\sum_{k=0}^{n} \binom{2k+2}{k+1}\binom{2n-2k}{n-k} x^{2k},$$

her results may be rendered concisely by

$$2^{2n} W_{2n}(x) = \tfrac{1}{2}\binom{2n}{n}(1+x^{2n}) + (x+x^2)f_{n-1}(x),$$

$$2^{2n+1} W_{2n+1}(x) = \binom{2n}{n} + \tfrac{1}{4}\binom{2n+2}{n+1}x^{2n+1} + 2x^2 f_{n-1}(x) + \tfrac{1}{2}xf_n(x).$$

The corresponding function $S_n(x)$ for the strict lead case is given by $S_n(x) = x^n W_n(x^{-1})$. This is a consequence of the relation, the author's equation (11), $P\{\Delta_n = k\} = P\{\Delta_n^* = n-k\}$, $k = 0, 1, \cdots, n$, with Δ_n (Δ_n^*) the number of strict (weak) lead positions with total vote n. A similar relation, in similar notation $P\{\Delta_{a,b}^* = k\} = P\{\Delta_{b,a} = a+b-k\}$, $a < b$, is central to the determination of $W_n(x)$.

The limiting distribution for the variable Δ_n/n is the arcsine law, which as the author notes is an instance of a theorem of P. Erdős and M. Kac [Bull. Amer. Math. Soc. **53** (1947), 1011–1020; MR **9**, 292]. The corresponding limit for weak leads is

$$\lim_{n\to\infty} P\left(\frac{\Delta_n^*}{n} \leqq \alpha\right) = 1 - \frac{2}{\pi} \arcsin \sqrt{(1-\alpha)}.$$

Finally, the case where A leads if either $\alpha_r > \beta_r$ or $\alpha_r = \beta_r$ and $\alpha_{r-1} > \beta_{r-1}$ (introduced by K.-L. Chung and W. Feller [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 605–608;

MR 11, 444]) is neatly related to preceding results for n even (the Chung-Feller case), but for n odd only the relations of the "unconditional" case probabilities to the probabilities for vote (a, b) and a symmetry relation appear. *J. Riordan* (Murray Hill, N.J.)

Kochen, Simon; Stone, Charles 4562
A note on the Borel-Cantelli lemma.
Illinois J. Math. **8** (1964), 248–251.

Let $\{X_n\}$ be a sequence of random variables with $E\{X_n\}>0$ and $E\{X_n^2\}<\infty$. Put $\alpha=\limsup E\{X_n\}^2/E\{X_n^2\}>0$ and $Y_n=\limsup X_n/E\{X_n\}$. Then $P\{Y_n\geqq 1\}>0$ and $P\{Y_n>0\}\geqq\alpha$. This is useful when some zero-one law is already known. The second conclusion above with $\alpha=1$ has been given by Rényi [*Wahrscheinlichkeitsrechnung*, p. 327, VEB Deutscher Verlag der Wiss., Berlin, 1962; MR **26** #5597]. Some known examples are discussed. The condition (5) of the paper appeared earlier in a paper of the reviewer and Erdős [Ann. of Math. (2) **48** (1947), 1003–1013; MR **9**, 292]. *K. L. Chung* (Stanford, Calif.)

Kinney, J. R.; Pitcher, T. S. 4563
The dimension of the support of a random distribution function.
Bull. Amer. Math. Soc. **70** (1964), 161–164.

In an earlier paper [same Bull. **69** (1963), 548–551; MR **26** #7008] L. E. Dubins and D. A. Freedman showed how to construct a random family of continuous distribution functions over the unit interval. The process was constructive, starting from a given probability measure over the unit square, and consisted of a sequence of horizontal and vertical segments converging with probability one to continuous distribution functions. These distributions are singular, and the present authors calculate specifically the constant Hausdorff-Besicovitch dimensions of their support. *D. A. Darling* (Ann Arbor, Mich.)

Kloss, B. M. 4564
Limit distributions on bicompact Abelian groups. (Russian. English summary)
Teor. Verojatnost. i Primenen. **6** (1961), 392–421.

This paper is a continuation of an earlier work [the author, Teor. Verojatnost. i Primenen. **4** (1959), 255–290; MR **22** #3761]. Given a bicompact Abelian group G, a characteristic function of a probability measure on G is the average with respect to this measure of a group character of G, and the usual properties of characteristic functions carry over, to a large extent, to these new characteristic functions.

The author is thus able to define stable and infinitely divisible measures, proving in particular that if G is a zero-dimensional or a compact Abelian Lie group, every infinitely divisible measure can be embedded in a continuous one-parameter semigroup of measures. The canonical form for the infinitely divisible measures is given, and certain limit theorems are proved for convolutions of measures. *D. A. Darling* (Ann Arbor, Mich.)

Kloss, B. M. 4565
Topology in a group and convergence of distributions. (Russian. English summary)
Teor. Verojatnost. i Primenen. **9** (1964), 122–125.

A group G is said to satisfy the distribution convergence principle (DCP) if, whenever $\xi_1, \xi_2, \cdots$ are independent G-valued random variables with partial products $\zeta_n=\xi_1\xi_2\cdots\xi_n$, it is possible to find constants $a_n\in G$ such that the translates $\eta_n=\zeta_n a_n$ converge in distribution. The author showed earlier [Teor. Verojatnost. i Primenen. **4** (1959), 255–290; MR **22** #3761] that compact groups satisfy DCP. He now shows that no other locally compact groups do so. *J. G. Wendel* (Aarhus)

Partasarati, K. R. [Parthasarathy, K. R.]; Sazonov, V. V. 4566
On the representation of infinitely divisible distributions on a locally compact Abelian group. (Russian. English summary)
Teor. Verojatnost. i Primenen. **9** (1964), 118–122.

Parthasarathy, Ranga, Rao and Varadhan [Illinois Math. J. **7** (1963), 337–369] obtained a Lévy-Khinchin formula for the characteristic functions of infinitely divisible distributions μ having no idempotent factors. The present note generalizes the result to all μ.
J. G. Wendel (Aarhus)

Kandelaki, N. P.; Sazonov, V. V. 4567
On the central limit theorem for random elements with values in a Hilbert space. (Russian. English summary)
Teor. Verojatnost. i Primenen. **9** (1964), 43–52.

Ist X eine zufällige Variable mit Werten in einem separablen Hilbertschen Raum H und $\mathbf{M}\|X\|^2<\infty$, die zur Vereinfachung als zentriert vorausgesetzt werde, so bezeichne Q_X ihre Verteilung in H und S_X den durch $(S_Xh, h')=\mathbf{M}((h, X)(h', X))$ gegebenen Operator. Es sei weiter (X_n) eine unabhängige Folge solcher Variabler und $Y_n=X_1+\cdots+X_n$. Eine Folge (A_n) beschränkter linearer Operatoren in H heißt eine normierende Folge für (X_n) in bezug auf S_X, wenn $\sup_n\sum_{j=1}^{\infty}(S_{A_nY_n}e_j, e_j)<\infty$ und $\lim_{k\to\infty}\sup_n\sum_{j=k}^{\infty}(S_{A_nY_n}e_j, e_j)=0$ für wenigstens eine, und damit für jede, orthonormale Basis (e_j) von H, sowie $\lim_{n\to\infty}(S_{A_nY_n}h, h)=(S_Xh, h)$ für jedes $h\in H$. Solche normierenden Folgen werden konstruiert. Ist nun X normal verteilt, so sind die folgenden Aussagen gleichwertig: (1) $Q_{A_nY_n}$ konvergiert schwach gegen Q_X und

$$\lim_{n\to\infty}\max_{1\leqq k\leqq n}\mathbf{P}\{\|A_nX_k\|\geqq\varepsilon\}=0$$

bei beliebigem $\varepsilon>0$.

$$\text{(2)}\qquad \lim_{n\to\infty}\sum_{k=1}^{n}\int_{\|A_nX_k\|\geqq\varepsilon}\|A_nX_k\|^2\,d\mathbf{P}=0$$

bei beliebigem $\varepsilon>0$. *K. Krickeberg* (Heidelberg)

Sazonov, V. V. 4568
Solution of a problem of R. L. Dobrušin. (Russian. English summary)
Teor. Verojatnost. i Primenen. **9** (1964), 180–181.

Author's summary: "An example of non-existence of a Markov distribution on a measurable space $(X\times Y\times Z, \mathscr{X}\times\mathscr{Y}\times\mathscr{Z})$ with given marginal distributions on $(X\times Y, \mathscr{X}\times\mathscr{Y})$ and $(Y\times Z, \mathscr{Y}\times\mathscr{Z})$ is constructed."
D. A. Darling (Ann Arbor, Mich.)

Tucker, Howard G. 4569
On continuous singular infinitely divisible distribution functions.
Ann. Math. Statist. **35** (1964), 330–335.

Let F be an infinitely divisible distribution function, i.e., F is for each positive integer n the n-fold convolution of a distribution function F_n. Then the characteristic function of F has the canonical representation

$$f(u) = \exp\left\{i\gamma u + \int_{-\infty}^{\infty}\left[e^{iux} - 1 - \frac{iux}{1+x^2}\right]\frac{1+x^2}{x^2}\,dG(x)\right\},$$

where γ is a constant and G is non-decreasing and bounded. It is known that if G is discrete, then F must be pure, i.e., discrete, continuous singular, or absolutely continuous.

The main result of this paper is a theorem which gives conditions on G so that F is continuous singular. The conditions are somewhat too complicated to list.

J. R. Blum (Albuquerque, N.M.)

Bergström, Harald 4570
★Limit theorems for convolutions.
Almqvist & Wiksell, Stockholm-Göteborg-Uppsala; John Wiley & Sons, New York-London, 1963. 347 *pp.* $15.00.

From the Preface: "In this monograph I present classical limit theorems in the theory of probability and generalizations of these theorems, and then I use a general method which permits even further generalizations. The limit theorems are stated in such a form that they give connections between the convergence in a certain norm, the so-called Gaussian norm, of a convolution product and a corresponding sum, in the same way that there is a correspondence between the convergence of an infinite product $\prod_{\nu=1}^{+\infty} a_\nu$ of numbers and the convergence of the infinite sum $\sum_{\nu=1}^{+\infty} [a_\nu - 1]$ under certain conditions. It would even have been possible to establish all limit theorems stated in this monograph as limit theorems for products of abstract elements belonging to a set with properties which correspond to some properties of the set of functions of bounded variation."

Thus, not only probability distribution functions but also functions of bounded variation are considered, in the latter case however they have to be dominated by bounded monotone functions. Instead of the usual method of Fourier transform, a Gaussian transform is used, although the former is also used as in the discussion of infinitely divisible distributions. A modified Riemann-Stieltjes integral is used so that any function of bounded variation can be integrated with respect to any other. This requires rather technical handling to which the author adds some unusual conventions, e.g., the symbol "∞" which satisfies the relations "$\infty + = -\infty$, $\infty - = +\infty$" and such that (∞, ∞), as well as the usual $(-\infty, +\infty)$, denotes the set of all real numbers. The exposition is quite detailed and presupposes only some elementary algebra and analysis, but the reader would have to be very determined to go through with it all. The author says, "Since I present a new method I need not quote many earlier works on the subject. I shall not give any historical background but refer to the well-known work of Gnedenko and Kolmogorov [*Limit distributions for sums of independent random variables* (Russian), GITTL, Moscow, 1949; MR **12**, 839].

{The expression for $\hat{f}_n(t)$ given on p. 94 is incorrect, due to a common misunderstanding of the "branches" of a multi-valued complex function.}

K. L. Chung (Stanford, Calif.)

Berman, Simeon 4571
Limiting distribution of the studentized largest observation.
Skand. Aktuarietidskr. **1962**, 154–161 (1963).

Let X_n be the largest of n independent observations, let $\Phi(x)$ be the asymptotic probability function for the normed largest observation $(x_n - b_n)/a_n$; then the studentized normed largest value has the same asymptotic probability under the condition that

$$(1) \qquad \lim_{n\to\infty} b_n/(a_n\sqrt{n}) = 0.$$

This holds for the first (exponential) type of initial distributions, for the second (Cauchy) type, provided the first two moments exist, but not for all distributions of the third (limited type). For the first type condition (1) states that the "characteristic product" $\alpha_n u_n$ increases more slowly than $\sqrt{n}$. The largest value of a uniform distribution has the third asymptotic distribution. But, and this is unexpected, the distribution of the largest studentized value has a normal distribution. Under the condition (1) the probability function of the largest absolute studentized deviate for symmetrical initial distribution converges to $\Phi^2(x)$.

E. J. Gumbel (New York)

Berman, Simeon M. 4572
Limit theorems for the maximum term in stationary sequences.
Ann. Math. Statist. **35** (1964), 502–516.

Let $\{X_n, n = 0, \pm 1, \cdots\}$ be a real-valued stationary stochastic process and let $Z_n = \max(X_1, \cdots, X_n)$. The purpose of this paper is to find conditions on the process such that Z_n (properly normalized) has a limiting distribution.

In Section 2 such conditions are found for a class of processes which includes a subclass of the exchangeable processes and a subclass of the Markov processes. It is also shown that Gaussian processes, with the exception of independent ones, never satisfy these conditions. Consequently, the remainder of the paper is devoted to a detailed study of conditions under which Z_n has a limiting distribution in the Gaussian case.

J. R. Blum (Albuquerque, N.M.)

Aumann, Robert J. 4573
On choosing a function at random.
Ergodic Theory (Proc. Internat. Sympos., Tulane Univ., New Orleans, La., 1961), *pp.* 1–20. *Academic Press, New York,* 1963.

Let X and Y be measurable spaces (i.e., sets with associated σ-rings of 'measurable' subsets) and let Y^X be the collection of all measurable functions from X into Y. The author discusses the problem of choosing a function f at random from Y^X and a point x at random from X so that $Z(f, x) = f(x)$ is a random variable. In particular, Z must be measurable.

In one approach Z is a random variable on $F \times X$ where $F \subset Y^X$. In this case the measurability of Z imposes a restriction on the collection of 'measurable' subsets of F. If F admits a collection of measurable subsets making Z

measurable, then F is called admissible. (Y^X is not necessarily admissible.)

In the other approach the author starts with a probability space (S, Σ, P), lets $\theta: S \to Y^X$, and defines $\varphi_\theta: S \times X \to Y$ by $\varphi_\theta(s, x) = [\theta(s)](x)$. In order for $Z = \varphi_\theta(s, x)$ to be a random variable φ_θ must be measurable. Define the range of a random variable φ_θ to be the range of θ.

The author discusses the relation between ranges of random variables and admissible subsets of Y^X. He proves Theorem 1: Every range is admissible and conversely, for every admissible set F there is a sample space S (there is no guarantee that S is 'nice') and a random variable φ_θ such that the range of φ_θ is F. In the case when X, Y, and S are copies of the unit interval he proves Theorem 2: Every range is a subset of some Baire class, and every Baire class is a subset of some range. (The author previously obtained results for admissible sets [Illinois J. Math. **5** (1961), 614–630; MR **25** #4053].) The author discusses some open questions.

D. L. Hanson (Columbia, Mo.)

Slobodenjuk, N. P. 4574

Some limit theorems for additive functionals of a sequence of sums of independent random variables. (Russian. English summary)

Ukrain. Mat. Ž. **16** (1964), 41–60.

Author's summary: "Let $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ be independent identically distributed random variables, $S_{n0} = 0$, $S_{nk} = n^{-1/2}(\xi_1 + \cdots + \xi_k)$, and $f_n(x, y)$ the sequence of measurable functions for which $\lim_{n\to\infty} \sup_{x,y} |f_n(x, y)| \to 0$. Limit theorems for random variables $\sum_{k=0}^{n-1} f_n(S_{nk}, S_{nk+1})$ are obtained." *D. A. Darling* (Ann Arbor, Mich.)

Ewens, W. J. 4575

The pseudo-transient distribution and its uses in genetics.

J. Appl. Probability **1** (1964), 141–156.

This paper discusses solutions $f(x)$ of the equation

$$-\frac{\partial}{\partial x}\{m(x)f(x)\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\{v(x)f(x)\} = 0$$

for $m(x) = -cx$, $v(x) = \beta x(1-x)$, arising from one-way mutation. This equation originates by setting $\partial f(x, t)/\partial t$ equal to zero in the Fokker-Planck diffusion equation, but the resulting solution $f(x)$ is not a stationary distribution, as none such exist. The function $f(x)$ is, however, given an interpretation and is called the pseudo-transient distribution. Some exact results for discrete analogues of the continuous model are obtained, by which the results for the continuous model can be seen by taking limits. The models considered are discrete time, discrete state; and continuous time, discrete state. Two genetic applications are given. *R. G. Stanton* (Waterloo, Ont.)

Blum, J. R.; Hanson, D. L.; Koopmans, L. H. 4576

On the strong law of large numbers for a class of stochastic processes.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 1–11 (1963).

The authors consider a sequence of random variables $X_1, X_2, \cdots$ which have, roughly, the property that the first m of them are asymptotically independent of X_{n+m} for $n \to \infty$. More precisely, they term the sequence "*-mixing" if there exist an N and a sequence c_n converging monotonically to zero such that $n > N$ implies

$$|\Pr\{AB\} - \Pr\{A\}\Pr\{B\}| \leq c_n \Pr\{A\}\Pr\{B\},$$

where $A = \bigcap_{1\leq k\leq m}\{X_k \in A_k\}$, $B = \{X_{n+m} \in A_{n+m}\}$ for real Borel sets A_j.

The authors give several theorems on the strong law of large numbers for such sequences. For example, if the X_i are identically distributed with mean 0 and if the characteristic function of X_i is analytic in a neighborhood of the origin, then the arithmetic mean of the first n X_i goes to zero with probability one exponentially fast. If the X_i are uniformly integrable, satisfy the Kolmogorov condition $\sum \operatorname{Var}(X_n/n) < \infty$, and $E(X_n) = 0$, the strong law of large numbers applies.

The condition for *-mixing appears to be very strong, e.g., if the sequence is Gaussian, then necessarily disjoint subsets of the X_i separated by a sufficiently long block of indices are independent. The authors conclude by giving conditions on a Markov chain to be *-mixing.

D. A. Darling (Ann Arbor, Mich.)

Dharmadhikari, S. W. 4577

Exchangeable processes which are functions of stationary Markov chains.

Ann. Math. Statist. **35** (1964), 429–430.

The author derives a property of exchangeable stochastic processes Y_n [cf. Loève, *Probability theory*, 2nd ed., p. 365, Van Nostrand, Princeton, N.J., 1960; MR **23** #A670]. If V_n, $-\infty < n < \infty$, is a process and $W_n = [V_k; -\infty < k \leq n]$, then V_n is a function of the Markov process W_n. The author shows that an exchangeable process with a countable state space is a function of a stationary countable state Markov chain if and only if it is a countable mixture of sequences of independent and identically distributed random variables.

D. Austin (Evanston, Ill.)

Feller, William 4578

On semi-Markov processes.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 653–659.

Consider a semi-Markov process with the matrix of transition probability functions (F_{ij}), where $F_{ij}(t)$ is the probability that a sojourn time in state i has a duration $\leq t$ and is followed by a transition into state j. Let Π be the matrix of Laplace transforms of the functions $P_{ij}(t)$, the conditional probability of the process being in state j at time t given that a sojourn in state i has just begun at time 0. Let $\Phi = (\phi_{ij})$ be the matrix of Laplace-Stieltjes transforms of F_{ij}. Let Σ be the diagonal matrix with entries $\sigma_i = \sum_j \phi_{ij}$. This paper begins with the natural generalization of the Kolmogorov backward equations, namely, $\Pi = s^{-1}[I - \Sigma] + \Phi\Pi$, where s is the argument of the transforms. This system of equations has a minimal solution which probabilistically has the same interpretation as in the Markov case.

The author shows that this minimal solution satisfies a system of "forward" equations,

$$\Pi = s^{-1}[I-\Sigma] + \Pi[I-\Sigma]^{-1}\Phi[I-\Sigma].$$

The main theorem of this paper is a characterization, in

terms of truncated moments of the F_{ij}, of when the minimal solution is strictly stochastic and hence unique.
R. Pyke (Seattle, Wash.)

Parry, William 4579
Intrinsic Markov chains.
Trans. Amer. Math. Soc. **112** (1964), 55–66.
The author investigates the structure of finite-state stochastic processes that are called intrinsically Markovian since they behave like Markov chains because "possible" sequences of the processes are determined by a chain rule. Necessary and sufficient conditions are established for a stochastic process to be intrinsically Markovian. An equivalence relation called compatibility between finite-state stochastic processes is defined. A sub-class of the compatibility class of an intrinsic Markov chain called piecewise linear is defined and shown to be equivalent to the sub-class of stationary Markov chains. A notion of absolute entropy of a finite-state stochastic process is defined and shown to be an invariant of compatibility which, within the compatibility class of an intrinsic Markov chain, dominates all the probability entropies of stationary processes [see Khinchin, *Mathematical foundations of information theory*, Dover, New York, 1957; MR **19**, 1148]. A unique stationary probability (whose entropy is equal to the absolute entropy) is shown to make the process a Markov chain which in turn is equivalent to a uniform piecewise linear process [see Halmos, *Entropy in ergodic theory*, Univ. Chicago, Chicago, Ill., 1959]. From this it is shown that for every positive number between zero and the absolute entropy, there exists a compatible stationary Markov chain (equivalent to a piecewise linear process) having this number for its entropy. A procedure for determining the absolute entropy and the chain having this maximal entropy is given. A process which behaves like a Markov chain in the information-theoretical sense is shown to be a Markov chain [see Wolfowitz, *Coding theorems of information theory*, Springer, Berlin, 1961; MR **26** #1227].
H. P. Edmundson (Pacific Palisades, Calif.)

Rao, K. Muralidhara 4580
On increasing Markov process.
J. Math. Kyoto Univ. **2** (1962/63), 335–364.
Let x_t be a Markov process with stationary transition probabilities on the real line such that almost all sample paths are right-continuous and non-decreasing, and let $n(a, A)$ ($\leqq +\infty$) be the expected time that the sample paths from the point a spend in the linear Borel set A. It is shown (Sections 3, 4) that, under certain regularity hypotheses on x_t, $n(\cdot, \cdot)$ satisfies the following conditions: (1) for each a, $n(a, (-\infty, b))$ is continuous, finite if $b<\infty$, vanishing if $b \leqq a$ and strictly positive if $b>a$; (2) (*) $\int n(a, db) f(b)$ is continuous in a if f is continuous and vanishes near $+\infty$; (3) if the integral (*) has a (non-negative) maximum at a point a_0, then $f(a_0) \geqq 0$; and finally (4) any continuous function vanishing at $+\infty$ can be approximated by the integral (*) uniformly on any right half-interval. Conversely, a general theorem of G. A. Hunt [Illinois J. Math. **1** (1957), 316–369, Section 15; MR **19**, 951], applied to this special case, tells us that if $n(\cdot, \cdot)$ satisfies the conditions (1)–(4), then it is the expected sojourn time of a (unique) increasing Markov process. The main result of this paper is that the condition (4) can be dropped, namely, that the conditions (1)–(3) on $n(\cdot, \cdot)$ are still sufficient for the existence of the corresponding increasing Markov process. The proof is based on a theorem of D. Ray [Ann. of Math. (2) **70** (1959), 43–72; MR **21** #6027] and it is shown, not directly but after the proof is completed, that the conditions (1)–(3) imply (4).
T. Watanabe (Urbana, Ill.)

Smith, Gerald 4581
Instantaneous states of Markov processes.
Trans. Amer. Math. Soc. **110** (1964), 185–195.
The author shows by example a rather curious analytic property of the transition functions $p_{ij}(t)$ for a countable state, continuous time Markov chain. The hypothesis (and conclusion) may be phrased independently of probability concepts; the construction, however, is probabilistic in nature. Let $P_t: p_{ij}(t)$ $(i, j=1, 2, \cdots)$ be a matrix of non-negative functions with row sum satisfying the semi-group condition $P_{t+s}=P_t P_s$. It is known that $p_{ij}(t)$ has a continuous derivative on $t>0$ and that the derivative exists at $t=0$ but may be $-\infty$ on the diagonal. The second derivative need not exist even when the first derivatives on the diagonal are all finite. Heretofore the only gap in the existence question for derivatives was the answer to the question of whether or not $\lim_{t \downarrow 0} p_{ii}'(t) = -\infty$ when $p_{ii}'(0)=-\infty$. The author shows, by example, that this continuity condition may fail.
D. Austin (Evanston, Ill.)

Wong, Eugene 4582
The construction of a class of stationary Markoff processes.
Proc. Sympos. Appl. Math., Vol. XVI, pp. 264–276. Amer. Math. Soc., Providence, R.I., 1964.
The author considers a diffusion process $X(t)$ with a linear interval as state space and having an invariant distribution $W(x)dx$, where W is a probability density from the Pearson system, $W'(x)=[(ax+b)/(cx^2+dx+e)]W(x)$. He discusses the general analytic description of the transition densities and works out the details for several specific choices of W. Also, he discusses the distribution of an additive functional $\int_0^t f[X(u)]\,du$ when f is related to W in a particular way.
R. M. Blumenthal (Seattle, Wash.)

Chatterji, S. D. 4583
A note on the convergence of Banach-space valued martingales.
Math. Ann. **153** (1964), 142–149.
For martingales $\{\xi_n, \mathcal{F}_n, n \geqq 1\}$ taking on values in a Banach space $\mathfrak{X}$, the author gives proofs of classical martingale convergence theorems by using methods of functional analysis. In particular, the author generalizes the mean convergence theorem of the reviewer [Trans. Amer. Math. Soc. **87** (1958), 439–446; MR **20** #1350] for the scalar-valued case and the mean convergence theorem of F. S. Scalora [Pacific J. Math. **11** (1961), 347–374; MR **23** #A684] for the case where $\mathfrak{X}$ is reflexive by allowing $\mathfrak{X}$ to be non-reflexive. A pointwise convergence theorem, previously proved by Scalora in the reflexive case, is obtained for martingales generated by taking conditional expectations of a random variable taking on values in a

possibly non-reflexive space $\mathfrak{X}$ relative to a monotone sequence of Borel fields. This theorem is proved by an elegant application of Banach's theorem. A pointwise convergence theorem is also proved for martingales not so generated, but taking on values in a reflexive $\mathfrak{X}$, under the hypothesis of uniform integrability. This latter result has been improved upon by A. Ionescu Tulcea and C. Ionescu Tulcea [Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 204–206; MR **24** #A2859]. *L. L. Helms* (Urbana, Ill.)

Capon, Jack 4584

Radon-Nikodym derivatives of stationary Gaussian measures.

Ann. Math. Statist. **35** (1964), 517–531.

A necessary and sufficient condition is given for the equivalence of a pair (P_0, P_1) of stationary Gaussian processes with different covariance functions (R_0, R_1) and zero mean functions. If $H(R_1)\otimes H(R_1)$ denotes the RKHS (reproducing kernel Hilbert space) corresponding to the kernel $R_1\otimes R_1$, the main condition for equivalence of P_0 and P_1 is that R_0-R_1 belong to $H(R_1)\otimes H(R_1)$. This modifies a related result of Parzen [Proc. Sympos. Time Series Analysis (Brown Univ., 1962), pp. 155–169, Wiley, New York, 1963; MR **26** #7119]. As a corollary, the author derives some known results of Feldman. The Radon-Nikodym derivative dP_1/dP_0 is computed generally as a limit of inner products in an RKHS and explicitly for the case of equivalent Gaussian autoregressive schemes. The proof relies on previous work on the equivalence problem by Hájek and on a theorem of Aronszajn.

G. Baxter (Minneapolis, Minn.)

Montroll, Elliott W. 4585

Random walks on lattices.

Proc. Sympos. Appl. Math., Vol. XVI, pp. 193–220. *Amer. Math. Soc., Providence, R.I.,* 1964.

This article is a review on results pertaining to random walks on lattices. The author shows that all of the probabilistic theory can be derived from the Green's function relevant to the particular lattice, and that many asymptotic results can be obtained by a consideration of the analytic structure of the Green's functions. Some of the topics covered are: Return to the origin, number of lattice points visited after n steps, and random walks on slightly defective lattices. *G. Weiss* (Queens, N.Y.)

Kingman, J. F. C. 4586

A martingale inequality in the theory of queues.

Proc. Cambridge Philos. Soc. **60** (1964), 359–361.

Using the well-known inequality $\lambda P[\max_{1\leq i\leq N} x_i \geq \lambda] \leq E[|x_N|]$, $\lambda>0$, for submartingales, the author obtains results concerning the stationary waiting time distribution of a single-server queue. Specifically, if w_n is the waiting time for the nth individual and μ_n is the difference between the service time of the nth individual and the elapsed time between the arrival of the nth and $(n+1)$st individual, the distribution of w_n is known to converge to the distribution of $w=\sup_n (\mu_1+\cdots+\mu_n)$ provided $E[\mu_n]<0$ [D. V. Lindley, same Proc. **48** (1952), 277–289; MR **13**, 759]. The μ_n's are assumed to be independent and identically distributed. The author shows that $P[w\geqq x]\leqq e^{-\varphi_0 x}$, where φ_0 is the unique non-zero root of $E[e^{\varphi\mu_n}]=1$, and that $E[w]=1/\varphi_0$. *L. L. Helms* (Urbana, Ill.)

Arora, K. L. 4587

Two-server bulk-service queuing process.

Operations Res. **12** (1964), 286–294.

Author's summary: "In the present paper a two-server queuing process fed by Poisson arrivals and exponential service time distributions has been considered under the bulk-service discipline. Time-dependent probabilities for the queue-length have been obtained in terms of Laplace transforms, from which different measures associated with the queuing process could be determined. The mean queue-length and the distributions of the length of busy periods for (i) at least one channel is busy and (ii) both channels being busy, are obtained."

Burke, P. J. 4588

The dependence of delays in tandem queues.

Ann. Math. Statist. **35** (1964), 874–875.

The author considers two single-server queueing systems in tandem; that is, customers proceed to the second queue after leaving the first counter. In the stationary case, with Poisson arrivals at the first queue, and exponential service times, it was shown by the reviewer [same Ann. **34** (1963), 338–341; MR **26** #1942] that the waiting times of the nth customer at the two queues are independent, provided the waiting times are defined so as to include the service times. The author now shows that if the service times are not included, then independence does not occur.

E. Reich (Stanford, Calif.)

Eisen, M.; Tainiter, M. 4589

Stochastic variations in queuing processes.

Operations Res. **11** (1963), 922–927.

Authors' summary: "In this paper we consider a queuing process in which there are two mean arrival and service rates. Not only does the system randomly change from one mode of service and arrival to the other, but units arrive at random and require varying amounts of service. Analytic expressions are obtained for the generating functions, the mean queue length, and the mean waiting time. Several practical illustrations are given."

Ghosal, A. 4590

Queues with finite waiting time.

Operations Res. **11** (1963), 919–921.

Author's summary: "The paper considers a single-server queueing system in which a customer does not wait more than a fixed time k, so that if he does not get his service within this time, he departs. Some results are obtained by applying the theory of storage."

Kendall, David G. 4591

Some recent work and further problems in the theory of queues. (Russian summary)

Teor. Verojatnost. i Primenen. **9** (1964), 3–15.

A selective review of some of the more interesting developments of queueing theory in recent years.

J. Kiefer (Ithaca, N.Y.)

Viskov, O. V. 4592
Two asymptotic formulae for queueing theory. (Russian. English summary)
Teor. Verojatnost. i Primenen. **9** (1964), 177–178.
For a $GI/G/1$ queueing system, an asymptotic distribution of waiting time for the nth customer in "over-critical" traffic (when the number of customers waiting for service increases indefinitely) is stated. Also, an asymptotic formula for the probability of the loss of a call in a multi-server system $M/M/s$ with unreliable servers (under the assumption of the increasingly "fast" repair of the out-of-order servers) is presented without proof.
S. Kotz (Toronto, Ont.)

Welch, Peter D. 4593
On pre-emptive resume priority queues.
Ann. Math. Statist. **35** (1964), 600–612.
Author's summary: "The following queueing problem is considered. Customers arrive at a service facility at r priority levels. At each priority level the input process is Poisson, and these processes are mutually independent. The service times have an arbitrary distribution function which depends upon the priority level. A single server serves under a pre-emptive resume discipline. Results are obtained which characterize the transient and asymptotic distribution of the queue sizes and the waiting times. The analysis proceeds through reductions of the processes of interest to corresponding processes in a simple generalization of an $M/G/1$ queue."
S. C. Port (Santa Monica, Calif.)

Yeo, G. F. 4594
Preemptive priority queues.
J. Austral. Math. Soc. **3** (1963), 491–502.
Author's summary: "In this paper priority queues with K classes of customers with a preemptive repeat and a preemptive resume policy are considered. Customers arrive in independent Poisson processes, are served, within classes, in order of arrival, and have general requirements for service. Transforms of stationary waiting time and queue size distributions and busy period distributions are obtained for individual classes and for the system; the moments of the distributions are considered."
T. Kawata (Washington, D.C.)

Grigelionis, B. 4595
On the degree of approximation of the composition of renewal processes by a Poisson process. (Russian. Lithuanian and English summaries)
Litovsk. Mat. Sb. **2** (1962), *no.* 2, 135–143.
Author's summary: "Let $X_n(t)=\sum_{r=1}^{kn} X_{nr}(t)$, where $X_{nr}(t)$ are independent renewal processes. In the note accuracy of approach of multidimensional distributions of the process $X_n(t)$ by means of the corresponding Poisson ones is considered."

Bartlett, M. S. 4596
Probability and statistics in the physical sciences. (French summary)
Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 3–21.
Author's summary: "Nous avons essayé de donner un exposé sommaire de l'emploi du calcul des probabilités et de la statistique dans les sciences physiques. Les applications de la théorie des processus stochastiques sont classées pour des raisons pratiques sous les rubriques: processus stationnaires, processus markoviens, processus additifs et processus ponctuels. Le rôle de la probabilité est ensuite brièvement examiné dans les deux domaines physiques importants: (1) mécanique quantique, (2) mécanique statistique par la théorie de l'information. Finalement, nous avons discuté quelques problèmes d'inférence statistique d'un type plus courant qui se sont présentés dans des contextes physiques."

STATISTICS

See also 3910, 4151, 4571, 4575, 4596, 4678, 4930, 4937, 4938, 4944, 4954, 4955, 4958, 4978.

Blanco Loizelier, E. 4597
The teaching of applied and industrial statistics in Spain. (Spanish. French and English summaries)
Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 285–300.

Bertschinger, André; Linder, Arthur; Soom, Erich 4598
Formation aux applications industrielles de la statistique en Suisse. (English summary)
Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 301–306.

Mothes, J. 4599
Enseignement des applications industrielles de la statistique en France. (English summary)
Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 307–314.

Sieben, J. W. 4600
Applied statistics training in the Netherlands. (French summary)
Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 315–318.

Teghem, J. 4601
Enseignement de la statistique industrielle. Rapport pour la Belgique. (English summary)
Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 319–324.

Palazzi, A. 4602
Enseignement des applications de la statistique dans les pays européens. Rapport pour l'Italie. (English summary)
Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 325–327.

Stange, K. 4603

Report on training in applied statistics in the Federal Republic of Germany. Facts and proposals. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 329–340.

Van Rest, E. D. 4604

The teaching of applied statistics in Great Britain. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 341–346.

Training in industrial statistics in Norway during 1958–1960. (French summary) 4605

Report prepared by Norsk Forening for Industriell Kvalitetskontroll.

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 347–349.

Janko, Jaroslav 4606

Applied statistics training in European countries. Special report for Czechoslovakia. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 351–366.

Mothes, J. 4607

Rapport général sur les enseignements des applications industrielles de la statistique en Europe. (English summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 367–378.

Owen, D. B. 4608

★Handbook of statistical tables.

Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1962. xii + 580 *pp.* $12.50.

The number of published tables in mathematical statistics is so great that a descriptive bibliography of them fills a volume larger than the present one [see Greenwood and Hartley, *Guide to tables in mathematical statistics*, Princeton Univ. Press, Princeton, N.J., 1962; MR **27** #4299]. The criteria used by the author for inclusion of tables are expressed in the following quotation from the preface: "Concerning the selection of tables for this book, the choice has been dictated largely by two considerations: (a) amount of space taken by the table vs. its usefulness, and (b) a desire to make this compilation all-inclusive enough so that it can be used as a supplementary handbook for most courses in statistics, and so that at the same time it would contain as many as possible of the unusual tables of merit which are not given in other compilations now on the market." Because of space limitations, the author has sometimes found it necessary to curtail the accuracy (number of decimal places or significant digits) and/or the extent (number of values of the parameters) of the tables included. In such cases, the reader who requires greater accuracy and/or extent can refer to the original source.

Information as to the coverage of this volume is given in the following statement on the jacket: "In addition to tables of the functions usually included in a handbook of this type (normal, F, t, chi-square, max F, range, etc.), the following functions are given: various order statistics from a normal distribution; noncentral t and chi-square; various multivariate normal tables including distribution functions, offset circle probabilities, quadratic forms and characteristic roots; confidence limits on functions of binomial parameters; many distribution-free statistics including the Birnbaum-McCarty upper confidence bound for $p = \Pr[y < x]$, the Mann-Whitney distribution, sign test, runs tests, Kendall and Spearman rank correlations, Friedman's chi-square, several tables of Smirnov and Kolmogorov statistics, ranks in analysis of variance, tolerance limits for finite and infinite populations, etc.; matching probabilities; multinomial distributions; various tables for determining sample size including those for the comparison of two proportions, for tolerance limits, for bounds on the end-points of confidence intervals on the standard deviation, etc.; random numbers; and many other tables. In all, over 100 different tables are included."

In order to minimize the possibility of error, tables were reproduced photographically from the output of digital computers whenever feasible.

H. L. Harter (Dayton, Ohio)

Ellison, Bob E. 4609

On two-sided tolerance intervals for a normal distribution.

Ann. Math. Statist. **35** (1964), 762–772.

A. Wald and the reviewer [same Ann. **17** (1946), 208–215; MR **8**, 478] gave a method of obtaining approximate two-sided tolerance intervals for a normal distribution. In their proof the estimate of the mean comes from N observations, the (independent) estimate of the variance has n degrees of freedom, n is held fixed and $N \to \infty$. The present author allows both n and N to vary. Using the method of the paper cited above in a more detailed analysis, the author obtains more general and improved results which do not lend themselves to short summary here. His results include general estimates of the rapidity of convergence.

J. Wolfowitz (Ithaca, N.Y.)

Gebhard, Richard F. 4610

A limit theorem for random interval sampling of a stochastic process.

Ann. Math. Statist. **35** (1964), 866–868.

A real-valued weakly stationary stochastic process $X_t(\omega)$ is sampled at random times $(0<)t_1<t_2\cdots$, where the t_i-t_{i-1} $(t_0=0)$ are independent, all with the same exponential distribution. $m = N^{-1}\sum_{i=1}^N X_{t_i}(\omega)$ is an estimator for $E\{X_t\}$. The author measures its efficiency by R_N, the ratio of $\operatorname{var}(m)$ and $N^{-1}\operatorname{var}\{X_t\}$ (which would be the variance at m if $X_{t'}$ and $X_{t''}$ were uncorrelated for $t' \neq t''$).

A theorem about convergence in probability of ER_N $(N\to\infty)$ and the value of the limit is given. It is not clear to the reviewer, however, what the author means by ER_N. In the terminology of this review it seems to be the conditional expectation of R_N (with respect to the $\{t_i\}$ process) given t_N.

H. Kesten (Ithaca, N.Y.)

Koopmans, L. H. 4611

On the coefficient of coherence for weakly stationary stochastic processes.

Ann. Math. Statist. **35** (1964), 532–549.

The coefficient of coherence $\rho(\lambda)$ is defined for a bivariate weakly stationary stochastic process

$$\mathbf{X} = \{X(t) \mid -\infty < t < \infty\},$$

$X(t)$ being the complex-valued column vector $(X_1(t), X_2(t))'$. If $\mathbf{F}(\lambda)$ is the spectral distribution function of $\mathbf{X}$ and $\mathbf{f}(\lambda)$ the spectral density function of $\mathbf{F}(\lambda)$ with respect to a fixed Lebesgue-Stieltjes measure μ,

$$\rho(\lambda) = |f_{12}(\lambda)|/[f_{11}(\lambda)f_{22}(\lambda)]^{1/2}$$

if the denominator does not vanish, and $\rho(\lambda)=0$ otherwise.

Since $0 \leq \rho(\lambda) \leq 1$ almost everywhere, $\rho(\lambda)$ is analogous to the modulus of the correlation coefficient for two variates X_1 and X_2. Analogues to the property of ρ^2 of measuring the proportion of the variance of X_1 attributable to regression on X_2 are developed.

The coefficient of coherence is shown to be invariant under a large class of linear transformations including those of physical interest. This invariance property characterizes the coefficient of coherence in a certain sense. The analogous property of the modulus of the correlation coefficient is precisely stated.

It is proved that $\rho(\lambda)$ is independent of the particular measure μ used. If μ_x dominates the spectral distribution of $\mathbf{X}$ and μ belongs to the class M^x of measures for which $\rho(\lambda)$ is defined (measures for which μ dominates μ_x), then $\rho_\mu(\lambda) = \rho_{\mu_x}(\lambda)$ almost everywhere.

E. S. Keeping (Edmonton, Alta.)

Guenther, William C. 4612

A generalization of the integral of the circular coverage function.

Amer. Math. Monthly **71** (1964), 278–283.

The center of a sphere, S, of assigned radius, has a spherically symmetrical n-dimensional normal distribution of unit standard deviation and mean at the origin. A point P is chosen with a uniform distribution in a ball of another assigned radius and center at the origin. The probability that S contains P is expressed in terms of integrals of the non-central chi-squared distribution. The case $n=2$ had previously been analyzed by Germond [Rand Res. Memorandum RM-330 (1950)], and the case $n=3$ by the author [SIAM Rev. **3** (1961), 247–251; MR **24** #A566].

I. J. Good (Princeton, N.J.)

Hartigan, J. 4613

Invariant prior distributions.

Ann. Math. Statist. **35** (1964), 836–845.

Author's summary: "The paper is mainly concerned with determining prior distributions on ignorance over parameter spaces, using invariance techniques similar to those of decision theory. Prior distributions are rarely determined exactly by such techniques and a number of less compelling methods for exact determination are given."

D. G. Chapman (Seattle, Wash.)

Khamis, Salem H. 4614

Sequel to a numerical solution of the problem of moments. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 481–490.

Author's summary: "Hartley et Khamis [Biometrika **34** (1947), 340–351; MR **9**, 623] ont obtenu, sous les conditions générales, une approximation numérique pour une fonction de distribution inconnue en utilisant ses $R+1$ premiers moments connus. La procédure est équivalente à une utilisation inverse de la correction de Sheppard des moments bruts (raw) basées sur le théorème d'Euler-Maclaurin. La procédure a été standardisée pour rendre possible le calcul de fréquences groupées en employant pour chaque ligne d'une matrice inverse (déterminée pour chaque R) les moments M_r standardisés qui ont été dérivés des moments μ_r de la distribution inconnue. Deux types de matrices inverses ont été discutés qui appartiennent aux distributions symétriques et asymétriques pour $R=6$ et $R=8$. Dans cet article, l'auteur a fait une courte revue de la méthode et des développements ultérieurs et a donné la matrice inverse pour $R=5$, 7 et 8 d'un cas asymétrique et pour $R=4$ et 6 pour un cas symétrique. La méthode de calcul employant les différences finies a été présentée pour obtenir les moments par rapport à une certaine origine en partant des moments par rapport à une autre origine. Cette technique est très utile pour réduire le travail en utilisant la présente méthode ainsi que les autres méthodes approximatives pour calculer les fonctions homogènes des moments, c'est-à-dire les cumulants β_1 et β_2."

Philipson, Carl 4615

On a class of distribution functions as applied to different stochastic processes.

Skand. Aktuarietidskr. **1961**, 20–54 (1962).

In this paper, the author defines a set of functions $f_r(x)$ in terms of the Gauss hypergeometric series ${}_2F_1(x)$. He discusses the properties of the integrals $G_r(x)$ of these functions, which can be regarded as distribution functions in stochastic processes.

The interconnections of the various well-known functions of statistics, all of which are special cases of these G-functions, are discussed, and some important new theorems concerning these compound Poisson processes are proved.

L. J. Slater (Cambridge, England)

Philipson, Carl 4616

An extension of the models usually applied to the theory of risk.

Skand. Aktuarietidskr. **1961**, 223–239 (1963).

In this paper, the author defines four characteristic functions for risk distributions. These functions have close connections with the confluent hypergeometric functions. He then defines generalizations of these risk functions and their corresponding distribution functions, such that $\Phi_0(\sigma) = \eta_1^{-q_1}\eta_2^{-q_2}\eta_3^{-q_3}$, and $\Phi_n = (-1)^n \partial[\Phi_0(s+t)]/\partial t$.

The theory of these new functions is worked out, and the corresponding characteristic functions for the compound Poisson distribution process are defined. The allied distribution functions in these general cases are also defined, and the paper concludes with some discussion of the application to actual statistical data of these compound Poisson processes.

L. J. Slater (Cambridge, England)

Srivastava, O. P.; Harkness, W. L.; Bartoo, J. B. 4617
Asymptotic distribution of distances between order statistics from bivariate populations.
Ann. Math. Statist. **35** (1964), 748–754.

The exact and asymptotic distribution of quantiles is well known in the univariate case. The reviewer [J. Res. Nat. Bur. Standards Sect. B **64B** (1960), 145–150; MR **25** #4591] obtained the exact and asymptotic distributions of quantiles and an auxiliary statistic in samples from a bivariate distribution. Further, he showed that the distances $X'_{i+l}-X_i'$, $X_i'-X'_{i-k}$ between quantities in the univariate case are asymptotically independently distributed as chi-square variates with $2l$ and $2k$ degrees of freedom, respectively, when k/n and $l/n\to 0$, $i/n\to F(\alpha)\neq 0, 1$, as the sample size $n\to\infty$. In this paper the authors extend the latter result of the reviewer and show that the distances between quantiles in the separate components of the sample from a bivariate population are independent asymptotically. *M. M. Siddiqui* (Fort Collins, Colo.)

Banerjee, K. S.; Federer, W. T. 4618
Estimates of effects for fractional replicates.
Ann. Math. Statist. **35** (1964), 711–715.

In a recent paper [same Ann. **34** (1963), 1068–1078; MR **27** #2053] the authors indicated how to adjust the treatment design matrix, X, to furnish estimates of effects as orthogonal linear functions of stochastic variates for any fractional replicate plan in which any treatment combination occurs at most one time. Results were obtained on the augmentation of X to $X_1=[X', X'\lambda]'$ such that $X_1'X_1$ is a diagonal matrix and also on transforming X_1 to another matrix $X_2=FX_1$. In the present paper, results are presented on the evaluation of variances of the estimated effects, on the existence and evaluation of the matrices λ and F used in the augmentation and adjustment on X, on the determination of aliases of effects, and on the calculation of the inverse of the information matrix. *S. Addelman* (Durham, N.C.)

Benedetti, Carlo 4619
A proposito dei rapporti tra differenza media e scostamenti medio quadratico, semplice medio e semplice medio dalla mediana.
Metron **21** (1961), 181–185.

Distribution-free bounds are derived for the ratio of Gini's coefficient of mean difference and the average (absolute) deviation about the median.
C. Villegas (Montevideo)

Breiman, Leo; LeCam, Lucien; Schwartz, Lorraine 4620
Consistent estimates and zero-one sets.
Ann. Math. Statist. **35** (1964), 157–161.

The authors derive a general result from which it follows that if $X_1, X_2, \cdots$ are independent random variables each with d.f. $F(x,\theta)$, $\theta\in[0,1]$, then there exists a function $f(x_1, x_2, \cdots)$ such that $P_\theta(f(X_1, X_2, \cdots)=\theta)=1$ for all $\theta\in[0,1]$ if and only if the σ-field generated on $[0,1]$ by the functions $F(x,\cdot)$ coincides with the usual Borel field. If there is an a priori distribution Q, this result entails the existence of estimators $f_n(X_1, \cdots, X_n)$ such that $P_\theta(\lim f_n=f(\theta))=1$ for almost all $\theta(Q)$.
J. Hájek (Prague)

Graybill, Franklin A.; Connell, Terrence L. 4621
Sample size required to estimate the parameter in the uniform density within d units of the true value.
J. Amer. Statist. Assoc. **59** (1964), 550–556.

Authors' summary: "A two-step procedure is given to find the sample size necessary to estimate the parameter θ in the uniform density within d units of the true value. It is demonstrated that an exact solution for all values of θ doesn't exist based on the maximum likelihood estimator. A table is presented which can be used to find the second sample size, n, such that $P[|y_n-\theta|<d]>1-\alpha$ is true, where y_n is the largest observation in the second sample, for $1-\alpha=.90, .95, .99$ and the preliminary sample size $m=2, 5\ (5)\ 30\ (10)\ 50\ (25)\ 100$. Also a table of comparisons between the expected sample size in this paper and two other solutions is given."

Huber, Peter J. 4622
Robust estimation of a location parameter.
Ann. Math. Statist. **35** (1964), 73–101.

The problem is to estimate the location parameter when the prototype distribution function F is known only approximately. For example, we have $F=(1-\varepsilon)\Phi+\varepsilon H$, where Φ is the standard normal cumulative and H is the unknown contaminating distribution. The class of estimators T_n is confined to maximum likelihood estimators for densities with convex $\rho(x)=-\log f(x)$. These estimators are shown to be consistent and asymptotically normal. Denote $\psi=d\rho/dx$, and let $V(\psi,F)$ be the asymptotic variance of $n^{1/2}(T_n-c)$ if T_n is based on ψ and the true distribution is F. If $F=(1-\varepsilon)G+\varepsilon H$, where G is a fixed symmetric distribution and H is an arbitrary symmetric distribution, then there is a saddle point (ψ_0, F_0) such that $\sup_F V(\psi_0,F)=V(\psi_0,F_0)=\inf_\psi V(\psi,F_0)$. In this sense ψ_0 gives the minimax solution of the robust estimation problem. The author gives an explicit formula for ψ_0 as well as for F_0. If H fails to be symmetric, the procedure ψ_0 will be biased, the asymptotic bias being dependent on ε. The author gives bounds above which the sample size should not be increased if we wish to keep the asymptotic bias below $\frac{1}{4}n^{-1/2}$ for $\varepsilon=0.1, 0.01, 0.001$, $G=\Phi$.

Then the paper deals with the minimax theory of a game of a statistician against Nature, containing the previous problem as a special case. The results are subsequently used to find the saddle point in estimating the location parameter for the family of symmetric distribution functions such that $\sup|F(t)-\Phi(t)|\leq\varepsilon$. The saddle point is first determined for distributions with finite Fisher information, and the extension of the validity is achieved by the previous minimax theory.

The problem of estimation of variance is reduced to that of estimating location by the transformation $Y=\log X^2$, but the results are less satisfactory since the restrictions imposed on contaminating distributions appear somewhat unnatural. The paper is concluded by heuristic proposals concerning the estimation of location if the scale parameter, as well as the extent of contamination,

is unknown, and by considering more general estimators based on minimizing the value of a U-statistic.

J. Hájek (Prague)

Kale, B. K. 4623

An extension of the Cramér-Rao inequality for statistical estimation functions.

Skand. Aktuarietidskr. **1962**, 80–89 (1963).

Consider the statistical estimation function $T(x_1, x_2, \cdots, x_n, \theta)$, where $x_1, x_2, \cdots, x_n$ is a random sample of n independent observations and θ is a parameter. Under suitable regularity conditions the author shows that

$$\operatorname{Var}(T) \geqq \frac{\left\{\dfrac{d\psi}{d\theta} - E\left(\dfrac{\partial T}{\partial \theta}\right)\right\}^2}{nI(\theta)},$$

where $\psi(\theta)$ is the expected value of T and $I(\theta)$ is Fisher's information about θ. If equality is attained, then T is a sufficient statistical estimation function as defined by Kimball [Ann. Math. Statist. **17** (1946), 299–309; MR **8**, 475]. The multi-parametric case is also considered.

S. Kullback (Washington, D.C.)

Sen, Pranab Kumar 4624a

On the properties of U-statistics when the observations are not independent. I. Estimation of non-serial parameters in some stationary stochastic process.

Calcutta Statist. Assoc. Bull. **12** (1963), 69–92.

Nandi, H. K.; Sen, P. K. 4624b

On the properties of U-statistics when the observations are not independent. II. Unbiased estimation of the parameters of a finite population.

Calcutta Statist. Assoc. Bull. **12** (1963), 124–148.

Hoeffding's results on U-statistics [Ann. Math. Statist. **19** (1948), 293–325; MR **10**, 134] are extended from the case of independent random variables (i) in Part I to that of m-dependent variables, (ii) in Part II to the case that the variables are the values obtained when drawing a sample (without replacement) from a finite population.

E. L. Lehmann (Berkeley, Calif.)

Thompson, W. A., Jr.; Remage, Russell, Jr. 4625

Rankings from paired comparisons.

Ann. Math. Statist. **35** (1964), 739–747.

In this interesting paper the authors study a simple stochastic model for ranking from paired comparisons. Let π_{ij} denote the probability that the ith item will be preferred to the jth; these probabilities, $\pi = \{\pi_{ij}\}$, determine a "comparison", $R(\pi)$ (an anti-symmetric, anti-reflexive relation), on the set of items. The authors begin by studying weak rankings ("partial rank-order", "semi-rank-order", "rank-order") determined by comparisons. They then discuss maximum likelihood estimation of π subject to the condition that $R(\pi)$ be circuit-free. Let $\hat{\pi}_{ij}$ denote the unrestricted maximum likelihood estimate of π_{ij} for the case where $R(\hat{\pi})$ includes a comparison between each pair of items; a method is provided of computing the maximum likelihood weak stochastic ranking. An interesting comment is that if each pair of items is compared exactly once, the principle of maximum likelihood applied to the author's model is equivalent to a criterion proposed by Kendall [Biometrics **11** (1955), 43–62; MR **17**, 758] and studied also by Slater [Biometrika **48** (1961), 303–312], namely, that the number of violations of expressed preferences be minimized.

H. D. Brunk (Columbia, Mo.)

Dempster, A. P. 4626

Tests for the equality of two covariance matrices in relation to a best linear discriminator analysis.

Ann. Math. Statist. **35** (1964), 190–199.

Consider two samples of n_1 and n_2 vectors drawn from two p-variate normal distributions $N(\mu_1, \Sigma_1)$ and $N(\mu_2, \Sigma_2)$ Two tests are proposed to test the null hypothesis $\Sigma_1 = \Sigma_2$. Consider, in p-dimensional space, the sample concentration ellipsoids E_1, E_2 centered at the sample means $\bar{X}_1, \bar{X}_2$. Let Ω be the family of parallel hyperplanes over which the usual sample best linear discriminator assumes constant values, and let π_1, π_2 be two members of Ω which are tangent to E_1 and E_2, respectively. Let d_i, r_i be the distances, measured in the direction of $\bar{X}_2 - \bar{X}_1$, from the center of E_i to π_i and to the surface of E_i, respectively. The two test statistics are $(n_1-1)d_1^2/(n_2-1)d_2^2$ and the weighted harmonic mean of $d_1^2/r_1^2 - 1$ and $d_2^2/r_2^2 - 1$ with weights $(n_2-1)d_2^2$ and $(n_1-1)d_1^2$. It is shown that, under the null hypothesis $\Sigma_1 = \Sigma_2$, the two test statistics are independently distributed as $\chi^2_{n_1-1}/\chi^2_{n_2-1}$ and $\chi^2_{p-1}/\chi^2_{n_1+n_2-2p}$, and are independent of the usual T^2 statistic.

C. Villegas (Montevideo)

Kagan, A. M.; Šalaevskiĭ, O. V. 4627a

The Behrens-Fisher problem for the existence of similar regions in an algebra of sufficient statistics. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 1250–1252.

Linnik, Ju. V.; Romanovskiĭ, I. V.; Sudakov, V. N. 4627b

A non-randomized homogeneous test in the Behrens-Fisher problem. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 1262–1264.

Both papers contain proofs of the following result. Let $\bar{x}, s_1^2$ and $\bar{y}, s_2^2$ be the means and the variances of two independent random samples (of sizes at least 2) from normal distributions with respective unknown parameters a_1, σ_1^2 and a_2, σ_2^2. If the total number of observations is odd, there exists a non-randomized Borel-measurable similar test of any size $\alpha \in (0, 1)$ for testing the hypothesis $a_1 = a_2$; the test depends only on $|\bar{x}-\bar{y}|/s_2$ and s_1/s_2. Both proofs make essential use of versions of a lemma attributed to I. V. Romanovskiĭ and V. N. Sudakov, whose demonstration is not included.

W. Hoeffding (Chapel Hill, N.C.)

Linnik, Ju. V. 4628

Statistically similar zones of linear type. (Russian)

Dokl. Akad. Nauk SSSR **144** (1962), 974–976.

The following theorem is stated, with a sketch of the proof. Let $X_1, X_2, \cdots, X_n$ be independent, identically distributed random variables whose common distribution is one among a set indexed by a parameter θ. Let $T = a_1X_1 + \cdots + a_nX_n$ have a distribution independent of θ,

and suppose that $|a_i| \neq |a_j|$ for some i, j. Let $E(|X_i|^k) < \infty$ for a sufficiently large k and for all θ, and suppose that for all θ the characteristic function of X_i does not vanish for real arguments. Then for any i, j, the distribution of $X_i - X_j$ is independent of θ.

D. A. Darling (Ann Arbor, Mich.)

Linder, A. 4629

Trennverfahren bei qualitativen Merkmalen.

Metrika **6** (1963), 76–83.

Aus der Einleitung: "Wie Fisher [*Statistical methods for research workers*, 12th ed., rev., Oliver & Boyd, Edinburgh, 1954] gezeigt hat, kann das von ihm eingeführte Trennverfahren auch bei qualitativen Merkmalen verwendet werden. Das von Fisher behandelte Beispiel ist nicht leicht zu verstehen. Es erscheint daher nützlich an einem Beispiel zu zeigen wie ein größeres Beobachtungsmaterial dank der erwähnten Methode zusammengefaßt dargestellt werden kann."

L. Schmetterer (Vienna)

Birnbaum, Allan 4630

Intrinsic confidence methods. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 375–383.

Author's summary: "Nous étudions la question très générale: Quelles techniques, quelles expressions et quelles idées servent proprement pour caractériser les traits des résultats des expériences qui constituent l'information et l'évidence relatives à des paramètres? Un principe de conditionnalité est formulé, qui a une relation avec la conception de statistique ancillaire de R. A. Fisher. Il est démontré que ce principe implique le principe de vraisemblance. Le dernier principe, énoncé par Fisher, est que la signification d'évidence des résultats d'une expérience quelconque est caractérisée complètement par la fonction de vraisemblance, sans autre relation avec l'expérience. Des conséquences de ce résultat sont discutées à propos des fondements de la statistique mathématique."

Roy, S. N. 4631

A survey of some recent results in normal multivariate confidence bounds. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 405–422.

Author's summary: "Cette étude passe en revue quelques résultats récents, dont la plupart sont disponibles imprimés et dont le reste n'a pas encore été publié, relatifs aux limites de confiance de fonctions paramétriques associées aux distributions normales à plusieurs variables, lesquels ont été développés dans un but particulier. La philosophie générale est examinée avec plus de détails que jusqu'à présent et la plupart des résultats obtenus jusqu'ici sont exposés dans cet esprit, le détail des démonstrations diverses ayant été omis. Chaque fonction paramétrique considérée est intéressante, soit en elle-même, soit comme mesure d'un écart de quelque hypothèse nulle usuelle en direction d'une alternative typique ou non typique. Parmi les résultats examinés se trouvent ceux se rapportant à une ou deux matrices de dispersion de population, aux hypothèses linéaires à plusieurs variables sous un modèle linéaire, à l'indépendance entre deux séries de variables aléatoires, à l'indépendance multiple dans une série p ou entre des séries k de $p_1, p_2, \cdots, p_k$, et aux vecteurs analogues du rapport des moyennes et du rapport des variances pour une distribution normale à deux variables corrélées."

Cunia, T. 4632

Least squares estimates and parabolic regression with restricted location for the stationary point.

J. Amer. Statist. Assoc. **59** (1964), 564–571.

Author's summary: "It is sometimes desired to fit a second degree polynomial $E(y|x) = \alpha + \beta x + \gamma x^2$ to a set of data so that its stationary point will fall in a given region. Methods are shown to calculate a, b, c, the least squares estimates of the regression coefficients α, β, and γ. An example is given in which this theory is applied to fit a parabolic regression to a set of data so that in a given contiguous and finite region of the independent variable, the regression function is strictly increasing and concave upwards."

Fisher, G. R. 4633

Iterative solutions and heteroscedasticity in regression analysis. (French summary)

Rev. Inst. Internat. Statist. **30** (1962), 153–159.

Author's summary: "Dans un article précédent [même Rev. **25** (1957), 52–55] nous avons suggéré que pour obtenir des estimations par la méthode du maximum de vraisemblance des coefficients dans une régression linéaire multiple avec des erreurs hétéroscédastiques, il faut renverser la matrice d'information à chaque étape du processus itératif. Ceci n'est pas nécessaire. Il faut seulement une inversion qui sert pour la procédure entière. Un plus grand nombre d'itérations est nécessaire que dans le processus où la matrice d'information est renversée à chaque étape, mais il y a une plus grande exactitude et souvent une économie du temps de calcul. Celui-ci dépend de quatre critères: (i) exactitude de l'ajustement; (ii) nombre des variables explicatives; (iii) différences de grandeur et de direction entre les coefficients; (iv) coût de l'inversion de la matrice. Là où il n'y a aucun avantage important dans la méthode par inversion unique, une approximation alternative est proposée. Ceci emploie des corrélatifs proches de la variable dépendante. Mais, puisqu'il y a une approximation, l'efficacité doit être normalement sacrifiée."

Whittle, P. 4634

On the convergence to normality of quadratic forms in independent variables. (Russian summary)

Teor. Verojatnost. i Primenen. **9** (1964), 113–118.

When examining the properties of parameter estimates and tests based upon least-square criteria, one often has occasion to consider the distribution of forms of the type

$$\mathscr{A} = x'Ax = \sum_1^n \sum_1^n a_{jk} x_j x_k,$$

where the variables x_j are independently distributed about zero. In particular, one is interested in determining conditions which would secure normal convergence of $\mathscr{A}$ with increasing n.

For any statistical variable ξ, the following definitions are set forth:

$$\mu_1(\xi) = E(\xi), \quad \lambda_1(\xi) = E|\xi|, \quad \mu_s(\xi) = E(\xi-\mu_1)^s,$$
$$\lambda_s(\xi) = E|\xi-\mu_1|^s \text{ for } s > 1, \quad \beta_j^2 = \mu_2(x_j).$$

The norm of a linear quadratic form is defined as

$$\|\mathscr{A}\| = \|x'Ax\| = (\sum_j \sum_k a_{jk}{}^2\beta_j{}^2\beta_k{}^2)^{1/2}.$$

The author proves the following two theorems. Theorem 1: The quadratic form $x'Ax$ will tend to normality in distribution with increasing n if the following conditions are fulfilled: (i) The x_j have zero expectation; (ii) $\lambda_{4+2\delta}(x_j)$ is finite and $\lambda_{4+2\delta}(x_j/\beta_j)$ uniformly bounded for all x_j and some δ in $(0, 1)$; (iii) There exists a positive constant d such that $\mu_4(x_j)/\mu_2{}^2(x_j) \geqq 1+d$ for all j; (iv) The integers $j = 1, 2, \cdots, n$ can be grouped into sets $g_1, g_2, \cdots, g_r$ such that

$$\text{(v)} \quad \lim_{n\to\infty} \{[\|\mathscr{A}\|^2 - \sum_{v=1}^r \|\mathscr{A}_v\|^2]/\|\mathscr{A}\|^2\} = 0,$$
$$\lim_{n\to\infty} \left\{\sum_{v=1}^r [\|\mathscr{A}_v\|/\|\mathscr{A}\|]^{2+\delta}\right\} = 0.$$

Theorem 2: Consider the form $\mathscr{A} = \sum_1{}^n \sum_1{}^n a_{j-k}x_jx_k$ in which the x_j are identically distributed with zero mean. This will tend to normality in distribution as $n\to\infty$ if the following conditions are fulfilled: (i) $\lambda_{4+2\delta}(x_j)$ is finite for some δ in $(0, 1)$; (ii) $\sum_{-\infty}{}^\infty a_j{}^2$ is finite.

S. Addelman (Durham, N.C.)

Whittle, P. 4635

On the fitting of multivariate autoregressions, and the approximate canonical factorization of a spectral density matrix.

Biometrika **50** (1963), 129–134.

The recursive method of Durbin [Rev. Inst. Internat. Statist. **28** (1960), 233–244] for the fitting of autoregressive schemes of successively increasing order is generalized to the fitting of multivariate autoregressions and schemes with rational spectral density function. For the recursion two autoregressions are fitted simultaneously: the usual retrospective autoregression where the time series is expressed in terms of its immediate past, and a prospective autoregression where the time series is expressed in terms of its immediate future. This recursive method is similar to the extension to the multichannel case by E. A. Robinson [J. Geophys. Res. **68** (1963), 5559–5567] of the method of N. Levinson [Appendix in N. Wiener, *Extrapolation, interpolation, and smoothing of stationary time series*, Tech. Press of M.I.T., Cambridge, Mass., 1949; MR **11**, 118] for the determination of a finite time-domain filter, where in the multi-channel case a prospective operator as well as a retrospective operator must be determined together with the filter operator for the recursion. In the last section of the paper under review, a proof is given that a multivariate autoregression fitted by the Yule-Walker relations, even if of insufficient order, is a minimum-delay operator.

H. Wold (Uppsala)

Ylvisaker, N. Donald 4636

Lower bounds for minimum covariance matrices in time regression problems.

[illegible] *Statist.* **35** (1964), 362–368.

Suppose we observe $Y(t) = \sum_{i=1}^n \beta_i f_i(t) + X(t)$, where the correlation function of $X(t)$, ρ^*, is a mixture of correlation functions $\rho(\lambda t)$, $0 < \lambda < \infty$. If the regressor functions f_i are linearly independent and belong to the reproducing kernel Hilbert space for each $\rho(\lambda t)$, then it is shown that the covariance matrix of the best linear unbiased estimators of $(\beta_1, \cdots, \beta_n)$ for ρ^* dominates the respective mixture of the same matrices for $\rho(\lambda t)$. It should be noted that this result, some of the applications and also the reproducing kernel Hilbert space for $\rho(t) = 1-|t|$, have been previously derived by the reviewer [Czechoslovak Math. J. **12** (87) (1962), 404–444; MR **27** #2070; ibid. **12** (87) (1962), 486–491; MR **26** #7096]. New are the considerations concerning the efficiency of least square estimators with respect to the best variance estimators; also the method of proving the main result is different.

J. Hájek (Prague)

Csorgo, M.; Guttman, Irwin 4637

On the consistency of the two-sample empty cell test.

Canad. Math. Bull. **7** (1964), 57–63.

Let $X_{(1)} < \cdots < X_{(n_1)}$ be the order statistics from a continuous distribution $F_1(x)$, and let $Y_1, \cdots, Y_{n_2}$ be a sample from a continuous distribution $F_2(x)$. Let S be the number of 'empty cells', i.e., the number of intervals $(X_{(i)}, X_{(i+1)}]$, $i = 0, \cdots, n_1$, which contain none of the Y's. (Here $X_{(0)} = -\infty$ and $X_{(n_1+1)} = +\infty$.) Suppose now that $F_1 = F_2$ and $\lim (n_2/n_1) = \rho > 0$. Then it can easily be shown that $S/(n_1+1)$ converges in probability to $1/(1+\rho)$. This fact may be used to construct a test of the hypothesis $F_1 = F_2$ by rejecting the hypothesis when S is 'too large'. The authors show that this test is consistent against a wide class of alternatives.

J. R. Blum (Albuquerque, N.M.)

Walsh, John E. 4638

★Handbook of nonparametric statistics. Investigation of randomness, moments, percentiles, and distributions.

D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1962. xxvi+549 *pp.* $15.00.

This is an impressive piece of work that will be even more so if the author is able to complete the second volume which will finish the task he has set for himself. This task was to cover thoroughly in handbook form, for reference but more especially for applications, nonparametric methods in statistics as developed up to 1958. The author is a well-known specialist in this field, to which he has made numerous and substantial contributions, and is therefore one of the best-qualified persons for this undertaking. But that he has proceeded so far and done it so well puts the statistical world in debt to him.

He does not attempt to give a precise definition of "nonparametric" (none has general acceptance). His criteria for inclusion of a procedure are that "it has properties which are satisfied to a reasonable approximation when some assumptions that are at least of a moderately general nature hold", that it is based on probability considerations and that it has importance for applications. Thus, in spite of its extent, the coverage is not exhaustive; for a good many procedures only bibliographical reference is given.

Though this is a handbook, it is not going to be readily used by the statistically unsophisticated. Even potential users who are well trained in mathematical statistics will find that they will need to give some time to becoming acquainted with the notation and the format used for the condensed presentation of procedures, both introduced for the purpose of keeping this book to a usable size. These are discussed in Chapters 2 and 3 and are summarized for ready reference on the inside covers. In addition, Chapter 4 is devoted to a general discussion of concepts, procedures and special terminology relevant to nonparametric statistics.

The remainder of the book, Chapters 5 through 11, is devoted to the description and discussion of specific procedures. Each chapter is concerned with a general class which is divided into subclasses within the chapter. Each chapter begins with a fairly lengthy discussion of the type of procedure, which is a bit repetitious among chapters and even sometimes within chapters. However, as a handbook, it is a useful feature that a user interested in a particular type of procedure be spared from reading too much of the whole book to get the information he wants. For each subclass, there is a brief discussion followed by a statement of the material that is common to all of the procedures in the subclass and will not be repeated in the description of each. Then for each procedure, in the concise format mentioned above, information is given under six headings: Data (type of observations used, operations performed preliminary to obtaining data and/or special features of data, notational statement, restrictions), Description (type of procedure or situation, population properties investigated), Assumptions (independence, randomness, qualitative and quantitative restrictions), Results (supplementary operations and special notation, statement of procedure, relevant tables and charts, supplementary information), Characteristics (level of computational effort required, efficiency and consistency properties, extent of symmetrical use of observations, sensitivity of procedure to the assumptions, difficulty of verifying the assumptions, additional remarks especially concerning practical usefulness) and Bibliography. The author remarks that under the third to the fifth of these headings he has often filled in gaps in the literature.

The headings for Chapters 5 through 11 are: Tests of randomness, Tchebycheff type inequalities, Estimates and tests for expected values, Estimates and tests for population percentiles, Distribution-free tolerance regions, Nonsequential results for distributions from ungrouped data, Sequential, decision, and categorical data results for distributions. To list the subclasses included within each chapter would require too much space. For example, under Chapter 5 one finds: runs above and below median, runs up and down, correlation coefficient statistics, statistics based on signs of differences, most powerful tests (specific parametric alternatives), sequential tests (bibliography), multivariate tests of randomness property (bibliography), relative efficiency summary.

An especially valuable feature is the extensive bibliography (602 items) at the end of the book. The projected second volume is "expected to contain material that is concerned with the two-sample problem, the several-sample problem, analysis of variance, regression and discrimination, multivariate analysis, matching and comparison problems, and tests of symmetry and extreme observations". *C. C. Craig* (Ann Arbor, Mich.)

d'Herbemont, G. 4639

Considérations géométriques sur les plans d'expérimentation. (English summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 145–154.

Linear models in the design of experiments are considered as subspaces of an N-dimensional vector space, N being the number of experiments. Assuming a linear model A, the least squares estimate of $M=E(X)$, the mean value of a vector X of observed results, is the orthogonal projection of X on the subspace A.

The author discusses various geometric properties of models in experimental design, such as orthogonality. The general structure of related analyses of variance is considered, and necessary and sufficient conditions given for the orthogonality of certain designs. *J. Gani* (E. Lansing, Mich.)

Hinkelmann, Klaus 4640

Extended group divisible partially balanced incomplete block designs.

Ann. Math. Statist. **35** (1964), 681–695.

In this paper the author investigates an m-associate class PBIB design which is called an extended group divisible PBIB design (EGD/m-PBIB). The definition and parameters of this design are presented and the uniqueness of its association scheme proved. For a design with incidence matrix N, the properties of NN' are explored. The eigenvalues of NN', its determinant and its Hasse-Minkowski invariants are obtained, and non-existence theorems presented. Examples illustrating the non-existence theorems and an example of an EGD/m-PBIB plan are given. *S. Addelman* (Durham, N.C.)

Mann, H. B. 4641

Balanced incomplete block designs and Abelian difference sets.

Illinois J. Math. **8** (1964), 252–261.

Let $g_1, \cdots, g_k$ be k distinct elements of a group G. If the elements $g_i g_j^{-1}$ $(i \neq j)$ represent every element of the group exactly λ times, the set (g_i) is said to be a difference set of order λ for the group G, and the sum of the permutation matrices representing the g_i is the incidence matrix of a balanced incomplete block design. The author gives a number of new results in the theory of Abelian difference sets and uses these to settle the existence of BIB's for a number of previously undetermined cases. *P. A. P. Moran* (Canberra)

Raghavarao, Damaraju 4642

Singular weighing designs.

Ann. Math. Statist. **35** (1964), 673–680.

If X is a matrix whose elements x_{ij} are $+1$, -1, 0 according as the jth object to be weighed is included in the ith weighing in the left-hand pan, the right-hand pan or not at all, the weighing design is defined to be singular if $S=X'X$ is singular. For such weighing designs, and under the assumptions of unbiasedness and homogeneity of variance, the objects which can have their weight estimated are determined. A general result is given for the further weighings required to estimate the weigh[illegible] [illegible] objects, and, in the particular case when th[illegible]

is one less than the number of objects, the additional weighing which gives most efficient estimates is derived.
R. M. Cormack (Old Aberdeen)

Seber, G. A. F. 4643
Orthogonality in analysis of variance.
Ann. Math. Statist. **35** (1964), 705–710.
From the introduction and conclusion: "In this paper we shall derive necessary and sufficient conditions for a general p-factor analysis of variance, with unequal observations per cell, to be orthogonal.... The literature on optimum designs is considerable and various criteria for optimality have been suggested. However, one feels intuitively that the role of orthogonality plays an important part in the structure of optimum designs...."
{In fact, the role of orthogonality in optimum design theory has been proved in the references cited by the author; the point, though, is that in most settings orthogonal designs do not exist, so that only a small part of optimum design characterization is settled by them.}
J. Kiefer (Ithaca, N.Y.)

Kitagawa, Tosio 4644
A mathematical formulation of the evolutionary operation programs. (French summary)
Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 293–309.
From the author's introduction: "The purpose of this paper is to give a mathematical formulation for the evolutionary operation programs (EVOP) developed by G. E. Box and his school and to discuss the statistical decision aspects associated with the EVOP under this formulation." *S. R. Searle* (Ithaca, N.Y.)

Kuperman, A. M. 4645
Application of statistical decision theory to certain problems related to level quantization. (Russian. English summary)
Avtomat. i Telemeh. **24** (1963), 1685–1691.
Previous work has considered special cases of level quantization when noise is taken into account. The present paper gives a general solution of the problem of optimal level quantization, taking noise into account, in the case of automatic measurements or of introduction of information into the controlling digital computer. The optimum positions of the quantization levels in analog-to-digital conversion are found, noise being taken into account. The analysis is based on statistical decision theory.
N. H. Choksy (Silver Spring, Md.)

Neyman, J. 4646
Two breakthroughs in the theory of statistical decision making. (French summary)
Rev. Inst. Internat. Statist. **30** (1962), 11–27.
Author's summary: "Dans ses deux papiers, l'un publié en 1951 et l'autre en 1956 [Proc. Second Berkeley Sympos. Math. Statist. and Probability, 1950, pp. 131–148, Univ. California Press, Berkeley, Calif., 1951; MR **13**, 480; Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 920–923; MR **18**, 606], Herbert Robbins a formulé deux catégories nouvelles de problèmes statistiques. Aussi il a donné des solutions de ces problèmes dans certains cas particuliers. Il semble que les deux catégories de problèmes auront une grande importance pour les applications. La première catégorie concerne le fameux problème de Bayes où les conditions impliquent qu'un paramètre θ à estimer est une variable aléatoire dont on ignore la distribution. M. Robbins indique une méthode d'utilisation des observations antérieures pour estimer l'estimateur de θ qu'on pourrait calculer si on connaissait la distribution a priori de θ. La deuxième catégorie de problèmes concerne le cas où on a à tester un grand nombre d'hypothèses statistiques $H_1, H_2, \cdots, H_n$ du même type. M. Robbins montre que, s'il est permis de faire le test d'une hypothèse H_i dépendre des observations relatives à toutes les hypothèses $H_1, H_2, \cdots, H_n$, même si ces hypothèses n'ont pas de relations logiques, la fréquence totale des erreurs peut être diminuée."

Guttman, Irwin; Tiao, George C. 4647
A Bayesian approach to some best population problems.
Ann. Math. Statist. **35** (1964), 825–835.
Consider a collection of k populations $\Pi_1, \cdots, \Pi_k$ where Π_i has probability density function $f(y|\theta_i)$. Associated with the population is a utility function $U(\theta_i)$. The problem is to choose the index i that maximizes $U(\theta_i)$ on the basis of sample observations from each of the populations Π_i. The approach is Bayesian in that it is assumed that the a priori distribution function of $\theta_i, \cdots, \theta_k$ is known. The authors first consider the utility function

$$U(\theta) = \int_{a_1}^{a_2} f(y|\theta)\, dy$$

for specified a_1 and a_2 and work out specific procedures for the cases where y's are normal or exponential. The a priori distributions are those usually assumed for the parameters of these distributions [cf. Jeffreys, *Theory of probability*, 2nd ed., Clarendon, Oxford, 1948]. Thus for the normal distribution with parameters μ, σ it is assumed that, a priori, μ and $\log \sigma$ are independent and locally uniformly distributed.
Also discussed are some other utility functions for these distributions, including the normal case where $U(\mu, \sigma^2) = \mu$. The procedures derived are in general the intuitively obvious "best" procedures.
D. G. Chapman (Seattle, Wash.)

Kraft, Charles H.; van Eeden, Constance 4648
Bayesian bio-assay.
Ann. Math. Statist. **35** (1964), 886–890.
Let F be a distribution function and let $Y = (Y_1, \cdots, Y_k)$ be a set of k independent random variables, each of which is binomial with parameters $(n_i, F(t_i))$. The experimenter knows the n_i and the t_i, can observe the Y_i, and wants to make some inference about F. The authors consider a priori distributions on a nonparametric space of F's, and for loss $\int (F-G)^2\, dW$ (G the estimator, W a fixed d.f.) show that LeCam's conditions [same Ann. **26** (1955), 69–81; MR **16**, 730] for completeness of the closure of Bayes estimates, are satisfied. Explicit estimates are computed in a particular case. {Independent related but non-overlapping papers are those of Freedman [ibid. **34** (1963), 1386–1403; MR **28** #1706] and Dubins and Freedman [Bull. Amer. Math. Soc. **69** (1963), 548–551; MR **26** #7008].}
J. Kiefer (Ithaca, N.Y.)

Pratt, John W.; Raiffa, Howard; Schlaifer, Robert 4649

The foundations of decision under uncertainty: An elementary exposition.

J. Amer. Statist. Assoc. **59** (1964), 353–375.

Authors' summary: "Bayesian rules for decision under uncertainty are derived constructively from two principles of consistent behavior and two principles asserting that the decision maker can scale his preferences for consequences and judgments concerning unpredictable events by reference to simple lotteries involving only two consequences and based on an imaginary experiment with subjectively equally likely outcomes. It is shown that the two principles of consistent behavior require the decision maker's scaled judgments to obey the axioms of probability, and by use of one further principle of consistent behavior it is shown that they should also agree with the usual definition of conditional probability and hence with Bayes' rule."

Sakrison, David J. 4650

A continuous Kiefer-Wolfowitz procedure for random processes.

Ann. Math. Statist. **35** (1964), 590–599.

The author considers a procedure for finding the minimum of the regression function on a hypercube $X=\{x\}$, where the observable process $\{Y_t(x), t \geqq 0\}$ is of the form $\sum_1^N g_j(x)V_{j,t}$ with $V_{j,t}$ bounded and ergodic. The full statements of the theorems are too long to give here, but the technique consists roughly of replacing with a differential equation the usual difference equation of the discrete time case [see Blum, same Ann. **25** (1954), 737–744; MR **16**, 382; Sacks, ibid. **29** (1958), 373–405; MR **20** #4886]. The author proves convergence in mean of order 2 and, for a particular choice of the parameters of the method, obtains a bound on the rate of convergence which yields an asymptotic minimax result (the least favorable case in dimension one being one in which the method reduces to a Robbins-Monro process; see Chung [ibid. **25** (1954), 463–483; MR **16**, 272]). The method is motivated by considerations of implementation, and an application is given to an optimum filtering problem. {Other continuous-time methods have been considered by Driml and Hanš [Trans. 2nd Prague Conf. Information Theory, pp. 113–122, Publ. House Czechoslovak Acad. Sci., Prague, 1960; MR **24** #A562], Driml and Nedoma [ibid. pp. 145–158; MR **23** #A4171], and Hanš and Špaček [ibid. pp. 203–213; MR **23** #A1401].}
J. Kiefer (Ithaca, N.Y.)

Schwarz, Gideon 4651

Sequential life testing. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 91–97.

Author's summary: "La théorie des tests séquentiels d'hypothèses statistiques, telle que développée par Abraham Wald, fournit un procédé optimum explicite seulement quand les hypothèses sont 'simples' et quand les répétitions d'une seule expérience fixe sont la seule chose dont dispose l'expérimentateur. Pour le cas des problèmes où ces deux conditions ne sont pas satisfaites, nous devons recourir à des approximations asymptotiques. Dans le présent article, nous appliquons deux résultats asymptotiques à deux problèmes relatifs aux cas exponentiels. Premièrement, nous établissons une méthode de choix des niveaux d'effort optimum pour les tests séquentiels accélérés de survie. En cela nous suivons un procédé d'Herman Chernoff. Dans le second problème, nous appliquons notre propre 'Méthode des formes asymptotiques' à la recherche d'une règle optimum pour l'arrêt des observations dans les tests séquentiels des distributions exponentielles."
S. Kullback (Washington, D.C.)

Cronholm, James N. 4652

A two-variable generating function for computing the sampling probabilities of a class of widely used statistics.

J. Amer. Statist. Assoc. **59** (1964), 487–491.

Author's summary: "The purpose of this paper is to present a method of finding the exact sampling probabilities of a class of statistics widely used in the analysis of frequency data. A two-variable generating function is described from which the exact sampling probabilities of statistics of the form

$$y = D\left[\sum_{i=1}^{k} E(n_i)\right]$$

can be obtained. In this expression, k is the number of categories, n_i is the number of independent observations in the ith category, D is a one-to-one function of its argument, and E is a single-valued function of n_i. The information measure, $\hat{H}$, is representative of this class of statistics. It is anticipated that presently unavailable tables of sampling probabilities can be assembled using the proposed generating function in conjunction with a digital computer."

Paulson, Edward 4653

A sequential procedure for selecting the population with the largest mean from k normal populations.

Ann. Math. Statist. **35** (1964), 174–180.

From the author's summary: "In the paper sequential procedures are given for selecting the normal population with the greatest mean when (a) the k populations have a common known variance or (b) the k populations have a common but unknown variance, so that in each case the probability of making the correct selection exceeds a specified value when the greatest mean exceeds all other means by at least a specified amount."
J. Sacks (Evanston, Ill.)

Durbin, J. 4654

Trend elimination by moving-average and variate-difference filters. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 131–141.

Author's summary: "Le sujet de cet article est l'élimination de la tendance dans une série chronologique par la méthode des différences de variables et celle des moyennes mobiles, considérée du point de vue de l'analyse des résidus. Il y est démontré que les résultats obtenus par ces deux méthodes sont les mêmes que ceux obtenus en soustrayant des observations un polynôme d'un degré peu élevé. On y décrit comment les valeurs de ce polynôme sont calculées à partir des valeurs des différences au début de la période. Il y est démontré que, bien que la variation aléatoire des estimations de la tendance obtenues de cette façon soit élevée, l'erreur systématique des estimations du

périodogramme, due aux composantes de la tendance qui n'ont pas été éliminées, est peu importante sauf peut-être pour des fréquences basses."

Parzen, Emanuel 4655

Spectral analysis of asymptotically stationary time series. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 87–103.

Series $X(t)$ of random variables with uniformly bounded fourth moments are considered for which $R_T(v) = T^{-1}\int_0^{T-v} X(t)X(t+v)\,dt$ is well-defined as a limit in mean square of approximating sums and for which, in addition, $\lim_{T\to\infty} E\{|R_T(v)-R(v)|^2\}=0$ for some function $R(v)$, which is called the covariance function of the series. Conditions for this last condition to hold are derived in terms of the covariance function of the sequence $X(t)X(t+v)$ (v fixed). It is then shown that $R(v)$ is positive definite so that it may be represented as the Fourier transform of a spectral distribution function which may be interpreted as providing an analysis of variance. Examples of such "asymptotically stationary" series are given which are an amplitude modulated signal, a frequency modulated signal and an autoregressive time series (discrete time) which is not assumed to have been generated so long ago as to have yet reached a stationary state. *E. J. Hannan* (Canberra)

Hanš, Otto; Křepela, Josef 4656

Producer-consumer aspects of acceptance sampling by attributes.

Trans. Third Prague Conf. Information Theory, Statist. Decision Functions, Random Processes (Liblice, 1962), pp. 223–264. *Publ. House Czech. Acad. Sci., Prague,* 1964.

Authors' summary: "Dependence between producer's and consumer's acceptance sampling plans is studied and the results are used for the construction of these plans and evaluation of the protection guaranteed. Classical one-step distribution functions of the fraction defective are replaced by modern methods using arbitrary distribution functions. Two essentially different and in practice most frequently used systems of expedition are studied, the distribution functions of the fraction defective in production being arbitrary or limited by the condition, that the expected value of the fraction defective is smaller than or equal to an a priori chosen value."

Milder, D. Michael 4657

Regression with systematic noise.

J. Amer. Statist. Assoc. **59** (1964), 422–428.

Author's summary: "This paper identifies a type of highly correlated noise—systematic noise—that often arises in applications of linear regression to observations of spacecraft. An explicit formula is given for the inverse of the covariance matrix. The inverse matrix has a simple series expansion in ε^2, where $1/\varepsilon$ is the amplitude of the systematic noise. When the leading term, the so-called filter matrix, is used in place of the exact inverse to estimate the regression parameters, the systematic noise is eliminated. The filter matrix is shown to be optimum among all matrices that eliminate systematic noise from the estimates. An example is discussed."

Wilson, Edwin B. 4658

Comparative experiment and observed association. II.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 539–541.

Part I [same Proc. **51** (1964), 288–293; MR **28** #2625] was listed earlier.

NUMERICAL METHODS

See also 4095, 4098, 4101, 4150, 4152, 4190, 4248, 4260, 4261, 4315, 4316, 4366.

Faddeev, D. K. [Фаддеев, Д. К.]; Faddeeva, V. N. [Фаддеева, В. Н.] 4659

★Computational methods of linear algebra [Вычислительные методы линейной алгебры].

Second, augmented edition.

Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow-Leningrad, 1963. 734 *pp.* 2.45 *r.*

This is a second, slightly enlarged, and slightly revised edition, of which the first appeared in 1960. A translation of the first edition appeared in 1963 [Freeman, San Francisco, Calif., 1963; MR **28** #1742]. It is not to be confused with a book of the same title but by only the second of the present two authors; this appeared in 1950 [GITTL, Moscow, 1950; MR **13**, 872].

Contents of the first edition were summarized in the review of the translation. The present edition contains 60 pages more of text, for a total of 676 pages, and 18 pages more of bibliography, for a total of 58 pages. Four sections are added to the text, all in Chapter 8, "Methods of iteration for the solution of the complete problem of characteristic values". These sections discuss the QR algorithm of Kublanovskaja and of Francis; the characteristic vectors of AA'; the polar resolution of a matrix; and the use of a "spectral analysis" of the sequence of iterations as discussed by Lanczos.

In the bibliography of the first edition the non-Russian names were transliterated into Russian and listed accordingly. The original spelling was also given but often incorrectly. In the second edition the non-Russian items are given in a separate listing, without transliteration, and most of the errors are corrected, but not all (e.g., Misses, R. and Geringer, H.).

This is an excellent, very complete, and very up-to-date treatment of the subject, with adequate consideration of non-Russian as well as of Russian contributions.

A. S. Householder (Oak Ridge, Tenn.)

Nielsen, Kaj L. 4660

★Methods in numerical analysis.

Second edition.

The Macmillan Co., New York; Collier-Macmillan, Ltd., London, 1964. xv + 408 *pp.* $9.00.

The first edition of this by now well-known book appeared in 1956 and was reviewed at that time [MR **17**, 897]. The second edition preserves the lucid account of numerical analysis from an elementary and practical standpoint as given in the original book. The major changes are: The analysis of empirical data which was formerly in Chapters 8 and 9 has been incorporated into one chapter and reorganized with additional elementary examples. A short new chapter on linear programming has been added. The number of problems has also been extended. Some minor

changes for the benefit of greater clarity have been made throughout the text. Unfortunately, a few minor and obvious printing errors of the first edition were also carried over to this edition (e.g., on p. 224, equation (68.4) should read: $F(x^2)=a_0{}^2(x^2-x_1{}^2)(x^2-x_2{}^2)\cdots(x^2-x_n{}^2)$, etc.). *W. J. Kotzé* (Montreal, Que.)

Wilkinson, J. H. 4661

★Rounding errors in algebraic processes.

Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963. vi+161 *pp.* $6.00.

Although much of the material here contained has appeared previously in scattered publications, this is the first reasonably complete and general treatment of the subject in book form. Consideration is given to both fixed-point and floating-point computation. There are three chapters, the first developing the general methods for making the analysis, the second dealing with specific methods for finding the zeros of polynomials, and the third with matrix computations. A brief summary of the results is out of the question, and the book itself will be indispensable to the practicing programmer.

A. S. Householder (Oak Ridge, Tenn.)

Hull, T. E.; Dobell, A. R. 4662

Mixed congruential random number generators for binary machines.

J. Assoc. Comput. Mach. **11** (1964), 31–40.

Authors' summary: "Random number generators of the mixed congruential type have recently been proposed. They appear to have some advantages over those of the multiplicative type, except that their statistical behavior is unsatisfactory in some cases. It is shown theoretically that a certain class of these mixed generators should be expected to fail statistical tests for randomness. Extensive testing confirms this hypothesis and makes possible a more precise definition of the unsatisfactory class. It is concluded that the advantages of mixed generators can be realized in special circumstances. On machines with relatively short multiplication times the multiplicative generators are to be preferred."

Stockmal, Frank 4663

Calculations with pseudo-random numbers.

J. Assoc. Comput. Mach. **11** (1964), 41–52.

Author's summary: "Two pseudo-random number generators are considered, the multiplicative congruential method and the mixed congruential method. Some properties of the generated sequences are derived, and several algorithms are developed for the evaluation of $x_i=f(i)$ and $i=f^{-1}(x_i)$, where x_i is the ith element of a pseudo-random number sequence."

Šafiev, R. A. 4664

On some iteration processes. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 139–143.

Two iterative procedures for the approximate solution of the equation $P(x)=0$ in a Banach space which make use of the first and second Fréchet derivatives of $P(x)$ are discussed: the method of tangent hyperbolas

$$x_{n+1} = x_n - \theta_n G_n P(x_n), \quad n = 0, 1, 2, \cdots,$$

$$\theta_n = \{I - \tfrac{1}{2}G_n P''(x_n) G_n P(x_n)\}^{-1}, \quad G_n = \{P'(x_n)\}^{-1},$$

as generalized by Mertvecova [Dokl. Akad. Nauk SSSR **88** (1953), 611–614; MR **15**, 39], and Čebyšev's method

$$x_{n+1} = x_n - \{I + \tfrac{1}{2}G_n P''(x_n) G_n P(x_n)\} G_n P(x_n), \quad n = 0, 1, 2, \cdots,$$

as generalized by Nečepurenko [Uspehi Mat. Nauk **9** (1954), no. 2 (60), 163–170; MR **15**, 801]. On the basis of the assumption that the third Gâteaux derivative of $P(x)$ exists and is bounded in a sphere about the initial approximation x_0, sufficient conditions are derived for a solution x^* of $P(x)=0$ to exist, and for each method to give a sequence $\{x_n\}$ which converges to it. Inequalities of the form

$$\|x^* - x_n\| \leqq AB^n C^{3^n},$$

$B<1$, $C\leqq 1$, are derived for the rate of convergence. A, B, and C are expressed in terms of bounds for the operators involved, and the bound of the third derivative of $P(x)$. *L. B. Rall* (Madison, Wis.)

Nahon, Fernand 4665

Sur les résidus de la méthode des moindres carrés.

C. R. Acad. Sci. Paris **257** (1963), 2965–2967.

The first part of the paper "L'inégalité de Boulon" gives essentially the classical result that the second-order moment exceeds or equals the square of the mean, applied to the residuals of a regression analysis with the regressors $\cos l_k$ and $\sin l_k$ and no constant term. Thereafter follows a geometrical interpretation of the solution as a projection in n-dimensional space, first for the case where the constant term is excluded, referred to as an analysis of the first order, and thereafter for the case where a constant term appears, referred to as an analysis of the second order. Finally, there follow a few comments on the application of the method of least squares to the expansion in a Fourier series. *E. Lyttkens* (Uppsala)

de Boor, Carl; Rice, John R. 4666

Chebyshev approximation by $a\prod \frac{x-r_i}{x+s_i}$ and application to ADI iteration.

J. Soc. Indust. Appl. Math. **11** (1963), 159–169.

In determining optimal parameters for the alternating direction implicit method for solving large matrix equations (with which the reader of this paper is assumed to be familiar) the problem of finding the best uniform approximation to a given continuous function on an interval by rational functions of the type displayed in the title arises. Results are obtained on the existence, uniqueness and characterization of these best approximations. A method of computing the optimal parameters is presented and also some conclusions based on computational experience.

T. J. Rivlin (Yorktown Heights, N.Y.)

Kjaer, Viggo A. 4667a

Harmonics by torsional vibration.

Acta Polytech. Scand. Math. and Comput. Mach. Ser. No. 8 (1963), 1–18.

Andersen, Chr. 4667b

The ruler method. An examination of a method for numerical determination of Fourier coefficients.

Acta Polytech. Scand. Math. and Comput. Mach. Ser. No. 8 (1963), 19–73.

Authors' summary: "The first author above has developed a special set of rulers to determine the approximate values of the Fourier coefficients for a function which is given by its graph. The use of the rulers and some results obtained by means of them are treated in the introduction, 'Harmonics by torsional vibration', written by the first author. In the second paper the 'ruler method' is examined in detail. The relation between the coefficients obtained by the use of the ruler method and the Fourier coefficients is established, and the ruler method is compared with the method of Runge. A great part of the examination consists of a numerical investigation. The numerical experiments were carried out on the Danish computer DASK."

P. J. Davis (Providence, R.I.)

Ghinea, Monique 4668

Sur la résolution des équations opérationnelles dans les espaces de Banach.

C. R. Acad. Sci. Paris **258** (1964), 2966–2969.

The problem is to solve the equation $f(x)=0$ where the domain of the function is a Banach space X and its range is contained in a Banach space Y. A unique solution is obtained by the Newton method under hypotheses on the first and second Fréchet derivatives in the neighborhood of a point where $\|f(x)\|$ is small. Rapid convergence is obtained if higher-order derivatives exist. The proof depends on a generalization of Taylor's formula with remainder for $f(x)$. The iteration procedure is of the form $x \to \Phi(x)=x+\lambda(x)f(x)$, where the domain of $\lambda(x)$ is X and its range is contained in Y, and it requires that $\lambda(x)$ satisfy a Lipschitz condition.

L. de Branges (Lafayette, Ind.)

Guittet, Jack 4669

Méthodes de directions alternées à n dimensions avec opérateur de perturbation.

C. R. Acad. Sci. Paris **257** (1963), 3557–3559.

This paper gives alternating gradient methods for the solution of $Ax=k$, where $A=\sum_{i=1}^{q} A_i+P$; the A_i are Hermitian and such that $[A_i+r_i]^{-1}$ is known, and P is a "small" matrix. The method is a trivial extension of that of a previous paper [same C. R. **257** (1963), 3282–3285; MR **28** #1749] in which was studied the case $q=2$.

L. M. Delves (Kensington)

Kunert, F. 4670

Some iterative methods of finding the eigenvalues of self-adjoint operators. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 143–145.

The author proves the equivalence of the LV_2-process of M. K. Gavurin [same Ž. **1** (1961), 757–770; MR **24** #B1758] for finding an isolated eigenvalue of a self-adjoint operator and the iterative method of R. E. von Holdt [J. Assoc. Comput. Mach. **3** (1956), 223–238; MR **18**, 418].

P. J. Davis (Providence, R.I.)

Fujii, Masatomo 4671

Remarks to accelerated iterative processes for numerical solution of equations.

J. Sci. Hiroshima Univ. Ser. A-I Math. **27** (1963), 97–118.

The author describes seven iterative processes which might be used for the solution of equations which may be written in the form $x=\phi(x)$. He calls these the classical iterative (CI), the Aitken's predictive (AP), the Aitken's iterative (AI), the modified Aitken's predictive (MAP), the modified Aitken's iterative (MAI), the simplified Aitken's predictive (SAP) and the simplified Aitken's iterative (SAI) processes. An error analysis is applied to all these methods and various conclusions are drawn. For example, for the effective use of the AP process the initial CI process must not be continued too long; furthermore, when using either the AP or the AI process the iteration must not proceed too far or overflow will occur; on the contrary, when the MAP or the MAI are used, there is no such restriction. Finally, it is shown that the SAP and the SAI processes are both inferior to the MAP and the MAI but that when a parameter k is such that $k \ll 1$, this inferiority is very slight and the simplicity of the computation is a compensating advantage.

G. N. Lance (Canberra)

Górecki, H.; Turowicz, A. 4672

Sur les équations algébriques trinômes.

Ann. Polon. Math. **14** (1963/64), 335–341.

Solutions of the equation $x^n+ax+b=0$ (a, b real; $n \geqq 3$) are given in two special cases: (i) If $a<0$,

$$|b| < (-[(n-1)^{n-1}/n^n]a^n)^{1/(n-1)},$$

and if n is odd, there are three real solutions of our equation; (ii) is n is even and $b<(|(n-1)^{n-1}/n^n|a^n)^{1/(n-1)}$, there are two real solutions. In both cases a real solution x_1 is given by the series

$$x_1 = -\frac{b}{a}\sum_{\nu=0}^{\infty}\binom{n\nu}{\nu}\frac{(-1)^{n\nu}}{(n-1)\nu+1}q^{\nu}, \quad q=\frac{b^{n-1}}{a^n}$$

(the conditions (i) and (ii) are sufficient, for example, to ensure logarithmic convergence of the series). As a by-product the following identity is obtained:

$$\sum_{\nu=0}^{\infty}\binom{n\nu}{\nu}\left(\frac{(n-1)^{n-1}}{n^n}\right)^{\nu}\frac{1}{(n-1)\nu+1} = \frac{n}{n-1} \quad (n \geqq 2).$$

A nomogram for the solutions is contained in the paper for the values $n=2, 3, 4, 5, 6, 7, 8, 9, 10, 14, 15, 20, 25, 30, 35$.

H. Gross (Bozeman, Mont.)

Grau, A. A. 4673

On the reduction of number range in the use of the Graeffe process.

J. Assoc. Comput. Mach. **10** (1963), 538–544.

Author's summary: "The Graeffe method for finding the moduli of zeros of polynomials and Laurent series requires an expanding relative number range which imposes stringent limits on its use on an automatic computer. A revised algorithm, which successfully deals with the basic problem involved, and which has additional advantages, is developed."

Bauer, F. L. 4674

La méthode d'intégration numérique de Romberg.

Colloq. Analyse Numér. (*Mons*, 1961), *pp.* 119–129. *Librairie Universitaire, Louvain*, 1961.

In Romberg's method [Norske Vid. Selsk. Forh. (Trondheim) **28** (1955), 30–36; MR **17**, 538] numerical integration operators (functionals) of a high degree of precision are formed as linear combinations of trapezoid rule operators using various numbers of points. Letting $T_n^{(2)}$ be the n-point trapezoid rule operator for the interval $(0, 1)$, Romberg defined the higher operators recursively by

$$T_n^{(2m+2)} = \frac{2^{2m}T_{2n}^{(2m)} - T_n^{(2m)}}{2^{2m}-1}.$$

The author notes that this method is very well adapted to machine calculation, and develops error formulas for these operators. Letting $R_{n,2m}$ denote the error of $T_{n\cdot 2^m}^{(2m+2)}$, he shows that

$$R_{n,2m} = \frac{(-1)^{m+1}B_{2m+2}}{2^{m(m+1)}(2m+2)!}f^{(2m+2)}(\xi),$$

where B is the Bernoulli number. In particular, $T_{2^m}^{(2m+2)}$, which involves 2^m+1 evaluations of the integrand, has an error which is approximately

$$\frac{2}{2^{m(m+1)}(2\pi)^{2m+2}}f^{(2m+2)}(\xi).$$

The author further remarks that a similar sequence of integration operators with similar properties may be obtained by starting from the midpoint rule. He gives an ALGOL program for Romberg's method, and the results of some calculations with it. *S. Haber* (Washington, D.C.)

Laurent, P.-J. 4675

Évaluations d'intégrales par une méthode de Monte-Carlo. Application à la résolution d'équations intégrales.

Deux. Congr. Assoc. Française Calcul et Traitement Information (*Paris*, 1961), *pp.* 123–135. *Gauthier-Villars, Paris*, 1962.

J. M. Hammersley and K. W. Morton considered [Proc. Cambridge Philos. Soc. **52** (1956), 449–475; MR **18**, 336] the problem of estimating by Monte Carlo methods the value of a definite integral and for this purpose developed a correlated stratified sampling technique. They pointed out that if the integrand can be made to satisfy certain conditions, the efficiency of their method will be increased. J. H. Halton and D. C. Handscomb [J. Assoc. Comput. Mach. **4** (1957), 329–340; MR **20** #1400] then constructed transformations which will make any well-behaved function satisfy these conditions. In the present paper alternative transformations are suggested, and it seems that these transformations will require less calculations than those of Halton and Handscomb. The procedure can immediately be extended to multiple integrals over the same interval. The author also considers the application of this method to the solution of integral equations with a symmetric kernel. *W. J. Kotzé* (Montreal, Que.)

Salihov, G. N. 4676

Cubature formulae on the 4-sphere which are invariant under transformations of the finite rotation groups. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 6, 25–29.

Une formule de cubature

$$(l, f) = \int_S f(\theta, \varphi, \psi)\,dS - \sum C_k f(x^k) \simeq 0$$

sur la sphère de l'espace à quatre dimensions est invariante par rapport aux transformations d'un groupe G de rotations, si $(l, f(\theta', \varphi', \psi')) = (l, f(\theta, \varphi, \psi))$, où θ', φ', ψ' s'obtiennent de θ, φ, ψ par les substitutions de G. On démontre le théorème suivant. Afin qu'une formule de cubature invariante par rapport au groupe G soit exacte pour toutes les harmoniques sphériques d'un ordre donné, il est nécessaire et suffisant qu'elle soit exacte pour toutes les harmoniques invariantes $Y_n(\theta, \varphi, \psi)$, c'est-à-dire pour celles qui restent invariantes à toutes les rotations de la sphère qui appartient à G.

La méthode de démonstration est analogue à celle utilisée par S. L. Sobolev [Dokl. Akad. Nauk SSSR **146** (1962), 310–313; MR **25** #4635]. On étudie aussi le nombre $S(n)$ des harmoniques invariantes par rapport à G et spécialement quand G est le groupe des rotations G_V et G_{XVI}. *A. Haimovici* (Iași)

Stroud, A. H.; Secrest, Don 4677

Approximate integration formulas for certain spherically symmetric regions.

Math. Comp. **17** (1963), 105–135.

This paper contains approximate integration formulas for integrals of a function $f(x)$, $x = (x_1, \cdots, x_n)$, multiplied by a weight function $w(x)$ of the forms $w(x) = \exp[-x\cdot x]$ and $w(x) = \exp[-\surd(x\cdot x)]$ over all of E^n.

Formulas exact for polynomials of degrees 2, 3, 5 and 7 are given for both weight functions. These were derived by elementary methods described by the reviewer and A. H. Stroud [MTAC **12** (1958), 272–280; MR **21** #970].

For higher-degree formulas the authors resort to the spherical product methods developed by W. H. Peirce [ibid. **11** (1957), 244–249; MR **20** #430] and by R. G. Hetherington [Ph.D. Diss., Univ. Wisconsin, Madison, Wis., 1961]. Supplementary tables are presented to enable the calculation of integrals by the product method up to degree 37 and 41, respectively, for $n = 2, 3, 4$.

The authors also discuss the still unsatisfactory state of affairs of the relationship between orthogonal polynomials in several variables and integration formulas. *P. C. Hammer* (Madison, Wis.)

Zubrzycki, S. 4678

Some approximate integration formulas of statistical interest.

Colloq. Math. **11** (1963), 123–136.

The author determines c's and r's so as to minimize

$$\sup_{f\in H}\left|\int_0^1 f(r)\,dr - (c_1 f(r_1) + \cdots + c_n f(r_n))\right|.$$

H designates the classes of non-decreasing functions $f(x)$ defined for $0 \leq x \leq 1$, $0 \leq f(x) \leq K$, and such that (1) f is continuous at $x = 0$ and $x = 1$, or such that

$$(2) \qquad |f(x') - f(x'')| \leq M|x' - x''|.$$

The results are cast in a mixed language of numerical analysis and game theory. For example, let B be the set of vectors $(\mathbf{c}, \mathbf{x}) = (c_1, \cdots, c_n; x_1, \cdots, x_n)$ such that $c_1 + \cdots$

$+c_n=1$ and $0<x_1<x_2<\cdots<x_n<1$. The first theorem is as follows. Consider an estimation game between Nature, who chooses f from H (with the first alternative above) and the statistician, who chooses $(\mathbf{c}, \mathbf{x})$ from B. Let

$$r(f, (\mathbf{c}, \mathbf{x})) = \left| \int_0^1 f(x)\,dx - (c_1 f(x_1) + \cdots + c_n f(x_n)) \right|$$

be the payoff in this game. Then the unique minimax strategy of the statistician is the vector $(\mathbf{c}_0, \mathbf{x}_0)$, $\mathbf{c}_0 = (1/n, \cdots, 1/n)$, $\mathbf{x}_0 = (1/2n, 3/2n, \cdots, (2n-1)/2n)$, and the minimax risk of the statistician is given by

$$\inf_{(\mathbf{c},\mathbf{x})\in B} \sup_{f\in H} r(f, (\mathbf{c}, \mathbf{x})) = K/2n.$$

P. J. Davis (Providence, R.I.)

Dahlquist, Germund G. 4679

Stability questions for some numerical methods for ordinary differential equations.

Proc. Sympos. Appl. Math., Vol. XV, pp. 147–158. *Amer. Math. Soc., Providence, R.I.*, 1963.

This is an expository paper on the author's fundamental results concerning stability of difference approximations for ordinary differential equations. However, he announces also the following new theorem: Let C be the class of differential equations $dx/dt = gx$, $g = \text{const}$, $\operatorname{Re} g < 0$, and let D be the class of linear multistep difference approximations whose solutions converge to zero for $t \to \infty$ when applied to any differential equation of class C. Then the trapezoidal rule belongs to D and it has the smallest local discretization error of all methods in D. A proof of this theorem and the generalization to nonlinear systems can be found in the author's paper [Nordisk Tidskr. Informations-Behandling **3** (1963), 27–43].

H. O. Kreiss (New York)

Gragg, William B.; Stetter, Hans J. 4680

Generalized multistep predictor-corrector methods.

J. Assoc. Comput. Mach. **11** (1964), 188–209.

Authors' summary: "The order p which is obtainable with a stable k-step method in the numerical solution of $y' = f(x, y)$ is limited to $p = k+1$ by the theorems of Dahlquist. In the present paper the customary schemes are modified by including the value of the derivative at one 'nonstep point'; as usual, this value is gained from an explicit predictor. It is shown that the order of these generalized predictor-corrector methods is not subject to the above restrictions; stable k-step schemes with $p = 2k+2$ have been constructed for $k \leqq 4$. Furthermore, it is proved that methods of order p actually converge like h^p uniformly in a given interval of integration. Numerical examples give some first evidence of the power of the new methods."

Lapeyre, Renée 4681

Résolution numérique d'un problème différentiel particulier.

C. R. Acad. Sci. Paris **256** (1963), 1441–1443.

The author proposes a method, related to the Runge-Kutta methods and also to multiple-step methods, for numerically solving the equation $y'' = f(x)y$. He compares it to a Runge-Kutta method, using as test equation $y'' = y$. The paper contains some misprints.

S. Haber (Washington, D.C.)

Urabe, M. 4682

Numerical study of periodic solutions of van der Pol's equation. (Russian summary)

Qualitative methods in the theory of non-linear vibrations (Proc. Internat. Sympos. Non-linear Vibrations, Vol. II, 1961), *pp.* 367–376. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

Die van der Polsche Gleichung $d^2x/dt^2 - \lambda(1-x^2)dx/dt + x = 0$ (mit $\lambda > 0$) hat für jedes λ eine eindeutig bestimmte stabile periodische Lösung $x(t)$, die für kleine λ und asymptotisch für große λ untersucht worden war. Verfasser untersucht nun den Zwischenbereich für λ, indem er durch Einführung von $dx/dt = y$ die Differentialgleichung in ein System überführt und dieses sehr genau numerisch integriert, wobei er eine eigens hierfür neu aufgestellte Integrationsformel mit der Adamsschen Extrapolationsformel kombiniert. Bei von Null an wachsendem λ wächst die Amplitude von $x(t)$ zunächst monoton bis zum Maximalwert 2,0235 (für $\lambda = 3{,}2651$), um dann für weiter wachsendes λ wieder monoton abzunehmen. In der Diskussion erklärt V. V. Kazakevich (U.S.S.R.) die Ergebnisse von Urabe für hochinteressant und wichtig.

L. Collatz (Hamburg)

Wright, K. 4683

Chebyshev collocation methods for ordinary differential equations.

Comput. J. **6** (1963/64), 358–365.

The solution of a differential equation can often be conveniently determined in the form of a Chebyshev series. If the equation is nonlinear, the algebraic system defining the coefficients is nonlinear, and it is expedient to employ iterative methods. The author considers the following methods for the first-order equation $y' = f(x, y)$: (i) Picard iteration, $y'_{i+1} = f(x, y_i)$, as suggested by the reviewer and Norton [same J. **6** (1963/64), 88–92; MR **27** #5367], (ii) the linearization $y'_{i+1} = f(x, y_i) + (y_{i+1} - y_i)(\partial f/\partial y)_i$, and (iii) a Taylor series method. In each case the author uses essentially the "selected points" principle [C. Lanczos, *Applied analysis*, Prentice-Hall, Englewood Cliffs, 1956; MR **18**, 823]. A step-by-step approach is also considered, and its stability discussed.

As the numerical examples indicate, the methods are promising, though more work is needed to determine their value. Other related experiments have been made by Norton [Comput. J. **7** (1964/65), 76–85].

C. W. Clenshaw (Teddington)

Douglas, Jim, Jr.; Gunn, James E. 4684

Alternating direction methods for parabolic systems in m space variables.

J. Assoc. Comput. Mach. **9** (1962), 450–456.

This paper generalizes results obtained in a previous paper of J. Douglas, Jr. [Numer. Math. **4** (1962), 41–63; MR **24** #B2122], in which he treated alternating direction methods for a single parabolic equation in three space variables. The authors treat the system

$$A\Delta\bar{v} = \frac{\partial \bar{v}}{\partial t} + \bar{\varphi}(x_1, \cdots, x_m, t, \bar{v}),$$

where $\bar{v}$ and $\bar{\varphi}$ are vectors with M components, Δ is the Laplacian, and A is a constant positive definite matrix. The boundary values of $\bar{v}$ are specified on the faces of a rectangular parallelepiped.

If $u(x)$ is a function defined on a rectangular lattice, let

$$\nabla_i u = h_i^{-1}[u(x_1, \cdots, x_{i-1}, x_i + h_i, x_{i+1}, \cdots, x_m) - u(x_1, \cdots, x_m)],$$

where h_i is the mesh width in the direction of the ith coordinate axis. Also, let

$$\bar{\nabla}_i u = \nabla_i u(x_1, \cdots, x_{i-1}, x_i - h_i, x_{i+1}, \cdots, x_m).$$

If K is the mesh width in the t-direction, then the alternating direction method used is given by the following scheme:

$$K^{-1}[\bar{u}_{n+1}^{(1)} - \bar{u}_n] = A[D^{(1)}\bar{u}_n + \delta^{(1)}\bar{u}_{n+1}^{(1)}] - \bar{\varphi}_{n+1/2}(\bar{u}_n),$$

$$K^{-1}[\bar{u}_{n+1}^{(k)} - \bar{u}_n] = A[D^{(k)}\bar{u}_n + \sum_{j=1}^{k} \delta^{(j)}\bar{u}_{n+1}^{(j)}] - \bar{\varphi}_{n+1/2}(\bar{u}_n),$$

$$k = 2, \cdots, m,$$

$$\bar{u}_{n+1} = \bar{u}_{n+1}^{(m)}.$$

Here $D^{(k)} = \frac{1}{2}\sum_{j=1}^{k} \bar{\nabla}_j \nabla_j + \sum_{j=k+1}^{m} \bar{\nabla}_j \nabla_j$, $\delta^{(k)} = \frac{1}{2}\bar{\nabla}_k \nabla_k$, $\bar{u}_n = \bar{u}(x_1, \cdots, x_m, nK)$ and

$$\bar{\varphi}_{n+1/2}(\bar{u}_n) = \bar{\varphi}(x_1, \cdots, x_m, (n+\tfrac{1}{2})K, \bar{u}_n).$$

This scheme for obtaining $\bar{u}_{n+1}$ from $\bar{u}_n$ is written in a form suggestive of its origin. A simpler system for computational purposes is obtained by subtracting the $(k-1)$st equation from the kth.

Several results are obtained giving the order of convergence of $\bar{u}_n$ to v in the mean square sense, assuming $\bar{v}$ to be sufficiently smooth up to the boundary of the region. If $\bar{\varphi}$ is independent of $\bar{v}$, the error is $O(h_1^2 + \cdots + h_m^2 + K^2)$. If $\bar{\varphi}$ depends on $\bar{v}$, the error is $O(h_1^2 + \cdots + h_m^2 + K)$; however, the scheme may be modified to a predictor-corrector variation of the above system to obtain an error $O(h_1^2 + \cdots + h_m^2 + K^2)$, even when $\bar{\varphi}$ depends on $\bar{v}$.

B. Frank Jones, Jr. (Houston, Tex.)

Durand, Émile 4685

Détermination du paramètre de relaxation dans la méthode de Frankel et Young pour de petites valeurs de la maille.

C. R. Acad. Sci. Paris **258** (1964), 3165–3167.

The author specially considers the nine-point Laplace difference equation and connects the determination of the relaxation factor ω with the study of the equation

$$\Delta X = \sigma/16h^2\left[13\sigma + 40\frac{2-\omega}{\omega}\right]X.$$

By this method the author obtains the same values of ω that P. Garabedian had obtained [MTAC **10** (1956), 183–185; MR **19**, 583], but without appealing to any hyperbolic partial differential equation. The author mentions that the method may be applied to a large class of problems.

A. Dou (Madison, Wis.)

Engeli, Max 4686

★Automatisierte Behandlung elliptischer Randwertprobleme.

Eidgenössische Technische Hochschule in Zürich zur Erlangung der Würde eines Doktors der Mathematik.

Schmidberger & Müller, Zürich, 1962. 133 *pp.*

A computer program in ALGOL is given for approximating the solution of linear elliptic boundary-value problems by finite-difference methods in two dimensions. For convenience a square grid is chosen. The program generates the difference equation at each point by means of Taylor series expansions and allows higher-order local approximations if desired. Theorems are presented which concern the generation of the difference equations. The discretization error and the convergence of the overrelaxation scheme used for solving the linear system are mentioned but no proofs are given.

B. Hubbard (College Park, Md.)

Forrington, C. V. D. 4687

A Fourier series method for the numerical solution of a class of parabolic partial differential equations.

Comm. ACM **7** (1964), 179–181.

Author's summary: "A Fourier series method is described which, when applied to a certain class of parabolic partial differential equations, reduces the problem to a system of ordinary differential equations. An application is given for which the method shows a considerable advantage over conventional finite-difference methods."

Gagua, M. B.; Culadze, M. G. 4688

On the numerical solution of the Dirichlet problem. (Russian)

Soobšč. Akad. Nauk Gruzin. SSR **24** (1960), 513–518.

Prihod'ko, E. M. 4689

On the numerical solution of the fundamental biharmonic problem for a large number of mesh-points. (Russian)

Ukrain. Mat. Ž. **15** (1963), 214–217.

Let R with boundary S be a rectangle in the xy-plane and consider the boundary-value problem $\Delta^2 u = f$ in R with u, $\partial u/\partial \nu$ given on S, where ν is the interior normal of S. This problem is discretized by introducing a rectangular net consisting of $(m+2) \times (n+2)$ mesh-points, of which one layer is on S and one layer is outside R. For interior mesh-points, the ordinary thirteen-point approximation of $\Delta^2 u$ is used, and for mesh-points on S, $\partial u/\partial \nu$ is replaced by a symmetric difference quotient. An explicit solution of the discrete problem is found in the form of a sum. To determine the values of the parameters in this sum, it is necessary to solve two systems of linear algebraic equations with $m-2$ unknowns each.

V. Thomée (Göteborg)

Uno, Toshio 4690

Problems of error propagation. (Japanese)

Sûgaku **15** (1963), 30–40.

This is an expository paper concerning the propagation of errors. In each step of digital computation, we cannot avoid some inaccuracy due to truncation, rounding or losing of significant figures in subtraction. In this paper, the author illustrates by examples how the errors caused by such inaccuracy are propagated as the computation proceeds and what effect it gives to the final result. In connection with such propagation of errors, the well-known facts are illustrated by citing the results of the author's numerical experiments made on evaluation of functions by means of recurrence formulas, numerical solution of ordinary differential equations by means of step-by-step methods and numerical solution of linear simultaneous algebraic equations by means of elimination

methods. But the theories obtained by theoretical approach to such propagation of errors are not described in detail. *M. Urabe* (Madison, Wis.)

Smirnov, S. V. 4691
Sufficiency of the generalized Gronwall conditions for the nomographability of functions of several variables. (Russian)
Sibirsk. Mat. Ž. **5** (1964), 130–146.
The author considers the generalization of the well-known Gronwall criterion for the representability of an equation with three variables by an alignment nomogram.

The paper contains the full proof of the sufficiency of the generalized criterion. The author gives three basic definitions and seven lemmas on which the proof of sufficiency is based. In the last section he proves a theorem on the uniqueness of the solution of a problem for a certain system of partial differential equations.
D. Mazkewitsch (Knoxville, Tenn.)

COMPUTING MACHINES

See also 4645, 4662, 4960.

Fujino, Sei-ichi 4692
Theory of computer mechanisms. (Japanese)
Sûgaku **15** (1963), 12–21.
The paper contains the following chapters: Lee computer, its programming, computers with restricted program structures, modified Lee computer, computability, representability, Minsky's machine, the finite automaton and its representability, and the universal Lee program. As the names of these chapters show, it is an introductory paper to the theory of computer mechanisms recently developed by H. Wang, C. Y. Lee, M. Minsky, S. Ginsburg and the reviewer. The author writes it systematically in terms of the Lee computer and its program structure.
S. Huzino (Fukuoka)

Kimura, Izumi 4693
Theory of asynchronous switching. (Japanese)
Sûgaku **15** (1963), 21–29.
In this paper the author explains how to extract mathematical concepts from the problems of design of asynchronous switching circuits and presents an introductory study to the theory of asynchronous circuits, developed first by the Illinois group as represented by D. E. Muller. By giving several elementary examples of asynchronous circuits he shows some concepts necessary for studying the behaviour of asynchronous circuits, e.g., speed independence, semi-modularity, etc. At the same time, he indicates the difficult points of investigation appearing in the design and analysis of the circuits.
S. Huzino (Fukuoka)

Klokačev, I. V. 4694
A refinement of the normalized floating point number notation on digital computers. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 192–194.

Ljubčenko, I. S.; Maĭzlin, I. E. 4695
Mathematical simulation of a certain technological process on a computer. (Russian. English summary)
Vestnik Moskov. Univ. Ser. I Mat. Meh. **1963**, *no.* **2**, 37–43. (1 *insert*)
Authors' summary: "The paper, dealing with industrial cybernetics, gives the construction of an algorithm simulating a technological process on an electronic digital computer. The state of all units of the technological line is analysed at certain specified intervals of time Δt."

Schroeder, R. 4696
Input data source limitations for real-time operation of digital computers.
J. Assoc. Comput. Mach. **11** (1964), 152–158.
Author's summary: "The purpose of this paper is to consider the limiting conditions associated with the handling of many unrelated input data sources by a digital computer which utilizes binary internal logic. Equations are derived which relate the number of distinct input sources to their input frequency through inequalities which specify the limiting conditions, i.e., the maximum number of data sources for a given input data frequency. Inequalities are derived for both serial and parallel input data configurations. The limiting conditions for real-time queueing are derived. The modification of the above conditions when program operations are taken into account is examined, and the information content associated with the limiting conditions is discussed. The absolute limit on computer capacity is derived by considering the limit on the information which can be stored in a given computer. Finally, theoretical conclusions are applied to an exemplary model."

Shimizu, Tatsujiro; Sugibayashi, Masutaro; Morimoto, Hiroshi 4697
Experiments in solving arithmetic problems given by sentences by electronic computers. (Japanese)
Sûgaku **15** (1963), 55–62.
The authors report in this paper the results of experiments by electronic digital computers for solving arithmetic problems by inserting problem-sentences written in a natural, not artificial, language (in particular, in English) as data for computing. The details of the programs reported in the paper have appeared in Math. Japon. **8** (1963), 47–57 (by T. Shimizu and M. Sugibayashi) [MR **29** #1772]; ibid. **8** (1963), 59–79 (by H. Morimoto) [MR **29** #1771]. *S. Huzino* (Fukuoka)

GENERAL APPLIED MATHEMATICS

Macdonald, J. Ross; Barlow, Carl A., Jr. 4698
Relaxation, retardation, and superposition.
Rev. Modern Phys. **35** (1963), 940–946.
Authors' introduction: "A physical process governed by a linear differential equation may exhibit a single retardation of relaxation time or a distribution of such times. The usual distinction between retardation and relaxation systems is based on whether the physical response variable considered primary increases toward a final value upon application of step-function stimulation or decreases

and relaxes toward such a final equilibrium or steady state. For example, when a constant mechanical stress is applied at $t=0$, the non-instantaneous progression of the resulting strain toward a finite or infinite final value is a retardation process. Alternatively, when a constant strain is applied and maintained, the stress relaxes toward a zero or nonzero final value. Clearly, either process may be considered, and the roles of stimulus and response become interchanged on going from one to the other.

"The development of mathematical methods of describing temporal and/or frequency response of linear distributed dielectric or mechanical systems has progressed in a parallel but infrequently tangential fashion for a long time. Many developments have been carried over freely from one field to the other, but this process has been retarded by the multiplicities of different nomenclatures in the two fields. In an effort to bring the notation and methods of the two fields into closer agreement, in the present work we shall first compare equivalent quantities in the two areas using reasonably well-standardized notation. We shall then show how retardation and relaxation processes can be described by a single set of constitutive relations and shall finally discuss representations of the superposition principle valid when impulse functions occur under the integral sign."

N. H. Choksy (Silver Spring, Md.)

MECHANICS OF PARTICLES AND SYSTEMS

See also 4196, 4199, 4446, 4698.

Thomas, Johannes 4699

Zur Statik eines gewissen Federsystems im E_n.

Math. Nachr. **23** (1961), 185–195.

A system is studied which consists of a finite number of helical springs, each anchored at a different point, but all having one end point in common. The problem posed is to determine the locus of possible positions of the common endpoint when the force on the system is prescribed. Existence of solutions to the problem is demonstrated. It is shown that the solutions lie on a certain hypersphere in a euclidian space with an appropriate number of dimensions. A procedure is given for obtaining solutions. Some remarks follow on the kinematic problem.

B. Bernstein (Washington, D.C.)

Pust, L. [Půst, L.] 4700

Transition through the resonance zone in oscillatory mechanical systems, considering the influence of the vibrator. (Russian. English summary)

Applications of the methods of non-linear vibrations to the problems of physics and technology (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. III*, 1961), *pp.* 398–408. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

The author discusses a technique for analyzing the behavior in resonance zones of oscillatory systems containing a nonlinear spring and damping. The basic method is a modification of that of Krylov-Bogoliubov-Mitropolsky [N. N. Bogoliubov and Y. A. Mitropolski, *Asymptotic methods in the theory of non-linear oscillations*, Hindustan, Delhi, 1961; MR **25** #5242]. No proofs are given or theorems listed, but a number of examples are worked out in detail.

C. S. Coleman (Baltimore, Md.)

Gohman, A. V. 4701

A geometric interpretation of the motion of a mechanical system of variable mass. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 28–39.

A motion of a mechanical system with variable masses depending on time and on position is interpreted as a motion of a point, with unit mass, in a space which is a particular case of the so-called rheonomic space introduced by A. Wundheiler [Prace Mat. Fiz. **40** (1932), 97–142].

S. Drobot (Columbus, Ohio)

Haseltine, W. R. 4702

Existence theorems for nonlinear ballistics.

J. Soc. Indust. Appl. Math. **11** (1963), 553–563.

The author extends and makes rigorous certain results of C. Murphy. These results concern steady yawing motions in ballistic problems. The techniques involve applications of classical results of Poincaré and the Krylov-Bogolyubov techniques.

S. P. Diliberto (Berkeley, Calif.)

Haseltine, William R. 4703

Existence theorems for nonlinear ballistics. II. Effect of gravity.

J. Soc. Indust. Appl. Math. **11** (1963), 1071–1077.

This is a continuation of the author's program [#4702 above] of making (for the first time) a precise application of the existence theorems for periodic surfaces to problems in ballistics which were either treated at best heuristically or incompletely.

S. P. Diliberto (Berkeley, Calif.)

ELASTICITY, PLASTICITY

See also 4139, 4943.

Caĭ, I. P. 4704

Separation of variables in the equations of statics in elasticity theory. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. **1961**, *no.* 3, 22–33.

It is shown that separation of variables is possible for the Lamé equations of the title in Cartesian, cylindrical and spherical coordinates.

Hayart, René 4705

Une application simple des formules de Murnaghan généralisées.

C. R. Acad. Sci. Paris **258** (1964), 1712–1715.

The elastic potential non-linear theory of continuum bodies with couple stresses is considered in view of deducing the generalized Murnaghan equations. For constant body couples and for a particular form of the elastic potential the author gives an application concerning the rectangular parallelepiped and proves that the equilibrium is achieved for a determined distribution of couple stresses. Other papers on the extension of the non-linear asymmetric theory of couple stresses are not mentioned.

M. Mişicu (Bucharest)

Ruhadze, Ž. A. 4706

Boundary-value problems of elasticity for piecewise non-homogeneous orthotropic bodies. (Russian)

Soobšč. Akad. Nauk Gruzin. SSR **30** (1963), 11–18.

The boundary-value problem of elasticity for piecewise non-homogeneous orthotropic bodies is reduced to a system of singular integral equations to which the Fredholm alternative does apply. The method used here follows that of V. D. Kupradze [same Soobšč. **22** (1959), 129–136; MR **21** #3967; ibid. **22** (1959), 265–271; MR **22** #5210]. *S. Drobot* (Columbus, Ohio)

Spillers, William R. 4707

A shrink fit problem.

J. Math. and Phys. **43** (1964), 65–71.

The problem consists of determining the surface stress in an infinitely long circular cylinder with radial surface displacement prescribed arbitrarily over a finite central band and the radial surface stress prescribed zero outside this band. The surface shear stress is zero. The solution uses the Fourier transform technique of Sneddon to produce dual integral equations which are solved by series substitution. No discussion is given of convergence of the various integrals and sums involved. Numerical values of the surface pressure are presented graphically for several values of Poisson's ratio and show sharp decay away from the band. Behaviour for zero Poisson's ratio is noted as being almost identical to the plane strain case. The reviewer remarks that the theory predicts exact identity. *R. J. Knops* (Newcastle upon Tyne)

Aleksidze, M. A. 4708

The calculation of freely supported plates. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **32** (1963), 23–26.

Zonenašvili, I. A. 4709

A problem on the bending of an elastic plate. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **31** (1963), 23–30.

The author considers the problem of plate deflection in a three-connected region. The plate is clamped on the first internal contour, free on the second one, and simply supported on the external contour. The solution to the problem is obtained by the application of the theory of singular equations of the first kind by means of N. P. Vekua's method. *Witold Nowacki* (Warsaw)

Sarkisjan, V. S. 4710

On the solution to the problem of the bending of anisotropic (non-orthotropic) plates. (Russian. Armenian summary)

Akad. Nauk Armjan. SSR Dokl. **37** (1963), 131–136.

In the case of general rectilinear anisotropy the derivatives in the differential equation of plate deflection can be grouped with respect to the order of the derivative, namely, the even derivatives and the odd derivatives multiplied by a coefficient μ which depends on the coefficients of the flexural rigidity. If the rectilinear anisotropy does not vary too much from orthotropy, then μ is smaller than unity and the small-parameter method can be applied in order to solve the differential equation. The author gives the solution by means of this method for a rectangular plate loaded sinusoidally. *Witold Nowacki* (Warsaw)

Buchwald, V. T.; Davies, G. A. O. 4711

Plane elastostatic boundary value problems of doubly connected regions. I.

Quart. J. Mech. Appl. Math. **17** (1964), 1–15.

If one of the boundaries of a doubly-connected region is a circle or can be mapped onto a circle, then, by analytic continuation a combination of the complex potentials (in terms of which the stress can be expressed) is obtained which satisfies the conditions on the circular boundary exactly. The remaining potential is then found either exactly or approximately by considering the conditions on the other boundary.

The method is applied to find an infinite series solution for the concentric circular annulus. Collocation is used to solve numerically the case of an elliptic plate with unstressed periphery and containing a concentric circular hole under uniform hydrostatic pressure. Here convergence considerations impose limitations on the relative dimensions of the plate when the proposed method is to succeed. *L. M. Milne-Thomson* (Tucson, Ariz.)

Verma, P. D. S. 4712

Symmetrical expansion of a hollow spherical dielectric. (French, German, Italian, and Russian summaries)

Internat. J. Engrg. Sci. **2** (1964), 21–26.

Author's summary: "Eringen's theory of homogeneous, isotropic hyperelastic dielectrics is applied to the symmetric expansion of a thick spherical shell. Electrostatic field and the stress tensor are determined. The effect of polarization is to increase the radial stress."

Teters, G. A. 4713

The postcritical stage of shells under plastic deformations and combined load. (Russian. Latvian and English summaries)

Latvijas PSR Zinātņu Akad. Vēstis **1964**, *no.* 2, 61–66.

Author's summary: "The article discusses post-critical deflections of shells. The geometrical and physical nonlinearities are taken into account. The combined load problem is solved using the local plastic deformation theory. Certain conclusions, as regards the relationship between the increase of the plastic deformations and the critical stresses and load carrying capacity of shells, are presented."

Vodička, Václav 4714

Forced vibrations of a composite annular membrane. (German summary)

Österreich. Ing.-Arch. **18** (1964/65), 1–7.

Author's summary: "The problem of vibrations of composite membranes produced by space- and time-dependent transversal forces is treated for the special case of symmetrical vibrations of an annular membrane."

Green, W. A. 4715

The growth of plane discontinuities propagating into a homogeneously deformed elastic material.

Arch. Rational Mech. Anal. **16** (1964), 79–88.

The author proves the interesting result that, in general, plane acceleration discontinuities, propagating into an isotropic elastic material in a state of homogeneous deformation, either become infinite in a finite time or decay to zero in an infinite time. An exception to this occurs for waves travelling along a direction of principal strain. Then, discontinuities transverse to the front are unchanged. It is also shown that plane discontinuities of third or higher order travel with constant amplitude.

M. Hayes (Newcastle upon Tyne)

Herrmann, G. 4716

On variational principles in thermoelasticity and heat conduction.

Quart. Appl. Math. **21** (1963), 151–155.

Author's summary: "The variational principles for displacements, for stresses and for both displacements and stresses in isothermal elasticity are extended to the coupled processes of thermoelasticity and heat conduction in a three-dimensional, anisotropic body. The character of these principles is examined and it is established that in a stable system one is concerned with a minimum, a maximum and a stationary value problem, respectively."

FLUID MECHANICS, ACOUSTICS

See also 4139, 4313, 4771, 4772.

Berker, Ratip 4717

Intégration des équations du mouvement d'un fluide visqueux incompressible.

Handbuch der Physik, Bd. VIII/2, pp. 1–384. *Springer, Berlin,* 1963.

The author gives a lengthy and exhaustive survey of the problems of the motion of a viscous fluid which have received an exact solution. Boundary layer problems are excluded from this survey: they are treated in another volume of this encyclopaedia. Problems discussed by the author falls into two classes. First of all the exact solutions of the Navier-Stokes equations and secondly exact solutions of the equations of Stokes and Oseen. The account of the solutions of the Navier-Stokes equations includes both steady and unsteady motions, motion in a plane, axi-symmetric motion and general motion in three dimensions. There is a most valuable account of the existence and uniqueness of the Navier-Stokes equations following the work of Leray and of Finn. The author has made a thorough search of the literature and the reviewer was not able to think of any problem which is not discussed in this majestic account. *G. Temple* (Oxford)

Weissinger, Johannes 4718

Theorie des Tragflügels bei stationärer Bewegung in reibungslosen, inkompressiblen Medien.

Handbuch der Physik, Bd. VIII/2, pp. 385–437. *Springer, Berlin,* 1963.

This article gives an account of the theory of aerofoils in steady flight, including the effects of viscosity and of compressibility. The author gives a competent account of the two-dimensional profile theory and a very satisfactory description of the various methods which have been proposed for the approximate solution of the problem of wings of finite span. There is a brief account of comparisons with observation and a useful appendix of mathematical formulae. *G. Temple* (Oxford)

Lin, C. C.; Reid, W. H. 4719

Turbulent flow, theoretical aspects.

Handbuch der Physik, Bd. VIII/2, pp. 438–523. *Springer, Berlin,* 1963.

This most valuable summary of the theory of turbulent flow gives a brief historical introduction and then a very thorough account of the mathematical representation of turbulence. The third section on the general nature of homogeneous turbulence describes the theories of Kolmogoroff. There follows an account of the physical transfer theories due to Heisenberg and Kolmogoroff and Lin. Section 5 gives a careful discussion of a hypothesis of a normal joint-probability distribution for the velocity field, and this is followed by a discussion of other accounts of the quasi-Gaussian hypothesis. The final section gives an account of the theories of turbulent diffusion.

G. Temple (Oxford)

Corrsin, S. 4720

Turbulence: Experimental methods.

Handbuch der Physik, Bd. VIII/2, pp. 524–590. *Springer, Berlin,* 1963.

This article gives an account of some of the methods used for the experimental study of turbulence in incompressible flow, and is chiefly concerned with the hot-wire anemometer, the production of simple turbulent flows and the use of grids. The production of turbulent shear flows and finally the production of uniform flows are described in some detail. In discussing the theory of the hot-wire anemometer, the author considers both the steady-state response and the dynamic response and gives full details of the mathematical theory. *G. Temple* (Oxford)

Schaaf, S. A. 4721

Mechanics of rarefied gases.

Handbuch der Physik, Bd. VIII/2, pp. 591–624. *Springer, Berlin,* 1963.

The author is concerned with the theory of rarefied gases where the continuum hypothesis is no longer valid. He summarises the elements of the kinetic theory of gases, giving, in particular, a valuable discussion of the interaction of gas molecules and solid surfaces. In the region where intermolecular collisions can be included we have free molecular flow discussed on the basis of Maxwellian distribution. When the mean free path in the gas is comparable with the dimensions of the body moving through the gas we have the phenomenon of slip flow which is briefly discussed by the author in connection with the equation set up by Burnett. The author also gives a rapid discussion of the principal experimental results. *G. Temple* (Oxford)

Scheidegger, A. E. 4722

Hydrodynamics in porous media.

Handbuch der Physik, Bd. VIII/2, pp. 625–662. *Springer, Berlin*, 1963.

The mechanics of fluids through porous media is a comparatively recent development of hydrodynamics which has been extensively studied in Russia. The present article discusses the rheology of porous media and the fundamental principles of statics and dynamics in fluids in such media. Here Darcy's law is the simplifying generalisation and the limitations of this are discussed with some care. The article concludes with a study of multiple phase flow.
G. Temple (Oxford)

Lai, Wei 4723

Flow of an inviscid fluid past a sphere in a pipe.

J. Fluid Mech. **18** (1964), 587–594.

By assuming a vortex sheet over a segment of the diameter of a sphere it is found that flow patterns can be obtained for the inviscid, swirling, incompressible flow about a large sphere held on the axis of a circular pipe. Numerical calculations are presented for a nearly spherical body having a diameter one-half that of the circular pipe for various flow patterns including irrotational flow, swirling flow with constant axial and angular velocities far upstream, and rotational flow with a paraboloidal velocity distribution far upstream. The possible effect of downstream waves on the swirling flow is also indicated for a nearly spherical body having a diameter ratio of 0.4.
E. V. Laitone (Berkeley, Calif.)

Ellingsen, Torbjørn 4724

On periodic motions of an ideal fluid with an elastic boundary.

Geofys. Publ. Norske Vid.-Akad. Oslo **25**, *no.* 4, 19 *pp.* (1964).

Author's summary: "A two-layer model of an ideal fluid over an elastic bottom layer is considered. The boundary conditions are developed for simple harmonic wave motions superposed on an arbitrary linear fluid flow. The frequency equations are discussed for surface waves on a uniform stream and for perturbation of a flow with constant shear. Two response functions defined for the boundary are found useful for a graphical discussion of the frequency equations."

Germain, Jean-Pierre 4725

Sur la théorie non linéaire des ondes longues.

C. R. Acad. Sci. Paris **256** (1963), 4588–4590.

In this paper, the steady irrotational flow of a fluid with a horizontal base and a free surface is examined. Let y be the vertical distance; x be the horizontal distance; ε be a parameter; $\theta=\varepsilon x$; $\psi(x, y)$ be a stream function such that $\psi=0$ is the free surface in (x, y, z)-space and $\psi=\psi_1$ is the horizontal base. In the (x, y) flow region the components of the velocity $u(\psi, x)$, $v(\psi, x)$ satisfy two first-order partial differential equations. Further, the following boundary data are assumed: $v=0$ on the horizontal and free surfaces; u and v satisfy a first-order ordinary differential equation on the free surface [Dubreil-Jacotin, J. Math. Pures Appl. (9) **13** (1934), 217–291; Gouyon, Ann. Fac. Sci. Univ. Toulouse (4) **22** (1958), 1–55; MR **24** #B621]. After expanding u, v in two power series in ε, the multipliers $u_n(\psi, \theta)$, $v_n(\psi, \theta)$ of ε^n are studied. Since u_n, v_n satisfy the same conditions as u, v, it can be shown that u_0, v_0, u_1, v_1, v_2 correspond to uniform flow and $u_2=u_2(\theta)$. The crucial question is whether: (a) $du_2/d\theta=0$, or (b) $du_2/d\theta\neq 0$. In the first case the author is led to the first approximation of shallow water theory. The second case leads to the theory of long waves [Friedrichs and Hyers, Comm. Pure Appl. Math. **7** (1954), 517–550; MR **16**, 413; Littman, ibid. **10** (1957), 241–269; MR **19**, 487]. For this second case, the author states that the following results have been obtained: (1) u_2 is unique and can be determined so that it is periodic; (2) u_2 can be reduced to the Friedrichs and Hyers second approximation to the solitary wave; (3) if u_n, v_n $(n>2)$ are bounded for all x, $-\infty\leqq x\leqq\infty$, then a unique periodic solution exists; (4) the period is given in terms of ψ_1, u_0 and an elliptic integral; (5) $v_{2n}=u_{2n+1}=0$, $u_{2n}(\psi, \theta)$ and $v_{2n+1}(\psi, \theta)$ are even and odd functions, respectively, of θ (this is the extension to long waves of a property noted by Gouyon [loc. cit.]); (6) a symmetry property of the velocity vector is obtained (this property, according to the author, was postulated in the previously cited paper of Friedrichs and Hyers and that of Littman); (7) a recurrence formula, involving the solution of a linear non-homogeneous third-order ordinary differential equation with variable coefficients, is given for u_{2n}; (8) v_{2n+1} are polynomials of degree $2n-1$ in $(\psi-\psi_1)$ whose coefficients are polynomials of degree $n-1$ in $u_2(\theta)$ multiplied by $du_2/d\theta$; (9) similarly, u_{2n} are polynomials of degree $2n-2$ in $(\psi-\psi_1)$ whose coefficients are polynomials in $u_2(\theta)$; (10) the solitary wave can be determined as a special case of the above formulas. The convergence of the above series in ε^n remains to be determined.
N. Coburn (Ann Arbor, Mich.)

Germain, Jean-Pierre 4726

Sur la théorie formelle des houles longues rotationnelles.

C. R. Acad. Sci. Paris **258** (1964), 1148–1150.

The author continues his previous study of water waves [#4725]. By expanding the "external force", which is assumed to be analytic in a parameter ε, in terms of a power series in ε (with first term ε^2), the author is able to determine the components of the velocity vector through the terms of order ε^3. It is noted that: (1) the approximation scheme leads to a unique determination of the higher-order terms of the velocity components in terms of the lower-order terms; (2) the solution is periodic; (3) a criterion is given for the determination of the single standing wave; (4) an expression for the free surface (up to terms of order ε^4) is given. The convergence problem remains to be discussed.
N. Coburn (Ann Arbor, Mich.)

Gouyon, René 4727

Sur les définitions du clapotis.

C. R. Acad. Sci. Paris **258** (1964), 1720–1721.

L'auteur montre que le clapotis-cuve et le clapotis-période sont équivalents. Dans le clapotis-période la condition relative à la seconde paroi verticale est remplacée par une condition de périodicité.
J. P. Guiraud (Paris)

Miles, John W. 4728

Free-surface oscillations in a slowly rotating liquid.

J. Fluid Mech. **18** (1964), 187–194.

From the author's summary: "Free surface oscillations of a liquid relative to an equilibrium state of uniform rotation about the vertical of an axisymmetric container are considered for small values of $\alpha=\omega^2 a/g$, where ω is the angular velocity of rotation and a the cylinder radius. A variational approximation is used to obtain explicit results, with an error of $O(\alpha^2)$, for axisymmetric gravity and inertial waves in a flat-bottomed circular cylinder. The results are found to be in agreement with observations reported by Fultz. The first-order (in ω) effects of rotation on asymmetric waves in a circular cylinder also are determined." *O. M. Phillips* (Cambridge, England)

Barenblatt, G. I.; Chernyi, G. G. [**Černyĭ, G. G.**] 4729

On moment relations on surfaces of discontinuity in dissipative media.

Prikl. Mat. Meh. **27** (1963), 784–793 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1205–1218.

The actions necessary to maintain certain discontinuities in viscous flow are determined. For example, if an incompressible fluid has velocity u_1 $(y>0)$ and u_2 $(y<0)$, the discontinuity can be maintained by applying couples along $y=0$ of moment $\mu(u_1-u_2)$ per unit length. This result is obtained by integrating the first moment $y\mu\partial^2 u/\partial y^2$ of the viscous forces across the discontinuity. Higher moments like $y^{n-1}\partial^n u/\partial y^n$ can be involved for non-Newtonian fluids.

Other examples considered are a discontinuity in a compressible boundary layer and a decaying discontinuity that occurs when the viscous force is $\mu\partial^2 u/\partial y^2+\eta\partial^3 u/\partial y^2\partial t$. *C. R. Illingworth* (Manchester)

Bendor, Edgar 4730

Rarefied viscous flow near a sharp leading edge.

AIAA J. **1** (1963), 956–958.

This is an extension of Oguchi's investigation by means of boundary-layer theory [L. Talbot (Editor), *Rarefied gas dynamics*, Academic Press, New York, 1961, pp. 501–524; MR **27** #3179] of the 'wedge' of viscous flow between a flat plate in a hypersonic stream and its bow shock. Three terms of a series solution are determined, the first one being Oguchi's solution. *C. R. Illingworth* (Manchester)

Brenner, Howard 4731

Effect of finite boundaries on the Stokes resistance of an arbitrary particle. II. Asymmetrical orientations.

J. Fluid Mech. **18** (1964), 144–158.

Part I appeared in same J. **12** (1962), 35–48 [MR **25** #1738]. It is established that the formula for the hydrodynamic force on a particle of any shape moving with velocity **U** in a bounded fluid of viscosity μ is $-6\pi\mu c[\boldsymbol{\phi}_\infty^{-1}-(c/l)\mathbf{k}+o(c/l)]^{-1}\cdot\mathbf{U}$, the force in an unbounded fluid being $-6\pi\mu c\boldsymbol{\phi}_\infty\cdot\mathbf{U}$. Here, c and l are characteristic dimensions of the particle and the boundaries, respectively, $\boldsymbol{\phi}_\infty$ is a tensor that depends only on the shape of the particle and **k** is a tensor that depends only on the shape of the boundaries and the position of the 'centre' of the particle relative to the boundaries.

The formula is easily modified to include the effect of movement of the boundaries or of a net flow at infinity. It is applied to find the velocity with which a uniform circular disk descends under gravity towards an inclined plane. It is also used to discuss the motion of two particles (one acting as a boundary for the other) in an unbounded fluid.

The possibility of rotation of the particle is also discussed. *C. R. Illingworth* (Manchester)

Datta, Subhendu K. 4732

Flow formation in Couette motion of an elastico-viscous Maxwell fluid in the presence of a transverse magnetic field.

J. Phys. Soc. Japan **18** (1963), 1667–1671.

Author's summary: "The unsteady flow of an incompressible elastico-viscous Maxwell fluid between two parallel plates, the upper one of which is fixed and the lower one is started impulsively from rest, is studied. It is assumed that a magnetic field perpendicular to the plates is present, no external electric field is imposed and the magnetic Reynold's number is very small. Laplace transforms have been used to get the velocity field in the fluid and the skin friction at the lower plate. It is found that the elasticity of the fluid retards the fluid motion and increases the skin friction."

Datta, Sunil 4733

Steady rotation of magnetized sphere in viscous conducting fluid.

J. Phys. Soc. Japan **19** (1964), 392–396.

Author's summary: "The solution is obtained for the steady rotation of a magnetized sphere in an incompressible viscous conducting fluid. The rotation is supposed to be slow enough to justify the neglect of inertia terms. An account has been taken of the electric field induced by the motion, and expressions up to second order perturbations in the velocity field have been obtained."

Mishra, Shankar Prasad 4734

Shear flow of an elastico-viscous fluid past a flat plate with suction.

J. Indian Math. Soc. (*N.S.*) **27** (1963), 27–32.

An exact solution is given for the flow of an elastico-viscous fluid past an infinite flat plate with uniform suction. *L. J. Crane* (Dublin)

Srivastava, P. N. 4735

Propagation of small disturbances in a semi-infinite visco-elastic liquid due to the slow angular motion of a disc.

J. Sci. Engrg. Res. **7** (1963), 343–350.

Author's summary: "The propagation of small disturbances in a semi-infinite visco-elastic liquid due to the slow angular motion of a disc is discussed in two cases: (i) the angular velocity of the disc is given by $\omega(t)=\omega_0\delta(t)$ (Dirac delta function); (ii) $\omega(t)=\omega_0 \sin nt$. The solutions as λ (the relaxation time parameter) tends to zero are shown to correspond to those for the ordinary viscous fluid. It has been observed that, unlike the ordinary viscous fluid, the

velocity of propagation of the disturbances in case (i) is finite and the disturbances do not reach all points of the fluid instantaneously."

Sastri, K. S. 4736

Heat transfer in the laminar flow between converging walls with and without heat sources.

J. Sci. Engrg. Res. 7 (1963), 380–387.

Author's summary: "The effect of internal heat generation on the heat transfer in the flow of a viscous incompressible liquid between converging plane walls is analysed. The energy equation is solved by using the Ritz method and the qualitative distinction between the behaviours of the heat transfer across the wall for various Prandtl numbers, with and without heat sources, is established."

Stewartson, K.; Roberts, P. H. 4737

On the motion of a liquid in a spheroidal cavity of a precessing rigid body.

J. Fluid Mech. **17** (1963), 1–20.

An incompressible viscous fluid completely fills a spheroidal cavity in a rigid body and rotates with it about the axis of symmetry with constant angular velocity ω. At time $t=0$ the body begins to precess with small, constant angular velocity Ω about an axis inclined to the axis of rotation. (In a related investigation, Bondi and Lyttleton [Proc. Cambridge Philos. Soc. **49** (1953), 498–515; MR **14**, 1131] assumed steady motion and encountered certain contradictions.)

In the present study it is assumed that both $(a^2-b^2)\omega/\Omega a^2$ and $(a^2-b^2)\omega/\nu$ are large, where a and b are the semi-major and semi-minor axes of the spheroidal cavity and ν is the kinematic viscosity. The solution for impulsive start of precession begins with inviscid perturbation flow and a vortex sheet at the boundary; i.e., a boundary layer, which can be studied for small time when the inviscid solution is known. This study provides the values of the normal velocity component at the outer edge of the boundary layer (sometimes called the "displacement effect") which gives rise to an additional inviscid flow of order of magnitude $\nu^{1/2}$. This part of the flow is also calculated here.

It is concluded that after a time of order a^2/ν the motion is independent of the initial conditions and consists of solid-body rotation plus a steady flow with closed elliptical streamlines and a boundary layer. The pertinence of all this to the geophysical problem of the flow in the Earth's core and the origin of the magnetosphere is discussed. *W. R. Sears* (Ithaca, N.Y.)

Wadhwa, Y. D. 4738

Unsteady stagnation flow towards a rotating lamina.

J. Sci. Engrg. Res. 7 (1963), 259–266.

Author's summary: "The steady flow of a viscous incompressible fluid caused by the rotation of an infinite plate, together with a stagnation flow at infinity towards the centre of the plate was discussed by Hannah. The same problem was treated with the Kármán-Pohlhausen method by Schlichting and Truckenbrodt. The solution is given here when the rotation of the plate and also the stagnation flow at infinity are started either impulsively or with a uniform acceleration."

Benney, D. J. 4739

Finite amplitude effects in an unstable laminar boundary layer.

Phys. Fluids 7 (1964), 319–326.

Author's summary: "The mean flows induced by the interaction of a two- and a three-dimensional wave are calculated for a profile consisting of two straight lines. Although this is a simplified model of an actual boundary layer, the results are consistent with the experimental work done at the National Bureau of Standards." *L. J. Crane* (Dublin)

Kraichnan, Robert H. 4740

Relation between Lagrangian and Eulerian correlation times of a turbulent velocity field.

Phys. Fluids 7 (1964), 142–143.

D'après Corrsin, le temps d'amortissement de la corrélation au sens de Lagrange dans un écoulement turbulent isotrope devrait être moins grand que celui de la corrélation temporelle locale au sens d'Euler.

L'auteur propose des arguments en faveur d'une conclusion opposée. Il les justifie d'abord par un exemple schématique, inspiré de la dissymétrie de structure entre les champs permanents au sens d'Euler, et les champs permanents au sens de Lagrange. Il indique ensuite une technique de calcul qui permet de comparer les corrélations au sens de Lagrange et d'Euler et de montrer que les corrélations de Lagrange peuvent s'amortir plus rapidement en fonction du temps que les corrélations temporelles d'Euler. *J. Bass* (Paris)

Lumley, J. L. 4741

Spectral energy budget in wall turbulence.

Phys. Fluids 7 (1964), 190–196.

Using the ideas of Reynolds number similarity and self-preservation, and other simplifying hypotheses, the spectral energy equations are developed for mechanical and thermal wall turbulence. The author has given physical interpretations of the various terms occurring in these equations. *P. C. Jain* (Madison, Wis.)

Saffman, P. G. 4742

The amplification of a weak magnetic field by turbulent motion of a fluid of large conductivity.

J. Fluid Mech. **18** (1964), 449–465.

This paper describes a natural development of a theory proposed by the author [J. Fluid Mech. **16** (1963), 545–572] describing the interaction of a stationary homogeneous turbulent velocity field with a magnetic field that is both convected and diffused. The magnetic diffusivity λ is supposed small compared with the kinematic viscosity ν of the fluid. The critical assumption in both papers is that the amplification of the field is ultimately limited by Ohmic diffusion, and an equation (related to this assumption) is proposed to describe the development and decay of the magnetic spectrum at very high wave numbers. In the paper under review, this equation is generalised to cover the equilibrium range of wave numbers, and also to cover the circumstance that Lorentz forces may be non-negligible. A weak field is supposed to be maintained on a large length scale by electromotive forces, and the

steady magnetic spectrum that results through its interaction with the velocity field is derived. The extent of the effect of the Lorentz forces depends on the relative magnitudes of the Reynolds number and the large parameter ν/λ, as well as on the magnitude of the electromotive forces, and the various possibilities are classified and discussed. *H. K. Moffatt* (Cambridge, England)

Sedunov, Yu. S. [Sedunov, Ju. S.] 4743

A calculation of the relative velocities of Stokesian particles in turbulent flow.

Izv. Akad. Nauk SSSR Ser. Geofiz. **1963**, 1747–1753 (*Russian*); *translated as Bull.* (*Izv.*) *Acad. Sci. USSR Geophys. Ser.* **1963**, 1057–1061.

The equations governing the motion of a small particle relative to surrounding turbulent fluid were given by Tchen and revised by Corrsin and Lumley [Appl. Sci. Res. A **6** (1956), 114–116, p. 115; MR **18**, 694]. This paper presents an evaluation of the relative importance of the various effects involved, such as Stokes drag, the particle inertia, gravitational settling and fluid pressure gradients. In doing so, the author is enabled to impose limits on the particle size within which the Lagrangian velocity autocorrelation is independent of the particle radius. He concludes that for small particles with density of the order unity in air, the inertia of the particle is the most important cause for its failure to 'follow the fluid'.

O. M. Phillips (Cambridge, England)

Chao, B. T. 4744

Turbulent transport behavior of small particles in dilute suspension.

Österreich. Ing.-Arch. **18** (1964/65), 7–21.

Author's summary: "An analysis is made of the important parameters which govern the turbulent transport behavior of small particles in dilute suspension under restrictive conditions. The Lagrangian equation of particle motion is treated as a linear, stochastic integro-differential equation to which the Fourier transform is applied. Expressions are obtained which relate the intensity, energy spectrum, double velocity correlation coefficient, etc., of the two phases. A measure is proposed to describe the degree of departure of the particle motion from that of the fluid. It is demonstrated that, under simplifying conditions, the results agree with those of Soo and of Friedlander. The implications of those authors' assumptions are pointed out."

O'Brien, Edward E.; Pergament, Stuart 4745

Note on the unphysical spectral predictions of the joint normal distribution hypothesis.

Phys. Fluids **7** (1964), 609.

Quelques remarques sur les relations entre la structure du spectre de la turbulence, son évolution en fonction du temps, et les hypothèses de normalité des lois statistiques de la turbulence en deux points. *J. Bass* (Paris)

Miele, Angelo; Saaris, Gary R. 4746

On the optimum transversal contour of a body at hypersonic speeds. (German and French summaries)

Astronaut. Acta **9** (1963), 184–198.

The body is assumed to be slender with a shape of the form $r=(z/l)^m R(\theta)$. It is also assumed that the pressure coefficient is given by the Newtonian approximation and that the skin-friction drag coefficient varies like $(l/z)^\alpha$. Essentially $R(\theta)$ is determined to give minimum total drag for a body of given length and base area by an interesting application of the calculus of variations. In terms of a friction parameter, K_f, that is proportional to the cube root of the ratio of the average friction coefficient to the average thickness ratio, the possible shapes vary according as to whether $K_f>1$ or $K_f\leqq 1$. An interesting variety of shapes is found. *H. C. Levey* (Perth)

Sapunkov, Ia. G. [Sapunkov, Ja. G.] 4747

Hypersonic flow past a circular cone at an angle of attack.

Prikl. Mat. Meh. **27** (1963), 930–939 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1422–1436.

L'auteur utilise la méthode de Cheng [J. Fluid Mech. **12** (1962), 169–191; MR **26** #987] de développements en puissances de $\varepsilon=(\gamma-1)/(\gamma+1)$ et $\sigma=\sin\alpha/\sin\tau$ (α: angle d'incidence, τ: angle du cône), mais il la modifie pour avoir une représentation valable dans la couche tourbillonnaire au voisinage de la surface du cône. La modification consiste essentiellement à exprimer l'entropie à l'aide de θ (angle du rayon vecteur avec l'axe du cône) et de la variable intermédiaire ζ qui s'identifie avec $-\sin\omega$ (ω angle du plan méridien avec un plan méridien de référence) hors de la couche tourbillonnaire et que l'on perturbe à l'intérieur de celle-ci. L'auteur obtient ainsi pour la masse spécifique en particulier une approximation uniformément valable à $O\{(\varepsilon+\sigma)^3\}$. À cet ordre d'approximation la modification est sans effet sur la pression à la surface du corps. La méthode est étendue au cas d'un cône sous incidence arbitraire, avec le seul petit paramètre ε. *J. P. Guiraud* (Paris)

Peyret, Roger 4748

Sur la structure du choc lent dans un schéma à deux fluides avec dissipation.

C. R. Acad. Sci. Paris **258** (1964), 3178–3181.

In a previous note [same C. R. **258** (1964), 2973–2976; MR **29** #1854], the author discussed the structure of a slow hydromagnetic shock wave when there was no dissipation. In the present note, the effects of dissipation are considered. *R. M. Gundersen* (Milwaukee, Wis.)

Filippov, I. G. 4749

The theory of linear three-dimensional unsteady problems of diffraction and certain nonlinear problems.

Prikl. Mat. Meh. **27** (1963), 708–714 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1079–1090.

If one assumes the flow of a plane weak shock wave past an infinite cylinder to be irrotational and isentropic, the linear boundary-value problem referred to in the title of this paper is the following. Solve the linear time-dependent wave equation subject to a mixed boundary condition on an infinite cylinder and a shock condition. A similar problem which gives rise to a non-linear wave equation is also discussed. *A. E. Heins* (Ann Arbor, Mich.)

Thomas, T. Y. 4750

The perfect relativistic gas.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 363–367.

In this paper the author investigates the propagation of sonic disturbances in an ideal, perfect relativistic gas. A special form is assumed for the material energy tensor, which, when the gravitational field is sufficiently weak and the flow velocity is small compared to the velocity of light, implies the approximation of the relativistic equations by the Euler equations, with the energy equation replaced by the conservation of entropy equation for a polytropic gas. The author obtains an explicit formula for the velocity of propagation of weak disturbances in a relativistic gas in terms of the flow quantities. In the Newtonian limit this formula reduces to that of the well-known expression for the velocity of propagation of a sonic disturbance for a polytropic gas.

J. K. Thurber (New York)

★Physical acoustics: Principles and methods. 4751

Vol. I—Part A: Methods and devices.

Edited by Warren P. Mason.

Academic Press, New York-London, 1964. xiii + 515 *pp.* $18.00.

Those papers of mathematical interest will be reviewed individually.

Lyamshev, L. M. [Ljamšev, L. M.] 4752

Scattering of sound by a cylindrical shell in a moving medium.

Dokl. Akad. Nauk SSSR **152** (1963), 1339–1341 (*Russian*); *translated as Soviet Physics Dokl.* **8** (1964), 993–995.

The author considers the problem of the scattering of a plane sound wave by a cylinder when a finite length of the cylinder surface is free to move about its axis. The boundary-value problem is reduced to the solution of an infinite system of equations, and the first approximation in the iterative solution of these equations is found.

W. E. Williams (Liverpool)

Brenner, H.; Cox, R. G. 4753

The resistance to a particle of arbitrary shape in translational motion at small Reynolds numbers.

J. Fluid Mech. **17** (1963), 561–595.

The matching technique of Proudman and Pearson [same J. **2** (1957), 237–262; MR **19**, 201] and Kaplun and Lagerstrom [J. Math. Mech. **6** (1957), 585–593; MR **19**, 1004] is used to calculate the force on a particle of arbitrary shape in a uniform stream. The calculations are carried as far as terms in $R^2 \log R$.

The force is also derived to $O(R)$ from the classical Oseen equations, and the authors conclude that this force gives the correct drag for arbitrary bodies but in general gives an incorrect lift component. It is also shown that a formula for the force due to Brenner [J. Fluid Mech. **11** (1961), 604–610; MR **23** #B2707] and Chester [ibid. **13** (1962), 557–569; MR **26** #2094] for bodies with certain types of symmetry is incomplete for arbitrary bodies. A detailed investigation is made of the conditions under which this formula is correct. Finally the results are applied to the specific example of a slightly deformed sphere.

{The analysis uses an elaborate index notation and the indices sometimes go astray, usually in an obvious way.}

D. R. Breach (Toronto, Ont.)

Riley, N. 4754

Magnetohydrodynamic free convection.

J. Fluid Mech. **18** (1964), 577–586.

Author's summary: "The flow of an electrically conducting fluid up a hot vertical plate in the presence of a strong magnetic field normal to the plate is considered. A solution is developed based on the idea of matching 'outer' and 'inner' solutions in the moving layer of fluid. An approximate Pohlhausen method of solution is also given which yields results in fairly good agreement with the exact analysis."

R. Ballabh (Lucknow)

Karpman, V. I.; Sagdeev, R. Z. 4755

The stability of the structure of a shock-wave front moving across a magnetic field in a rarefied plasma.

Ž. Tehn. Fiz. **33** (1963), 805–814 (*Russian*); *translated as Soviet Physics Tech. Phys.* **8** (1964), 606–611.

According to the equations of ordinary magnetohydrodynamics, one may investigate the structure of a steady shock-wave in a rarefied plasma where the static pressure p is negligibly low as compared with the magnetic pressure: $p \ll H^2/8\pi$. The profile of the shock-wave is ordinary (aperiodic) due to the magnetic viscosity if the frequency of collision between electrons and ions denoted by ν is sufficiently higher than a critical frequency ν_{cr} defined by

$$\nu_{cr} = \frac{u}{a}\left(\frac{H_1}{6(H_2 - H_1)}\right)^{1/2},$$

where H_1 is the magnetic force before the shock, H_2 the force after the shock, u the gas velocity within the shock and a the "electron-scattering length" given by $a =$ (light speed)/(frequency of plasma oscillation due to electrons). If ν is not sufficiently higher than ν_{cr}, the front structure has an oscillatory profile which is given by

$$u - u_2 \sim \exp\left(-\frac{\nu}{u_\partial} x\right) \cos\left[\left\{\frac{1}{2}(1 - M^{-2})\right\}^{1/2} \frac{x}{a}\right],$$

where x is the coordinate along the flow and M the Mach number. For investigating the stability of the front structure in this case, the ordinary equations of magnetohydrodynamics do not seem valid, because the frequencies of the disturbances may be of the order of or greater than the ion Larmor frequency. The authors propose "magnetohydrodynamics with ion scattering" which differs from the usual magnetohydrodynamics by terms representing the scattering effect. For example, the effect results in an additional term $(\text{curl}\,\mathfrak{H}) \times \mathfrak{H}/4\pi$ in the ordinary Euler equation of the medium. Here $\mathfrak{H}$ is the total magnetic force including the disturbances. When $\nu < \nu_{cr}$, it is shown that the oscillatory profile of a shock-wave front is unstable because of the lack of sufficient energy decay. Oblique shock-waves are also considered briefly.

T. Koga (Raleigh, N.C.)

Shafranov, V. D. [Šafranov, V. D.] 4756

Equilibrium state of a toroidal pinch.

Ž. Tehn. Fiz. **33** (1963), 137–144 (*Russian*); *translated as Soviet Physics Tech. Phys.* **8** (1963), 99–103.

The author considers the stability of a thin toroidal pinch with finite conductivity. He disputes the conclusion, reached by Ju. V. Vandakurov [same Ž. **30** (1960), 1134–1136; MR **22** #9025; ibid. **31** (1961), 907–915; MR **25** #3670], that there is no equilibrium configuration for such a pinch. He goes on to construct an approximate equilibrium solution, by using the known solution for a cylindrical pinch. *F. D. Kahn* (Manchester)

Silven, Saul 4757

Application of operational methods to the analysis of uniform plasmas.

J. Mathematical Phys. **5** (1964), 557–560.

The derivation of the trajectories $\boldsymbol{\tau}(t)$ of electrons in a neutral gas, in the presence of a static homogeneous magnetic field (neglecting the motions of the ions, but including collision effects), is reduced here to the determination of a special tensor $\mathbf{s}$. The latter is defined by the equation of motion $d\boldsymbol{\tau}/dt = \mathbf{s}\boldsymbol{\tau}$ according to which the matrix expansion $e^{\mathbf{s}t} = \sum_{n=0}^{\infty} ((\mathbf{s}t)^n/n!)$ determines the trajectory if $\boldsymbol{\tau}(0)$ is known. The equation of motion under the influence of some electric field $\mathbf{E}$ then fixes the tensor $\mathbf{K}$ for the dielectric constant of the medium when the polarization of the latter (proportional to the electronic density) is taken into account. For time-harmonic situations this tensor (well-known in magnetio-ionic theories) here appears as the sum of three contributions connected with unit vectors parallel and perpendicular to the magnetic field. The motion of an electron in the absence of an external field is computed next. A very simple equation then results for the undamped motion which occurs in the absence of the collision effects.

H. Bremmer (Eindhoven)

Stewartson, K. 4758

On the Kutta-Joukowski condition in magneto-fluid dynamics.

Proc. Roy. Soc. Ser. A **277** (1964), 107–124.

This is another step in the author's investigations of steady, aligned-fields magnetohydrodynamic flows. The question arises: What is the correct condition to fix the value of the circulation in two-dimensional flow about an obstacle? The author seeks to obtain the answer by studying small-perturbation flow of a resistive, inviscid fluid, following Lary [J. Fluid Mech. **12** (1962), 209–226; MR **25** #4746]. Numerical solutions of the integral equation of this theory are obtained for flow about a flat plate at incidence, for various values of β, the ratio of magnetic to dynamic pressure of the undisturbed stream, and of R, the magnetic Reynolds number. Asymptotic expansions for $\alpha\to\infty$ are also obtained, where α denotes $|1-\beta|R$. Unfortunately, the integral equation solution is ambiguous (except at $\alpha=0$) as regards circulation, so that conclusions can only be based on plausibility of the results and comparison with the asymptotic results.

The results are summarized as follows. For $\beta=\frac{1}{2}$, the Kutta-Joukowski condition of finite pressure may be imposed at either leading or trailing edge; in either case the solution is rendered unique and appears to join to a corresponding asymptotic solution having the same property. However, if $\beta=2$ and the Kutta-Joukowski condition is imposed at the trailing edge, the leading-edge singularity appears to weaken with increasing α, and for large α the Kutta-Joukowski condition is satisfied by virtue of singular behavior of the pressure distribution near the trailing edge; in the limit $\alpha=\infty$ the condition has moved to the leading edge. If the condition is imposed at the leading edge, there are computational difficulties and the asymptotic solution does not seem to exist. Finally, for $\beta=1$ the solution requires no auxiliary condition, for it is analogous to a calculation by Oseen's approximation.

The conclusion would seem (to this reviewer) to be that in applying Lary's approximations the Kutta-Joukowski condition should always be applied at the trailing edge; the lift at positive incidence is then positive. In the limiting case of the ideal conductor, then, the condition should be imposed at the leading edge for $\beta>1$, as conjectured by Ring [ibid. **11** (1961), 417–439; MR **24** #B1084], by McCune and Resler [J. Aero/Space Sci. **27** (1960), 493–503; MR **23** #B1312], and by Lary.

The author does not discuss the physical mechanism by which the Kutta-Joukowski condition is actually achieved in a flow; it is presumably a process involving unsteady flow and boundary-layer separation. He does mention that the inviscid resistive "wake" in aligned-fields flow is forward-running when $\beta>1$. The relation of Lary's theory to boundary-layer theory is not completely clear, while the reviewer believes that the relation of ideal-fluid theory ($\alpha=\infty$) to boundary-layer theory is clear. This has been studied by the reviewer in a paper [Astronaut. Acta **7** (1961), 223–236; MR **26** #2131] in which the additional complicating effects of viscosity are also considered.

W. R. Sears (Ithaca, N.Y.)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 4169, 4314, 4656, 4742, 4929, 4976.

Jones, D. S. 4759

★The theory of electromagnetism.

International Series of Monographs on Pure and Applied Mathematics, Vol. 47. A Pergamon Press Book.

The Macmillan Co., New York, 1964. xvi+807 *pp.* \$15.00.

This book is divided into eleven chapters as follows: (1) The representation of the electromagnetic field; (2) The special theory of relativity; (3) Radiation; (4) Cavity resonators; (5) Theory of waveguides; (6) Refraction; (7) Surface waves; (8) Scattering by obstacles without edges; (9) Diffraction by obstacles with edges; (10) Aperiodic phenomena; (11) Miscellaneous topics.

The starting point of the book is Maxwell's equations and the first chapter is primarily concerned with the general properties of these equations. It includes most of what is generally regarded as the standard material in electromagnetic theory such as vector and scalar potentials, Hertz vectors, retarded potentials, Green's functions and tensors (for harmonic time dependence). There is also what appears to the reviewer to be an original derivation of a representation of the electromagnetic field in terms of two scalar functions. A clear, concise account is also presented of the macroscopic properties of dielectrics and ferrites. This first chapter also contains some of the basic mathematical apparatus generally used in electromagnetic theory and the particular topics treated are generalised

functions, curvilinear coordinates, special functions (Bessel, Legendre, Mathieu, parabolic cylinder, spheroidal and ellipsoidal functions).

The second chapter is concerned entirely with the fundamentals of the theory of special relativity and the electrodynamics of moving media. The necessary results and definitions of tensor calculus are presented in the initial part of the chapter.

Chapter 3 starts off with the classical theory of the Lienard-Wiechert potentials for a moving electron and the Čerenkov effect. The author then goes on to discuss the electromagnetic field produced by a time-dependent fixed charged distribution and the theory of linear antenna systems. The chapter concludes with a lucid account of the antenna boundary-value problem.

The main feature of the fourth chapter is, in the reviewer's opinion, a very lucid and concise account of the concepts of Lebesgue integration and Hilbert space and of the theory of linear operators. This theory is applied to derive general results concerning the eigenvalues of differential operators. Some of the typical eigenvalue problems occurring in cavity resonators are discussed and the chapter concludes with a discussion of the effect of imperfectly conducting walls, boundary perturbations and apertures on the eigenfrequencies of cavities.

The initial part of Chapter 5 is devoted to an exposition of the material generally found in university undergraduate courses on waveguide theory. The greater part of the chapter, however, is concerned with the more practical aspects of waveguide theory such as the construction of equivalent circuits. The methods available for calculating equivalent circuits are illustrated by the detailed solution of the particular problems of the inductive post and capacitive iris in a waveguide, the bifurcated waveguide and the waveguide of discontinuous cross-section. Problems involving dielectrics and ferrites in waveguides are also considered and the chapter concludes with a discussion of the radiation from waveguides and horns.

The chapter on refraction, in addition to treating refraction at a plane interface for both isotropic and anisotropic media, contains a detailed discussion of methods of treating the problem of propagation in an inhomogeneous medium. An account is given of the W.K.B. method, Langer's extension of this method, and Langer's method is illustrated by applying it to the determination of uniformly valid asymptotic expansions for Bessel functions of large order.

Chapter 7 is the shortest in the book and is concerned with the problems of the propagation of waves on the exterior of a cylinder and the excitation of surface waves on a plane surface.

A detailed account is given in Chapter 8 of some of the particular cases for which the problem of scattering of electromagnetic waves (assuming harmonic time-dependence) can be solved exactly. In particular, a detailed account is given of the determination of the low- and high-frequency solution for the problem of the scattering of a plane wave by a conducting circular or parabolic cylinder and by a conducting sphere. The approximate solution of scattering problems for arbitrary curved obstacles is also discussed in some detail.

A detailed account is given in Chapter 9 of the three main approaches that have to be employed in dealing with scattering problems involving edges, viz., transform techniques, separation of variables method and approximate methods. An account is presented of the basic theory of the Laplace transform and the Wiener-Hopf approach is applied to the half-plane problem and to the problem of radiation from a semi-infinite circular cylinder. The use of the Wiener-Hopf method in determining approximate solutions is illustrated by application to the problem of high-frequency scattering by a strip. The use of the Kontorovich-Lebedev transform in diffraction problems is also discussed. The method of separation of variables is illustrated by application to the problem of diffraction by a disc or a strip. The approximate methods available for low-frequency diffraction by a disc and a strip are considered and an account given of the approximate methods available for treating high-frequency diffraction by an aperture.

Chapter 10 examines certain particular features of the problem of electromagnetic wave propagation with arbitrary time dependence. The problem of refraction of a pulse at a plane interface is treated and a general solution presented for the initial-value problem for the wave equation in an arbitrary number of dimensions. A detailed account is given of the asymptotic evaluation of integrals of the type which occur in wave propagation theory and which cannot be evaluated in a closed form.

Chapter 11 is devoted to a consideration of various miscellaneous topics which are of relevance in the study of electromagnetic phenomena. The scope of this chapter seems best illustrated by some of the headings and subheadings: Theory of electrons, Fluid motion (equations of motion, shock waves, boundary layers), Magnetohydrodynamics (equations of motion, small-amplitude waves, shock waves, steady flow of a viscous fluid), Plasma dynamics (Boltzmann's equation, plasma waves).

The author states in his Preface that the book is intended to take the reader from Maxwell's equations to the frontiers of research, and the reviewer feels that this has been achieved to a very great extent and in a remarkably clear fashion. The scope of each chapter is extremely wide, research sources are given in some detail and particular attention paid to experimental work where relevant, and the exposition throughout is extremely clear. The only drawback is one which must inevitably arise with a project of this magnitude, namely, that most of the research work of the last two or three years is not discussed at all. The general standard of the printing is very high, though there are a (very) few places where the incorrect type seems to have been used.

This book is one which can be regarded as a valuable contribution to the literature on electromagnetic theory and one which will prove extremely valuable to both the research student and the experienced research worker. The author is to be complimented for both attempting such an ambitious project and for completing the task in such an admirable fashion. *W. E. Williams* (Liverpool)

Grinberg, È. Ja. 4760

On the determination of the properties of certain potential fields. (Russian)

Akad. Nauk Latv. SSR Trudy Inst. Fiz. **12** (1961), 147–154.

The author considers the problem of determining the force on an electrolyte due to a current field and a magnetic field. The electrodes and magnetic poles are strips bounded

by generators on a cylindrical surface. The field intensities are then perpendicular to the generator and the flow of electrolyte is parallel to the generator. The force on the electrolyte is proportional to the magnitude of the vector product of the fields. Assuming that the equipotential lines in a plane perpendicular to the generator form an orthogonal grid, the desired force can be expressed as the modulus of an analytic function. Numerical calculations are made for the case in which the boundary surface is a plane. *G. Johnson* (Houston, Tex.)

Hellman, O. 4761

A generalization of the equations of the electromagnetic field of a moving electron.

Proc. Phys. Soc. **83** (1964), 391–395.

The author replaces in Maxwell's equations for the four-potential, A_μ, the $\Box A_\mu$ term by the difference

$$\mathrm{Trace}[A_\mu(x+2l_0\gamma)-2A_\mu(x+l_0\gamma)+A_\mu(x)]/l_0^2,$$

and the $\partial A_\mu/\partial x_\mu$ term by $\mathrm{Trace}[\gamma_\mu A_\mu(x+l_0\gamma)]/l_0$; the γ_μ's are the Dirac matrices, and l_0 is a scalar characteristic length. The resulting equations for the field of a point charge have solutions with no singularities. Equating the energy content of the field generated by a point charge at rest with the rest energy of the charge, one finds the numerical value of l_0 to be of order 10^{-13} cm. (The author gives no physical reasons for the introduction of the Dirac matrices, or for the modifications presumably necessary if the field is generated by charged Bosons.)

N. L. Balazs (Stony Brook, N.Y.)

Jääskeläinen, Paavo 4762

On orthogonal functions in microwave theory. (Finnish)

Arkhimedes **1961**, *no.* 1, 10–17.

Keller, Joseph B. 4763

A theorem on the conductivity of a composite medium.

J. Mathematical Phys. **5** (1964), 548–549.

The author considers a system of identical parallel cylinders of any cross-section and of conductivity σ_2, arranged in a rectangular x-y lattice, and imbedded in a medium of conductivity σ_1. When an electric field E_x in the x-direction is applied to the medium, the resultant average current density will be in the x-direction also. The author defines the effective conductivity $\Sigma_x(\sigma_1, \sigma_2)$ of the system in the x-direction to be j_x/E_x, where j_x is the average current density.

Using the theory of harmonic functions, he shows that the effective conductivities Σ_x and Σ_y in the x- and y-directions are related by the formula:

$$\sigma_1/\Sigma_x(\sigma_1, \sigma_2) = \Sigma_y(\sigma_2, \sigma_1)/\sigma_2.$$

He considers special cases of this formula for square lattices with cylinders symmetric in the line $y=x$, as well as generalizations to a statistically homogeneous, isotropic random distribution of cylinders in a conducting medium.

O. Frink (University Park, Pa.)

Morgunov, B. I. 4764

On the motion of charged particles in a magnetic field. (Russian)

Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1964**, *no.* 2, 3–7.

Ostrovskii, L. A. [Ostrovskii, L. A.] 4765

Electromagnetic waves in nonlinear media with dispersion.

Ž. Tehn. Fiz. **33** (1963), 905–908 (*Russian*); *translated as Soviet Physics Tech. Phys.* **8** (1964), 679–681.

A discussion of the propagation of plane electromagnetic waves through a medium with a given non-linear connection between the transverse components $B_\perp$ and $H_\perp$ of the magnetic induction and field, respectively. The approximation of Maxwell's equations for the special case of quasi-monochromatic waves leads to two simultaneous partial differential equations for the dependence of the amplitudes B_0 and H_0, and that of the associated phase ψ, on the variables Z and t. The most general solution of this set of equations is obtained by assuming that the non-linear relation between $B_\perp$ and $H_\perp$ may also be applied to the corresponding amplitudes B_0 and H_0 (quasi-static approximation). The consequences for a weakly non-linear medium are discussed next. The paper is written in a very concise form. The reader should be warned about inaccuracies in the formulas presented.

H. Bremmer (Eindhoven)

Rohrlich, F. 4766

Solution of the classical electromagnetic self-energy problem.

Phys. Rev. Lett. **12** (1964), 375–377.

It is well known that the conventional versions of classical electrodynamics exhibit inconsistencies when these formalisms are applied to problems concerning radiation reaction. As a result a good deal of controversy has accumulated with respect to various remedial proposals: First of all, there have been arguments on the consistency and technical soundness of various first-aid devices; further, there have been spirited "frame-theoretical" disputations concerning the orientation and nature of possible remedies; and finally, there have been heated debates as to whether there is any use at all—within the framework of a classical approach—to bother with attempts at resuscitation.

The author of the present note is of the firm persuasion that a consistent classical theory is a most desirable perquisite to the construction of a consistent quantum electrodynamics. He has discussed these questions in a number of previous papers [e.g., Ann. Physics **13** (1961), 93–109; MR **26** #5847]; and in the present note gives an action integral, manifestly covariant, free of cut-offs and convergent, which leads to the non-runaway solutions of the Lorentz-Dirac equation. In line with some older developments [e.g., Fokker, Z. Physik **58** (1929), 386–393; Schönberg, Phys. Rev. (2) **69** (1946), 211–224; MR **8**, 428] these felicities have been secured through a judicious mixing of retarded and advanced solutions of Maxwell's equations. As noted in another place [MR **27** #3248: review of Prigogine and Henin, Physica **28** (1962), 667–688], this leaves ample ground for further contention.

T. Erber (Chicago, Ill.)

Sobel'man, I. I. [Sobel'man, I. I.]; Tyutin, I. V. [Tjutin, I. V.] 4767

Induced radiative processes in quantum and classical theories.

Uspehi Fiz. Nauk **79** (1963), 595–616 (*Russian*); *translated as Soviet Physics Uspekhi* **6** (1963), 267–278.

The authors attempt to develop the classical theory of radiative processes and bring out some specific features of the behaviour of classical systems as compared with quantum systems. An attempt is made for a general treatment of the classical theory of the interaction between the radiation field and non-linear oscillators with frequency depending on the energy. The authors conclude that under certain conditions classical systems can strengthen incident radiation instead of absorbing it. To the reviewer, it appears that the term separated out and called 'spontaneous emission' in the paper is nothing but the emission of radiation by virtue of the motion of the oscillator. *N. D. SenGupta* (Bombay)

Tai, C. T. 4768
On the electrodynamics in the presence of moving matter.
Proc. IEEE **52** (1964), 307–308.
This note considers two apparently different approaches in formulating the equations describing the electrodynamics in the presence of moving matter, the first being based upon the special theory of relativity. In the second approach (developed by L. J. Chu) two generalized current-density densities are introduced as the sources of the local fields, and then the currents due to the moving polarized and magnetized matter are identified. An interpretation is given of these two models, based upon the method of motional flux.
J. A. Morrison (Murray Hill, N.J.)

Tassie, L. J. 4769
Gauge invariance and localizability in electromagnetic theory.
Phys. Rev. (2) **133** (1964), B1351.
Two examples of classical theories of electromagnetic interaction are briefly stated. The first is the theory of McManus [Proc. Roy. Soc. London Ser. A **195** (1948), 323–336; MR **10**, 664] which is gauge-invariant and charge-conserving but non-local. The second is the Lyttleton-Bondi theory [ibid. **252** (1959), 313–333; MR **22** #10764] which is local but gauge-dependent and not charge-conserving. It is therefore concluded that there is no connection between gauge invariance and localizability as claimed by Aharonov and Bohm [Phys. Rev. (2) **130** (1963), 1625–1632; MR **27** #1164].
F. Rohrlich (Syracuse, N.Y.)

Kritikos, Haralambos N. 4770
Radiation from an aperture in an infinite conducting wedge.
IEEE Trans. Antennas and Propagation **AP-12** (1964), 96–101.
From the author's summary: "In this paper the problem of the far-field radiation of an infinite aperture occupying one side of an infinite conduction wedge is solved. For uniform illuminations, as well as for illuminations tapered to zero at the edges, it is shown that the radiated field can be effectively represented by line sources situated along the edge of the wedge. Also, for the special case of illuminations which taper to zero at the edges within a few wavelengths, it is shown that the shadow region radiation is proportional to the rate of taper."
A. E. Heins (Ann Arbor, Mich.)

Ang, D. D.; Knopoff, L. 4771
Diffraction of scalar elastic waves by a clamped finite strip.
Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 471–476.
As the authors state, their problem corresponds to that of two-dimensional diffraction of a plane acoustic wave obliquely incident on a rigid strip. The (known) integral equation is derived and solved in the low-frequency case to the first-order term only. The corresponding far-zone field is indicated.
{Reviewer's remarks: In equation (14), the first factor $\log z$ should be removed; in equation (22), the constant $\pi/2$ should read π; the equation immediately above equation (22) should read $f(t) = -\exp(i\alpha t)$. Further, various integrals should be taken in the sense of principal value.} *C. J. Bouwkamp* (Eindhoven)

Ang, D. D.; Knopoff, L. 4772
Diffraction of scalar elastic waves by a finite crack.
Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 593–598.
The problem treated is the same as that of the preceding paper [see #4771], except that now the strip is acoustically soft. Again, only first-order low-frequency results are indicated. *C. J. Bouwkamp* (Eindhoven)

Ivanov, E. A. 4773
On the diffraction of a plane wave by two parallel elliptic cylinders. (Russian)
Vesci Akad. Navuk BSSR Ser. Fiz.-Tèhn. Navuk **1963**, *no.* 4, 5–13.
The author approximates the solution obtained by him and A. M. Rodov in two previous papers [the author and A. M. Rodov, same Vesci **1960**, no. 2, 27–36; the author, ibid. **1962**, no. 1, 34–41]. This solution is the sum of four infinite series involving Mathieu functions. The coefficients are given by infinite linear systems of equations. The approximations hold provided $kl \gg 1$, where k is the wave number and l is the distance between the axes of the cylinders. *N. D. Kazarinoff* (Ann Arbor, Mich.)

Kravcov, V. V. 4774
On a method of solving a diffraction problem (the two-dimensional case). (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 354–358.
From the Kirchoff-Weber formula for the external field of a plane monochromatic wave incident upon a simply connected region in the plane bounded by a Ljapunov curve l, the author obtains a Fredholm equation of the first kind for each of three types of boundary condition: Dirichlet, Neumann and mixed. The solution is given explicit form as a convergent series by use of the addition theorem for the Hankel functions in the kernel. This series is used also as a basis for a second representation in terms of functions orthonormal on l and for an asymptotic representation in the case of large wave number.
R. N. Goss (San Diego, Calif.)

Volland, Hans 4775
Die Streumatrix der Ionosphäre. (English summary)
Arch. Elek. Übertr. **16** (1962), 328–334.

It is shown that, for very long electromagnetic waves, the ionosphere behaves like a microwave four-port (not necessarily symmetric). The scattering matrix of a horizontally stratified ionosphere is developed for incident plane waves, a general solution of the scattering matrix is found, and the reflection matrix is calculated for an ionosphere composed of an arbitrary number of homogeneous layers of finite thickness.

I. Stakgold (Evanston, Ill.)

Kacenelenbaum, B. Z. [Каценеленбаум, Б. З.] 4776

★Theory of irregular wave-guides with slowly varying parameters [Теория нерегулярных волноводов с медленно меняющимися параметрами].

Izdat. Akad. Nauk SSSR, Moscow, 1961. 216 *pp.* 0.96 *r.*

Klyatskin, I. G. [Kljackin, I. G.] 4777

An inconsistency in the antenna integral equation.

Dokl. Akad. Nauk SSSR **152** (1963), 1089–1091 (*Russian*); *translated as Soviet Physics Dokl.* **8** (1964), 1005–1006.

The author discusses how a contradiction in the "induced-emf" method should be corrected in connection with the antenna integral equation. However, the discussion is unsatisfactory in the mathematical sense. Furthermore, from the point of view of physics, the author seems not to be aware of many studies done along this line since 1950 [T. T. Wu and R. W. P. King, J. Appl. Phys. **30** (1959), 74–76; R. W. P. King, *Fundamental electromagnetic theory*, 2nd ed., Dover, New York, 1963; E. C. Jordan, *Electromagnetic waves and radiating systems*, Prentice-Hall, New York, 1950].

Y. Hayashi (Ann Arbor, Mich.)

Yeh, C. 4778

An application of Sommerfeld's complex-order wave-functions to an antenna problem.

J. Mathematical Phys. **5** (1964), 344–350.

The antenna radiation problem for a circular slot on a semi-infinite dielectric coated conducting cone is solved by separation of variables in the radial direction. This leads to an infinite set of equations for coefficients of the series solution. These equations are solved approximately by effectively truncating the system.

I. Kay (Ann Arbor, Mich.)

Boeri, Fernand; Sideriadès, Lefteri 4779

Résonance non linéaire, du type Duffing, d'un ensemble de deux circuits électriques couplés.

C. R. Acad. Sci. Paris **258** (1964), 3828–3831.

Authors' summary: "On applique la méthode du premier harmonique à l'étude du régime permanent de deux résonateurs série électriques, couplés par mutuelle inductance, dans l'hypothèse où le circuit primaire comporte une non linéarité du type Duffing. On montre la possibilité d'obtention de courbes de résonance avec un ou deux sauts, et l'on l'effectue une discussion suivant la valeur des paramètres."

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 4140, 4306, 4716, 4736.

Coleman, Bernard D.; Mizel, Victor J. 4780

Existence of caloric equations of state in thermodynamics.

J. Chem. Phys. **40** (1964), 1116–1125.

This paper concerns restrictions on the constitutive relations for the stress, heat flux, internal energy, and entropy as functions of the temperature, temperature gradient, specific volume, and velocity gradient that can be deduced from the second law of thermodynamics. It is shown, for example, that neither the internal energy nor the entropy density can depend on the temperature gradient or the velocity gradient. It is also shown that the rate of working of the extra stress and the inner products of the heat flux and the negative of the temperature gradient need not be separately positive when a continuum is deforming. The existence of a caloric equation of state is proved. Some special consequences of linearity are derived.

R. A. Toupin (Yorktown Heights, N.Y.)

Goldenberg, H. 4781

Derivation of three-dimensional from two-dimensional solutions of transient heat conduction problems.

Quart. J. Mech. Appl. Math. **16** (1963), 483–505.

Le résultat essentiel est le suivant: soit dans le cylindre $R(x, y) \leq 0$, $0 \leq z \leq l$, la solution $V(x, y, z, t)$ du problème de conduction transitoire, avec chauffage constant A dans le cylindre, pour la condition initiale $V(x, y, z, 0) = 0$ et les conditions aux limites: $\alpha_i \partial V_i/\partial n_i + \beta_i V = \gamma_i$ ($i = 0, 1, 2$ se rapportant aux extrémités et à la face latérale du cylindre). Alors $V(x, y, z, t)$ s'exprime par la formule

$$(1) \qquad V = \int_0^t \left(\Phi \frac{\partial f}{\partial t} + f_1 \frac{\partial \Phi_1}{\partial t}\right) dt$$

au moyen des solutions Φ, Φ_1, f, f_1, des problèmes aux limites à une et à deux dimensions suivants: $f(x, y, t)$—cylindre infini $R(x, y) \leq 0$, chauffage A, $f(x, y, 0) = 0$, $\alpha_2 \partial f/\partial n_2 + \beta_2 f = \gamma_2$ sur $R(x, y) = 0$. $\Phi(z, t)$—plaque $0 \leq z \leq l$, pas de chauffage, $\Phi(z, 0) = 1$, $\alpha_i \partial \Phi/\partial n_i + \beta_i \Phi = 0$ ($i = 0, 1$) sur $z = 0$ et $z = l$. $f_1(x, y, t)$—cylindre infini $R(x, y) \leq 0$, pas de chauffage. $f(x, y, 0) = 1$, $\alpha_2 \partial f_1/\partial n_2 + \beta_2 f_1 = 0$ sur $R(x, y) = 0$. $\Phi_1(z, t)$—plaque $0 \leq z \leq l$, pas de chauffage, $\Phi_1(z, 0) = 0$, $\alpha_i \partial \Phi_1/\partial n_i + \beta_i \Phi_1 = \gamma_i$ ($i = 0, 1$) sur $z = 0$ et $z = l$. Si $\gamma_0 = \gamma_1$ la solution se simplifie car $\partial \Phi_1/\partial t = 0$. La formule (1) est encore valable pour la région de la plaque extérieure au cylindre et elle est étendue à des cas où le chauffage est variable dans l'espace et dans le temps.

R. Gerber (Grenoble)

Vodička, Václav 4782

Steady temperature in a composite corner.

Acta Phys. Austriaca **17** (1963/64), 227–230.

Author's summary: "The present paper brings new results on the steady temperature distribution in a composite rectangular corner, a boundary of which is kept at the given temperature and the other face at zero. General results are applied to the special case of a homogeneous solid, which is of physical and technical interest."

QUANTUM MECHANICS

See also 4359, 4397, 4398, 4761, 4769.

Bethe, Hans A. 4783
★Intermediate quantum mechanics.
Notes by R. W. Jackiw.
W. A. Benjamin, Inc., New York-Amsterdam, 1964. x + 276 *pp.* $9.00.

From the author's preface: "This book is intended to serve as a text for a second course in quantum mechanics for graduate students in both theoretical and experimental physics."

It begins with a concise summary of non-relativistic perturbation and variation methods and a discussion of the constants of motion, spin and symmetry considerations. An unusual amount of space is devoted to calculation of two-electron atoms, to the self-consistent field method, the Thomas-Fermi statistical model, and to the theory of multiplets. A brief chapter on molecules follows, and the semi-classical theory of radiation is presented in concise but adequate detail. Collision theory is barely touched on. A short chapter on the Klein-Gordon equation precedes a more extensive treatment of the formal theory of the Dirac equation and of its solutions. The book ends with an introduction to field theory which includes the analytical mechanics of fields, second quantization of several matter fields, and the elements of quantum electrodynamics.

This volume originated as a set of notes by a student in the author's course, and this origin occasionally manifests itself by a rather low ratio of words to equations. This is true in spite of the fact that the author continually stresses the connection with experimental information and with the physical picture rather than the formal development of the theory. Indeed, there are many illuminating insights which are not always found in texts on this subject. As an example, there is an enlightening discussion of why the physical solutions to the Schrödinger equation for n identical particles should be either totally symmetric or antisymmetric, even though mathematically many other degenerate solutions of lower symmetry occur. There is at least one amusing, repeated misprint, when reference is made to the attempts to replace the probabilistic prediction of quantum mechanics by "casual" descriptions. There is nothing casual in the author's assignment that such hidden variables cannot have any observable consequences and are therefore empty.

H. Lustig (New York)

Baktavatsalou, M. 4784
Étude sur l'interprétation physique des opérateurs de la mécanique quantique relativiste des particules à spin.
Cahiers de Phys. **17** (1962/63), 265–303.

This is a careful review of the derivation of a Schrödinger equation from a relativistic wave equation and of the problem of definition of observables for such a quantum mechanical system, particularly the position operator. A detailed bibliography is given including some of the author's own contributions in the field. The author seems to be unaware of the elegant paper of Foldy [Phys. Rev. (2) **122** (1961), 275–288].

E. C. G. Sudarshan (Syracuse, N.Y.)

Baktavatsalou, M. 4785
Sur l'opérateur de position des particules chargées de spin 0 ou $\hbar/2$ placées dans un champ magnétique constant.
C. R. Acad. Sci. Paris **258** (1964), 474–476.

The author investigates the position operator for a spin $\frac{1}{2}$ particle in a constant magnetic field for the non-relativistic and extreme relativistic cases. Spin 0 particles are treated in the non-relativistic limit.

V. de Alfaro (Princeton, N.J.)

Bose, A. K. 4786
Solvable potentials.
Phys. Lett. **7** (1963), 245–246.

The problem of constructing potentials for which two solutions can be written down in "closed" form is studied with particular reference to the Riemann and the Whittaker equations. The method is that of suitable transformation of both the dependent and independent variables. A unified method of obtaining a class of solvable potentials (which includes most known solvable potentials) is thus obtained. *E. C. G. Sudarshan* (Syracuse, N.Y.)

Mano, Koichi 4787
Representations for the nonrelativistic Coulomb Green's function.
J. Mathematical Phys. **5** (1964), 505–508.

Author's summary: "The derivation of the representations for the non-relativistic Coulomb Green's function is discussed. It is shown that the recently published representations, both in integral and closed forms, can be obtained from an expression for the difference of the diverging and converging wave solutions of the Green's function which is given in terms of the wavefunctions summed by Gordon."

Mandel, L. 4788
Statistical properties of light and the Sudarshan representation.
Phys. Lett. **7** (1963), 117–119.

Glauber had pointed out [Phys. Rev. Lett. **10** (1963), 84–86] that the eigenstates of the annihilation operators of the various photon modes are useful in the quantum theory of partial coherence; this observation was used by the reviewer to transcribe classical coherence theory into a quantum theory. The present note shows how to compute the multivariate field amplitude distributions and applies it to an electromagnetic field at a fixed temperature.

E. C. G. Sudarshan (Syracuse, N.Y.)

Glauber, Roy J. 4789
Coherent and incoherent states of the radiation field.
Phys. Rev. (2) **131** (1963), 2766–2788.

This is an elegant and lucid presentation of the quantum theory of optical coherence. The pace is leisurely: the choice of mode functions and the quantum theory of a collection of harmonic oscillators are discussed. The eigenfunctions of the annihilation operators (the "coherent states") are introduced and shown to be relevant to quantum coherence theory. The properties of these states and the expansion of arbitrary states and arbitrary

operators in terms of these states are discussed at considerable length. States which are sums of projections to coherent states are studied and are referred to as the P-representation (though in previous literature the representation was called the "diagonal representation"). A heuristic discussion of the field density matrix is given, and applied to the study of correlation and coherence properties. The author definitely but simply disagrees with an existing demonstration of the equivalence between classical and quantum descriptions making use of the diagonal representation [the author, Phys. Rev. Lett. **10** (1963), 84–86]; it is not clear whether the author feels that the diagonal representation of the density operator is not generally valid, or that he disagrees with the terminology. There seems to be some lack of clarity, in contrast to the rest of the paper, about the relation between the demonstration of an equivalence and the approach to a limit. (Similarly, there is a curious reference to the Planck distribution when the Bose-Einstein distribution is discussed.) Despite these, this paper is a valuable survey of this fascinating field. The author seems unaware of the basic review article on the mathematical aspects by N. Aronszajn [Trans. Amer. Math. Soc. **68** (1950), 337–404; MR **14**, 479].
E. C. G. Sudarshan (Syracuse, N.Y.)

Maris, Th. A. J.; Herscovitz, Victoria E.; Jacob, Gerhard 4790

Quantum electrodynamics with zero bare fermion mass.
Phys. Rev. Lett. **12** (1964), 313–315.

The Dyson equation which relates the electron propagator S_F to the photon propagator D_F and the vertex function Γ is discussed. If D_F and Γ are known, then this is a nonlinear integral equation for S_F. As in previous papers [R. Haag and Th. A. J. Maris, Phys. Rev. (2) **132** (1963), 2325–2330; MR **28** #857; K. Johnson, M. Baker and R. S. Willey, Phys. Rev. Lett. **11** (1963), 518–520] the zero order expressions for D_F and Γ are used and the "bare mass" of the electron is put equal to zero, thus making the theory dilatation invariant. In contrast to Haag and Maris [loc. cit.] and in keeping with Johnson, Baker and Willey [loc. cit.] the Landau gauge was used. This allows a finite solution without any subtractions in the equation. The authors give an approximate solution in analytic form which is close to the true solution everywhere except in the neighborhood of the singularity of S_F. In contrast to Johnson, Baker and Willey [loc. cit.], they find that the equation poses no eigenvalue problem for the fine structure constant. *R. Haag* (Urbana, Ill.)

Salija, R. N. 4791

The formalism of gauge-invariance and the mass of vector particles. (Russian. Georgian summary)
Soobšč. Akad. Nauk Gruzin. SSR **33** (1964), 57–60.

It is shown that gauge-formalism is insufficient to explain the existence of the rest mass of a vector particle.
A. H. Klotz (Liverpool)

Smorodinskiĭ, Ya. A. [Smorodinskiĭ, Ja. A.] 4792

The theory of spiral amplitudes.
Ž. Èksper. Teoret. Fiz. **45** (1963), 604–609 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 415–418.

Author's summary: "The method of spiral amplitudes developed by Wick and Jacob becomes very lucid when formulated in a relativistic velocity space having a Lobachevsky metric. In this space the relativistic properties of the spiral amplitudes become obvious. As an example we treat the problem of transforming spin amplitudes from one channel of a binary reaction to another (the crossing transformation of field theory)."

{Reviewer's remark: The crossing relations obtained in this article do not give the known result for πN scattering. As the author does not specify the details of the analytic continuation, the root of a possible mistake cannot be easily traced. Fortunately, a similar approach to this problem is available in the form of a detailed and lucid paper by T. L. Trueman and G. C. Wick [Ann. Physics **26** (1964), 322–335; MR **28** #5746].} *P. Singer* (New York)

Strocchi, F. 4793

The relativistic limit of the theory of vector mesons. (Italian summary)
Nuovo Cimento (10) **31** (1964), 884–889.

Author's summary: "The analogue of the Cini-Touschek transformation (relativistic limit) in the case of vector mesons is considered. The spinor form of Maxwell equations is obtained after this transformation is performed."
A. H. Klotz (Liverpool)

Beckers, J. 4794

Sur une identité dans la théorie des collisions.
Bull. Soc. Roy. Sci. Liège **32** (1963), 886–898.

Author's summary: "In this note, we obtain a general formula similar to Kato's identity for the phase shifts, but concerning the scattering amplitudes. Our formula is not restricted to the case of central potentials and can be extended to complex collisions. It gives indications about the approximation made in variational methods for scattering amplitudes as, for example, in Kohn's method. If $V(r)$ is central, the correspondence between our identity and that of Kato is established."

Cerulus, F.; Martin, A. 4795

A lower bound for large angle elastic scattering at high energies.
Phys. Lett. **8** (1964), 80–82.

Following the experimental indications that the elastic cross-section at 90° in the center of mass system is a fast decreasing function of the total center of mass energy, whereas the total cross-section is at most slowly decreasing or possibly constant, the author shows, using analyticity properties of the scattering amplitude in the variable $z = \cos\theta$, that too fast a decrease of the 90° elastic scattering cross-section is not compatible with a slowly varying total cross-section. *J. N. Chahoud* (Bologna)

Džibuti, R. I.; Ratišvili, I. G. 4796

Polarization effects in (α, d) reactions. (Russian. Georgian summary)
Soobšč. Akad. Nauk Gruzin. SSR **32** (1963), 319–326.

The aim of this paper consists in the theoretical investigation of compound-particles angular distribution and polarization in process of direct stripping reactions. As an example the authors use the reaction (α, d). The general behaviour of the angular distribution and the polarization

dependence on the scattering angle are expressed in a correct manner in the relations obtained.

In this contribution only the polarization vector is investigated. However, by means of the relations obtained, it is possible to obtain the form of the polarization tensor elements which determine the quadrupolarization of the deuteron beam. The authors recall that the polarization will have the basic type of the deuteron beam spin ordering at low and medium energy direct reactions and that the effect of quadrupolarization is seen at high energies. *M. Blažek* (Bratislava)

Ford, William F. 4797
Anomalous behavior of the Coulomb T matrix.
Phys. Rev. (2) **133** (1964), B1616–B1621.

The infinite range of the unscreened Coulomb potential has been the source of a variety of mathematical difficulties for many scattering problems. These same quirks (non-uniformities, etc.), of course, underlie all of the more recent formal rephrasings of the problem. In particular, there have been indications that the Coulombic two-body scattering matrix (T-matrix) exhibits a discontinuity on the energy shell. The present paper provides some direct computational support for this assertion. Inasmuch as the essential concern of the entire calculation is a strictly mathematical point, it would have been desirable to exhibit full rigor at each step. *T. Erber* (Chicago, Ill.)

Henley, E. 4798
Time reversal in nuclear forces.
Lectures on field theory and the many-body problem, pp. 189–197. *Academic Press, New York*, 1961.

The question of establishing time-reversal invariance in strong interactions is discussed by studying the theoretical implications of detailed balance experiments, polarization phenomena, beta-decay, and correlations in successive transitions. A clear analysis in terms of collision theory is given and difficulties and limitations are pointed out.
P. Roman (Boston, Mass.)

Kaus, P.; Pearson, C. J. 4799
Jost functions and determinantal method in potential scattering. (Italian summary)
Nuovo Cimento (10) **28** (1963), 500–527.

The first and second order approximations in the Fredholm expansion of the partial wave amplitudes in potential scattering are considered. The analytic properties of the corresponding approximate Jost functions for real and complex angular momentum are compared with those of the exact ones. Numerical results are presented for s and p waves in the case of a square-well potential. The Fredholm approximations to the phase shifts are found to be significantly better than the Born approximation at low energies when bound states or resonances are present. The second approximation exhibits some unphysical features.
H. M. Nussenzveig (Rio de Janeiro)

Calogero, Francesco; Charap, John M.; Squires, Euan J. 4800
The continuation in total angular momentum of partial-wave scattering amplitudes for particles with spin.
Ann. Physics **25** (1963), 325–339.

The problem of continuation to the complex angular momentum variable of the partial wave amplitudes for scattering of particles with spin is considered by using the helicity scattering amplitude and by assuming suitable analyticity properties in the $\cos\theta$ variable. The same techniques previously used by Froissart and by Gribov for the spinless case are then used, and very similar results are obtained for what concerns the analytic properties in the angular momentum variable and the asymptotic Regge-like behaviour. The existence of kinematic zeros of the amplitude, which corresponds to the decoupling of the so-called "sense" and "nonsense" channels is also investigated to finally show that the "nonsense-nonsense" amplitudes, although present, are still dominated asymptotically by the Regge pole term.

It should be noticed that part of the above findings stays on the potential model analogy although some other results, such as those due to the unitarity requirement in the crossed channel, have no counterpart in potential theory. *E. Predazzi* (Turin)

de Alfaro, V.; Predazzi, E.; Rossetti, C. 4801
Analyticity in the angular momentum in potential scattering. (Italian summary)
Nuovo Cimento (10) **31** (1964), 42–55.

The radial wave function is represented as a superposition of solutions of the problem without potential and with different angular momenta (Bessel functions). This representation is shown to be a convenient one in order to study the continuation to complex angular momenta. A representation for the Jost function as an expansion over meromorphic functions of angular momentum is obtained. A perturbative form of the Regge trajectories is also given.
F. Calogero (Rome)

Martin, A. 4802
A lower bound for elastic cross-sections in the high-energy region. (Italian summary)
Nuovo Cimento (10) **29** (1963), 993–996.

Under the validity of a certain spectral representation for the forward scattering amplitude (which could be derived from, but does not require, the Mandelstam representation), by making use of a certain inequality obeyed by the Legendre function of the second kind, it is shown that the ratio of the elastic cross-section to the square of the total cross-section is bounded below by a quantity of the order of the inverse square of the logarithm of the energy.
E. C. G. Sudarshan (Syracuse, N.Y.)

Menzel, Donald H. 4803
Generalized radial integrals with hydrogenic functions.
Rev. Modern Phys. **36** (1964), 613–617.

Certain (non-relativistic) matrix elements between states in a Coulomb field are evaluated by means of special devices involving partial differentiations with respect to an auxiliary variable. Some didactic advantages vis-à-vis the original treatment by Gordon [Ann. Physik (5) **2** (1929), 1031–1056] are claimed. It should be noted that a very elegant and general treatment of these questions has already been given by K. Alder and A. Winther [Danske Vid. Selsk. Mat.-Fys. Medd. **29** (1955), no. 18; MR **17**, 566]. *T. Erber* (Chicago, Ill.)

Prokhorov, L. V. [Prohorov, L. V.] 4804

Analytic properties of the forward scattering amplitude in the nonlocal theory.

Ž. Èksper. Teoret. Fiz. **45** (1963), 791–796 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 542–545.

If we postulate that the matrix elements of the commutator of a field and the current decrease exponentially in large space-like directions, then a forward dispersion relation can be proved for the elastic scattering amplitude. The method follows that of K. Symanzik [Phys. Rev. (2) **105** (1957), 743–749] with suitable modification to allow for the non-locality of the field. By an example it is shown that if the commutator does not decrease sufficiently fast, then complex singularities might appear.

R. F. Streater (London)

Summerfield, G. C. 4805

On the Fermi approximation in thermal neutron scattering.

Ann. Physics **26** (1964), 72–80.

In this paper the author extends the results of Lippman and Schwinger [Phys. Rev. (2) **76** (1950), 469–480; MR **12**, 570] on the scattering of neutrons by nuclei bound to molecular or crystalline structures with the aim of investigating the validity of the Fermi approximation when the scatterer is a compound system of many atoms. In particular, the author looks for an approximation of a higher order than that of Fermi so as to explain multiple scattering and many-particle excitation effects even when no dynamical correlation exists between the particles of the system. To this end a particular perturbation expansion of the T-matrix, whose squared modulus appears in the expression of the differential cross section, is suggested. Qualitative considerations, tending to justify the use and the convergence of this perturbative expansion, are proposed. The first order of this expansion yields the Fermi approximation; the second order gives the Lippman-Schwinger results in the case of the scattering of a single bound proton.

The author points out that in the limit of infinite neutron wave length (energies much weaker than thermal) the Fermi approximation fails if the scattering is due to macroscopic aggregates of atoms. Effects of this type should, however, be observable only if the energy of the neutrons is much less than thermal and when the scatterers are of almost macroscopic dimensions. The construction of the perturbation expansion and its justification seem to be the critical point of this paper and would deserve, perhaps, a more detailed discussion.

P. Caldirola (Milan)

Van Hove, L. 4806

High-energy collisions of strongly interacting particles.

Rev. Modern Phys. **36** (1964), 655–665.

A theoretical analysis of high-energy collisions, relying heavily on experimental indications, is presented. The main feature of the work is that it establishes logical connections between the salient features of high-energy collisions on the one hand, and various properties of jets on the other. It is assumed that there are no correlations in the inelastic final state between the secondaries emitted in different intervals of longitudinal momentum, except for the requirement of energy-momentum conservation.

After a general introduction, a proof is presented to show the relationship between the approximate energy independence of cross-sections and the imaginary character of the scattering amplitude. In the following section a class of rather realistic jet models is discussed. Using the results here obtained, the shadow scattering is calculated and compared with experiment. Then the multiplicity- and momentum-distribution of uncorrelated jets is studied, and the paper terminates with suggestions for further experimental work.

P. Roman (Boston, Mass.)

Streater, R. F.; Wightman, A. S. 4807

★PCT, spin and statistics, and all that.

W. A. Benjamin, Inc., New York-Amsterdam, 1964. viii+181 *pp.* $9.00.

This delightful and clearly written book will be of great value to both graduate students and research physicists who wish to become acquainted with the basic notions and fundamental theorems of the axiomatic relativistic quantum field theory. The book is divided into two parts of equal length, each containing two chapters. The first part provides the mathematical notions and tools. In particular, Chapter 1 is a survey of relativistic transformation laws in quantum theory, and Chapter 2 is concerned with the theory of distributions, holomorphic functions, and some comments on Hilbert space. The second part of the book develops the modern axiomatic theory of quantized fields. In particular, Chapter 3 introduces the properties of the vacuum expectation values, and formulates the reconstruction theorem, which enables one to recover a field theory from its vacuum expectation values. The last chapter applies the previously developed notions and methods to give a full account of the well-established general theorems of relativistic quantum field theory. In particular, the global nature of local commutativity, the PCT theorem, the connection between spin and statistics, Haag's theorem, and the equivalence classes of local fields are discussed. Each chapter is followed by a bibliography with comments.

Apart from its intrinsic scientific and pedagogic value, the book is commendable for its charming good sense of humor, which is illustrated here by quoting the preface in full: "The idea of this book arose in a conversation with H. A. Bethe, who remarked that a little book about modern field theory which contained only Memorable Results would be a Good Thing. In the field of historical research this approach led to the publication of a treatise [W. C. Sellar and R. J. Yeatman, *1066 and All That*, Dutton, New York, 1931] which has become a standard text for serious students. Although it is often dangerous to use the tried and true methods of one subject in another field of research, the application to physics of the principles of that book has led to at least one good result: we have eliminated all theorems whose proofs are non-existent."

P. Roman (Boston, Mass.)

Bialynicki-Birula, I.; Śniatycki, J.; Tatur, S. 4808

Functional methods in the Thirring model.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 479–482.

The authors discuss ambiguities which may arise in quantum field theory due to the singular behaviour of propagators at coinciding points. This is done for the

Thirring model, for which two different solutions are obtained depending on the prescription used in evaluating singular expressions.
John G. Taylor (Cambridge, England)

Białynicki-Birula, I. 4809
Gauge covariant and gauge invariant Green functions. Connection between Zumino's and Mandelstam's formulations.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 483–485.
Author's summary: "It is shown that Zumino's and Mandelstam's formulations of quantum electrodynamics are completely equivalent. The gauge freedom of Green functions in Zumino's formulation corresponds to the arbitrariness in the definition of Green functions in Mandelstam's formulation."
John G. Taylor (Cambridge, England)

Białynicka-Birula, Z. 4810
Redundant zeros and poles on the unphysical sheet in a model of field theory.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 301–304.
Unstable particles may be described either by poles on the unphysical sheet of a scattering amplitude or by redundant zeros. It is shown that in a fixed fermion model every pair of redundant zeros in the two-particle scattering amplitude gives rise to four complex poles on the first unphysical sheet. *John G. Taylor* (Cambridge, England)

Borchers, H. J.; Zimmermann, W. 4811
On the self-adjointness of field operators. (Italian summary)
Nuovo Cimento (10) **31** (1964), 1047–1059.
A basic concept in quantum field theory is a field which is in general an unbounded operator in a Hilbert space (after being smeared out by a test function). On the other hand, it is possible to develop a relativistic quantum theory, taking a system of von Neumann algebras of bounded local observables as a basic concept. It has been conjectured that a system of von Neumann algebras of bounded local observables can be associated with a given field under certain general conditions, and hence the former theory can be transcribed into a form of the latter theory. The paper under review is concerned with a mathematical sufficient condition under which this conjecture is true.

In addition to the Wightman axioms, the authors assume that the vacuum Ω is an analytic vector for every smeared-out field $A(f)$. Explicitly, it is assumed that $\sum \|A(f)^n\Omega\| z^n/n!$ has a nonvanishing radius of convergence. The authors then prove that $A(f)$ is essentially self-adjoint if f has compact support (Theorem 2) and that the spectral projections of the closures of $A(f)$ and $A(g)$ commute with each other if the supports of f and g are mutually space-like (Theorem 3).

The authors point out an example of a symmetric but non-selfadjoint operator A with a cyclic vector Ω such that $(\Omega, A^n\Omega) = O(n^{n(1+\varepsilon)})$, $\varepsilon > 0$. From this the authors conclude that their assumption cannot be improved in a simple manner.

Independently of the above assumption, the authors prove that $A(f)$ and $A(f)_G$ have the same closure (Theorem 1), where $A(f)_G$ is the restriction of $A(f)$ to a domain D_G and D_G is spanned linearly by $A(g_1)\cdots A(g_n)\Omega$, n arbitrary and the support of g_i in G. *H. Araki* (Kyoto)

Frank, William M. 4812
Convergence of Yukawa theories with a finite number of interacting boson modes.
J. Mathematical Phys. **5** (1964), 363–372.
The author has been engaged for some time in trying to extend and confirm the tentative results on the convergence of perturbation expansions in field theory [the reviewer, Proc. Cambridge Philos. Soc. **48** (1952), 625–639; MR **14**, 607; W. E. Thirring, Helv. Phys. Acta **26** (1953), 33–52; MR **14**, 708]. In this paper he presents a rigorous discussion of the case of Yukawa type theories involving either one boson and two fermions, or three bosons. In both cases one of the bosons is capable of taking only a finite number of modes. He finds that for interaction with a quantized fermion field the radius of convergence of the perturbation serves as a function of the coupling constant is non-zero but finite if the (regularized) fermion propagators are in L^2, and is actually infinite when the fermion field also only has a finite number of modes. By contrast the three-boson field has zero radius of convergence. The proof depends on a transformation of the perturbation expansion which resembles that of Borel summability, and, as it stands, makes no comment on the physically interesting case of an infinite number of modes. However, the marked difference that the statistics of the interacting particles makes on the behaviour of the expansion is very clearly shown here. It leads one to suspect that results for a three-boson interaction may not be very relevant to that of a fermion-boson interaction, a result long conjectured [M. Fierz, High Energy Nuclear Physics (Proc. Fifth Annual Rochester Conf., 1955), p. 67, Interscience, New York, 1955; G. Baym, Phys. Rev. (2) **117** (1960), 886–888; MR **22** #3513]. However, as yet nothing has been established by the infinite mode case. *C. A. Hurst* (Adelaide)

Galindo, A.; Pascual, P. 4813
Some remarks on the relativistic wave equations. (Italian summary)
Nuovo Cimento (10) **31** (1964), 132–139.
A general method is given to construct realizations of relativistic elementary particles. To circumvent the difficulties due to the concomitant algebraic relations which arise when the electromagnetic interaction is introduced for particles with spins greater than one, a new method is presented which generalizes, in a natural way, the two-component electron theory of Feynman and Gell-Mann [Phys. Rev. (2) **109** (1958), 193–198; MR **19**, 813].
S. Azuma (Fukusima)

Geshkenbeĭn, B. V. [Geškenbeĭn, B. V.]; Ioffe, B. L. 4814
Restrictions on the values of the coupling constants and the vertex part for the interaction of three particles in quantum field theory. II.
Ž. Èksper. Teoret. Fiz. **45** (1963), 555–564 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 382–388.

In an earlier paper [same Ž. **44** (1963), 1211–1227; MR **28** #853; cf. also, Phys. Rev. Lett. **11** (1963), 55–58] the authors have derived a certain upper bound for the coupling constant for the interaction between three particles. Their result was obtained using rather strong assumptions about the analyticity domain of the vertex function $\Gamma(\kappa^2)$, viz., that it is analytic in the whole complex κ^2-plane except for a cut along the positive real axis. Further, they assumed that the two-particle Green's function has no zero. In the summary of the paper to be reviewed here the authors state: "It is shown that the omission of the assumption made earlier for the fermion case that there is no zero for the Green's function does not alter the conclusion about the restriction on the coupling constant. The boson case is treated with the same assumption as earlier and it is shown that the vertex part $\Gamma(\kappa^2)$ lies between $\Gamma_{\min}(\kappa^2) \leqq \Gamma(\kappa^2) \leqq \Gamma_{\max}(\kappa^2)$. Explicit expressions are obtained for the functions $\Gamma_{\min}(\kappa^2)$ and $\Gamma_{\max}(\kappa^2)$ which depend on the value of the renormalized coupling constant". The reviewer should like to remark that C. J. Goebel and B. Sakita have shown by explicit examples [ibid. **11** (1963), 293–296] that the combined analyticity assumptions on the vertex part and the Green's function are very essential for the result of the authors. If these assumptions are sufficiently relaxed, no upper bound is obtained for the coupling constant.

G. Källén (College Park, Md.)

Woronowicz, S. 4815

On a theorem of Mackey, Stone and v. Neumann.

Studia Math. **24** (1964/65), 101–105.

Author's summary: "Araki [H. Araki, J. Mathematical Phys. **1** (1960), 492–504; MR **23** #B866] and Gel'fand [I. M. Gel'fand and N. Ja. Vilenkin, *Generalized functions* (Russian), Part 4, Fizmatgiz, Moscow, 1961; MR **26** #4173], using Bochner's theorem for nuclear spaces, determined all cyclic representations of the commutation rules for systems with infinite degrees of freedom (quantum theory of fields). The present paper gives a corresponding generalization of Mackey's representation theorem [G. W. Mackey, Duke Math. J. **16** (1949), 313–326; MR **11**, 10]."

Kaneko, Takao; Ohnuki, Yoshio; Watanabe, Keiji 4816

Diagonalization of the particle mixture interaction.

Progr. Theoret. Phys. **30** (1963), 521–535.

The existence of particles having similar quantum numbers has suggested the idea of particle mixing due to various interactions. In this article, the case of n spinor fields among which direct conversion is assumed possible is considered in the realm of quantum field theory. As these fields do not correspond to one-particle states, a transformation is required in order to obtain the fields for which one can make use of the asymptotic condition. The authors deal with this problem by diagonalizing the Fourier transform of the propagator matrix

$$T\langle 0|\psi_i(x)\bar{\psi}_j(y)|0\rangle,$$

where $\psi_i(x)$ is the unrenormalized Heisenberg field operator. Then, using the Lehman representation for the propagator, it is shown that the previously accomplished diagonalization is equivalent to a renormalization procedure. The formalism developed in the paper is applied to the problem of $\Lambda - N$ parity non-conserving mixing, which has been considered in connection with the weak non-leptonic decays of strange baryons. {For a detailed treatment of the mixing of vector fields by using similar techniques we refer to the articles of G. Feldman and P. T. Matthews [Phys. Rev. (2) **132** (1963), 823–830; MR **28** #897] and S. Coleman and H. J. Schnitzer [ibid. (2) **134** (1964), B863–B872].} *P. Singer* (New York)

Kita, Hideji; Kawai, Yōko 4817

Quantized free unstable particle field.

Progr. Theoret. Phys. **31** (1964), 269–299.

Authors' summary: "The quantized field of unstable particles without charge and spin is introduced. With this purpose, c-number solutions of the Klein-Gordon equation with complex mass $m = m_0 - (i/2)\gamma$ are investigated, and it is proved that a new set of non-scalar elementary wave solutions, which represent a direct idealization of states of a freely moving unstable particle, exists for the values of $\gamma/2m_0$ smaller than the critical value which is evaluated to be $0.8\cdots$. With the aid of this set of elementary wave solutions, the quantized free unstable particle field is defined mathematically well. The commutator of the fields is evaluated, which turns out to be an invariant function of coordinates-difference in spite of the non-simple transformation property of the elementary waves used, and vanishes for the space-like separations. The resulting causal Green function Δ_F has a good asymptotic behaviour at large distances. The set of infinitesimal generators of the inhomogeneous Lorentz group is found in terms of the creation and annihilation operators of particles, and it is shown that there exists a representation, which is non-unitary, of the inhomogeneous Lorentz group. The quantized field and the generators P_μ and $M_{\mu\nu}$ have no simple transformation property like scalar, vector and tensor." *S. Deser* (Waltham, Mass.)

Kurdgelaidze, D. F. 4818

Two-parameter non-linear field theory. (Russian)

Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1964**, *no.* 1, 11–17.

The stability problem for a particle described by a classical spinor field obeying a non-linear equation is investigated. It is shown that a stable solution exists for particles confined to a finite volume when at least two non-linear terms are present in the Lagrangian.

I. Białynicki-Birula (Warsaw)

Matinjan, S. G.; Perel'man, M. E. 4819

The generalized Ward identity for electromagnetic transitions between different particles. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **32** (1963), 301–305.

This paper is completely wrong. The authors use the same symbol T_μ to denote two different operators. This can be clearly seen from their formula (3) and the wave equation for the electromagnetic potential given after formula (7). Due to this mistake all further results are incorrect.

I. Białynicki-Birula (Warsaw)

Newton, R. G. 4820

Angular momentum continuation in the three-body problem without cuts.

Phys. Lett. **8** (1964), 210–211.

The total angular momentum j of the three-body system is written as $j = l + L$ when l and L are angular momenta of subsystems. The angular momentum j is made complex by making either l or L complex and fixing the other. The analyticity in j is different depending on whether l, which refers to a two-particle subsystem, or L, which involves all three, is chosen. With l complex there are two-particle poles that are integrated over to give cuts, while this does not happen with L.

F. R. Halpern (La Jolla, Calif.)

Nishijima, K. 4821

Theorem on the product of field operators.

Phys. Rev. (2) **133** (1964), B204–B208.

The construction of a local field operator for a composite particle introduced by Haag, Nishijima, and Zimmermann is proved to reproduce the original field operator when it is applied to an elementary particle field. The proof of this theorem is given for a simple model using a technique which is described in the author's earlier paper [same Rev. (2) **122** (1961), 298–306, Appendix; MR **22** #11908]. The importance of the divergent character of field theory for the validity of the theorem is stressed. Then, if the self-energy integrals in the Feynman diagrams converge, the proof fails and we can no longer obtain the theorem in a manner described in this paper.

S. Aramaki (Urbana, Ill.)

Schlitt, D. W. 4822

A soluble two-dimensional field theory satisfying unitarity and crossing symmetry. (Italian summary)

Nuovo Cimento (10) **31** (1964), 858–873.

A two-dimensional model is considered in which scattering amplitudes are assumed to be boundary values of analytic functions with only those singularities which are required by unitarity and crossing symmetry. An explicit expression containing one free parameter is obtained for the elastic scattering amplitude on the assumption that all inelastic amplitudes vanish. It is shown that the solution obtained is consistent with third-order perturbation theory with $\lambda\Phi^4$ interaction. The effects of crossing symmetry and inelastic scattering are discussed, and it is shown that while the scattering in the low-energy region is insensitive to these, the position of bound states and resonances depends strongly on crossing symmetry and inelastic scattering.

J. N. Islam (College Park, Md.)

Takahashi, Yasushi; Umezawa, Hiroomi 4823

Relativistic quantization of fields.

Nuclear Phys. **51** (1964), 193–211.

The authors propose a method of covariant quantization of free fields without reference to the Lagrange formalism. Properties of the wave functions satisfying the general relativistic equation of motion without interaction are discussed, together with the normalization condition of them. By using the normalization condition, the quantization of the fields is considered. The interactions of quantized fields are discussed especially in connection with the asymptotic limit of field operators under the assumption that the limit exists in the sense of the weak convergence.

Y. Kato (Kobe)

Weidlich, W. 4824

On the inequivalent representations of canonical commutation relations in quantum field theory. (Italian summary)

Nuovo Cimento (10) **30** (1963), 803–829.

Quantum field theory has often been formulated in the past in terms of an infinite set of operators q_k, p_k satisfying the canonical commutation relations (1) $[q_k, p_l] = i\delta_{kl}$ ($k, l = 1, \cdots, \infty$). These canonical quantities are linearly related to the field quantities at one fixed time. The Hamiltonian on the other hand is given as a non-linear function of the p_k, q_k. It was recognized later that the commutation relations (1) allow a tremendous number of unitarily inequivalent representations and that the choice of the appropriate representation already involves a dynamical problem. Various methods of dealing with this question have been previously discussed. In the paper under review the author proposes and illustrates one method which allows, for instance, a straightforward and systematic perturbation expansion. The essential idea is to use "pseudo-operators", i.e., formal expressions in the p_k, q_k which are not considered as operators in the Hilbert space but as algebraic objects. In particular, a pseudo-operator V is called pseudo-unitary if (due to the relations (1) alone) it satisfies $V^\dagger V = VV^\dagger = 1$. Given the Hamiltonian $H(p_1 \cdots, q_1 \cdots)$, the dynamical problem consists in finding a pseudo-unitary operator $V(p_1 \cdots, q_1 \cdots)$ which transforms H into the Hamiltonian of a free field: (2) $VHV^\dagger = H_0$. To make the solution unique a certain boundary condition ("asymptotic condition") must be taken into account. If $H = H_0 + gH'$, a perturbation treatment is obtained by putting (3) $V = e^{iF}$, where F is "pseudo-Hermitian" and expanding F in powers of the coupling constant g. In this manner one obtains ultimately a perturbation series for the actual field in terms of the incoming field. Of course, the customary renormalization procedure still remains to be done as in the Feynman-Dyson expansion for the S-matrix. The crucial step in avoiding difficulties due to the inequivalent representations is equation (3), i.e., the fact that one expands F and not V in a power series.

R. Haag (Urbana, Ill.)

Fano, U. 4825

Precession equation of a spinning particle in non-uniform fields.

Phys. Rev. (2) **133** (1964), B828–B830.

The general equation governing the changes of orientation of a spinning particle is expressed in a compact tensorial form. This form is an extension of the vector equation describing the Larmor precession of a particle's magnetic moment about the direction of a uniform magnetic field to the case of a particle under the influence of arbitrary non-uniform fields.

The time derivative of each component of a mean multipole moment of the particle is found to be a linear combination of the multipole moments of all orders, subject to selection rules which express the interesting features of the motion. Various particular motions are discussed.

I. M. Barbour (Rome)

Jasselette, P. 4826
Remarks on the generators of the $SU3$ group. (Italian summary)
Nuovo Cimento (10) **32** (1964), 136–140.
Author's summary: "It is suggested that the sign of the λ_8 matrix among the generators of the $SU3$ group be changed. The rotation senses around the three axes of the group basic space would thereby be made coherent. Furthermore, various relations, in particular commutation rules, would become symmetrical. Relations between two representations of Weyl's subgroup of $SU3$ are given."
Y. Ne'eman (Pasadena, Calif.)

Klein, Lewis (Editor) 4827
★Dispersion relations and the abstract approach to field theory.
International Science Review Series, Vol. 1.
Gordon and Breach, Publishers, Inc., New York, 1961. xii+273 *pp.* $4.95.
This book is a collection of 15 reproductions of papers which have been published in various journals from 1954 to 1960. Some of the papers are translated from the German original. A short preface by the editor explains the common ideas which link the various papers, and groups them into logical categories. While the selection of the material certainly reflects the personal taste of the editor, it can be said that the papers here included are now all classics, and they give a fair and appropriate cross-section of basic research in axiomatic quantum field theory and the attempts at proving dispersion relations in this framework, for the period mentioned above.
P. Roman (Boston, Mass.)

Brown, Lowell; Fivel, Daniel I.; Lee, Benjamin W.; Sawyer, Raymond F. 4828
Fredholm method in potential scattering and applications to complex angular momentum.
Ann. Physics **23** (1963), 187–220.
The formulation of complex angular momentum problems in potential scattering is here given by using the Lippmann-Schwinger integral equation rather than the Schrödinger differential equation. Techniques of functional analysis are then widely used in treating problems concerning the introduction of complex angular momenta. Integral equations for the scattering amplitudes and expressions for the Jost functions are given in the frame of Fredholm theory. The analytic properties of the Jost functions are then discussed and the T-matrix is proved to be meromorphic both in the coupling constant g and in l for $\mathrm{Re}\, l > -\frac{3}{2}$ for Yukawa potentials. The asymptotic behaviour of the T-matrix for large values of $|k|$ and $|l|$ is considered including the very delicate case $|\mathrm{Im}\, l| \to \infty$ while $|\mathrm{Re}\, l| = \mathrm{const}$. The poles g_i of the T-matrix are investigated by considering them as the eigenvalues of the homogeneous counterpart of the scattering integral equation. Finally the leading pole in the complex angular momentum variable is discussed in the high energy-weak coupling limits and the Watson-Sommerfeld representation is derived.
T. Regge (Turin)

Eden, R. J.; Taylor, J. R. 4829
Poles and shadow poles in the many-channel S matrix.
Phys. Rev. (2) **133** (1964), B1575–B1580.
Authors' summary: "The connection between the partial-wave S matrix on different Riemann sheets is obtained from unitarity and analyticity. Under the assumption that coupling between channels can be varied analytically, it is shown that a resonance pole or bound-state pole may lead also to 'shadow poles' on other Riemann sheets. The existence of shadow poles is illustrated by a unitary resonance model based on a sum of Feynman diagrams. In general, the number of shadow poles that can be deduced from an observed resonance depends on the number of channels that still have a particular resonance pole in the absence of coupling between channels. If the pole still appears in all channels, then shadow poles occur on every Riemann sheet; if it appears in only one channel, then shadow poles appear on half the sheets. If the resonance disappears in the absence of channel coupling, our method leads to no conclusions. In connection with the unitary symmetry scheme we note that the existence of shadow poles would permit a simple changeover from the separated poles of a resonance multiplet with broken symmetry to the coincident poles of the multiplet that must occur when symmetry breaking interaction is switched off."
S. F. Tuan (Lafayette, Ind.)

Gell-Mann, M.; Goldberger, M. L.; Low, F. E.; Marx, E.; Zachariasen, F. 4830
Elementary particles of conventional field theory as Regge poles. III.
Phys. Rev. (2) **133** (1964), B145–B160.
Part II (by Gell-Mann, Goldberger, Low and Zachariasen) appeared in Phys. Lett. **4** (1963), 265–267 [MR **27** #3329].
Authors' summary: "It is shown that an elementary particle of conventional field theory may, under certain conditions, lie on a Regge trajectory. These conditions are that the system contain a 'nonsense' channel at the angular momentum of the particle and that the Born approximation scattering amplitude factor in a well-defined way. They are satisfied by a spin $\frac{1}{2}$ fermion interacting through a conserved current with a spin one neutral boson. The particle in question is the fermion."
E. Predazzi (Turin)

Gell-Mann, M.; Goldberger, M. L.; Low, F. E.; Singh, V.; Zachariasen, F. 4831
Elementary particles of conventional field theory as Regge poles. IV.
Phys. Rev. (2) **133** (1964), B161–B174.
Authors' summary: "The usual field theory of spin 0 'nucleons' coupled to vector mesons (or heavy photons) is studied in order to find whether the nucleon lies on a Regge trajectory. Photon-nucleon scattering is examined, to each order in the coupling constant, with the highest power of $\ln\cos\theta$ retained. It is found that a suitable Regge trajectory is generated, but the nucleon does not lie on it. The nucleon pole term in the scattering amplitude corresponds to a fixed singularity in angular momentum. The spin 0 'nucleon' thus behaves differently from a particle of spin $\frac{1}{2}$."
E. Predazzi (Turin)

Khalfin, L. A. [Halfin, L. A.] 4832
The hypothesis of Regge poles in quantum field theory and the threshold singularities of inelastic processes.
Ž. Èksper. Teoret. Fiz. **45** (1963), 631–636 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 433–436.

The scattering amplitude is written as a sum of infinitely many inelastic amplitudes with thresholds at $s_1, s_2, s_3, \cdots$. It is assumed that each inelastic amplitude has a representation in terms of poles in angular momentum plus a background integral. If it is required that the amplitude in this form give the correct threshold singularities at each inelastic threshold, certain restrictions on the inelastic amplitudes are obtained. In particular, the author argues that the partial cross-sections of each inelastic process are bounded at infinite energy by a power (negative) of the energy. *A. O. Barut* (Boulder, Colo.)

Kolkunov, V. A. 4833
Nonrelativistic Regge trajectories. I.
Ž. Èksper. Teoret. Fiz. **45** (1963), 1123–1132 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 775–781.

Author's summary: "Using the exact solution of the Schrödinger equation, we study the motion of Regge poles for potentials of the type $r^{2\varepsilon-2}\exp[-(r\mu)^{2\tau}]$, with $\varepsilon>0$, $\tau>0$, as a function of the coupling constant g and the energy $E=k^2/2m$."

Muradyan, R. M. [Muradjan, R. M.] 4834
A study of the analytic properties of ladder diagrams by the method of complex orbital angular momenta.
Dokl. Akad. Nauk SSSR **149** (1963), 80–83 (*Russian*); *translated as Soviet Physics Dokl.* **8** (1963), 271–274.

The author discusses the contribution to the nth-order ladder diagram in perturbation theory when the horizontal or vertical lines of the ladder are on the mass shell. These contributions enable certain sums of products of Legendre functions to be expressed in terms of integrals. The methods used to obtain these results are those of continuation into the complex angular momentum plane and use of the Watson-Sommerfeld transformation.
John G. Taylor (Cambridge, England)

Olive, D. I. 4835
Unitarity and the evaluation of discontinuities. II. (Italian summary)
Nuovo Cimento (10) **29** (1963), 326–335.

Part I appeared in Nuovo Cimento (10) **26** (1962), 73–102 [MR **26** #1103]. The contribution to unitarity from a 2-particle normal threshold is found, using hermitian unitarity of S-matrix elements. The result requires the use of 'extended unitarity', which is given a mathematical justification. This contribution is shown to agree with that coming from perturbation theory. A general formalism is set up for evaluating a normal threshold discontinuity on an unphysical sheet by means of solutions of linear Fredholm equations. Difficulties in the way of extending the results to multiparticle processes are discussed, the main problem arising from singularities in physical regions. This is stated to be avoided for the lowest three-particle threshold.
John G. Taylor (Cambridge, England)

Otokozawa, Jun; Masuda, Naohiko; 4836
Yamazaki, Miwae
Analyticity of the production amplitude in the higher order diagram.
Progr. Theoret. Phys. **30** (1963), 647–656.

Authors' summary: "Partial wave s-singularity of the production amplitude is analyzed in the cases of $\pi+N\to 2\pi+N$ and $2\pi+N\to 2\pi+N$. It is shown that the complex singularities of the fourth-order diagram coincide completely with those obtained from the corresponding pole diagram. Although the left-hand and right-hand singularities simultaneously appear on some region of the s-t plane, they are shown to be separated unambiguously. Detailed discussions about more general diagrams are also made and similar results are obtained."
J. N. Chahoud (Bologna)

Patashinskiĭ, A. Z. [Patašinskiĭ, A. Z.]; 4837
Pokrovskiĭ, V. L.; Khalatnikov, I. M. [Halatnikov, I. M.]
Quasiclassical scattering in a centrally symmetric field.
Ž. Èksper. Teoret. Fiz. **45** (1963), 989–1002 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 683–692.

Authors' summary: "The amplitude is obtained for quasiclassical scattering in a centrally symmetric field into classically unattainable angles. The position of the Regge poles and the asymptotic behavior of the scattering matrix in the complex angular momentum plane are investigated for a certain class of potentials $U(r)$ at energies E satisfying the conditions: $E\gg|U(r)|$ along the real axis and $\lambda\ll a$ (λ is the wavelength of the incident particle)."

Petráš, M. 4838
A note on the de Alfaro, Regge and Rossetti dynamical equation for Jost functions.
Nuovo Cimento (10) **31** (1964), 247–249.

The lines of a shorter derivation of a non-linear equation for the Jost function are outlined in this paper. The author will publish elsewhere the detailed calculations.
V. de Alfaro (Princeton, N.J.)

Rajasekaran, G. 4839
Scattering amplitudes on unphysical sheets and resonance poles.
Nuovo Cimento (10) **31** (1964), 697–700.

This is an extension of an earlier work of Dalitz and the author [Phys. Lett. **7** (1963), 373–377] on resonance poles and mass differences within unitary multiplets. The author here derives a simple relation between multichannel two-particle scattering amplitudes defined on two unphysical sheets; he also obtains useful width relationship between total and partial width of a given channel.
S. F. Tuan (Lafayette, Ind.)

Polkinghorne, J. C. 4840
The complete high-energy behavior of ladder diagrams in perturbation theory.
J. Mathematical Phys. **5** (1964), 431–434.

The high-energy behavior of the sum of ladder Feynman graphs is shown to yield the Regge behavior $s^{\alpha(t)}$. An equation for the trajectory function $\alpha(t)$ is obtained which is analogous to that obtained by the Fredholm method for Yukawa potentials [M. Cassandro, M. Cini, G. Jona-Lasinio and L. Sertorio, Nuovo Cimento (10) **28** (1963), 1351–1374]. The investigation is based on a modified form

of the Mellin transform method [J. D. Bjorken and T. T. Wu, Phys. Rev. (2) **130** (1963), 2566–2572; MR **28** #886; T. L. Trueman and T. Yao, ibid. (2) **132** (1963), 2741–2748; MR **28** #3724], and it displays the connection between high-energy behavior and bound-state poles of the S-matrix, without explicit reference to complex angular momenta. *F. Calogero* (Rome)

Shirkov, D. V. [Širkov, D. V.] 4841
Invariant charge and Regge asymptotic behavior.
Dokl. Akad. Nauk SSSR **148** (1963), 814–817 (*Russian*); *translated as Soviet Physics Dokl.* **8** (1963), 164–166.

The method of the renormalization group is used to predict the asymptotic behaviour of the scattering amplitude. By making approximations which maintain the exact relationship between renormalized and unrenormalized quantities, and the renormalization constants, Regge behaviour is predicted. It is argued that the analysis is more reliable than the summing of usual perturbation graphs, because this necessarily leads to the omission of some graphs which might affect the high-energy behaviour.

{Reviewer's remark: While the renormalization group method is interesting and can probably be refined, the method of summing Feynman graphs [J. C. Polkinghorne, #4840 above] has become a fine art and is a more convincing method.} *R. F. Streater* (London)

Simonov, Yu. A. [Simonov, Ju. A.] 4842
Integral representation of a square diagram with anomalous mass ratio.
Ž. Èksper. Teoret. Fiz. **44** (1963), 1622–1627 (*Russian. English summary*); *translated as Soviet Physics JETP* **17** (1963), 1092–1095.

This is a continuation of previous work [same Ž. **43** (1962), 2263–2272; MR **27** #6531] for writing down integral representations of scattering amplitudes in both energy and momentum transfer in cases when the Mandelstam representation breaks down. The general method is to use the Bergmann-Weil representation. The particular example considered here is the lowest-order scattering amplitude with arbitrary internal masses. The representation is composed of three terms in general, one of these being identical to the Mandelstam representation, the last two vanishing on transition to the normal mass case. To be of practical value such a representation will have to be extended to higher-order diagrams, and some rules given for calculating the weight function outside perturbation theory. *John G. Taylor* (Cambridge, England)

Ter-Martirosyan, K. A. [Ter-Martirosjan, K. A.] 4843
The asymptotic values of the amplitudes of inelastic processes.
Ž. Èksper. Teoret. Fiz. **44** (1963), 341–354 (*Russian. English summary*); *translated as Soviet Physics JETP* **17** (1963), 233–241.

Author's summary: "A simple method is proposed for expanding any many-point amplitude in relativistic theory in terms of partial waves. The expansions obtained are used for analyzing the asymptotic behaviour of inelastic processes by the Regge and Gribov method."
T. Regge (Turin)

Warburton, A. E. A. 4844
The analytic properties of Regge trajectories. I. (Italian summary)
Nuovo Cimento (10) **32** (1964), 122–126.

Author's summary: "The analytic continuation into each other of the first six Regge trajectories for a repulsive Yukawa potential is demonstrated by numerical integration of the Schrödinger equation for complex angular momentum and energy. The corresponding complex and real branch points are located."

Yamamoto, K. 4845
Asymptotic behavior of the scattering amplitude for particles with arbitrary spins.
Nuovo Cimento (10) **27** (1963), 1277–1280.

The author extends Froissart's results [Phys. Rev. (2) **123** (1961), 1053–1057] on the limiting behavior of the total cross-section and elastic scattering amplitude at high energy for spin-zero particles to particles of arbitrary spin. For the total cross-section, Froissart's bound of $(\text{const})\log^2 W$ remains; while for the elastic scattering amplitude (in conventional relativistic normalization), the bound is $(\text{const})W^{3/2}(\log^{3/2} W)f(\theta)$, where W and θ are the total energy and scattering angle in the center of mass frame, and $f(\theta)$ is a function which depends on the particle spins and generalizes the factor $\sin^{-1/2}\theta$ which Froissart found for spinless particles. The author does not evaluate $f(\theta)$ in general, but gives $f(\theta)=\sin^{-3/2}\theta$ for pion-nucleon scattering. The author points out that high-energy bounds can also be given for inelastic scatterings in which both the initial and final states have two particles. *O. W. Greenberg* (College Park, Md.)

Kabir, P. K. (Editor) 4846
★The development of weak interaction theory.
International Science Review Series, Vol. V.
Gordon and Breach Science Publishers, New York-London, 1963. xxv+286 *pp*. $4.95.

This book is a reproduction of 40 articles on the development of the theory of weak interactions. A somewhat unusual feature of the collection is that it covers a rather long and particularly vigorous period: the first article is Fermi's classical paper of 1934 and the last developments reviewed are some of the interesting contributions of the year 1960. There is a short, but well-organized, preface by the editor which puts the subject matter covered into correct perspective. *P. Roman* (Boston, Mass.)

Fried, H. M. 4847
Crossed graphs in the Feinberg-Pais theory of weak interactions.
Phys. Rev. (2) **133** (1964), B1562–B1564.

Author's summary: "A possible damping mechanism is suggested to prevent the occurrence of essential singularities, such as that found on the light cone by Bardakci, Bolsterli and Suura [same Rev. (2) **133** (1964), B1273–B1275], when finite-order expansions of the irreducible Bethe-Salpeter amplitude are iterated in configuration space without prior regularization. An infinite number of irreducible Feynman graphs are considered and approximated by a 'peratization' method; a simple example is found in which the light cone damping, obtained by

Feinberg and Pais [ibid. (2) **131** (1963), 2724–2761; MR **27** #6537] by summing over the regularized ladder graphs, is reproduced by this crossed graph method."
Y. Kato (Kobe)

Ogievetskiĭ, V. I. [Ogieveckiĭ, V. I.]; Polubarinov, I. V. 4848
On the theory of the neutral vector field with spin 1.
Ž. Èksper. Teoret. Fiz. **45** (1963), 709–712 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 487–489.
Authors' summary: "The most general form of the interaction between a neutral vector field and a spinor field is derived. Such interactions turn out to be necessarily invariant with respect to certain phase transformations."
S. L. Glashow (Berkeley, Calif.)

Bialynicki-Birula, I. [Białynicki-Birula, I.] 4849
Elementary particles and generalized statistics.
Nuclear Phys. **49** (1963), 605–608.
It is explicitly shown that the relation $[a_k, N_l]=\delta_{kl}a_k$, $N_k=\frac{1}{2}(a_k{}^+a_k \pm a_k a_k{}^+)$ can be satisfied if and only if the a_k's satisfy the commutation relations for ordinary or "generalized" statistics. An explicit representation for the a_k's is presented [see H. S. Green, Phys. Rev. (2) **90** (1953), 270–273; MR **14**, 1046], in which they are written as linear combinations of independent fermion [boson] annihilation operators $b^{(i)}$ which commute [anticommute] with each other. It is argued that generalized statistics can be appropriate only for composite systems (the present reviewer disagrees with this statement, since it is in the spirit of generalized statistics that only the a's have physical significance, the b's being at most a mathematical tool).
G. F. Dell'Antonio (Naples)

Evans, N. T. 4850
A discrete subgroup for higher-symmetry models. (Italian summary)
Nuovo Cimento (10) **31** (1964), 414–419.
This is the generalization of the reflections about the three axes in the root diagrams of $SU(3)$ [C. A. Levinson, H. J. Lipkin, and S. Meshkov, Nuovo Cimento (10) **23** (1962), 236–244; MR **26** #1109] recently used as a shortcut in place of coefficient manipulation. The discrete symmetries of the Weyl group are presented for an arbitrary compact Lie group, with some examples with respect to C_2 and G_2, etc. *Y. Ne'eman* (Pasadena, Calif.)

Ezawa, Hiroshi; Muta, Taizo; Umezawa, Hiroomi 4851
An approach to the elementarity of particles.
Progr. Theoret. Phys. **29** (1963), 877–892.
This is a discussion of a field-theoretic model illustrating the notion of composite particle (sufficiently mutilated to be simply a two-particle scattering system with possible bound states). An "off-the-mass-shell transition matrix" is introduced which is simply related to the conventionally defined wave function. Off-the-energy-shell unitarity conditions are written down (which have the same content as the orthogonality and completeness relations on the wave functions). The familiar $Z=0$ condition is the criterion for a non-elementary particle. Using a suitable model the deuteron is tested for being a bound state of the neutron and proton, and it is found to be so.
E. C. G. Sudarshan (Syracuse, N.Y.)

Fonda, L.; Ghirardi, G. C.; Rimini, A. 4852
Interpretation of the normalizable state in the Lee model with form factor.
Phys. Rev. (2) **133** (1964), B196–B203.
It is found that in the Lee model with sufficiently convergent form factor, both bound states and resonances in the V-sector can be present simultaneously. For the unstable case, $m_V > m_N + m_\theta$, it is shown by an explicit example that the bound state which appears for sufficiently strong coupling is not related to the bound V state at zero coupling, but rather represents a composite particle. This happens when the form factor has a sufficiently sharp peak. *W. M. Frank* (Silver Spring, Md.)

Gerstein, I. S. 4853
On Moravcsik's principle of equivalence.
Nuovo Cimento (10) **31** (1964), 694–696.
The author finds that Moravcsik's "principle of equivalence" [M. J. Moravcsik, Nuovo Cimento (10) **30** (1963), 466–468; MR **28** #1911] is consistent with the unitary symmetry model of strong interactions, agreeing with the reviewer's review of Moravcsik's work [MR **28** #1911].
S. L. Glashow (Berkeley, Calif.)

Horn, D.; Ne'eman, Y. 4854
Unitary symmetry and the weak currents. II. (Italian summary)
Nuovo Cimento (10) **31** (1964), 879–883.
Part I appeared in Nuovo Cimento (10) **29** (1963), 760–770 [MR **28** #895]. The structure of the weak interaction current is discussed from the point of view of unitary symmetry. Vector currents are assumed to be conserved in the SU_3 limit and to transform like $\pi^+ + K^+$. Axial vector currents are determined by a rather mysterious appeal to the Goldberger-Trieman relations. Agreement with experiment is poor unless further assumptions are made. *S. L. Glashow* (Berkeley, Calif.)

Okubo, S. 4855
Intermediate vector mesons and unitary symmetry.
Phys. Lett. **8** (1964), 362–364.
The weak vector and axial vector currents can be described in the context of unitary symmetry [M. Gell-Mann, Phys. Rev. (2) **125** (1962), 1067–1084; MR **25** #1861; the reviewer, Nuclear Phys. **26** (1961), 222–229; MR **23** #B2856] by a certain linear combination of octet components ($\cos\theta$ "j_{π^+}" $+\sin\theta$ "j_{K^+}") and its charge-conjugate [N. Cabibbo, Phys. Rev. Lett. **10** (1963), 531–533]. The present paper presents a theory in which the nonleptonic strangeness-violating effective four-fermion Hamiltonian also behaves like an octet component, and thereby ensures $|\Delta I| = \frac{1}{2}$. A previous theory of that type [B. D'Espagnat, Phys. Lett. **7** (1963), 209–210] required 6 intermediate bosons, behaving like the 3 and $\bar{3}$ representations of $SU(3)$; the present theory requires only 4 intermediate bosons. For hyperon decays, assuming only scalar and pseudo-scalar (i.e., non-derivative) matrix

elements, the theory gives an explicit prediction with respect to Λ, Ξ^-, and Σ^+ decays; agreement with experiment is rather good. *Y. Ne'eman* (Pasadena, Calif.)

Klein, Abraham; Lee, Benjamin W. 4856

Does spontaneous breakdown of symmetry imply zero-mass particles?

Phys. Rev. Lett. **12** (1964), 266–268.

It has been asserted—but not rigorously proved—that the spontaneous breakdown of elementary particle symmetries leads to the appearance of zero-mass mesons [J. Goldstone, Nuovo Cimento (10) **19** (1961), 154–164; MR **23** #B1417; J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. (2) **127** (1962), 965–970; MR **28** #3712; S. Bludman and A. Klein, ibid. (2) **131** (1963), 2364–2372; MR **27** #5556]. In this paper, a non-relativistic model is considered for which the usual careless arguments indicate (falsely) the existence of a zero-energy excitation. A loophole is found which the authors claim also applies to Lorentz-invariant field theories. It is concluded that "there exists no general proof, independent of model and method of calculation, which establishes the existence of zero-mass particles in field theories with the spontaneous breakdown of symmetry."

S. L. Glashow (Berkeley, Calif.)

Macfarlane, A. J.; Mukunda, N.; Sudarshan, E. C. G. 4857

Generalized Shmushkevich method: Proof of basic results.

J. Mathematical Phys. **5** (1964), 576–580.

Clebsch-Gordan coefficients are most familiar for the rotation group in three dimensions but have been defined also for certain more general classes of groups [see, for example, M. Hamermesh, *Group theory and its application to physical problems*, Addison-Wesley, Reading, Mass., 1962; MR **25** #132]. In this paper it is shown that for any compact group one can define Clebsch-Gordan coefficients that satisfy natural orthogonality conditions. The results are applied to justify a generalization of the Shmushkevich method to SU_3. The symmetries of the general Clebsch-Gordan coefficients are not treated in detail.

W. T. Sharp (Toronto, Ont.)

Fisher, Michael E. 4858

Magnetism in one-dimensional systems. The Heisenberg model for infinite spin.

Amer. J. Phys. **32** (1964), 343–346.

Author's summary: "It is observed that the free-energy, susceptibility, and correlation functions for a linear chain of N spins with nearest-neighbor isotropic Heisenberg coupling can be calculated explicitly in the (classical) limit of infinite spin. The results are compared briefly with those for Ising and Heisenberg chains of spin $\frac{1}{2}$."

Krieger, T. J.; Porter, C. E. 4859

Reduced width amplitude distributions and random sign rules in R-matrix theory.

J. Mathematical Phys. **4** (1963), 1272–1279.

Under the assumption that the reduced width distribution functions are subject to level independence and invariance under orthogonal transformation between channels, they are shown to be products of multivariate gaussian distributions, the channel-channel correlation coefficients being level-independent. This result (which has nothing to do with the connotations of "reduced width", "channel" or "level", and which may be deduced in a straightforward manner) is used as a basis "for a more precise statement of the 'random sign' rules of R-matrix theory than hitherto given. ... The influence of the energy eigenvalue distribution and of variable reduced width amplitude statistics is discussed". *E. C. G. Sudarshan* (Syracuse, N.Y.)

Ullah, Nazakat 4860

Reduced width amplitude distributions for the unitary ensemble.

J. Mathematical Phys. **4** (1963), 1279–1282.

The reduced width amplitude distributions (appropriate to level independence) are determined under the condition that they are invariant under arbitrary unitary transformation between the channels. Some implications are deduced. *E. C. G. Sudarshan* (Syracuse, N.Y.)

Richardson, R. W.; Sherman, N. 4861

Exact eigenstates of the pairing-force Hamiltonian.

Nuclear Phys. **52** (1964), 221–238.

Authors' summary: "The problem of determining the eigenstates of the pairing-force Hamiltonian is reformulated in terms of the eigenstates of a many-boson system with an N-body interaction. The N-body interaction includes the effects of the Pauli principle on the eigenstates of the pairing-force Hamiltonian. Explicit expressions for four types of eigenstates are derived. These four types are the eigenstates of N pairs in one or two multiply degenerate single-particle levels, the one-pair eigenstates and a new restricted class of N-pair eigenstates."

Biedenharn, L. C.; Swamy, N. V. V. J. 4862

Remarks on the relativistic Kepler problem. II. Approximate Dirac-Coulomb Hamiltonian possessing two vector invariants.

Phys. Rev. (2) **133** (1964), B1353–B1360.

Part I (by Biedenharn) appeared in same Rev. (2) **126** (1962), 845–851 [MR **25** #1878]. Authors' summary: "The Dirac-Coulomb Hamiltonian is shown to contain a 'fine structure interaction' which, when removed, defines a new Hamiltonian differing from the Dirac-Coulomb Hamiltonian in order $(\alpha Z)^2/|\kappa|$. The solutions of this new Hamiltonian, as well as its complete set of invariant operators, are explicitly given. This 'symmetric Hamiltonian' possesses a larger symmetry group than the R_4 group structure of the nonrelativistic Coulomb Hamiltonian. The simplicity of the complete orthonormal set of solutions of the symmetric Hamiltonian lends itself to several useful applications which are briefly indicated. The relation between the solutions of this new Hamiltonian and the Sommerfeld-Maue-Meixner-Furry wave functions is discussed."

Mills, D. L. 4863

Ground-state occupancy of an ideal Bose-Einstein gas confined to a finite volume.

Phys. Rev. (2) **134** (1964), A306–A308.

Author's summary: "The number of particles in the ground state has been computed as a function of temperature for an ideal Bose-Einstein gas confined to a box of finite volume by evaluating the discrete sum over states on a computer. Large deviations from London's bulk-gas result are found when the length of the box is much greater than its width for the range of dimensions investigated here. It is shown that the deviations occur because in this limit the system tends to behave like a one-dimensional system."

Pandres, Dave, Jr. 4864

Tensor virial theorems for variational wave functions.

Phys. Rev. (2) **131** (1963), 886–887.

Author's summary: "It is shown that the tensor virial theorem and certain generalisations of it are satisfied by optimum energy variational wave functions in which the variational parameters are elements of a square matrix that scales the column matrix of particle co-ordinates."

D. F. Mayers (Oxford)

Tietz, T.; Florczak, J. 4865

Atomic energy levels and eigenfunctions for the atomic field.

Nuovo Cimento (10) **31** (1964), 691–693.

An approximate analytic method is described for solving the equation

$$\frac{d^2R}{dr^2}+\left[-\alpha^2-2V(r)-\frac{l(l+1)}{r^2}\right]R = 0$$

when the potential function $V(r)$ is given. Constants Z^*, A, n^* and l^* are introduced (to be determined by a variational method) so that

$$-2V(r)-\frac{l(l+1)}{r^2} \simeq \frac{2Z^*}{r}\frac{1+Cr+Dr^2}{(1+Ar)^2(1+Br)}-\frac{l^*(l^*+1)}{r^2},$$

$$C+n^*(n^*+1)A^2/2Z^* = A+B,$$

$$D+n^*(n^*+1)A^2B/2Z^* = A\times B.$$

Then a solution of the form

$$R(r) = r^{(l^*+1)}(1+Ar)^{-n^*}\exp[-\alpha r]f(r)$$

exists with $f(r)=\sum_{i=0}^{m} a_i r^i$, $m=n^*-l^*-1=0, 1, 2, \cdots$. The constants α and a_i are solutions of systems of nonlinear algebraic equations. *C. Froese* (Cambridge, Mass.)

Rapp, Donald 4866

Vibrational energy exchange in quantum and classical mechanics.

J. Chem. Phys. **40** (1964), 2813–2818.

Author's summary: "The probability of vibrational energy exchange in a molecular collision can be calculated using (1) a wave-mechanical treatment using the method of 'distorted waves', (2) a semiclassical time-dependent perturbation procedure in which the perturbation energy is obtained as a function of time from the classical collision trajectory, and (3) a purely classical calculation of the energy transferred to a classical vibrator. These methods are reviewed, related, and compared."

Aizu, Ko; Dell'Antonio, Gianfausto; Siegert, Arnold J. F. 4867

Field operators for bosons with impenetrable cores. II. Equations of motion and general operator formalism.

J. Mathematical Phys. **5** (1964), 471–489.

This paper is a continuation of a paper by the last-named author [Phys. Rev. (2) **116** (1959), 1057–1062; MR **22** #6497]. In the present paper the equations of motion are derived in the Fock representation, and the non-relativistic field theory of hard sphere bosons is discussed.

D. ter Haar (Oxford)

Bonometto, Silvio 4868

Invarianza di misura per le interazioni forti con simmetria isobarica tetradimensionale.

Ist. Veneto Sci. Lett. Arti Atti Cl. Sci. Mat. Natur. **120** (1961/62), 327–344.

The author analyzes the role of generalized gauge transformations in a scheme for strongly interacting particles [J. Tiomno, Nuovo Cimento (10) **6** (1957), 69–83; N. Dallaporta, ibid. (10) **7** (1958), 200–214].

V. de Alfaro (Princeton, N.J.)

Dietrich, Klaus 4869

On the connection of the independent-pair model with a variational principle.

Z. Physik **178** (1964), 335–341.

Author's summary: "It is shown that Brenig's method of deriving the basic equations of the independent-pair model can also be understood from a variational principle applied to the pseudo-expectation value $\langle\Pi|H|\Psi\rangle$ (Π = model wave function, Ψ = true wave function). Deviations from a vanishing first and second variation of $\langle\Pi|H|\Psi\rangle$ with respect to the model wave function Π are shown to be due only to the independent-pair approximation, and thus permit us to make a qualitative estimate of the accuracy of this method." *K. Kumar* (Canberra)

Kobe, Donald H. 4870

Green's function derivation of the method of approximate second quantization.

Ann. Physics **25** (1963), 121–131.

Let the quantity r be the number of annihilation operators and s the number of creation operators. The Green's functions $G_{r,s}$ form coupled integral equations. However, if we neglect the interaction Hamiltonian for all Green's functions for which $r+s$ is greater than some arbitrarily chosen number, the set of coupled integral equations terminates. At the same time, the Feynman type graphs are reduced to simpler forms. (The graphical reduction was considered by the author and W. B. Cheston [same Ann. **20** (1962), 279–305].) If the two-line Green's function is substituted into the reduced equations of the four-line Green's function, it yields $G_{02}=G_{20}=0$. Therefore, in this approximation, the "dangerous diagrams" [N. N. Bogoljubov, V. V. Tolmačev and D. V. Širkov, *A new method in the theory of superconductivity* (Russian), Izdat. Akad. Nauk SSSR, Moscow, 1958; English transl., Consultants Bureau, New York, 1959; MR **25** #5748] are completely eliminated.

In the present paper, the author used the reduced coupled equations for G_{22} and G_{04} to derive the equations of the "method of approximate second quantization"

(MASQ) as obtained by Bogoljubov, Tolmačev and Širkov. However, the equations derived by the present author are of wider scope in that there is no restriction about the spin direction, nor about the type of matrix elements of the potential. Thus the author claims that "the MASQ could be derived from the Green's function equations, but it would be more difficult to reverse the steps and derive the Green's function equations from the MASQ". It is interesting to observe that two apparently different methods lead to the same equation.

{It seems strange that the abstract does not reflect the contents in any respect.} *T. Sasakawa* (Kyoto)

McGuire, J. B. 4871
Study of exactly soluble one-dimensional N-body problems.
J. Mathematical Phys. **5** (1964), 622–636.

The formal similarity between the motion of a single particle in two dimensions and the internal motion of three one-dimensional particles interacting through delta-function potentials is exploited to describe some exactly soluble one-dimensional three-body problems. When the delta functions are of infinite strength, the equivalent two-dimensional problem is analogous to a known electromagnetic diffraction problem. When these solutions are interpreted in terms of the three-body system, it is found that they describe a redistribution of energy among the particles, a result which cannot be achieved with any sequence of two-body interactions. Finite equal-strength delta-function potentials are introduced to obtain the effects of binding and rearrangement. The only general solutions obtainable are of the non-diffracting type, which generate no new velocities. These solutions can only be obtained for equal-mass particles. The S-matrix for this system is obtained and rearrangement collisions and scattering from bound states are described. A diagrammatic technique is presented which illustrates the construction of the S-matrix for N equal-mass one-dimensional particles interacting through finite equal-strength delta-function potentials. An N-particle bound state is found with energy decreasing as N^3, and the scattering of a particle from an $(N-1)$-particle bound state is illustrated. The author points out that solutions for inelastic processes can be related to solutions of generalized diffraction problems in multi-dimensional spaces.

M. Eisner (College Station, Tex.)

Pluhař, Z.; Úlehla, I. 4872
Perturbation theory for system of many-fermions starting from degenerate state.
Czechoslovak J. Phys. **13** (1963), 861–870.

The Van Hove-Hugenholtz resolvent method of perturbation theory applied to single-particle energies [N. M. Hugenholtz, *The many body problem*, pp. 1–45, esp. pp. 26–31, Methuen, London, 1959; MR **21** #2503] is extended to the system of $A+1$ fermions which is n-tuply degenerate in the unperturbed state. The secular equation for the energy eigenvalue is derived. *T. Sasakawa* (Kyoto)

Mikhaylov, I. N. [Mihaĭlov, I. N.] 4873
Partially projected functions in the superfluid model of a nucleus.
Acta Phys. Polon. **24** (1963), 419–425.

Author's summary: "A new set of trial functions is suggested for solving variational problems arising in the superfluid model of a nucleus. These functions satisfy additional invariance conditions which follow from the particle-number-conservation law. From another point of view the new functions contain only a small part of these superfluous components which are present in functions of standard kind. The properties of the new functions are investigated. The energy of a system in the state described by the function which satisfies the stationarity condition is found."

Sass, A. R. 4874
Image properties of a superconducting ground plane.
J. Appl. Phys. **35** (1964), 516–521.

Author's summary: "The Green's-function technique is utilized in the determination of the field distribution of an infinitely long current carrying conductor of arbitrary but constant cross-section above a superconducting ground plane of finite thickness. It is assumed that the superconductor can be described by the phenomenological London equations. The integral expressions that are obtained are solved analytically for a few special cases of interest. Under conditions that are often encountered in a physical system, a modified image method can be utilized in order to calculate the field distribution to within 2% of the computer solution."

D. Mattis (Yorktown Heights, N.Y.)

STATISTICAL PHYSICS, STRUCTURE OF MATTER

Burshtein, A. I. [Burštein, A. I.] 4875
External relaxation.
Fiz. Tverd. Tela **5** (1963), 1243–1257 (*Russian*); *translated as Soviet Physics Solid State* **5** (1963), 908–917.

Integro-differential equations are derived describing relaxation due to a random perturbation. These are applied to the case of interactions of atoms with radiation and with thermal inhomogeneities in solids.

D. ter Haar (Oxford)

Caspers, Willem J. 4876
Dispersion relations for nonlinear response.
Phys. Rev. (2) **133** (1964), A1249–A1251.

Some dispersion relations are discussed for the example of quadratic response of a magnetic system to which two high-frequency fields are applied perpendicular to a constant field. The susceptibility for the response parallel to the constant field is derived from a response function, and some general sum rules are obtained.

R. Kubo (Tokyo)

Dullemond, C. 4877
Coupled Schrödinger equations and statistical boundary conditions.
J. Mathematical Phys. **4** (1963), 1415–1432.

A model is considered in which boundary conditions of the following form are placed on the radial solution of the Schrödinger equation $\partial\psi(r)/\partial r|_{r=a} = F\psi(a)$. When the equations are coupled, ψ becomes a unicolumnar matrix

and F a square one. The author considers the effect of making F statistical. This gives rise to a statistical behavior of the phase shift. Under certain and rather well-known circumstances, the distribution of the phase shift can be considerably narrower than that of the boundary-condition parameter.
H. Feshbach (Cambridge, Mass.)

Lebowitz, J. L.; Percus, J. K. 4878
Integral equations and inequalities in the theory of fluids.
J. Mathematical Phys. **4** (1963), 1495–1506.
The method of functional Taylor expansions, previously employed by these authors [same J. **4** (1963), 116–123; MR **26** #5925] to study non-uniform classical systems, is here developed into a technique for studying the equilibrium distribution functions of a uniform system. The method furnishes a new derivation of a set of exact linear integral equations, originally found by Mayer, and including the Mayer-Montroll and Kirkwood-Salsburg equations; it also furnishes a new set of equations of similar structure, but non-linear, involving conditional probabilities.

If the infinite Taylor series used in the derivation is replaced by a finite series with remainder term, further integral equations are obtained. For non-negative pair interaction potentials, all these integral equations can be turned into integral inequalities of the type first studied by Lieb [ibid. **4** (1963), 671–678; MR **27** #6566]; from these, by a method of successive elimination which is outlined here, one can obtain a set of upper and lower bounds on the density and distribution functions. As an illustration the authors consider the hard-sphere system: the best upper and lower bounds on the density are plotted against fugacity, and the best available approximation to the density fits snugly between the bounds. A similar comparison for the pair distribution function is also made.

Potentials which do take negative values are also discussed briefly, and the paper closes with a section on some further inequalities which are not obtainable by the functional Taylor series method. *O. Penrose* (London)

Smith, V. H., Jr.; Gersch, H. A. 4879
Cluster expansions and the theory of many-boson systems.
Progr. Theoret. Phys. **30** (1963), 421–434.
Authors' summary: "The previous configuration-space cluster integral treatment of the properties of the ground state of a many-boson system is modified by including those higher order diagrams consistent with the pair excitation approximation, which were previously omitted. The cluster expansions involve a parameter, analogous to the fugacity in classical expansions, whose definition automatically accounts for the depletion of the free-particle ground state. The resulting expectation value for the Hamiltonian is in agreement with that obtained by field-theoretic methods for states of pair-excitation type."
D. Mattis (Yorktown Heights, N.Y.)

Temperley, H. N. V. 4880
On the mutual cancellation of cluster integrals in Mayer's fugacity series.
Proc. Phys. Soc. **83** (1964), 3–16.
An attempt is made to evaluate the cluster integrals b_l for large l by starting from the complete diagram of l points and $\frac{1}{2}l(l-1)$ lines and dropping lines successively. For the "Gaussian model", defined by putting $f_{12} = -\exp(-r_{12}^2)$, a theorem can be proved, and with some additional assumptions the tentative conclusion is reached that the Mayer fugacity series has no singularity on the positive real axis. For the rigid-line and rigid-square models a similar treatment leads to rather inconclusive results. *K. Schram* (Utrecht)

Weinstock, Jerome 4881
Cluster formulation of the exact equation for the evolution of a classical many-body system.
Phys. Rev. (2) **132** (1963), 454–469.
An exact non-Markoffian equation is derived for the evolution of an infinite homogeneous system. This equation—which may be viewed as a time-dependent analogue of the equilibrium virial expansion—may be readily applied when the forces between particles include infinite repulsions. The derivation of this equation from Liouville's equation is analogous to Mayer's derivation of the virial expansion from the partition function. In this way the formal development of non-equilibrium statistical mechanics is placed on a similar footing to that of equilibrium statistical, and a many-body problem is reduced to understanding the dynamics of isolated groups of particles. Fourier expansions and expansions in powers of the interaction potential are avoided by dealing with s-body Green functions (propagators) which are always convergent functions of the interaction potential. These functions correspond to multiplet collisions in ordinary configuration space between s isolated particles and are time-dependent analogues of the irreducible clusters well known in equilibrium statistical mechanics. The kernel (memory) of the equation of evolution consists of a linear sum of the time-dependent irreducible clusters. The non-Markoffian behavior of the equation of evolution is thus directly given by the time dependence of these clusters, and is explicitly related to incompleted collisions. The equation of evolution is solved in the asymptotic limit of long times. In this limit it is found (because the kernel rapidly vanishes) that the equation reduces to a Markoffian master equation involving a scattering operator for both completed and incompleted collisions in configuration space. *G. F. Dell'Antonio* (Naples)

Weinstock, Jerome 4882
Three-body scattering operator in nonequilibrium statistical mechanics.
Phys. Rev. (2) **132** (1963), 470–482.
A method is presented for calculating the time-dependent irreducible clusters $\beta_s(t)$ which appear in the kernel of the equation of evolution derived in the preceding article [#4881]. The clusters $\beta_1(t)$ and $\beta_2(t)$—which correspond to binary and ternary collisions, respectively—are calculated in detail. They are each found to divide into the two following parts: (1) a "completed" collision part which corresponds to collisions which are eventually completed (scattering processes) and (2) an "incompleted" part which corresponds to those collisions not completed by time t. The incompleted collision parts contribute to the "memory" of the equation of evolution and are shown to

be relatively small when t is large. The completed collision parts, which play a central role in the theory of transport coefficients, are time-independent scattering operators in momentum space and do not contribute to the memory. By means of the "binary-collision expansion" a systematic method is presented for the calculation of the three-body scattering operator $[\lim_{t\to\infty} t^{-1}\beta_2(t)]$ which is directly applicable to interaction forces with infinite repulsions. An approximate formula is then derived for this scattering operator in a form which can be readily used to calculate the density correction to transport coefficients which arise from ternary collisions. *G. F. Dell'Antonio* (Naples)

Zalewski, Kacper 4883

Long time behaviour of the response function.

Acta Phys. Polon. **23** (1963), 801–809.

The asymptotic behavior of the function

$$\varphi(t) = \int_{-\infty}^{\infty} f(\omega)e^{i\omega t}\,d\omega$$

is discussed by extending Saito's treatment [Phys. Rev. (2) **117** (1960), 1163–1173; MR **22** #6138]. As an example this function is calculated for a model of a system of coupled oscillators which has been discussed previously by Kogure [J. Phys. Soc. Japan **16** (1961), 14–23; MR **23** #B2414]. *R. Kubo* (Tokyo)

Bronk, Burt V. 4884

Accuracy of the semicircle approximation for the density of eigenvalues of random matrices.

J. Mathematical Phys. **5** (1964), 215–220.

Certain statistical properties of the energy levels of complex physical systems have been found to coincide with those for distributions of eigenvalues derived from ensembles of random matrices. However, if ensembles of random matrices give a fair representation for the Hamiltonian of a complex physical system, the density of the characteristic values at the lower end of the spectrum should show some similarity with the exponential dependence found in nuclear spectra. The limiting distribution of the density for very high-dimensional random matrices is a semicircle, i.e., concave from below if plotted against the characteristic value which represents, in this case, the energy.

In this paper the deviations from the limiting distribution are investigated and it is shown that there is a region, at the very lowest part of the spectrum, where the density is convex from below, similar to an exponential function. The region of convexity is called the tail of the distribution. It is shown, however, that the average number of roots in the tail is very small, of the order of 1. It is concluded that those ensembles of random matrices which have been studied up to now do not give a fair representation of Hamiltonians of complex systems.

I. M. Barbour (Rome)

Fujita, S. 4885

Remarks on Kubo's formula and on Kadanoff-Baym's equations.

Phys. Lett. **9** (1964), 119–121.

The author points out that Kubo's formula of electrical conductivity can be calculated from a generating function which is the expectation of current for the modified density operator, $\rho' = C\exp[\alpha N - \beta H + \beta Jx]$. For evaluation of this generating function, the Green function method may be used; an evolution equation is derived for this purpose. *R. Kubo* (Tokyo)

Korringa, J. 4886

Dynamical decomposition of a large system.

Phys. Rev. (2) **133** (1964), A1228–A1229.

Author's summary: "The equations for the statistical matrix ρ_θ and the correlation matrix $\rho_\theta(t_1, t_2)$ of a large system subject to time-dependent external forces are cast in a new form based on a continuation of Schrödinger's equation to complex times. It is shown that these equations also apply when the time dependence of the Hamiltonian is due to a coupling with another system (heat bath) which is at equilibrium at a temperature θ', but that such time dependence must be expressed in terms of random functions of the complex argument $z' = t - i/2\theta'$. A generalization to the case that the external time-dependent fields and the coupling with a heat bath exist simultaneously is straightforward and leads to a Schrödinger-type equation with non-Hermitian Hamiltonian which describes the dynamical and statistical aspects of the motion." *S. Fujita* (Brussels)

Korringa, J.; Motchane, J.-L.; Papon, P.; Yoshimori, A. 4887

Derivation of the modified Bloch equations for spin systems.

Phys. Rev. (2) **133** (1964), A1230–A1234.

Authors' summary: "A new form of the equations for the time-dependent statistical matrix of a spin system is used to derive the modified Bloch equations, without considering details of the relaxation mechanism. A number of restrictive conditions are imposed on the system, most of which agree well with known limitations of the applicability of the modified Bloch equations. In order to avoid complications, only spins one-half are considered. The theory does not apply to the case where the relaxation is anisotropic but where the constant field and an applied rotating field are comparable in magnitude."

S. Fujita (Brussels)

Wertheim, M. S. 4888

Analytic solution of the Percus-Yevick equation.

J. Mathematical Phys. **5** (1964), 643–651.

The properties of the Percus-Yevick equation for the pair distribution function in classical statistical mechanics are studied for pair potentials consisting of a hard core plus a short-range tail. It is surmised that one dimension shows no phase transition. In three dimensions the analysis is complicated, but a phase transition is no longer prohibited.

D. ter Haar (Oxford)

Derrick, G. H. 4889

Simple variational bound to the entropy.

Phys. Rev. (2) **133** (1964), A1215–A1217.

A variational principle is used to find for the entropy $S(E)$ of a system with energy E the inequality $S(E) \geq -k \ln \operatorname{Tr} U^2$ for non-negative normalised density matrices.

D. ter Haar (Oxford)

Khachaturyan, A. G. [Hačaturjan, A. G.] 4890
Application of the Green's function method to the thermodynamics of interstitial solutions.
Fiz. Tverd. Tela **5** (1963), 15–20 (*Russian*); *translated as Soviet Physics Solid State* **5** (1963), 9–12.
The Green's function technique is applied to the problem of equilibrium of interacting interstitial atoms in solutions. A nonlinear equation of the Hammerstein type is obtained for the occupation numbers, and symmetry results are used to analyze the possible solutions of this equation. A procedure is outlined by which it is possible to make a unique theoretical determination of the points where the high-temperature phase loses stability, and of the symmetry of the new phase. *J. Zak* (Cambridge, Mass.)

Khachaturyan, A. G. [Hačaturjan, A. G.] 4891
Nonlinear equations of integral type and their applications to the problem of ordered alloys.
Fiz. Tverd. Tela **5** (1963), 26–35 (*Russian*); *translated as Soviet Physics Solid State* **5** (1963), 16–22.
A method of solving nonlinear equations of integral type is developed. The method is applied to the problem of binary solid solutions and in particular to the ordering of a disordered solid solution in a face-centered cubic lattice. Results are obtained taking into account the interactions in all coordination spheres. *J. Zak* (Cambridge, Mass.)

Khachaturyan, A. G. [Hačaturjan, A. G.] 4892
Nonlinear equations of integral type and their application to the study of the crystal symmetries of interstitial solutions.
Fiz. Tverd. Tela **5** (1963), 750–758 (*Russian*); *translated as Soviet Physics Solid State* **5** (1963), 548–553.
Nonlinear equations of integral type are applied for studying the equilibrium distribution of interacting interstitial atoms in solutions. A method previously developed by the author for solving nonlinear integral equations is applied to the case of complicated lattices which contain more than one atom in the unit cell. Ordered structures of interstitial atoms in face-centered and body-centered cubic lattices are treated. *J. Zak* (Cambridge, Mass.)

Guénault, A. M.; MacDonald, D. K. C. 4893
Non-linear Brownian movement of a generalized Rayleigh model. II. Extension of the model to include 'sticky' collisions.
Proc. Roy. Soc. Ser. A **275** (1963), 175–189.
This paper is an interesting sequel to one by Alkemade, van Kampen, and MacDonald [same Proc. **271** (1963), 449–471; MR **27** #6567], in which the nonlinear statistical behavior of the Rayleigh piston model was investigated. In that paper the gas-piston interaction consisted of instantaneous collisions, so that one could speak of the "private energies" of the gas and the piston ("weak coupling"). In the present paper the authors consider the effect produced by gas molecules which may stick to the piston for finite periods of time. This produces an element of strong coupling between the gas and piston. By requiring that the sticking and emission occur instantaneously, they retain the Markovian nature of the interaction. The expansion of the master equation and the analysis for both models are carried to one higher order of nonlinearity than in the previous paper. In particular, they discuss the calculation of the auto-correlation function beyond the purely linear approximation. Also other differences between the two models, which appear in the nonlinear approximation, are discussed.
E. A. Jackson (Urbana, Ill.)

Montgomery, David; Tidman, Derek A. 4894
Secular and nonsecular behavior for the cold plasma equations.
Phys. Fluids **7** (1964), 242–249.
Authors' summary: "The origin of 'secular' behavior for the nonlinear cold plasma equations is studied. The equations involved are closely related to the Klein-Gordon equation with a small nonlinear term. A method is developed for arriving at perturbation-theoretic solutions of this equation, and the method is then applied to the case of the higher-order effects of an electromagnetic wave propagating in the cold electron plasma. An explicit expression for the second-order frequency shift is calculated." *E. A. Jackson* (Urbana, Ill.)

Weitzner, Harold 4895
Radiation from a point source in a plasma.
Phys. Fluids **7** (1964), 72–89.
At the initial time, a point disturbance is set in a plasma where the state is assumed to be spatially homogeneous and time-independent, with no electric or magnetic fields. The point disturbance set at the initial time will be a source of radiation which is the main object of interest in this paper. The particle (electron) distribution is considered as subject to the linearized relativistic Vlasov equation, and the fields to the Maxwell equations. The initial disturbances are given as appropriate to the construction of Green's functions, that is, they are given by δ-functions of the position vector $\mathbf{r}$. For example, the perturbed part of the distribution function is given by $f(\mathbf{r}, \mathbf{p}, t)_{t=0} = \delta(\mathbf{r})\tilde{g}(\mathbf{p})$, where $\mathbf{p}$ denotes momentum. The method of solution was proposed earlier by the author [Phys. Fluids **5** (1962), 933–946; MR **26** #4747]. Functions are processed by means of Fourier transformations with respect to $\mathbf{r}$ and Laplace transformations with respect to t. The electric field, both the transverse and longitudinal parts, are well traced. The oscillations of the plasma are investigated. No signal travels faster than the speed of light. {The reviewer has a question: Is the dynamical effect of the magnetic field on particles to be neglected in relativistic cases where high velocities of particles may be expected to appear?} *T. Koga* (Raleigh, N.C.)

Kessel', A. R. [Kessel', A. R.]; Morocha, A. K. [Moroča, A. K.] 4896
Equations of motion for effective spin.
Fiz. Tverd. Tela **5** (1963), 1528–1536 (*Russian*); *translated as Soviet Physics Solid State* **5** (1963), 1110–1115.
Representation theory is used for the derivation of a system of $4S^*(S^*+1)$ Bloch-type equations from the system of the usual $4S(S+1)$ equations. S^* is the effective spin while S is the spin. The former system contains usually a smaller number of equations and is therefore more convenient to deal with. The derived equations are applied to rare earth ions and solutions are found for

different possible interactions between the paramagnetic ion and the lattice. Expressions are also derived for the relaxation time. *J. Zak* (Cambridge, Mass.)

Klein, Max 4897

Approximations to the pair correlation function for a hard-sphere fluid.

Phys. Fluids **7** (1964), 391–401.

This paper is a continuation of the earlier work by the author [the author and M. S. Green, J. Chem. Phys. **39** (1963), 1367–1387; the author, ibid. **39** (1963), 1388–1397] on the diagram analyses of the classical pair correlation function. After a review of the diagram summation techniques which lead to the so-called convoluted-hypernetted chain approximation, the author introduces a new series of approximations. He considers the form of the pair correlation function, $G(R)$, given by the CHNC theory,

$$G(R) = \exp[-\beta V(R)+S(R)]-1,$$

where $S(R)$ is the sum of all nodal or series diagrams and $V(R)$ is the pair potential function, expands $\exp[S(R)]$, and retains only a finite number of terms. Thus

$$G(R) = \exp[-\beta V(R)] \sum_{j=0}^{n} \frac{[S(R)]^j}{j!} - 1$$

for $n = 1, 2$ yields the two new approximations considered.

Numerical results are then presented for a number of approximations, including the Percus-Yevick approximation, for systems of rigid spheres. Some of this numerical work is superfluous since an analytic solution for the Percus-Yevick approximation is now known [M. S. Wertheim, Phys. Rev. Lett. **10** (1963), 321–323; MR **27** #1251].

{In the reviewer's opinion, these diagram summation techniques have proved to be sterile as applied to the theory of the radial distribution function. The approximations, such as the ones introduced in this paper, are arbitrary with no physical or mathematical basis. The CHNC and PY approximations can now best be understood in terms of the functional Taylor series expansions introduced by Percus [ibid. **8** (1962), 462–463]. Numerical work on all the infinite number of formal diagram manipulations now seems pointless unless these approximations can be given some physical content.}

Z. Salzburg (Houston, Tex.)

Lowry, B. A.; Davis, H. T.; Rice, S. A. 4898

Perturbation calculation of mixed pair correlation functions.

Phys. Fluids **7** (1964), 402–406.

In recent years there has been an interest in perturbation expansions of statistical mechanical functions. These theories are usually based upon a Taylor series development in powers of a perturbation potential divided by kT. This article applies this technique in a straightforward manner to the pair correlation function. Similar treatments can be found in articles by Mazo [J. Chem. Phys. **29** (1958), 1122–1128], Buff and Schindler [ibid. **29** (1958), 1075–1081] and Smith and Alder [ibid. **30** (1959), 1190–1199; MR **21** #2387]. The major difference in these developments is that Lowry et al. first factor out the Boltzmann weighting factor due to the direct perturbation between the pair of particles before carrying out the formal Taylor expansion. The same result can be achieved by simply expanding the potential of mean force between pairs of molecules, and this is certainly a more physically meaningful expansion technique.

{This type of perturbation expansion has very little meaning for strongly repulsive potentials. The authors fail to make this clear, but the only application they consider in detail is concerned with variations in the attractive region of the potential field.}

Z. Salzburg (Houston, Tex.)

Birss, R. R. 4899

Magnetic symmetry and "forbidden" effects.

Amer. J. Phys. **32** (1964), 142–152.

The distinction between magnetic and nonmagnetic crystal classes is illustrated by reference to cubic lattices. The appearance of "forbidden" effects in various crystal classes is discussed and recent experimental work on pyromagnetism, the magnetoelectric effect and piezomagnetism in antiferromagnetic materials is reviewed.

Werner Nowacki (Bern)

Izyumov, Yu. A. [Izjumov, Ju. A.] 4900

The Green's function method in the theory of ferromagnetism.

Fiz. Tverd. Tela **5** (1963), 717–723 (*Russian*); *translated as Soviet Physics Solid State* **5** (1963), 523–527.

The Green's function method of Bogoljubov and Tjablikov [Dokl. Akad. Nauk SSSR **126** (1959), 53–56; MR **21** #7628] in magnetism is extended by allowing the axis of quantization of each spin to point along an arbitrary direction. This enables their decoupling approximation to be applied to antiferromagnets and spiral antiferromagnets, the poles of the Green's function yielding the magnon spectrum in each case.

D. Mattis (Yorktown Heights, N.Y.)

Bellman, Richard; Kalaba, Robert; Vasudevan, Ramabhadra 4901

Invariant imbedding theory of neutron transport: Correlation functions.

J. Math. Anal. Appl. **8** (1964), 225–231.

The authors consider a transport model consisting of a rod of length x in which particles can move to right or left. Moving particles cannot collide with each other but may collide with the (fixed) nuclei of the rod. Both macroscopic cross-section and speed are constant. Each collision results in binary fission. A single particle is injected into the system at time $t=0$.

Let $u_1(x,t)dt$ be the probability that a particle is reflected from the rod in time interval $(t, t+dt)$. Denote by $u_2(x, t_1, t_2)dt_1dt_2$ the probability that one particle is reflected in time interval (t_1, t_1+dt_1) and another is reflected in (t_2, t_2+dt_2). It is shown that the particle-counting invariant-imbedding technique, previously used to obtain an integro-differential equation for u_1, may be extended to obtain an equation satisfied by u_2. The Laplace transform is used to connect these equations with known results concerning the mean square number of particles reflected over all time.

A similar investigation is made for energy-dependent particle distributions. *G. M. Wing* (Albuquerque, N.M.)

Gallone, S.; Ghilardotti, G. 4902

On the use of Case's general solution of the transport equation in neutron transport problems. (Italian summary)

Nuovo Cimento (10) **31** (1964), 203–218.

Case's method for the representation of solutions for the one-velocity neutron transport equation in plane geometry is summarized and applied to the asymptotic solution in a non-capturing medium, and the extrapolation distance in Milne's problem. Probably the most valuable section is on the comparison of Case's solution with the solution by spherical harmonics which is given in a more elegant form than has been seen by the reviewer so far.

E. H. Bareiss (Argonne, Ill.)

Michel, K. H.; Verboven, E. 4903

On the interference between streaming and collision in transport equations.

Phys. Lett. **8** (1964), 176–177.

When conduction electrons are accelerated by an external field, the interacting phonons also feel the field. Therefore the customary simple drift term in a Boltzmann-Bloch type equation must be modified if one goes to higher-order approximations of the interaction. Starting from a Kubo formula, the authors analyse this situation in terms of the Green's function approach. *R. Kubo* (Tokyo)

Moskalev, O. B. 4904

The critical regime of a reactor with a non-linear relation between temperature and neutron flow in the approximation of the one-speed transport equation. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 166–168.

Shure, Fred; Natelson, Michael 4905

Anisotropic scattering in half-space transport problems.

Ann. Physics **26** (1964), 274–291.

The objective of this paper is to extend the results of Case [same Ann. **9** (1960), 1–23; MR **22** #357] for half-space transport problems to include anisotropic scattering. The results again are conveniently presented in terms of an X-function. Three identities involving the X-function are generalized to incorporate the scattering anisotropy. A table of the outgoing neutron angular density is given for five different values of an anisotropy parameter.

W. Sangren (San Diego, Calif.)

Zezula, Rostislav 4906

Über eine Näherungsmethode zur Lösung gewisser Randwertaufgaben der Reaktortheorie.

Comment. Math. Univ. Carolinae **3** (1962), *no.* 2, 37–60.

The eigenvalue problem

$$\Delta\varphi + B^2\varphi = 0,$$

$$\varphi(R, z) = \varphi(r, \pm H/2) = 0,$$

$$(*) \qquad \left.\frac{\partial\varphi}{\partial r}\right|_{r=a} = K(z)\varphi(a, z),$$

where $K(z)$ is a step function, arises in the study of a bare homogeneous finite reactor with an axial control rod. Standard arguments lead to a solution of the form

$$\varphi(r, z, B^2) = \sum_{k=1}^{\infty} \xi_k h_k(r, B^2) g_k(z),$$

and condition (*) yields an infinite set of equations

$$(**) \qquad C_j(B^2)\xi_j = \sum_{k=1}^{\infty} \xi_k K_{jk}, \quad j = 1, 2, \cdots,$$

where the K_{jk} are the matrix elements of $K(z)$ with respect to the orthonormal set $\{g_k\}$ and the C_j are known.

The author investigates, in a somewhat more abstract setting, the possibility of solving (**) by placing $\xi_1 = 1$, solving the inhomogeneous system for $j \geqq 2$ for arbitrary B, and then so choosing B as to satisfy the equation corresponding to $j = 1$. *G. M. Wing* (Albuquerque, N.M.)

RELATIVITY

See also 4768, 4939, 4940, 4942.

Noll, Walter 4907

Euclidean geometry and Minkowskian chronometry.

Amer. Math. Monthly **71** (1964), 129–144.

The term "Minkowskian chronometry" here implies the study of the structure of space-time associated with the special theory of relativity, and it is the purpose of the present paper to axiomatize the subject by means of direct coordinate-free methods without restriction on the number of dimensions. A brief enumeration of definitions and results concerning vector spaces $\mathfrak{V}$ whose inner products $\mathbf{u}\cdot\mathbf{v}$ ($\mathbf{u}, \mathbf{v} \in \mathfrak{V}$) are not necessarily positive definite is given. A vector $\mathbf{v}$ is said to be time-like if $\mathbf{v} \in \mathfrak{V}_{-1}$, where $\mathfrak{V}_{-1} = \{\mathbf{v} | \mathbf{v}^2 < 0 \text{ or } \mathbf{v} = 0\}$. The maximal dimension of subspaces contained in $\mathfrak{V}_{-1}$ is called the index ind $\mathfrak{V}$ of $\mathfrak{V}$. Certain inequalities valid when ind $\mathfrak{V} = 1$ are derived: the most notable of these are the reversed triangle and Schwarz inequalities. The axiomatic treatment of pseudo-Euclidean geometry is based on the fact that the structure of the latter is determined by its separation function (which is the square of the metric in the Euclidean case). Associated with each pseudo-Euclidean space is a unique vector space, the so-called translation space. When the index of the latter is zero or unity, the pseudo-Euclidean geometry reduces respectively to Euclidean geometry or Minkowskian chronometry. An axiomatic treatment of Minkowskian chronometry based on notions such as observers, clock-readings, signals, etc., indicates that these data uniquely determine the separation function. The index of the translation space is shown to be unity. The resulting concept of temporal order is discussed in detail.

H. Rund (Pretoria)

Mould, Richard A. 4908

Internal absorption properties of accelerating detectors.

Ann. Physics **27** (1964), 1–12.

A Lorentz-invariant detector is operationally defined. It can be expressed as a line integral over a closed path in Minkowski space which transforms like a four-vector, p^μ. The definition can be extended to more general flat spaces describing apparent gravitational fields. The spacelike or timelike nature of p^μ is offered as an invariant criterion

for the existence of radiation corresponding to absorption of energy if and only if p^μ is timelike. It is then shown that such a detector, when uniformly accelerated past a Coulomb field, will register radiation. The difference in behavior of this detector for accelerated and non-accelerated motion is consistent with the predictions on the basis of the principle of equivalence.
F. Rohrlich (Syracuse, N.Y.)

Treder, Hans-Jürgen 4909
Singuläre Einstein-Räume und die Feldgleichungen von Einstein und Rosen.
Math. Nachr. **26** (1963), 167–174.
The author considers space-times which are such that the metric tensor has a determinant g which vanishes in a four-dimensional coordinate patch of finite extent. These space-times are required to satisfy a modification of the Einstein field equations and to have coordinate systems in which the metric tensor satisfies $g_{\mu 0}=0$, $\det g_{ik}=-\gamma^2<0$ ($\mu=0,1,2,3$; $i,k=1,2,3$). In addition, the vacuum field equations are replaced by $g^m R_{\mu\nu}=0$, where m is positive and $R_{\mu\nu}$ is the Ricci tensor formed from the $g_{\mu\nu}$. The case $m=2$ was discussed by Einstein and Rosen [Phys. Rev. (2) **48** (1935), 73–77]. Particular attention is paid to the static case, and the nature of the behavior of the $g_{\mu\nu}$ as functions of the coordinates in the vicinity of the boundary of the singular region is determined.
A. H. Taub (Berkeley, Calif.)

Popovici, Andrei 4910
Séparation conforme des états de spin de la particule de spin maximal 2.
C. R. Acad. Sci. Paris **258** (1964), 3207–3209.
This paper concerns itself with the reduction of nonlinear field equations of the type of general relativity [A. Popovici, A. Hristev, and J. Popovici, same C. R. **257** (1963), 3120–3122; MR **28** #1565], imposed in the framework of a six-dimensional "conformal Riemannian manifold" $C_6{}^1$, into sets of field equations in four-dimensional Riemannian space describing fields of spin 2, 1, and 0.
R. Ingraham (University Park, N.M.)

Bonnor, W. B.; Swaminarayan, N. S. 4911
An exact solution for uniformly accelerated particles in general relativity.
Z. Physik **177** (1964), 240–256.
An exact solution of the vacuum field equations is exhibited and interpreted as the field of two pairs of particles. In a special coordinate system they appear to move collinearly in opposite directions with constant acceleration. In all cases save one, the particles are connected by "wires" along the axis; in the exceptional case, two particles' masses are equal and positive, and two are equal but negative. The authors show that a "potential" function occurring in the metric arises from $\frac{1}{2}$(advanced + retarded) contributions from the particles. These appear to be point-like singularities, and they are shown to move freely in the fields of the remaining particles. (That self-fields can be identified and subtracted unambiguously follows from the above-mentioned property of the "potential".) In the special coordinates, the metric is Minkowskian at infinity, except along null-lines asymptotic to the particles' world-lines. The solution describing four free particles is a completion of some earlier work of Bondi [Rev. Modern Phys. **29** (1957), 423–428; MR **19**, 814].
R. H. Boyer (Liverpool)

Edelen, Dominic G. B. 4912
Transformation-theoretic problems in variant field theory. I. Fundamentals.
J. Math. Mech. **13** (1964), 201–214.
This article is the first installment of a gloss on the author's book [*The structure of field space*, Univ. California Press, Berkeley, Calif., 1962].
The author's fundamental space is a differentiable manifold with a symmetric affine connexion. He also postulates a Lagrangian density containing additional, physically significant, field variables. The author is at pains to assert that these fields need not transform covariantly under coordinate transformations. The form and consequences of the variational equations associated with such a Lagrangian density constitute the burden of this rather formal article. *R. H. Boyer* (Liverpool)

Edelen, Dominic G. B. 4913
Material momentum-energy tensors and the calculus of variations.
Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 367–372.
Author's summary: "The purpose of this note is to demonstrate certain intrinsic limitations imposed on the Einstein theory of general relativity by the requirement that the field equations obtain from a homogeneous variational principle. These limitations result in significant constraint as to the nature of physical processes that can be examined. In particular, generalized irreversible processes are disallowed. Thus, if one attempts to develop relativistic fluid mechanics from a variational statement, any idea of entropy is highly artificial. On the other hand, if a variational statement is not required, the existence of intrinsic quantities with the properties of temperature and entropy can be rigorously established."
D. K. Sen (Toronto, Ont.)

Finzi, Arrigo 4914
On the hypothesis of the variation of the gravitational constant, and on some of its consequences.
Ann. Physics **26** (1964), 411–417.
The author further examines the hypothesis that the "constant" of gravitation varies according to the law $k=k_0[1+a(V/c^2)]$, where a is a constant and V is the gravitational potential. He suggests how observations of white dwarfs, and the problem of the critical mass of neutron stars, could throw light on this hypothesis. The author considers that the secular variation of some other "constants" of nature would imply the secular divergence of various standards of length and time which would lead to conflicting metrics, whence "general relativity would be left without much physical content", whereas "there are not equally cogent reasons for ruling out a variation of . . . k". This reviewer disagrees: the variable k invalidates Einstein's field equations, whereas Marzke and Wheeler [*Gravitation and relativity*, pp. 40–64, Benjamin, New York, 1964; MR **28** #970] have shown that general relativity allows the units of length and time to be compared

geometrically at a distance without the need for extraneous clocks and rods, and thus the possible variation of physical standards of length and time does not affect the theory. *W. Rindler* (Dallas, Tex.)

Goel, G. C. 4915

Generalized Riemann spaces of general relativity.

Tensor (*N.S.*) **15** (1964), 5–11.

This paper is concerned with the unified field theory of Einstein in which g_{ij} and $\Gamma_{ij}{}^h$ are asymmetric and such that

$$g_{ik;l} \equiv g_{ik,l} - g_{sk}\Gamma_{il}{}^s - g_{is}\Gamma_{lk}{}^s = 0,$$

a subscript comma denoting partial differentiation. The author considers the special case for which

$$g_{sk}\Gamma_{\check{il}}{}^s + g_{is}\Gamma_{\check{lk}}{}^s = 0,$$

where the inverted circumflex as usual denotes the skew-symmetric part of the symbol to which it is attached. He obtains a variety of formulae, including

$$g_{\check{ij},k} + g_{\check{jk},i} + g_{\check{ki},j} = 0,$$

which leads him to identify $g_{\check{ij}}$ with the electromagnetic field tensor. Other equations obtained are

$$(*) \qquad \Gamma^i_{\check{ik}/i} = 0,$$

$$(**) \qquad R_{(jk)} + \Gamma_{ij}{}^h\Gamma_{hk}{}^i = 0,$$

where the solidus denotes covariant differentiation with respect to the symmetric $\Gamma_{ij}{}^h$ and $R_{(jk)}$ is the Ricci tensor formed from these Γ's. He concludes with the remark that equation (*) "may explain purely electromagnetic phenomena" and that equation (**) "is a combined representation of gravitation and radiation (or some other field)". *H. S. Ruse* (Leeds)

Hély, Jean 4916

Remarques sur les états de radiation de type intégrable.

C. R. Acad. Sci. Paris **258** (1964), 1415–1418.

The author has considered [same C. R. **249** (1959), 1867–1868; MR **22** #9251; ibid. **251** (1960), 1981–1982; ibid. **252** (1961), 3754–3756; MR **23** #B1530; ibid. **257** (1963), 2083–2085; MR **27** #5584] gravitational radiation defined by the metric

$$g_{\alpha\beta} = a\delta_{\alpha\beta} + e_\alpha b_\beta + e_\beta b_\alpha,$$

where e_α is a gradient and a nul-vector. Here it is shown that this general form includes the Robinson-Trautman metric, which includes all integrable metrics of types IIIa and IIb [see E. T. Newman and T. W. J. Unti, J. Mathematical Phys. **3** (1962), 891–901; MR **26** #5946; E. T. Newman and L. A. Tamburino, ibid. **3** (1962), 902–907; MR **26** #5947], and that in an infinity of ways. The hypothesis is made that the above form may include all metrics of integrable total radiation, with the additional condition $e^\alpha\partial_\alpha a = 0$. *C. W. Kilmister* (London)

Jankevič, Č. [Jankiewicz, Cz.] 4917

The stationary gravitational field in a conformal space. (Russian)

Acta Phys. Polon. **24** (1963), 13–22.

The author considers the solutions of Einstein's field equations of a stationary gravitational field. Writing the Riemannian metric of the continuum in the form

$$ds^2 = -e^{2\Phi}h_{ij}\,dx^i\,dx^j + e^{-2\Phi}(dx^0 + f_i\,dx^i)^2 \quad (i, j = 1, 2, 3),$$

he defines a three-dimensional space with the metric $d\sigma^2 = h_{ij}\,dx^i\,dx^j$ to be conformal corresponding to the space-time manifold. In the case when the latter is stationary, he shows that the only solution of the field equations of an empty world is a flat space-time. The harmonic coordinates of it become then the cartesian coordinates. *A. H. Klotz* (Liverpool)

Liebscher, Ekkehard 4918

Zur gravitativen Wechselwirkung von Photonen.

Math. Nachr. **26** (1963), 175–179.

This paper discusses the equations of motion of a test dipole of zero mass (a test particle with spin) in the gravitational field of a pencil of light. The metric of the space-time used is that given by Brinkmann [Math. Ann. **91** (1924), 269–278; ibid. **94** (1925), 119–145]. A complete discussion of the equations of motion cannot be given until these equations are augmented by a supplementary condition [cf. E. Corinaldesi and A. Papapetrou, Proc. Roy. Soc. London Ser. A **209** (1951), 259–268; MR **13**, 695; A. H. Taub, #4923 below]. Nevertheless, the author is able to show in this more exact treatment the result obtained by Tolman [*Relativity, thermodynamics and cosmology*, pp. 274–279, Clarendon, Oxford, 1934] in the approximate treatment of a test monopole of zero mass in the gravitational field of a pencil of light, namely, that when the test system is moving parallel to the pencil of light, its external motion is not affected by the pencil. *A. H. Taub* (Berkeley, Calif.)

Namysłowski, Józef 4919

Symmetrical form of Dirac matrices in general relativity.

Acta Phys. Polon. **23** (1963), 673–684.

The author investigates explicit forms of the solutions of the equations $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\cdot 1$ for the generalized Dirac matrices γ_μ, where $g_{\mu\nu}$ is the metric tensor of a general space-time. Special coordinate systems are used in the spin spaces. The author does not state an invariant characterization of these coordinate systems, but such a characterization can be given, namely: The coordinate systems are those in which the generalizations of γ_5, the charge conjugation spinor C, and the spinor which maps the spin space onto its dual space have particular values. *A. H. Taub* (Berkeley, Calif.)

Papapetrou, A. [Papapetrou, Achilles] 4920

Quelques identités dans la relativité générale et dans la théorie du champ unifié.

Ann. Inst. H. Poincaré **18**, 99–108 (1963).

In a space with (non-symmetric) metric tensor $g_{\mu\nu}$ the curvature tensor $R^\alpha{}_{\beta\mu\nu}$ is defined in the usual manner by means of the connection coefficients $\Gamma_\mu{}^\lambda{}_\nu$ and their derivatives, it being assumed, however, that the latter are independent of the $g_{\mu\nu}$. Elementary operations involving $R^\alpha{}_{\beta\mu\nu}$ and $g^{\mu\nu}$ give rise to tensors $K^{\alpha\cdots}_{\beta\cdots}$, and these in turn are used to define a scalar density $\mathbf{S} = \mathbf{A}^{\alpha\cdots}_{\beta\cdots}K^{\beta\cdots}_{\alpha\cdots}$, where $\mathbf{A}^{\alpha\cdots}_{\beta\cdots}$ denotes an arbitrary tensor density with reciprocal valencies. Thus $\mathbf{S}$ can be regarded as the integrand of an

invariant integral, and the application of Noether's theorem gives rise to sets of fairly complicated identities which are satisfied by $K^{\alpha}_{\beta\ldots}$ and its derivatives. The special case $K^{\alpha}_{\beta\ldots} = R^{\alpha}{}_{\beta\mu\nu}$ is considered: for symmetric $\Gamma_{\mu}{}^{\lambda}{}_{\nu}$ the first of these relations reduces to the Bianchi identity, while some of the others assume a structure which indicates quite clearly that they are the analogues of the identities which are supposed to express conservation of energy-momentum in the general theory of relativity. The paper concludes with the generalization of these results to the case of non-symmetric connections.

H. Rund (Pretoria)

Papapetrou, Achille [Papapetrou, Achilles] 4921
Champs gravitationnels stationnaires à symétrie axiale.
C. R. Acad. Sci. Paris **258** (1964), 90–93.

In 1953 [Ann. Physik (6) **12** (1953), 309–315; MR **13**, 391] the author obtained a class of solutions of the equations $R_{ik}=0$ for the axially symmetric stationary metric,

$$ds^2 = -e^{\mu}(d\rho^2+dz^2)-l\,d\phi^2-2m\,d\phi\,dt+f\,dt^2,$$

where μ, l, m and f are functions of ρ and z. He now shows how to obtain, by an ingenious method, two further classes of solutions.

{Reviewer's remark: One of these is related to the solution of R. Tiwari and M. Misra [Proc. Nat. Inst. Sci. India Part A **28** (1962), 771–779; MR **27** #2323].} Unfortunately, neither of the classes seems to refer to the field of rotating bodies, because the correct boundary conditions at infinity are not satisfied.

W. B. Bonnor (London)

Roll, P. G.; Krotkov, R.; Dicke, R. H. 4922
The equivalence of inertial and passive gravitational mass.
Ann. Physics **26** (1964), 442–517.

The general theory of relativity depends significantly on the assumption that the ratio of gravitational to inertial mass of a particle is a universal constant (depending only on the units involved). The present paper is a most impressive report of an elaborate and lengthy series of experiments designed to test this assumption to a very much higher degree of accuracy than that afforded by the famous experiments of R. v. Eötvös (1890, 1922) and P. Zeeman (1917). The null results of the latter with regard to the difference between these ratios for different materials are now confirmed with remarkable precision.

Let $(M/m)_A$, $(M/m)_B$ denote the ratios of the passive gravitational to the inertial masses of two materials A and B, respectively. These determine the parameter

$$\eta(A,B) = \{(M/m)_A-(M/m)_B\}[\tfrac{1}{2}\{(M/m)_A+(M/m)_B\}]^{-1}.$$

It is concluded with 95% confidence that $|\eta(\mathrm{Au},\mathrm{Al})| < 3\times10^{-11}$, or, stated more exactly, various measurements of η give a substantially Gaussian distribution with mean value $\eta(\mathrm{Au},\mathrm{Al}) = (1.3\pm1.0)\times10^{-11}$, in which the probable error is based on the observed scatter in results from individual data runs, a Gaussian distribution being assumed. (An analysis of the results of Eötvös yields $|\eta| < 9\times10^{-9}$ with 95% confidence.)

These results are derived from torsion balance experiments, the gravitational acceleration towards the sun being used. The design of the experiment is described in great detail with particular reference to special effects which had to be considered (such as field gradient effects, magnetic contaminants, electrostatic effects, gas pressure and Brownian motion effects, etc.). Some 28 pages are devoted to the description of the apparatus alone. An extensive and scrupulous analysis of the tabulated data concludes the paper.

H. Rund (Pretoria)

Taub, A. H. 4923
Motion of test bodies in general relativity.
J. Mathematical Phys. **5** (1964), 112–119.

The theory of a pole-dipole test body given by Papapetrou is given a concise and coordinate independent formulation by the use of Schwartz's theory of distributions. Two related distributions, one tensor-valued to describe the energy-momentum properties of the test body and one scalar-valued to describe the material properties, are introduced. The method for treating test bodies with higher-order poles is briefly discussed.

G. L. Clark (London)

Białas, Andrzej 4924
On the continuity of the Petrov classification.
Acta Phys. Polon. **23** (1963), 699–703.

The vacuum Riemann tensor (or, more generally, the Weyl conformal curvature tensor) can be uniquely characterised by a totally symmetric four-index spinor. This spinor is the symmetrized product of one-index spinors which define the four "gravitational principal null directions" associated with the flow of "gravitational density" [R. Penrose, Ann. Physics **10** (1960), 171–201; MR **22** #6563; R. Debever, C. R. Acad. Sci. Paris **249** (1959), 1324–1326; MR **21** #7060]. The Petrov type of the Riemann curvature tensor [A. Z. Petrov, Kazan. Gos. Univ. Učen. Zap. **114** (1954), no. 8, 55–69; MR **17**, 892] can also be determined from this arrangement.

The author investigates the connection between the algebraic properties of the Weyl conformal curvature tensor in two neighbouring points of space-time and proves the following theorem: "If the Weyl tensor is continuous at point P, then in some neighbourhood of this point the Weyl tensor is not more special (in the sense of Penrose's diagram) than at the point P".

B. K. Dutta (Calcutta)

Białas, Andrzej 4925
Electromagnetic waves in general relativity as the source of information.
Acta Phys. Polon. **24** (1963), 465–469.

Recently, A. Trautman [*Recent developments in general relativity*, pp. 459–463, Pergamon, Oxford, 1962; MR **29** #2004] proposed a method of obtaining wave-like solutions of the gravitational field equations. The idea is that a solution representing wave motion has to have the possibility of propagation of information. This can be achieved if such a solution depends on an arbitrary function of a scalar parameter. Physically, this means that we can modulate the amplitude of the wave and therefore the wave can propagate information.

In this paper, the author applies the Trautman method to the case when the space-time is not empty, but is filled with electromagnetic radiation. It is shown that the

Einstein-Maxwell field equations lead to the well-known solution corresponding to the electromagnetic null field in general relativity [H. W. Brinkmann, Math. Ann. **94** (1925), 119–145; A. Peres, Phys. Rev. (2) **118** (1960), 1105–1110; MR **23** #B1527]. *A. Peres* (Haifa)

Uhlmann, Armin 4926

Remark on the future tube.

Acta Phys. Polon. **24** (1963), 293.

The "future tube" considered by the author is the set of points in complexified Minkowski space whose position vectors have their imaginary part timelike and future pointing. The author shows that the future tube is holomorphic to a certain symmetric domain of Cartan and that its biholomorphic self-transformations correspond to the orthochronous conformal transformations of the Minkowski space. *R. Penrose* (Austin, Tex.)

Vyšín, V. 4927

The possibility of an interpretation of Kantor's direct first-order experiment on the propagation of light from a moving source.

Phys. Lett. **8** (1964), 36–37.

An account is given of an optical test carried out by Kantor in 1961. In this experiment a beam of light was split into two beams which traversed the apparatus in opposite directions recombining to produce interference fringes. At first sight this experiment seemed to be in contradiction with Einstein's postulate that the speed of light is independent of the uniform motion of the source, and good agreement between experiment and theory was only achieved with the aid of a classical interpretation. In this note the author points out that the experiment is one on a rotating system and derives the formula for the shift with the aid of general relativity. Since the experiment is on a rotating system, the result is the same as the result obtained by classical theory.

G. L. Clark (London)

Brill, Dieter R. 4928

Electromagnetic fields in a homogeneous, nonisotropic universe.

Phys. Rev. (2) **133** (1964), B845–B848.

The author solves the combined Maxwell-Einstein field equations for a space-time admitting a certain three-parameter group of motions and thus generalizes a solution of the Einstein vacuum field equations obtained by the reviewer in 1951 [Ann. of Math. (2) **53** (1951), 472–490; MR **12**, 865] and recently extended analytically by Newman, Tamburino and Unti [J. Mathematical Phys. **4** (1963), 915–923; MR **27** #2325]. The author claims that his generalization describes a space-time with gravitational and electro-magnetic radiation but does not describe the former radiation or give its energy density.

A. H. Taub (Berkeley, Calif.)

Chambers, Ll. G. 4929

Comparison of two generalizations of Maxwell's equations involving creation of charge.

J. Mathematical Phys. **4** (1963), 1373–1375.

The author considers two modifications of Maxwell's equations in flat space-time which allow creation of charge. The first is that of Lyttleton and Bondi [Proc. Roy. Soc. London Ser. A **252** (1959), 313–333; MR **22** #10764] and the second is that of W. H. Watson [Proc. Sympos. Appl. Math., Vol. II, pp. 49–57, Amer. Math. Soc., Providence, R.I., 1950; MR **11**, 293]. He obtains in both theories the solutions corresponding to the creation of a charge at the origin at time $t=0$. He finds the solutions are approximately equal but that of Watson's equations is formally much simpler. A similar result follows for the solutions corresponding to the creation of a current, which the author also derives. He concludes that Watson's theory is to be preferred. *W. B. Bonnor* (London)

ASTRONOMY

Nahon, Fernand 4930

Sur l'application de la méthode des moindres carrés à l'analyse des vitesses radiales.

C. R. Acad. Sci. Paris **257** (1963), 3305–3308.

The geometrical interpretation of the least squares method, given in the author's paper [#4665 above], is applied to the problem of the analysis of radial velocities of stars. The radial velocities R_k are assumed to be independent of the galactic latitude so that

$$R_k = u\cos l_k + v\sin l_k + w + \varepsilon_k,$$

where l_k is the galactic longitude, u, v, and w constants, the last one referring to the K-effect, and ε_k is the residual. Estimators of the constants and their variances are given.

E. Lyttkens (Uppsala)

Živko, Madevski 4931

A contribution to the study of certain special cases of the four-body problem. (Macedonian. French summary)

Bull. Soc. Math. Phys. Macédoine **13** (1962), 33–38.

Author's summary: "On étudie ici, par la méthode vectoriel, les cas spéciaux du problème des 4 corps: de tétraèdre équilatéral, de losange, $m_1=m_3=m$, $m_2=m_4=m'$ et

$$m' = \frac{8\cos^3\frac{\varphi}{2}-1}{8\sin^3\frac{\varphi}{2}-1}\cdot\frac{\sin^3\frac{\varphi}{2}}{\cos^3\frac{\varphi}{2}}\cdot m,$$

de carré, $m_1=m_2=m_3=m_4$, et de triangle équilatéral, les trois corps égaux aux sommets et le quatrième dans son centre. On donne aussi les équations du mouvement dans ces cas-là."

Živko, Madevski 4932

On the center of attraction in the three-body problem. (Macedonian. French summary)

Bull. Soc. Math. Phys. Macédoine **13** (1962), 39–44.

Giacaglia, Giorgio E. O. 4933

The influence of high-order zonal harmonics on the motion of an artificial satellite without drag.

Astronom. J. **69** (1964), 303–308.

Author's summary: "Results [King-Hele, Cook and Rees, Nature 197 (1963), 785] concerning the computation of the amplitudes of zonal harmonics for the potential field of the earth up to the 12th order make necessary the extension of the theory by Brouwer [Astronom. J. 64 (1959), 378–397; MR 23 #B1549] on the subject by including such terms. The power of Von Zeipel's method, as used by Brouwer, is once more proved to be great in such an extension. The practical implication of the present work will be concerned mainly with the effects of the inclusion of higher zonal harmonics on terms present when low-order zonal harmonics alone are considered. In the present paper the secular and long-period variations in Delaunay's elements of an artificial satellite are found for zonal harmonics of any order. The solution is given in such a way that the contribution of any prescribed zonal harmonic can be added to Brouwer's solution [loc. cit.] without any change in notation and in a straightforward manner. The completeness of the solution is the same; it includes secular perturbations to the second order in J_2, short-period and long-period perturbations to the first order; all coefficients J_p for $p>2$ are assumed to be of order J_2^2."

Lur'e, A. I. [**Lur'e, A. I.**] 4934

Free fall of a material point in a satellite cabin.

Prikl. Mat. Meh. **27** (1963), 3–9 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1963), 1–9.

The author presents a mathematical analysis of a material point in a satellite in gravitational space. The equations of motion of its inertial center are considered and later modified to the inclusion of the material point. Expressions are derived for the elliptic orbit, and for ease of operation, they are put in matrix form. The velocity expressions for the material point are then derived. Considering a circular orbit, expressions for the angular velocity vector are obtained by means of the Euler transformation (small angles) and direction cosines. The paper concludes with the derivation of the equation for non-gravitational forces acting on a satellite. This is more complicated than the preceding case. The case of aerodynamic resistance results in a closed-form solution, but for other types of forces one must solve the resulting differential equations by numerical integration methods. The reviewer was impressed by the paper's content and believes that this is an interesting contribution to orbital mechanics. *H. Saunders* (Philadelphia, Pa.)

Parker, E. N. 4935

Dynamical properties of stellar coronas and stellar winds. I. Integration of the momentum equation.

Astrophys. J. **139** (1964), 72–92.

Author's summary: "The hydrodynamic momentum and mass flow equations are integrated for a stellar corona with spherical symmetry about the center of the star. The coronal temperature is taken to be a known function $T(r)$ of radial distance. The general properties of the equations are discussed, and it is shown that if the corona is tightly bound by the solar gravitational field, so that the thermal velocity is small compared to the gravitational escape velocity, and if the coronal temperature declines outward more slowly than r^{-1}, then the corona is quasi-static at its base and expands to supersonic velocity in space. No alternative is available. The stellar mass loss resulting from the expansion is determined by the coronal temperature between the base of the corona and the point at which the flow becomes supersonic. The amount by which the velocity of the resulting stellar wind exceeds the thermal velocity in the corona depends principally upon the temperature beyond the point where the flow becomes supersonic." *H. K. Moffatt* (Cambridge, England)

Parker, E. N. 4936

Dynamical properties of stellar coronas and stellar winds. II. Integration of the heat-flow equation.

Astrophys. J. **139** (1964), 93–122.

From the author's summary: "The temperature $T(r)$ in a stellar corona is computed under the circumstances that energy is supplied outward from the base of the corona only by thermal conduction. The heat-flow equation is solved analytically under a variety of circumstances. It is shown that the energy flow to infinity is non-vanishing for finite coronal density and thermal conductivity. The temperature declines less rapidly than r^{-1}, and a supersonic stellar wind is the only available solution of the equations compatible with negligible pressure at $r=\infty$.

"A variety of asymptotic cases are worked out to illustrate some of the temperature profiles $T(r)$ to be expected under various circumstances. For instance, in a corona of very low density the energy consumed by expansion of the corona can be neglected and $T(r)\propto r^{-2/7}$, as in Chapman's original static coronal model. The result is a supersonic stellar wind with a velocity $v(\infty)$ of the same order as the gravitational escape velocity $2^{1/2}w$. In a corona with medium density and sufficiently low temperature that $v(\infty)$ is small compared to w, a near region, in which $T(r)\propto r^{-4/7}$, extends for some distance outward from the star before the far region, $T(r)\propto r^{-2/7}$, takes over. The result is a supersonic stellar wind velocity $v(\infty)$ of the same order as the characteristic thermal velocity c_0 at the base of the corona. In a corona which is exceedingly dense, an intermediate region, in which $T(r)\propto r^{-1}$, appears between the near and the far regions, which has the result of extending to a large distance the point at which the coronal expansion becomes supersonic. In a corona which is exceedingly hot ($c_0\approx w$) the expansion becomes so violent that thermal conduction becomes negligible and the behaviour of the corona is approximately adiabatic." *H. K. Moffatt* (Cambridge, England)

Neyman, Jerzy; Scott, Elizabeth L. 4937

Field galaxies: luminosity, redshift, and abundance of types. I. Theory.

Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. III, pp. 261–276. *Univ. California Press, Berkeley, Calif.*, 1961.

In three earlier papers the authors have developed the theory of space distribution of galaxies [Neyman and Scott, Astrophys. J. **116** (1952), 144–163; MR **14**, 803; Neyman, Ann. Inst. H. Poincaré **14** (1955), 201–244; MR **17**, 420; Neyman and Scott, J. Roy. Statist. Soc. Ser. B **20** (1958), 1–43; MR **21** #4051]. In the present investigation their general formulae are adapted to the field galaxies, that is, no clustering tendency is present. Compared to the classical theory of the distribution of stars, when the space density is constant, the essential

new feature is the introduction of the selection probability $\phi_t(m)$ that a galaxy of type t and apparent magnitude m, situated in the investigated region will be included in the catalogue. The catalogue numbers of different types of galaxies are Poisson-distributed variables, of which the mean values depend on the distribution of the absolute magnitudes as well as the above-mentioned selection probability.

The joint frequency function of the apparent magnitude and the distance of the galaxies in a catalogue differs from that obtained from the theory of distribution of stars with constant space density essentially by the fact that the selection probability enters as a factor. The catalogue distribution of the apparent magnitude has a frequency function proportionate to $\phi(m)10^{0.6m}$. Also in this case we obtain the classical expression with the factor $\phi(m)$ added.

It is an interesting feature that the distribution of the absolute magnitudes of the catalogue is obtained from a formula of the same form as in the classical case, where the selection probability has unit value for objects brighter than a certain apparent magnitude, while it is supposed to be zero for fainter objects. The same feature holds true for the conditional frequency function of the distance for a given apparent magnitude. When the conditional frequency function of the apparent magnitudes for a given distance (or for a given value of the redshift) is concerned, however, the selection probability $\phi(m)$ again enters as a factor. *E. Lyttkens* (Uppsala)

Neyman, Jerzy; Scott, Elizabeth L. 4938

Magnitude-redshift relation of galaxies in clusters in the presence of instability and absorption.

Astronom. J. **66** (1961), 581–589.

In 1954 the authors introduced a new test for the stability of a system of galaxies [Studies in mathematics and mechanics, pp. 336–345, Academic Press, New York, 1954; MR **16**, 869]. The present paper gives a heuristic explanation of the test and reports on the empirical results on ten smaller groups of galaxies.

The test is a kinematic one, the more precise question being whether the regression coefficient of the radial velocity on the distance from us for galaxies belonging to a given cluster has a value which differs from zero or not. In practice some distance indicator independent of the radial velocity must be used. Therefore the regression coefficient of the radial velocity on the apparent magnitudes for galaxies belonging to a given cluster is used. A closer investigation shows, however, that effects of the order of the Hubble constant will likely not be detected. In the empirical investigations much larger effects are found. The interpretation of the results meets the difficulty that correlation between radial velocity and magnitude within a cluster of galaxies may be due to systematic errors in the measurement of the radial velocities or to systematic selection effects when the galaxies are assigned to different clusters. But the authors find that these factors are not too important. These questions were taken up by Page and Holmberg in the discussion afterwards. In answer to G. Burbidge and Page, Scott argued that for ten small groups of clusters the test should have a power of about 0.9 if the expansion were ten times as large as the general expansion indicated by the Hubble constant.

E. Lyttkens (Uppsala)

Avez, André 4939

Sur un modèle d'univers stationnaire sans section d'espace globale.

C. R. Acad. Sci. Paris **254** (1962), 3984–3985.

The author examines the theorem of D. Aufenkamp [same C. R. **232** (1951), 213–214; MR **12**, 546] that there could not exist a static world-model with compact spatial cross-section in the absence of the cosmological constant. He shows that the hypothesis on the spatial cross-section is necessary by constructing a static world-model with zero cosmological constant without a global spatial cross-section and containing closed world-lines.

G. J. Whitrow (London)

Brezhnev, V. S. [Brežnev, V. S.] 4940

A relativistic model universe.

Astronom. Ž. **40** (1963), 280–283 (*Russian*); *translated as Soviet Astronom. AJ* **7** (1963), 215–217.

The author investigates an expanding homogeneous isotropic world-model composed of a uniform relativistic gas subject to the condition $p=\frac{1}{3}\rho c^2$, relating pressure p and density ρ. He takes the cosmological constant to be positive and the curvature to be positive, and solves the gravitational field equations. He shows that the model is free from singularities, the expansion factor R having a non-zero minimum value. A comparison is made with the Friedmann-Lemaître model filled with material particles. In this model there is an initial singularity. The author attributes this singularity to the condition $p=0$ ascribed to the model, which he argues is not compatible with large values of ρ. *G. J. Whitrow* (London)

Layzer, David 4941

A preface to cosmogony. I. The energy equation and the virial theorem for cosmic distributions.

Astrophys. J. **138** (1963), 174–184.

This paper seeks to justify on a quasi-Newtonian basis the idea of the author that self-gravitating systems in cosmology (clusters of galaxies) are formed through gravitational clustering in an expanding cosmic distribution, rather than through the fragmentation of finite gas clouds. The peculiar motions of galaxies are attributed to the local variations of the gravitational field.

The author first derives an equation relating the rate of change of the local peculiar velocity of a "particle" (galaxy) to the gradient of a Newtonian type potential function. This function is defined in terms of the relative mass density arising from clustering and anti-clustering.

Next, a cosmological energy equation is obtained, confirming a result previously derived by W. M. Irvine [Ph.D. Diss., Harvard Univ., Cambridge, Mass., 1961], involving the time derivatives of T_m and U_m, the kinetic and potential energies, respectively, arising from the peculiar motions and positions of the particles.

Finally, a cosmological virial theorem is derived by the author. By it the author infers that T_m and U_m would each be nearly constant in time as the universe expands, provided that non-gravitational forces and the effects of radiation can be neglected. However, the theorem depends on physical assumptions and details of proof which were not clear to this reviewer. *W. Davidson* (London)

Sengupta, Priyansy [Sen Gupta, Priyansu] 4942
Behavior of light in a pulsating universe.
Astronom. Ž. **40** (1963), 277–279 (*Russian*); *translated as Soviet Astronom. AJ* 7 (1963), 213–214.

The author considers the behaviour of light rays in a pulsating universe of Birkhoff-Friedmann type with the following metric: $ds^2 = -e^{\mu} d\sigma^2 + e^{\nu} dt^2$, where $e^{\mu} = (1-c^2a^2)^2/(1-c^2r^2)^2\rho^{2/3}$, $e^{\nu} = \dot{\mu}/A(t)$, $\rho = \rho_0(1-\alpha\cos^2 pt + \lambda t^2)$, $c = (\alpha/a \cos pt)$, a is the maximum radius of the universe, $A(t)$ is an arbitrary function and α is a constant.

He finds that under specific conditions the behaviour of light rays is the same as that of an expanding cosmological model. *D. K. Sen* (Toronto, Ont.)

GEOPHYSICS

See also 4737.

Knopoff, L.; Hudson, J. A. 4943
Transmission of Love waves past a continental margin.
J. Geophys. Res. **69** (1964), 1649–1653.

Green's theorem technique is employed to obtain transmission coefficients for Love waves propagating in an elastic layer of changing thickness. Numerical values of transmission coefficients are calculated for some special cases. It is also found that the energy flux is inversely proportional to the thickness of the layer for small frequencies. *A. Ben-Menahem* (Pasadena, Calif.)

ECONOMICS, OPERATIONS RESEARCH, GAMES

See also 4269, 4656, 4678, 4961.

Barlow, Richard E.; Proschan, Frank 4944
Comparison of replacement policies, and renewal theory implications.
Ann. Math. Statist. **35** (1964), 577–589.

In the theory of equipment maintenance a block replacement policy is one in which a unit is replaced upon failure and at times T, $2T$, $3T$, $\cdots$, and an age replacement policy is one in which a unit is replaced at failure or at age T, whichever comes first. If the failure probability density is denoted by $f(t)$ and the hazard ratio by $h(t)$, then an important class of failure distributions are those for which $h(t)$ is monotone increasing or decreasing. The authors obtain a number of inequalities on the counting function, $E(N(t)) = M(t)$, where $N(t)$ is the number of replacements in $(0, t)$, assuming the underlying distributions to be either IHR or DHR and assuming different replacement policies. As an example, if $\bar{F}(t) = \int_t^\infty f(x)\,dx$ and $F(t) = 1-\bar{F}(t)$ is IHR [DHR], then $M(t) \leq [\geq] tF(t)/\int_0^t \bar{F}(x)\,dx \leq [\geq] t/\mu_1$ for all $t \geq 0$, where μ_1 is the mean time between failures. *G. Weiss* (Queens, N.Y.)

Hammersley, J. M. 4945
The mathematical analysis of traffic congestion. (French summary)
Bull. Inst. Internat. Statist. **39** (1962), *livraison* 4, 89–108.

An expository paper, with an extensive bibliography.

Glicksman, A. M. 4946
★An introduction to linear programming and the theory of games.
John Wiley and Sons, Inc., New York-London, **1963**. x+131 *pp.* $2.25.

This excellent book combines simplicity of presentation with rigor and some generality. It could only have been conceived, and successfully completed, by one with long familiarity both with the potential, and with the needs, of bright high school students. Interested laymen and college undergraduates would also find it very useful.

Part I contains three simple linear-programming word problems. It illustrates the algebraic representation of a linear programming problem and introduces graphical solutions, open and closed half-planes, polygonal regions, boundaries, vertices, multiple optima, and so on. Part II treats convex sets in the Cartesian plane and defines relation, linear relation, distance, endpoints, convex sets, linear forms, extreme points, and so on. In it is proven the fundamental extreme point theorem that justifies a search among extreme points for the extreme value of a linear form defined on their convex hull. Part III discusses the simplex method. It gives an example of the Gauss-Jordan complete elimination procedure and then discusses and gives examples of extended and condensed simplex tableaux. Part IV provides elementary aspects of the theory of games, using a two-by-two example to illustrate the basic elements of the theory. It then gives a graphical analysis of $m \times 2$ and $2 \times n$ matrix games. Part V relates matrix games and linear programming. It reduces both player A's and player B's problems to linear programming problems, describes the dual relation between these problems, and sketches the fundamental duality theorem and its corollary, the minimax theorem. The book has interesting problems after each chapter; these are often word problems requiring algebraic formulation before solution. *J. J. Stone* (White Plains, N.Y.)

Balas, Egon 4947
Un algorithme additif pour la résolution des programmes linéaires en variables bivalentes.
C. R. Acad. Sci. Paris **258** (1964), 3817–3820.

Author's summary: "Un algorithme est proposé pour la résolution des programmes linéaires en variables bivalentes (0 ou 1). Il commence par rendre toutes les variables 0 et continue en égalant systématiquement certaines variables à 1, de telle manière qu'en testant une (petite) partie de toutes les 2^n combinaisons possibles, on arrive, soit à une solution optimale, soit à la conclusion qu'il n'y a pas de solution réalisable."

Ben-Israel, Adi 4948
The geometry of solvability and duality in linear programming.
Israel J. Math. **1** (1963), 181–187.

A restatement of well-known theorems of duality in linear programming. There is a classification into 8 mutually exclusive cases. *G. Tintner* (Los Angeles, Calif.)

Bernholtz, B. 4949
A new derivation of the Kuhn-Tucker conditions.
Operations Res. **12** (1964), 295–299.

Author's summary: "The Kuhn-Tucker necessary conditions for a local maximum of a differentiable function subject to differentiable inequality constraints are derived by showing that the classical necessary conditions for a local maximum of a differentiable function of N variables subject to less than N equality constraints are also meaningful when there are N or more equality constraints, if appropriate assumptions, similar to Kuhn and Tucker's 'constraint qualification', are made."

Kuhn, Harold W.; Quandt, Richard E. 4950

An experimental study of the simplex method.

Proc. Sympos. Appl. Math., Vol. XV, pp. 107–124. *Amer. Math. Soc., Providence, R.I.*, 1963.

It is not known why the simplex method is efficient. This paper reports on empirical tests made by the authors comparing nine different criteria used for selecting the pivot element. The best criterion is not known. The test linear programs were created with randomly generated coefficients. The efficiencies of the methods were compared with respect to the number of iterations required and the amount of computer time used. The results equally favored four methods referred to as the η gradient type. Essentially these first "normalize" the coefficients in each column of the canonical form before selecting the column with the largest coefficient in the objective row. In addition to the information sought, the study unexpectedly gave rise to two new conjectures in stochastic programming. A proof of one of the conjectures is given and the other still remains an open question.

R. Wollmer (Berkeley, Calif.)

Witzgall, Christoph 4951

An all-integer programming algorithm with parabolic constraints.

J. Soc. Indust. Appl. Math. **11** (1963), 855–871.

The author solves in a finite number of steps the integer programming problem of minimizing a linear objective function subject to linear and parabolic constraints using a modification of Gomory's all-integer algorithm. A parabolic constraint is an inequality

$$a_{00}-L_0(X)-b_1(L_1(X))^2-\cdots-b_k(L_k(X))^2 \geqq 0,$$

where $b_s>0$ and $L_0(X),\cdots,L_k(X)$ are $k+1$ linearly independent, homogeneous linear forms of n variables. The principal modification is a restoration step following a pivot step to restore homogeneity to the linear forms within a parabolic constraint.

E. L. Johnson (Berkeley, Calif.)

★Advances in game theory. 4952

Edited by M. Dresher, L. S. Shapley, and A. W. Tucker. Annals of Mathematics Studies, No. 52.

Princeton University Press, Princeton, N.J., 1964. x+679 *pp.* $8.50.

This volume is essentially a continuation of four preceding volumes entitled *Contributions to the theory of games* [Annals of Math. Studies No. 24 (1950); No. 28 (1953); No. 39 (1957); No. 40 (1959), Princeton Univ. Press, Princeton, N.J.]. The papers will be reviewed individually.

Maschler, Michael 4953

n-person games with only 1, $n-1$, and n-person permissible coalitions.

J. Math. Anal. Appl. **6** (1963), 230–256.

A payoff configuration (p.c.) in an n-person game G in characteristic function form is a pair consisting of a partition of the players into coalitions and a payoff vector that divides the payoff to each coalition in the partition among its members. For every G there are defined 6 sets of p.c.'s (generally all different) denoted $\mathcal{M}_0, \mathcal{M}, \mathcal{M}_1, \mathcal{M}_0^{(i)}, \mathcal{M}^{(i)}, \mathcal{M}_1^{(i)}$ and called bargaining sets [the reviewer and the author, *Advances in game theory*, pp. 443–476, Princeton Univ. Press, Princeton, N.J., 1964]; they are characterized by certain stability properties of the p.c.'s. For the games G described in the title of this paper, the author characterizes all 6 bargaining sets in terms of the so-called $(n-1)$-quota of G; this is the unique vector $(\omega_1,\cdots,\omega_n)$ such that for each i, $v(N-\{i\})=\sum_{j\in N-\{i\}}\omega_j$, where v is the characteristic function and $N=\{1,\cdots,n\}$. It turns out that if there are no weak players (i with $\omega_i<v(\{i\})$), then the characterization is relatively uncomplicated; for example, for each $i\in N$ each of the bargaining sets contains precisely one p.c. whose player partition is $(N-\{i\},\{i\})$, namely, the one with payoff vector $(\omega_1,\cdots,\omega_{i-1},v(\{i\}),\omega_{i+1},\cdots,\omega_n)$. When there are weak players, it is necessary to use an inductive method of deleting players until no weak players are left. For all games described in the title, $\mathcal{M}_0=\mathcal{M}=\mathcal{M}_1$ and $\mathcal{M}_0^{(i)}=\mathcal{M}^{(i)}=\mathcal{M}_1^{(i)}$.

R. J. Aumann (Jerusalem)

BIOLOGY AND BEHAVIORAL SCIENCES

See also 4255, 4256, 4575.

van der Vaart, H. R. 4954

The role of mathematical models in biological research. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 31–59.

Author's summary: "On analyse plusieurs aspects du traitement mathématique des problèmes biologiques. On démontre qu'un grand nombre de questions concernant ce traitement mathématique sont de nature méthodologique générale. Quelques exemples sont analysés à fond et, grâce à ceux-ci, il est établi que la méthode empirique, consistant dans l'ajustement de certaines courbes standardisées, et la méthode théorique, consistant dans la construction des modèles mathématiques, ne sont, en réalité, que deux membres (plus ou moins extrêmes) de toute une série graduée de méthodes. Beaucoup de modèles mathématiques ont la forme générale d'une équation, $Tg=a$, ou $Tg(\xi)=a$, où l'ensemble de définition de T est défini comme un ensemble $\mathcal{G}$ de fonctions (inconnues) g des conditions expérimentales, ou bien comme l'ensemble des valeurs de telles fonctions g (les valeurs des fonctions g étant de nature quelconque: scalaires, vecteurs, fonctions du temps t, 'time-series', etc.). L'équation différentielle qui détermine la courbe logistique, est un des exemples donnés afin de montrer que des modifications, quoique petites en apparence, de T ou de a, peuvent entraîner des modifications assez importantes des propriétés de la solution de l'équation citée plus haut. Enfin, la plupart des catégories d'épreuves (au sens de M. Fréchet) en biologie sont telles que les résultats d'observation ne sont pas

déterminés ni reproductibles dans une catégorie donnée. C'est pourquoi il faut qu'un modèle complet en biologie soit un modèle probabiliste. Or, entre un modèle mathématique et son extension probabilisée, il y a, assez fréquemment, des différences frappantes."

Kraemer, Helena Chmura 4955

Point estimation in learning models.

J. Mathematical Psychology **1** (1964), 28–53.

Author's summary: "The problem of point estimation for several forms of the all-or-none learning models is examined. Possible estimation procedures leading to estimators satisfying small sample optimality criteria are presented, and asymptotic distribution theory for large sample estimators is described. Several approaches to the problem are presented, and, when possible, comparisons are made among the estimators obtained."

INFORMATION, COMMUNICATION, CONTROL

See also 3910, 3933, 3936, 3937, 3938, 4404, 4542, 4596, 4645, 4657.

Kozin, F.; Bogdanoff, J. L. 4956

A comment on "The behavior of linear systems with random parametric excitation".

J. Math. and Phys. **42** (1963), 336–337.

This brief paper shows that the Fokker-Planck equation derived by Caughey and Dienes [same J. **41** (1962), 300–318; MR **25** #5549] for a linear first-order differential equation with a white-noise parameter is at variance with classical results. The reason advanced by the authors for this variance is that Caughey and Dienes did not take due account of the subtleties involved in taking the mathematical expectation of a certain integral.

J. C. Samuels (Lafayette, Ind.)

Razumovskii, S. N. [Razumovskiĭ, S. N.] 4957

On the transformation of programs for the solution of complicated logical problems to optimal form.

Dokl. Akad. Nauk SSSR **139** (1961), 562–565 (*Russian*); *translated as Soviet Physics Dokl.* **6** (1962), 555–557.

The author observes that existing computer programs for the analysis of (or involving) complicated logical problems are not optimal. That is to say, these programs do not most effectively utilize the possibilities for obtaining information contained in the logical conditions inherent in such problems. Therefore the author proposes transformations of problem structures in such a manner that the amount of information obtainable from the logical (i.e., {0, 1}-valued) functions involved be increased. Such transformations consist in decompositions or compositions of logical functions, in a change of the order in which the program realizes them, etc.

The initial information on a problem is assumed to be given in the form of a "schema" (i.e., computational routine) Σ consisting of "logical acts" and "resulting acts" of some definite standard types. It is assumed that n given logical functions determine the content of corresponding logical acts $A_1, \cdots, A_n$ and of m resulting acts $R_1, \cdots, R_m$. Consider the set $\Xi = \{\xi\}$ of all conjunctions of logical functions such that in each ξ each function occurs exactly once without or with negation. Identically false conjunctions are excluded from Ξ outright. Each ξ uniquely determines the ordered n-tuple of zeros or ones (as truth values for the constituent logical functions) that make ξ come out true.

By dint of Σ a 1-1 correspondence is established between the elements of Ξ and the resulting acts. In particular, ξ_{jk} corresponds to R_k whenever the truth of ξ_{jk} leads to resulting act R_k. Thus, the specification of Σ determines a subset $\Theta = \{\theta\}$ of Ξ, where Θ contains exactly those ξ to which certain resulting acts correspond. Φ_k, the logical function of a resulting act R_k, is the disjunction (alternation) of functions θ corresponding to the same R_k, and Φ_k is true whenever, as a result of realizing Σ, one is led to R_k.

A logical act A_1 corresponding to some function is singled out and called an "initial act" of Σ. This Σ is supposed to include prescriptions for operations to be carried out after the realization of resulting acts. A "terminating resulting act" is called one which, after being realized, causes a transfer back to A_1 or calls for another routine altogether. Various identities involving Φ's and arbitrary logical functions are given as criteria for terminating resulting acts.

Further, let p_k be the probability that Φ_k is true and P_i the probability that θ_i is true. If Φ_k involves ν many constituents of θ_i and θ_i has n_i constituents, then

$$S_k = (1/p_k)(n_{k_1}P_{k_1} + \cdots + n_{k_\nu}P_{k_\nu})$$

is the expected value of the number of logical acts in function Φ_k, provided Φ_k is true.

The amount of information about a true Φ_k that one obtains on the average from one logical act of schema Σ is given as

$$V(\Sigma) = -\sum_{k=1}^{m} p_k \log_2 p_k \Big/ \sum_{k=1}^{m} p_k S_k.$$

Theorem 1: The absolute maximum of $V(\Sigma)$ is unity. It is attained if and only if $S_k = -\log_2 p_k$; $k = 1, \cdots, m$.

If $\Pi(\theta)$ is the result of applying transformation Π to the functions θ of the original Σ and $\Phi_k(\theta) = \Phi_k(\Pi(\theta))$ $(k = 1, \cdots, m)$, then Π is called "identical for the schema as a whole", and this is the optimizing transformation. Theorem 2: The optimizing transformation maximizes $V(\Sigma)$ for given p_k.

{Because of the complete absence of references, the extreme concision of the paper and its idiosyncratic nomenclature (purposely retained above and also in the translation), the author's achievements are needlessly hard to appreciate. It would seem, e.g., that the author's set Ξ had already been investigated by R. Carnap [*Logical foundations of probability*, §§ 18, 19 et passim, Univ. Chicago Press, Chicago, Ill., 1950; MR **12**, 664] and J. G. Kemeny [J. Symbolic Logic **13** (1948), 16–30; MR **9**, 487]; it is the set of consistent state-descriptions or models. As to the intertwinement of state-description combinatorics with probability considerations, certainly standard sources like R. M. Fano [*Transmission of information*, M.I.T. Press, Cambridge, Mass., 1961; MR **24** #B442] and S. Kullback [*Information theory and statistics*, Wiley, New York, 1959; MR **21** #2325] provide essential stepping stones.}

E. M. Fels (Munich)

Root, William L. 4958

Stability in signal detection problems.

Proc. Sympos. Appl. Math., Vol. XVI, pp. 247–263. *Amer. Math. Soc., Providence, R.I.*, 1964.

Let $y(t)=s(t, \alpha)+x(t)$ be an observed waveform, $s(t, \alpha)$ be a signal, and $x(t)$ be a sample function from a stochastic process (the noise). Many detection systems form a functional $f(y)$ to test a hypothesis about the parameter α or to estimate α. Ordinarily the form of the functional is determined partly by an assumption about the statistical properties of the noise. It is an important practical question to ask how the test or estimate is affected if the noise does not have the assumed properties. This paper is concerned with some aspects of this question.

When the noise is gaussian, a fairly simple criterion is derived which ensures that the probability of error in a hypothesis test approaches that for the assumed noise when the actual noise distribution approaches the assumed distribution. A similar criterion is derived which ensures that the distribution of a linear estimate of α approaches the nominal distribution when the distribution of the noise approaches the nominal distribution.

When the noise is not gaussian or the estimator is not linear, the problem is less tractable. However, the author does develop a condition which is sufficient to ensure that the distribution of the functional $f(y)$ converges to its nominal distribution as the actual probability measure of the noise converges to the nominal measure.

L. L. Campbell (Kingston, Ont.)

Volkonskiĭ, V. A. 4959

An optimal estimate of the time of origin of a signal in the presence of multiplicative high-frequency Gaussian noise. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 79–95.

In the observation interval $|t|<t_0$ the data are $\xi_\lambda(t)=S(t-\tau)\nu(\lambda t)$, where $S(t)$ is a known function and τ is a location parameter to be estimated, $|\tau|<t_0$. $\nu(t)$ is a stationary band-limited Gaussian noise process of mean zero. In the limit $\lambda\to\infty$ the estimate $\hat{\tau}$ is to be asymptotically unbiased and of minimum variance among the class of processes $\xi_\lambda(t)$ having as $\lambda\to\infty$ equal variances and equal correlation times. The estimate is obtained by applying the data to a quadratic detector, passing its output through a certain linear filter, and observing the time when the output of the filter is a maximum. The asymptotic minimum variance of $\hat{\tau}$ is derived by means of a Cramér-Rao inequality.

C. W. Helstrom (Los Angeles, Calif.)

Winograd, S. 4960

Redundancy and complexity of logical elements.

Information and Control **6** (1963), 177–194.

Author's summary: "This paper deals with the problem of increasing the reliability of gate-type logical circuits through the use of redundancy. We will derive a lower bound on the amount of redundancy necessary to achieve a certain error correcting ability and show how this bound varies with the complexity of the elements used in the design of the redundant circuit, measured by the number of inputs. The complexity of encoders of block codes for transmission of information is defined. A bound similar to the one mentioned above on the error correcting ability of codes is derived which depends on the codes' rate of transmission and on the complexity of their encoders. Finally, we establish a connection between the bound on the error correcting ability of a redundant circuit and the bound on the error correcting ability of a block code."

A. Nerode (Ithaca, N.Y.)

Herdan, Gustav 4961

★The calculus of linguistic observations.

Mouton & Co., The Hague, 1962. 271 *pp.* *Dfl.* 42.00.

This book is a diversified collection of essays, some of which are theoretical, some empirical, and some mystical. The author's ideas are challenging and interesting: challenging in the sense that the reader is obliged to detect and correct misstatements and ignore nonsense; and interesting in the sense that the author presents a large collection of data on language behavior and an equally large collection of conjectures on linguistic theory. It is not intended to serve as a text but must be regarded as the author's highly personal view of mathematical linguistics. However, the author confuses mathematical linguistics with statistical linguistics, thus ignoring algebraic linguistics with its theory of formal grammars. After crediting Zipf-Yule and Shannon for the first and second stages in the development of mathematical linguistics, the author claims that his "theory of linguistic duality as the general pattern of distribution" is the third stage.

The author's title is borrowed from the book *The calculus of observations* [Blackie, London, 1924] by the English mathematicians E. T. Whittaker and G. Robinson. However, the author presents no calculus as we customarily understand the term. *The calculus of linguistic observations*, like the author's previous works [*Language as choice and chance* (LCC), Noordhoff, Groningen, 1956; MR **18**, 708, and *Type-token mathematics* (TTM), Mouton, The Hague, 1960], is divided into several "books", which, in turn, are divided into several parts. Book One, Language in the Mass—The Universe of Language Statistics: Part I, Statistics of Phonemic (Alphabetic) Systems; Part II, Vocabulary and Stylo-statistics; and Part III, Spectrography of Speech on the Morphemic Level. Book Two, The Relation between the Language in the Mass and Language in the Line: Part IV, The Statistical Theory of Linguistic Coding—Information Theory; Part V, Quantum Theory of Language. Book Three, Language in the Line—The Universe of Language Discourse: Part VI, The Universe of Discourse as a Line—Linguistic Duality; Part VII, Language and Mathematics. A medium length biography is included. The book of 264 pages is unnecessarily long because some is a repetition of material previously given in the 352 pages of his LCC and 433 pages of TTM, and the remainder could have been said once and succintly.

Both the author and the publisher must accept responsibility for the high incidence of typographical errors in the mathematical expressions. The errors include missing, wrong, and misplaced letters, wrong font, reversed signs of inequality, etc. This is all the less understandable since most of these same expressions appeared (either correctly or incorrectly) in his earlier books LCC and TTM. The mathematical misprints are too numerous to list, so the reader is cautioned not to accept blindly the mathematical formulas as they stand.

The author is usually on solid ground as long as he

stays close to the data and elementary theorems, but is in trouble when he generalizes or reasons by analogy. For example, his discussion of Zipf's law (p. 59 et seq.) is just as apt and correct as when it appeared in his other two books, but he presents no convincing argument that "The conformity of the form-function pattern of grammar with Desargue's Theorem (a basic theorem of duality) has, therefore, its roots in the economy of linguistic coding" (p. 234). Despite the fact that the arguments for some of his conjectures are incomplete or unconvincing, they (or correctly stated versions of them) are interesting and should be re-examined rigorously. However, this task will fall to others. *H. P. Edmundson* (Pacific Palisades, Calif.)

Barbašin, E. A. 4962

Construction of periodic motion as one of the problems of programming control theory. (Russian. English summary)

Applications of the methods of non-linear vibrations to the problems of physics and technology (Proc. Internat. Sympos. Non-linear Vibrations, Vol. III, 1961), pp. 34–40. Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963.

This report is virtually identical with the author's paper in Prikl. Mat. Meh. **25** (1961), 276–283 [MR **26** #1560].

H. A. Antosiewicz (Los Angeles, Calif.)

Bocharov, I. N. [Bočarov, I. N.]; Fel'dbaum, A. A. [Fel'dbaum, A. A.] 4963

An automatic optimizer for the search for the smallest of several minima (A global optimizer).

Avtomat. i Telemeh. **23** (1962), 289–301 (*Russian. English summary*); *translated as Automat. Remote Control* **23** (1962), 260–270.

The general optimization problem is to determine the inputs $x_1, \cdots, x_n$ of an object B so that a certain quantity $Q(x_1, \cdots, x_n)$ is minimized (or maximized: the treatment is analogous) under the constraints $H_j(x_1, \cdots, x_n) \leq 0$, $j=1, \cdots, m$. The problem in the paper is to design a device A that automatically seeks the smallest minimum of Q. The functions Q and H_j are the outputs of B and are fed into the automatic optimizer A. The outputs of A are the $x_1, \cdots, x_n$ and are fed into B. The automatic optimizer under arbitrarily given initial values goes through cycles of search in accordance with prescribed algorithms and determines the smallest minimum. This optimizer differs from those previously described in the literature in that the latter were applicable to cases where there is only one minimum in the region of admissible values of the x_i.

After pointing out problems in which the knowledge of the smallest minimum is essential, the authors state three (among many) algorithms by which the search can be carried out. Based on these, the block diagram of such a global optimizer is designed. The paper ends with an account of experimental results obtained by a five-channel global optimizer designed according to the principles stated in the paper. *N. H. Choksy* (Silver Spring, Md.)

Butkovskiĭ, A. G. 4964

Necessary and sufficient optimality conditions for sampled-data control systems. (Russian. English summary)

Avtomat. i Telemeh. **24** (1963), 1056–1064.

The usual maximum principle of Pontrjagin is well known and is applied to optimal control problems for systems described by ordinary differential equations. However, when the system is a sampled-data one, it is described by difference equations. Discrete analogues of the maximum principle have been given, but the present author gives an example where such an analogue is not valid. The author establishes the necessary condition for optimality in discrete systems as a principle of local minimum. The system is described by the vector difference equation $x(k+1)=f(x(k), u(k))$, $k=0, 1, \cdots, N-1$, with an initial state $x(0)=a$. The problem is to determine an optimal control $u(k)$ from a given admissible set so as to maximize the functional $I=cx(N)$ (c is a constant vector). As usual, the function $H(x, \psi, u)=\psi f(x, u)$ is introduced, where $\psi(l-1)=\psi(l)F(l)$, $l=1, 2, \cdots, N-1$, and $F=\partial f^i/\partial x^j$, $i, j=1, 2, \cdots, n$. Then we have Theorem 1: Let there exist the optimal control $u^*(k)$ and the optimal trajectory $x^*(k)$ corresponding to it with the given initial state. Then when $u=u^*(k)$ and $x=x^*(k)$ there exists a solution $\psi^*(k)$ satisfying $\psi(k-1)=\psi(k)F(k)$ with $\psi^*(N-1)=c$ such that for fixed $x=x^*(k)$ and $\psi=\psi^*(k)$, the function $H(x^*(k), \psi^*(k), u)$ has a local minimum when $u=u^*(k)$.

The author then gives a sufficient condition for optimality in the form of an extended maximum principle. He introduces the function $\Pi(k, x, u)=\psi(k)f(x, u)-\psi(k-1)x$, where $\psi(k)$ is some function of the discrete argument and, by definition, $\psi(-1)=0$ and $\psi(N-1)=c$. Then we say that the functions $u(k)$ and $x(k)$ satisfy the extended maximum principle if for all $k=0, 1, \cdots, N-1$, $\Pi(k, x, u)$ is maximized over all u and x from their respective admissible regions. We then have Theorem 2: In order that an admissible control $u^*(k)$ and the trajectory $x^*(k)$ corresponding to it be optimal, it is sufficient that the functions $u^*(k)$ and $x^*(k)$ satisfy the extended maximum principle.

The author shows that the latter theorem is a necessary and sufficient condition for optimality when the sampled-data system is a linear one.

N. H. Choksy (Silver Spring, Md.)

Cypkin, Ja. Z. 4965

Frequency criteria for absolute stability of non-linear sampled-data systems. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 281–289.

Author's summary: "Frequency criteria for the absolute stability of the nonlinear sampled-data automatic control systems are described which are similar to the frequency criteria for the linear sampled-data systems. The proposed frequency criteria of the absolute stability permit one to solve problems concerning the investigation of the absolute stability of stationary and a certain class of nonstationary sampled-data automatic control systems. It is shown that to provide the absolute stability of the nonlinear automatic control systems it is sufficient to provide the proper stability margin of corresponding linearized sampled-data automatic control systems."

Mil'šteĭn, G. N. 4966

An application of successive approximations for solving an optimal problem. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 321–329.

Author's summary: "The convergence of the successive

approximations to the optimum control is considered. It is found that the optimum control approximations can be assumed optimum in the sense of a certain functional. The notions of Liapounoff method of functions and of the theory of dynamic programming are used."

Pallu de La Barrière, Robert 4967
Extension du principe de Pontrjagin au cas de liaisons instantanées entre l'état et la commande.
C. R. Acad. Sci. Paris **258** (1964), 3961–3963.
Let $f: R^n \times R^m \to R^n$ be continuously differentiable, and consider the system

$$(1)\quad \frac{dX(t)}{dt} = f(X(t), U(t)) \quad (t \in [0, T], X(0) = x_0).$$

The control U is admissible if: (a) it is piecewise continuous; (b) the solution $X(t)$ of (1) is defined in $[0, T]$; (c) $p(X(t), U(t)) \equiv 0$, where $p: R^n \times R^m \to R^k$ is continuously differentiable and $p_u(x, u)$ is of rank k. U is optimal if it is admissible and maximizes $w'X(T)$ for given $w \in R^n$. A necessary condition that U be optimal is derived; a similar result was obtained by L. D. Berkovitz [J. Math. Anal. Appl. **3** (1961), 145–169; MR **25** #2470].
W. M. Wonham (Baltimore, Md.)

Pjatnickiĭ, E. S. 4968
On the structural independence of single-loop control systems. (Russian. English summary)
Avtomat. i Telemeh. **24** (1963), 1213–1216.
A solution is given for the problem of the structural independence of single-loop control systems without derivative action. Let the characteristic equation of the system be $D(p)+k=0$, where $k=\text{const}>0$ and $D(p)$ is a product of m terms of the form $(a_\nu p^2+b_\nu p+c_\nu)$, $\nu=1, \cdots, m$. For the polynomial $D(p)$ let n be the degree, σ the number of zero roots, τ the number of real positive roots, r the number of factors of the form ap^2+c, λ the number of roots with negative real parts, and s the minimum possible number of real negative roots. A control system is said to be structurally independent if there exist values of a_ν, b_ν, c_ν, k such that the characteristic equation has any set of preassigned negative real roots. The main result is: In order that the system with the above characteristic equation be structurally independent, it is necessary and sufficient that the system be structurally stable and that the following two conditions be satisfied: (1) $r<2$; (2) if $\sigma \leqq 1$ and $\tau=0$, then the polynomial $D(p)/p^\sigma$ should be hurwitzian and we should have $s=2-\sigma$ for even n and $s=1-\sigma$ for odd n. If $\tau=1$ we should have $s=1$ for even n and $s=0$ for odd n.
N. H. Choksy (Silver Spring, Md.)

★Proceedings of the First International Congress of the International Federation on Automatic Control [Труды I. Международного Конгресса Международной Федерации по Автоматическому Управлению]. 4969
Moscow, 27 June–7 July 1960. Theory of continuous systems. Special mathematical problems. Edited by V. A. Trapeznikov and Ja. Z. Cypkin.
Izdat. Akad. Nauk SSSR, Moscow, 1961. 754 *pp.* 3.60 *r.*
The papers of mathematical interest will be reviewed individually.

Dreyfus, Stuart E.; Elliott, Jarrell R. 4970
An optimal linear feedback guidance scheme.
J. Math. Anal. Appl. **8** (1964), 364–386.
Essentially, this is the problem: Given the criterion function $\phi(\mathbf{x}(t_1), t_1)$, the system differential equation $\dot{\mathbf{x}}=f(\mathbf{x}, \mathbf{u}, t)$, the terminal constraints $\psi(\mathbf{x}(t_1), t_1)=0$, and a proposed control $\mathbf{u}(t)^*$, and the resultant trajectory $\mathbf{x}(t)^*$ which is assumed to be optimal, one is required to find the neighboring optimal controls and trajectories. This is accomplished by expanding the system differential equation to first order, the criterion function to second order, and the transversality conditions on the optimal trajectory to first order. The result is that one has an auxiliary problem of minimizing a quadratic criterion subject to linear differential constraints and linear terminal constraints. This problem can then be solved via the integration of the matrix Riccati equation with the proper boundary conditions. The present paper discusses the formulation and solution of this problem from the viewpoint of dynamic programming. The explicit rule for the computation of a second-order fit to the optimal return function (Equations (3.6) and (3.8)) is simply the discrete version of the Riccati equation. The authors also describe two methods of calculating the boundary conditions of the Riccati equation. A simple analytical example and a numerical example of injecting a rocket into orbit are included to demonstrate the application of the method.
Y.-C. Ho (Cambridge, Mass.)

Kushner, Harold J. 4971
On the dynamical equations of conditional probability density functions, with applications to optimal stochastic control theory.
J. Math. Anal. Appl. **8** (1964), 332–344.
The author considers the one-dimensional stochastic control problem: $\dot{x}(t)=f(x, u, t)+z(t)$, where it is required to determine an optimal $u(t)$ in order to minimize $E\int_0^T k(x(\sigma), u(\sigma), \sigma)\,d\sigma$, where the initial value $x(0)$ is a random variable with a given distribution and $u(t)$ is based on the observation $x(\sigma)+\varepsilon(\sigma)=b(\sigma)$, where $z(\sigma)$ and $\varepsilon(\sigma)$ are Gaussian white noise processes. It is shown that the conditional probability density $P[x(t) \mid \int_0^t b(\sigma)\,d\sigma, 0 \leqq \sigma \leqq t]$ is a Markov process, and a differential representation for this process is obtained, as well as for the moment $E[x(t) \mid \int_0^t b(\sigma)\,d\sigma, 0 \leqq \sigma \leqq t]$. Using dynamic programming, a functional equation is also derived which yields qualitative information concerning the optimum control.
A. V. Balakrishnan (Los Angeles, Calif.)

Krotov, V. F. 4972
Methods for solving variational problems on the basis of the sufficient conditions for an absolute minimum. I.
Avtomat. i Telemeh. **23** (1962), 1571–1583 (*Russian. English summary*); *translated as Automat. Remote Control* **23** (1963), 1473–1484.
This paper is a survey in the language of control theory of the Hamiltonian approach to the derivation of sufficient conditions for an absolute minimum in a general class of variational problems. The author shows the close relation, by now well known, between Bellman's dynamic programming and the Hamilton-Jacobi partial differential equation. It is the opinion of the reviewer that such a paper would have been more complete by including a

discussion of Carathéodory's method of "geodätisches Gefälle" which is very similar to the subject treated in the paper. *H. Halkin* (Whippany, N.J.)

Krotov, V. F. 4973

Methods for solving variational problems. II. Sliding regimes.

Avtomat. i Telemeh. **24** (1963), 581–598 (*Russian. English summary*); *translated as Automat. Remote Control* **24** (1963), 539–553.

In the preceding paper [#4972] the author introduces and discusses certain sufficient conditions for a minimum in a class of variational problems. In the present paper he is particularly concerned with problems for which the minimum is achieved by "sliding regimes" (or generalized curves) and not by originally admissible curves. He discusses certain procedures for applying his sufficient conditions in such cases and illustrates one of them by an example. He concludes by stating the conjecture that his sufficient conditions are also necessary.

J. Warga (Brookline, Mass.)

Mucenieks, V. A.; Rastrigin, L. A. 4974

Extremal control by random search. (Russian. Latvian and English summaries)

Latvijas PSR Zinātņu Akad. Vēstis **1964**, *no.* 2, 93–104.

Authors' summary: "A comparison of random search and the gradient method for the extremal control of multidimensional systems has been carried out. Several characteristics of the controlled system are defined where random search is found more effective than the gradient method."

Beneš, V. E. 4975

Markov processes representing traffic in connecting networks.

Bell System Tech. J. **42** (1963), 2795–2837.

The principal problem treated here is the theoretical calculation of the grade of service in terms of blocking probability on the structure of a connecting network corresponding to a physical communication system. Traffic models in connecting networks are constructed as a class of Markov stochastic processes x_t based on the following assumptions: (i) the holding-times of calls are mutually independent random variables, each with the negative exponential distribution of unit mean, and (ii) the calling rate per idle inlet-outlet pair is constant (called the traffic parameter), such that only one parameter, λ, needs to be specified. The operation of the network is described in terms of the handling of calls to busy terminals, blocked calls, and selection of routes. In other words, the traffic models take into account the set of permitted states of the connecting networks, the random epochs at which the attempt at new calls and the termination of calls in progress occur, and the placement of routing calls. Next, the blocking probability is defined and the formula for it is given. Further aspects of this formula are investigated, such as a one-sided connecting network and a two-sided network with N inlets and M outlets, to derive algebraic relationships between the blocking probability, traffic parameters, and other parameters with respect to the operation of the networks. Three formulas for the calculation of the state probabilities (stationary) are also proposed. *W. H. Kim* (New York)

Kaphengst, Heinz 4976

Beschreibung von Kontaktschaltungen durch Äquivalenzen auf der Knotenmenge.

Z. Math. Logik Grundlagen Math. **10** (1964), 147–150.

Angegeben wird eine mathematische Beschreibung für allgemeine Kontaktschaltungen mit beliebig vielen Knotenpunkten und Polen, die sich auf gewissen Äquivalenzrelationen über der Menge der Knotenpunkte stützt. Knotenpunkte gehören bei einer solchen Relation genau dann zu einer Äquivalenzklasse, wenn sie elektrisch leitend miteinander verbunden sind.

H. Rohleder (Leipzig)

Teverovskiĭ, V. I. 4977

A method of analyzing periodic states of a relay system containing arbitrary switching relays. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 339–346.

Author's summary: "A method for determining periodical states in a relay system with two on-off relays is considered. The method is based on plotting relay system characteristics according to Ya. Z. Tsypkin's method. The analysis of the stability of periodical states in systems with two relays is described. The results obtained are applicable to the analysis of the two-loop or single-loop systems with known frequency characteristics of the linear parts."

Hellgren, Gösta 4978

Some characteristics of two simple linear and stochastically time-varying control systems of interest in angle and range measurement and tracking by monopulse radar.

Svenska Aeroplan A.B. Tech. Note TN 47 (1960), 87 *pp.*

The author studies the characteristics of the output signal of a tracking servo for both deterministic and stationary stochastic noise (including large variations). Random amplification can be generated either by a stationary Gaussian process with arbitrary power spectrum or by a quasi-stationary (periodic) process with a power spectrum at will. The results obtained are applied to an angular tracking monopulse radar with slow AGC. A similar discussion is applied to a linear stochastically time-varying control system of AGC-servo type. The results of these investigations are used to examine the AGC influence upon the angular accuracy of a non-tracking angle-measuring monopulse radar. The output of the tracking servo is investigated under various input conditions (unit step, stationary and quasi-stationary stochastic, with additive noise to the output, etc.). The time function of response, the stability of the tracking and the dynamic error are calculated and presented in generalized numerical calculations. In a similar fashion, the AGC-servo is studied. In the Appendix, block diagrams for equivalent circuits are developed.

P. R. Arendt (Eatontown, N.J.)

Igarashi, Shigeru 4979

On the logical schemes of algorithms. (Japanese)

J. Information Processing Soc. Japan **3** (1962), 66–72.

In this paper the author tries to improve formulations in the theory of algorithms defined and studied by Ju. I. Janov [Problemy Kibernet. **1** (1958), 75–127; MR **24** #B1735; Problems of Cybernetics **1** (1960), 82–140]. He points out that the disadvantage in the definition of Janov's logical schemes of algorithms lies in the definition of 'shift distribution', and he defines a new logical scheme of algorithms without shift distribution which contains Janov's definition of a scheme as a special case; he excludes from the definition of scheme the restriction to repetitive uses of an operator which is essential to Janov's definition.

The author also reduces the problems of equivalence of logical schemes of algorithms to the problems of equivalence of finite automata which simulate the given schemes, and he gives a method of solving the above problems by using a digital computer. *S. Huzino* (Fukuoka)

AUTHOR INDEX

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GENERAL

Dumas de Rauly, Daniel 4980

★Problèmes de mathématiques.

Gauthier-Villars Éditeur, Paris, 1963. 221 *pp.*

According to its preface, this collection of mathematical problems is intended as an introduction to courses in mathematics given in the Grandes Écoles (institutes of technology and their like). Its range is indicated by the titles (and subtitles) of its fourteen chapters, which are, in order: combinatorial analysis; probability; algebra; linear algebra (vector spaces, linear equations, determinants); matrices (diagonalization—eigenvalues, transformations); integral calculus (simple integrals, differentiation and integration under the integral sign, geometrical applications, double and triple integrals); integration using analytic functions; the gamma function; the beta function; orthogonal functions; convergence of series; special series; trigonometric series; functions of many variables (partial derivatives, local behavior, homogeneous functions, maximum and minimum, scalar fields—force fields). The problems are not numbered; a rapid count by the reviewer makes the total about 230 (Chapters VIII, IX and X, on the gamma, beta, and orthogonal functions, contain no problems). Each problem is followed by a solution, deliberately compressed so that the reader may be forced to work out the details. "Modern" mathematics is stressed as well as uniform methods. Although significant and brief characterization of the problems in general seems impossible, a few remarks on particular problems may be useful. The first problem in the book is to evaluate the sum on the left of

$$\sum_{k=0}^{n-2}\binom{n}{k}\binom{n}{k+2} = \binom{2n}{n-2},$$

which is a slightly disguised form of the Vandermonde identity, but curiously the latter never appears, although a version of it is hidden in the twentieth of the 22 problems in the chapter on probability. The last five probability problems are on Banach's matchbox problem. The first asks for a probability distribution associated with finding one match in a box for the first time, the second for a distribution associated with finding a box empty for the first time. If $p_n(x, 1)$ is the generating function for the first distribution when there are n matches in each of the two boxes initially, $p_n(x, 0)$ the similar function for the second, then $p_n(x, 1) = xp_{n-1}(x, 0)$. Thus the proofs that $p_n(1, 1) = 1$, $p_n(1, 0) = 1$ must be essentially the same, but the author gives separate proofs. Indeed, it is strange that although a probability generating function actually appears in the first problem on page 19, it is not identified as such, and further there are no problems on finding means. On the other hand, there is a geometric problem on cubes whose solution may be a welcome aid to the reader. In the chapter on algebra, the congruence $(a+b)^p \equiv a^p + b^p \pmod p$ is identified as the binomial formula for a ring with characteristic p, p a prime, which is number theory in modern dress. The first problems in the chapter on integral calculus ask for the proofs of the inequalities of Schwarz, Hölder, and Minkowski. The fourth proves the interesting formula

$$\int_0^\infty [(x^2+a^2)(x^2+b^2)]^{-1/2}\,dx = \tfrac{1}{2}\pi m(a, b),$$

where $m(a, b)$ is the arithmetic-geometric mean of (a, b), that is, the common limit of the sequences a_n, b_n when $a_{n+1} = \sqrt{(a_n b_n)}$, $b_{n+1} = (a_n + b_n)/2$, $a_0 = a$, $b_0 = b$. The fifth is devoted to the A. Hurwitz proof that the number e is transcendental. The chapter on special series deals with such oddities as $u_n = (np^a)^{-1}$, where $a > 0$ and p is the number of digits in n (with base 10).

None of these remarks should be read as casting judgment on the book; they reflect only the reviewer's tastes. However, it should be noticed that the reader must puzzle out for himself the author's notational usage, and there are a number of typographical errors.

J. Riordan (Murray Hill, N.J.)

★Séminaire Delange-Pisot, 3^e^ année: 1961/62. 4981

Théorie des nombres.

Faculté des Sciences de Paris.

Secrétariat mathématique, Paris, 1963. ii + 187 *pp.*

Exp. (1) G. Rauzy, Approximations diophantiennes linéaires homogènes; Exp. (2) F. Bertrandias, Fractions continues; Exp. (3) F. Bertrandias, Théorèmes de Markov [Exposé du chapitre 2 de l'ouvrage: J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge Univ. Press, New York, 1957; MR **19**, 396]; Exp. (4) F. Badiou, Propriétés métriques des fonctions continues; Exp. (5) E. Brisse, Géométrie des nombres [Cet exposé n'a pas été rédigé, et ne sera pas multigraphié]; Exp. (6) H. Saffari, Problèmes non homogènes de la géométrie des nombres; Exp. (7) J. Chauvineau, Equirépartition et équirépartition uniforme modulo 1; Exp. (8) F. Badiou, Théorèmes métriques; Exp. (9) G. Rauzy, Approximation des nombres algébriques [Cet exposé n'a pas été rédigé, et ne sera pas multigraphié; voir G. Rauzy, *Approximation diophantienne des nombres algébriques*, Secrétariat mathématique, Paris, 1961; MR **23** #A3116]; Exp. (10) J. Hily, Sur les propriétés arithmétiques des fonctions entières; Exp. (11) J.-P. Bertrandias, Répartition modulo 1 des puissances successives des nombres réels; Exp. (12) J.-P. Bertrandias, Applications des nombres θ; Exp. (13)

J.-L. Krivine, Fonctions récursives [Cet exposé correspond au chapitre 1 de R. M. Smullyan, *Theory of formal systems*, Princeton Univ. Press, Princeton, N.J., 1961; MR **22** #12042]; Exp. (14) M. Mendès-France, Transcendance de a^b; Exp. (15) P. Barrucand, Le problème de Waring et la méthode de Hardy [dissection de Farey]; Exp. (16) H. Delange, Application de la méthode du crible à l'étude des valeurs moyennes de certaines fonctions arithmétiques.

HISTORY AND BIOGRAPHY

See also 5449.

Samplonius, Yvonne 4982

Die Konstruktion des regelmässigen Siebenecks nach Abu Sahl al-Qûhî Waigan ibn Rustam.

Janus **50**, 227–249 (1963).

This study gives a facsimile reproduction of Cairo MS V213,6, the Arabic text of al-Qūhī's construction for the regular heptagon, of which there are two other extant copies. A parallel translation into German accompanies the text, and both are preceded by an introduction. The latter lists the works of al-Qūhī (or al-Kūhī), a mathematician and astronomer from northern Iran who worked in tenth-century Baghdad.

His solution involves the construction of a parabola and an hyperbola whose intersection determines a triangle $P_1P_2P_4$, where $P_1, P_2, \cdots, P_7$ are the vertices of a regular heptagon. *E. S. Kennedy* (Beirut)

Hofmann, Jos. E. 4983

Über die *Exercitatio Geometrica* des M. A. Ricci.

Centaurus **9** (1963/64), 139–193.

Michel Angelo Ricci (1619–1682), an able though almost forgotten mathematician and divine, co-editor (from 1668 to 1675) of the first scientific journal (the Giornale de'Letterati) in Italy, ended his life as cardinal. He had been a pupil of Torricelli and remained in close contact with him until Torricelli's death in 1647. Ricci was in fact the mediator who made available to Torricelli the extremum problems of the French (Fermat, Roberval); Ricci's own studies are inspired by discussions with Mersenne on the same topic in 1644/45.

The only published work of Ricci is his *Exercitatio Geometrica* [Rome, 1666; reprinted, London, 1668] which was held in high esteem by his contemporaries. In it Ricci determined the tangent lines to the "higher conic sections" $(y/b)^{p+q}=(x/a)^p[(c\pm x)/(c\pm a)]^q$ by means of a reduction to extremum problems in an elementary way without using infinitesimal procedures. In one of his letters he gave the first proof of the so-called Bernoulli inequality. Most remarkable is the author's reconstruction of Ricci's quadrature of the cissoid (1647) based on an integral theorem of Roberval. Hitherto the first quadrature of this curve had been attributed to Huygens, who achieved it in 1658.

The paper is divided into five sections: (1) Ricci's life; (2) The printing history of the *Exercitatio*; (3) The correspondence between Torricelli and Ricci (1644/46) on extremum and tangent problems; (4) The *Exercitatio*; (5) The quadrature of the cissoid. There are 1 portrait, 22 diagrams, a list of the letters used and an index of names and publications. *C. J. Scriba* (Hamburg)

Selenius, Clas-Olof 4984

Kettenbruchtheoretische Erklärung der zyklischen Methode zur Lösung der Bhaskara-Pell-Gleichung.

Acta Acad. Abo. Math. Phys. **23**, *no.* 10, 44 *pp.* (1963).

Verfasser gibt einen Existenzbeweis für das von den alten Indern geübte zyklische Verfahren zur Behandlung der Gleichung $x^2-py^2=1$, der mit jenem in J. E. Hofmann [Abh. Preuss. Akad. Wiss. Math.-Natur. Kl. **1944**, no. 7, 19 pp.; MR **8**, 305] übereinstimmt (diese in den letzten Kriegsmonaten erschienene Abhandlung war dem Verfasser unzugänglich). Zusätzlich stellt Verfasser den Zusammenhang mit den einschlägigen Kettenbruchentwicklungen her, die durch das zyklische Verfahren wesentlich verkürzt werden. *J. E. Hofmann* (Ichenhausen)

LOGIC AND FOUNDATIONS

See also 5007, 5016, 5889, 5891.

Feys, Robert 4985

Logique formalisée et raisonnement juridique.

Essays on the foundations of mathematics, *pp.* 312–321. *Magnes Press, Hebrew Univ., Jerusalem*, 1961.

The author of this paper died April 13, 1961. During the last years of his life one of his principal interests was the application of logic to jurisprudence. This paper is a general statement of the aims and possibilities of such application. There are no mathematical results.

H. B. Curry (University Park, Pa.)

Faris, J. A. 4986

★Quantification theory.

Monographs in Modern Logic.

Routledge & Kegan Paul, London, 1964. vii+147 *pp.* 12*s*. 6*d*.

This is an elementary exposition. The aim is to expound techniques for constructing particular formal proofs, including proofs of invalidity by counter-examples (mostly finite). There is a lot of explanation, including discussion of "meaning" in terms of ordinary discourse, examples proved in detail, and even cook-book directions for carrying out certain of the processes. The formulation is a mixture of various types; it contains a larger number of basic rules than usual. The exposition is marred by a tendency to introduce technical terms without defining them (or even indicating that they will be defined later) and by the introduction of superfluous complications.

H. B. Curry (University Park, Pa.)

da Costa, Newton C. A. 4987

Sur un système inconsistant de théorie des ensembles.

C. R. Acad. Sci. Paris **258** (1964), 3144–3147.

Let NF be the version of Quine's "new foundations" given by Rosser [*Logic for mathematicians*, pp. 212–213, McGraw-Hill, New York, 1953; MR **14**, 935]. The author constructs a formal system NF_1 for set theory which is

based upon a non-classical propositional logic, but otherwise resembles the system NF. It is stated that: (1) NF_1 is simply inconsistent; (2) If NF_1 is non-trivial (i.e., it is not the case that every formula is provable), then NF is consistent. Since a non-classical propositional logic is used, it makes sense to speak of a simply inconsistent but non-trivial formal system. This paper depends on three previous papers of the author [C. R. Acad. Sci. Paris **257** (1963), 3790–3792; MR **28** #1123; ibid. **258** (1964), 27–29; MR **28** #2960; ibid. **258** (1964), 1366–1368].
H. J. Keisler (Madison, Wis.)

Waligórski, Stanisław 4988
Calculation of prime implicants of truth functions.
Algorytmy **1**, *no.* 2, 77–89 (1963).
Author's summary: "A method of calculating prime implicants of the truth function on the digital computer is presented. Using this method, computations can be performed as recursive procedures. Three main operations are used for calculating prime implicants, decomposition of zero-one sequences, alternative of elements of one set with elements of another set, taking minimal elements of a set of zero-one sequences. An additional reduction of intermediate results depending on the cost function of expressions may also be performed."

Monteiro, Antonio [Monteiro, António] 4989
Linéarisation de la logique positive de Hilbert-Bernays.
Rev. Un. Mat. Argentina **20** (1962), 308–309.
A pseudo-Boolean algebra H (here called "algèbre de Heyting") is called linear if the intersection L of all filters D in H for which H/D is completely ordered, is $\{1\}$. The author states, without proof, several necessary and sufficient conditions for H to be linear. H/D is linear if $L \leqq D$. The Lindenbaum algebra of the positive logic of Hilbert-Bernays, with the axiom $(a \to b) \vee (b \to a)$ added, is a free linear algebra A. For A a simple characteristic matrix is given. *A. Heyting* (Amsterdam)

Maslov, S. Ju. 4990
On certain methods of prescribing sets in generating bases. (Russian)
Dokl. Akad. Nauk SSSR **153** (1963), 266–269.
Es werden auf der Grundlage der in denselben Dokl. **152** (1963), 272–274 [MR **27** #4753] eingeführten Begriffe weitere Resultate mitgeteilt. Eine erzeugende Basis B (vgl. loc. cit) wird stark entscheidbar genannt, wenn es einen Algorithmus gibt, der für jedes Schema H von B, jede Folge $P_0, P_1, \cdots, P_m$ von Wörtern und jede Folge $i_0, i_1, \cdots, i_m$ von Zahlen $i_j \in \{0, 1\}$ $(j = 0, \cdots, m)$ festzustellen gestattet, ob man eine Folge $Q_0, Q_1, \cdots, Q_m$ von Wörtern so konstruieren kann, daß Q_0 durch einmalige Anwendung von H aus $Q_1, \cdots, Q_m$ abgeleitet werden kann und

$$(\forall j)(0 \leqq j \leqq m \to ((i_j = 0 \to Q_j = P_j) \wedge (i_j = 1 \to Q_j > P_j)))$$

gilt. Es wird ein Beweis folgenden Satzes skizziert: In jeder stark entscheidbaren und aufzählbaren erzeugenden Basis B über einem Alphabet A kann man zu jeder entscheidbaren Menge $\mathfrak{W}$ von A-Wörtern mit unendlichem Komplement eine unendliche entscheidbare Menge $\mathfrak{M}$ von A-Wörtern konstruieren, so daß $\mathfrak{M} \cap \mathfrak{W} = \emptyset$ und $\mathfrak{M} \cup \mathfrak{W}$ nicht streng darstellbar in B ist (das analoge Resultat ist für schwach entscheidbare und entscheidbare Basen allgemein nicht richtig).

Es bezeichne F einen (partiellen) Algorithmus, der jedem Wort P über einem Alphabet A^*, auf das er anwendbar ist, eine Menge $F(P)$ von Wörtern über dem Alphabet A zuordnet. Es wird gesagt, daß die Menge $\mathfrak{N}$ von A-Wörtern bezüglich F der Menge $\mathfrak{M}$ von A^*-Wörtern entspricht, wenn folgendes gilt: (1) Zu jedem $P \in \mathfrak{M}$ gibt es ein $Q \in \mathfrak{N}$, so daß $Q \in F(P)$; (2) Zu jedem $Q \in \mathfrak{N}$ existiert ein A^*-Wort P, so daß $Q \in F(P)$, und für jedes A^*-Wort S mit $Q \in F(S)$ gilt $S \in \mathfrak{M}$. Eine Menge $\mathfrak{M}$ von A^*-Wörtern heißt streng darstellbar in der Basis B (über dem Alphabet A) mittels des Algorithmus F, wenn man in der Basis B einen Kalkül $\mathfrak{L}$ angeben kann, so daß der Menge $\mathfrak{M}$ bezüglich F eine Menge $\mathfrak{N}$ von A-Wörtern entspricht, die in $\mathfrak{L}$ streng darstellbar ist. Neben einer Reihe von grundsätzlichen Bemerkungen über die strenge Darstellbarkeit von Wortmengen mittels Algorithmen bzw. gewissen Familien von Algorithmen findet sich in der Arbeit eine sinngemäße Verallgemeinerung der Sätze (1) und (2) aus der zitierten Arbeit auf Darstellbarkeit mittels Algorithmen. *G. Asser* (Greifswald)

Mycielski, Jan 4991
On the axiom of determinateness.
Fund. Math. **53** (1963/64), 205–224.
This paper discusses some consequences and alternative formulations of the so-called axiom of determinateness introduced by the author and Steinhaus [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 1–3; MR **25** #3850]. The paper is based upon Zermelo-Fraenkel-Skolem set theory without the axiom of choice. Let X be a non-empty set and P a subset of the set X^ω of all countable sequences of elements of X. The game $G_X(P)$ has two players, I and II, and is played as follows: I chooses an element $x_0 \in X$, then II chooses an element $x_1 \in X$, and so on, I making the even choices and II the odd choices. Player I wins the game if $(x_0, x_1, x_2, \cdots) \in P$, and II wins otherwise. A_X denotes the statement "for every $P \subseteq X^\omega$, one of the players has a winning strategy for the game $G_X(P)$". The axiom of determinateness is the statement A_2, where $2 = \{0, 1\}$. The author and Świerczkowski [#4992 below] proved that A_2 implies that every subset of the real line is Lebesgue measurable (and consequently A_2 contradicts the axiom of choice). In the paper under review, other consequences of A_2 are proved; we list four of them below. (1) For every countable family F of non-empty subsets of a set of power $\leqq 2^{\aleph_0}$, there exists a choice set (due to the author, Scott, and Świerczkowski). (2) Every subset of a separable metric space has the property of Baire. (3) Every non-denumerable separable metric space has a compact perfect subset. (4) There is no cardinal $\mathfrak{n}$ such that $\aleph_0 < \mathfrak{n} < 2^{\aleph_0}$. Several modifications of A_2 are considered, and some implications are proved; for instance, A_2 and A_ω are equivalent to each other. D. Scott showed that for a certain cardinal $\mathfrak{t}$, $A_\mathfrak{t}$ is inconsistent, and Scott's argument is modified here to show that A_{ω_1} is inconsistent. Using results of Gödel [Proc. Nat. Acad. Sci. U.S.A. **24** (1938), 556–557], it is shown that some weaker hypotheses than A_2 (e.g., "for every analytic set $P \subseteq 2^\omega$, one of the players has a winning strategy for the game $G_2(P)$") already contradict Gödel's axiom of constructibility. *H. J. Keisler* (Madison, Wis.)

Mycielski, Jan; Świerczkowski, S. 4992

On the Lebesgue measurability and the axiom of determinateness.

Fund. Math. **54** (1964), 67–71.

The following (infinite) game is considered. Given a set P ($\subset N^N$), players π_0, π_1 choose alternately natural numbers m_{2p+0}, m_{2p+1}, and π_0 wins if $(m_0, m_1, \cdots) \in P$, otherwise π_1. A winning strategy for π_0 is a function f on finite sequences such that, for all $m_1, m_3, \cdots, m_{2p+1}, \cdots$ and $m_0 = f(\langle\ \rangle)$, $m_{2p+2} = f(\langle m_0, \cdots, m_{2p+1}\rangle)$, $(m_0, m_1, \cdots) \in P$. Analogously for π_1. Note that f itself can be coded by an element of N^N. Then P is called determined if either π_0 or π_1 has a winning strategy. The main result is this: Given a non-measurable set S of real numbers, there is a P, constructed in a perspicuous way from S, which is not determined. The terminology: axiom of determinateness seems as odd as would be, e.g., axiom of commutativity, when one wants to speak of commutative groups. More formally, what the authors are after is to study closure properties of the class $\mathfrak{D}$ of determined P, i.e., statements $G_{\mathfrak{D}}$ in which the quantifiers in G over subsets of N^N are relativised to $\mathfrak{D}$. The result above shows that not all P are determined, and it seems dubious whether it is even consistent to assume this (together with the usual axioms of set theory and, say, the principle of dependent choices). In view of this, it is more informative to prove $\vdash G_{\mathfrak{D}}$ than $\vdash [\forall P(P \in \mathfrak{D})] \to G$; for, if $\vdash G_{\mathfrak{D}}$, we automatically have the latter, but not conversely. *G. Kreisel* (Stanford, Calif.)

Hájek, Petr; Sochor, Antonín 4993

Ein dem Fundierungsaxiom äquivalentes Axiom.

Z. Math. Logik Grundlagen Math. **10** (1964), 261–263.

Given Gödel's axiomatic set theory (Axioms A, B, C, but not Axiom D or the Axiom of Choice). Define an a-chain to be any function f defined on a finite ordinal, whose range is a subset of a, and such that $f(n+1) \in f(n)$ for any $n+1$ in the domain of f. Define $\gamma(x)$ to hold if, for any set a, every a-chain whose first term is x can be extended to a maximal a-chain. Let $\Gamma = \{x | \gamma(x)\}$. Take von Neumann's function ψ defined by $\psi(0) = 0$, and, for $\alpha > 0$, $\psi(\alpha) = \mathfrak{P}(\bigcup_{\beta<\alpha} \psi(\beta))$, where $\mathfrak{P}$ is the power set function. Let $\Pi = \bigcup_{\alpha \in On} \psi(\alpha)$. The authors prove that $\Gamma = \Pi$. It follows that Axiom D is equivalent to $\Gamma = V$. It is noted that $\gamma(x)$ implies that there is no infinitely descending $\in$-sequence beginning with x. The converse holds under the assumption of the axiom of choice. In addition, it is impossible to prove from Axioms A, B, C the equivalence of $\gamma(x)$ and $(\exists u)(u \in x \mathbin{\&} u \cap x = 0)$. {Misprint: In Definition 2a, "$\alpha \geqq \omega_0$" should be replaced by "$\alpha \leqq \omega_0$".}

E. Mendelson (Flushing, N.Y.)

Lachlan, A. H. 4994

Standard classes of recursively enumerable sets.

Z. Math. Logik Grundlagen Math. **10** (1964), 23–42.

Myhill [same Z. **1** (1955), 97–108; MR **17**, 118] introduced a striking new use of the recursion theorem in order to prove that every recursively enumerable set is 1-1 reducible to every creative set; he also proved that the 1-1 degrees coincide with the recursive isomorphism types. Rogers [J. Symbolic Logic **23** (1958), 331–341; MR **21** #2585] gainfully exploited these ideas to show that any two standard enumerations of the partial recursive functions are recursively isomorphic. The author defines standard class (of n-ary recursively enumerable relations) and standard enumeration, and demonstrates that any two standard enumerations of a standard class are recursively isomorphic. He generalizes various notions of productivity and develops results of Myhill, Smullyan and Cleave as results concerning standard classes. It is not clear to the reviewer that it is worthwhile to generalize a portion of recursion theory without obtaining any new results or any new insights into old results.

G. E. Sacks (Ithaca, N.Y.)

McLaughlin, T. G. 4995

On contraproductive sets which are not productive.

Z. Math. Logik Grundlagen Math. **10** (1964), 49–52.

This paper strengthens results of the author's earlier paper [Math. Scand. **11** (1962), 175–178; MR **27** #4741]. The principal theorem states that if α is a contraproductive, non-productive set of numbers, P is a partial recursive function contraproductive for α, e is a number such that $\omega_e \subseteq \alpha$, then the set $\Sigma_e = \{j | \omega_j = \omega_e \mathbin{\&} j \notin \delta(P)\}$ is neither immune nor recursively enumerable. Several familiar results follow elegantly as corollaries.

A. R. Anderson (New Haven, Conn.)

Ohashi, Kempachiro 4996

Enumeration of some classes of recursively enumerable sets.

Z. Math. Logik Grundlagen Math. **10** (1964), 1–6.

This paper generalizes the well-known fact (and its proof) that the recursive sets form a recursively enumerable class. For any two recursively enumerable binary relations R_1, R_2, $\langle R_1, R_2\rangle$ is defined to be the class of recursively enumerable sets S for which there exists a partial recursive function f satisfying (i) S is the range of f, (ii) if $f(k)$ is defined, then for each $n < k$, $f(n)$ is defined and $R_1(f(n), f(n+1))$, (iii) if $f(n)$ is defined, then $(\exists y)R_2(f(n), y)$. The main theorem states that $\langle R_1, R_2\rangle$ always possesses a $\Sigma_1{}^0$ enumeration. Several corollaries and additional enumeration results are given. *D. L. Kreider* (Hanover, N.H.)

Machover, M. 4997

A note on sentences preserved under direct products and powers.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **8** (1960), 519–523.

Let A_k, A_∞ be the classes of sentences of a simple applied functional calculus L with equality, which are preserved under direct products of k, respectively infinitely many systems and B_k, B_∞ the corresponding classes preserved under powers. Using a generalized form of a lemma by Mostowski the author proves that the above mentioned classes are recursively enumerable. If L has only one-place predicate letters then they are recursive and if either L has two predicate letters, one of which has at least two arguments or L has a predicate letter of more than two arguments, then the classes are non-recursive. The results for A_∞ and B_∞ extend previous results by A. Oberschelp [cf. Arch. Math. Logik Grundlagenforsch. **4** (1958), 95–123; MR **21** #6330] and S. Feferman [Amer. Math. Soc. Notices **6** (1959), 619–620]. The case where L has exactly one two-place predicate letter is left open. It is observed that the method applies in other cases as well.

B. van Rootselaar (Zbl **95**, 8)

Rabin, Michael O.; Wang, Hao 4998

Words in the history of a Turing machine with a fixed input.

J. Assoc. Comput. Mach. **10** (1963), 526–527.

"Word" here means the content of the minimal segment of tape (of the Turing machine) containing all marked cells and the initially scanned cell. From a result of Ullian the authors easily show that, given any word W, there exists a suitable Turing machine such that the set of words which it generates (during the course of computation) from W is not recursive. However, the authors show also the following: if a Turing machine is non-erasing, then it is decidable whether, from a given input, a given word occurs as one of those generated (during the course of computation) by this machine.

R. M. Baer (Berkeley, Calif.)

Rabin, Michael O. 4999

Non-standard models and independence of the induction axiom.

Essays on the foundations of mathematics, pp. 287–299. *Magnes Press, Hebrew Univ., Jerusalem,* 1961.

This paper gives further information about non-standard models of arithmetic along the lines of the author [Logic, Methodology and Philos. of Sci. (Proc. 1960 Internat. Congress), pp. 151–158, Stanford Univ. Press, Stanford, Calif., 1962; MR **27** #3540]. The main result (for a wide class of systems) is this. There is an open formula $D(x, y)$ such that in every non-standard model M there is m, $m \in M$, and $(Ey)D(m, y)$ false in M, but in some extension $M' \supset M$, $(Ey)D(m, y)$ is true. From this he gets the result (not previously published): If R_n is a consistent set of formulae in first-order arithmetic Z of logical complexity n, there is a theorem T_n of Z which is not a logical consequence of R_n. Actually, T_n depends only on n, since one can enumerate in a single expression E_n all predicates of complexity n, and can take for T_n the statement: The axiom of induction holds for E_n.

G. Kreisel (Stanford, Calif.)

Kreisel, G. 5000

On weak completeness of intuitionistic predicate logic.

J. Symbolic Logic **27** (1962), 139–158.

This paper adds new results to the author's paper [same J. **23** (1958), 317–330; MR **21** #2591]. The main tool is Gödel's result (of which a detailed proof is published here for the first time): For each primitive recursive relation $A(n, \alpha)$ between natural numbers and free choice sequences there is a prenex formula $\mathfrak{A}$ such that weak completeness of HPC (the intuitionistic predicate calculus) for $\neg\,\mathfrak{A}$ implies

$$(\alpha)_B \neg\neg (En)A(n, \alpha) \rightarrow \neg\neg (\alpha)_B(En)A(n, \alpha).$$

(The subscript B restricts α to the binary fan.) Let us denote this formula by $\mathfrak{B}(A)$. If A does not contain α, then $\mathfrak{A}$ can be chosen as a negative formula, i.e., a formula which contains no existence or disjunction symbols and in which every prime formula is preceded by $\neg$.

Let $\mathfrak{A}^* = (E\mathfrak{x})\mathscr{A}(\mathfrak{x})$ denote the formula associated with $\mathfrak{A}$ in Gödel's interpretation [the author, Constructivity in Math. (Amsterdam, 1957), pp. 101–128, North-Holland, Amsterdam, 1959; MR **21** #5568]. If $A_K(n, \alpha)$ denotes Kleene's $A(\bar{\alpha}(n))$ [Proc. Internat. Congr. Mathematicians (Cambridge, Mass., 1950), Vol. 1, pp. 679–685, Amer. Math. Soc., Providence, R.I., 1952; MR **13**, 422], then $(\mathfrak{B}(A_K))^*$ is intuitionistically refutable. On the other hand, let $\mathrm{HA}_{NF\omega}$ denote the negative fragment in the sense defined above of intuitionistic arithmetic of finite order with the fan theorem as an axiom, and let KA be Kleene's system of intuitionistic analysis [Constructivity in Math. (Amsterdam, 1957), pp. 285–289, North-Holland, Amsterdam, 1959; MR **21** #2590], extended to finite types and with axioms of choice. If $\mathfrak{A}$ is provable in KA, the $\mathfrak{A}^*$ is derivable in $\mathrm{HA}_{NF\omega}$ (more exactly, there is a constant φ such that $\mathscr{A}(\varphi)$ is provable in $\mathrm{HA}_{NF\omega}$). Thus, if $\mathfrak{B}(A_K)$ were provable in KA, then $(\mathfrak{B}(A_K))^*$ would be intuitionistically valid, which is impossible. Thus weak completeness of HPC cannot be proved by the methods formalized in KA. An analogous results holds if the fan theorem in KA is replaced as an axiom by the "bar theorem"

$$(n)[n \leq \varphi_0^{\tau_0}(\Phi, \alpha) \rightarrow \beta(n) = \alpha(n)] \rightarrow \Phi(\alpha) = \Phi(\beta).$$

However, in order to have a continuous φ_0, the functionals cannot be supposed to be extensional.

As a remarkable additional result it is proved that if a negative or prenex formula of HA is provable in KA, then it is provable in HA. *A. Heyting* (Amsterdam)

Makkai, M. 5001

On a generalization of a theorem of E. W. Beth.

Acta Math. Acad. Sci. Hungar. **15** (1964), 227–235.

Consider a first-order predicate logic L with an identity symbol, $\mu(\lambda)$-ary predicate symbols P_λ, $\lambda < \rho = \text{domain } (\mu)$, and an extra n-ary predicate symbol P. Let $\mathfrak{A} = \langle A, R_\lambda \rangle_{\lambda<\rho}$ be a variable for structures of type μ, and for each $R \subseteq A^n$ let $(\mathfrak{A}, R) = \langle A, R_\lambda, R \rangle_{\lambda<\rho}$. Beth [Nederl. Akad. Wetensch. Proc. Ser. A **56** (1953), 330–339; MR **15**, 385] proved the well-known theorem that, roughly speaking, a set Σ of sentences of L defines P implicitly if and only if Σ defines P explicitly. In this paper the following infinite version of Beth's theorem is proved, and it is shown that Beth's original theorem is an easy consequence. Assume the GCH (generalized continuum hypothesis). Let Σ be a set of sentences of L. Then (a) and (b) below are equivalent. (a) For any $\mathfrak{A}$, the set of all $R \subseteq A^n$ such that $(\mathfrak{A}, R)$ is a model of Σ has power at most $|A|$ (where $|A|$ = power of A). (b) There are finitely many formulas $F_k(v_0, \cdots, v_{n-1}, \cdots, v_m)$, $k = 1, \cdots, N$, not containing P, such that the disjunction

$$\bigvee_{k=1}^{N} (\exists v_n)\cdots(\exists v_m)(v_0)\cdots(v_{n-1})(P(v_0, \cdots, v_{n-1}) \leftrightarrow F_k(v_0, \cdots, v_m))$$

is a consequence of Σ. The above theorem also has been proved by C. C. Chang. Chang announced his proof by private communication in November, 1962, and the paper of Makkai under review was submitted for publication on April 29, 1963. From these dates it appears that the actual dates of discovery were reasonably close together. Chang and Makkai gave quite similar proofs, using saturated (i.e., replete) models. Vaught [J. Symbolic Logic **27** (1962), 480; Fund. Math. **54** (1964), 303–304] showed by an indirect method that the above theorem is true even without the GCH. More recently, Chang found a direct proof without assuming the GCH, which used special models instead of saturated models. He

also obtained some stronger conclusions; for instance, the inequality "at most $|A|$" may be replaced by "less than $2^{|A|}$" in the above statement of the theorem. Chang's results are stated in "Some new results in definability" [Bull. Amer. Math. Soc. (to appear)].

H. J. Keisler (Madison, Wis.)

Shepherdson, J. C. 5002

A non-standard model for a free variable fragment of number theory.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 79–86.

The free variable fragment considered consists of the successor axioms, the basic recursion equations for predecessor (P), $+$, $\cdot$, and cut-off subtraction

$$[x \dot{-} 0 = x, x \dot{-} y' = P(x \dot{-} y)],$$

and the rule of induction applied to quantifier-free formulae (in the notation above). If $x \leqq y$ is defined to be $x \dot{-} y = 0$, the usual properties for order can be derived. The domain of the non-standard model M_0 consists of polynomials $a_p t^{p/q} + \cdots + a_1 t^{1/q} + a_0$, where p, q and a_0 are integral ($p \geqq 0, q > 0$), a_i are real algebraic and $a_p \geqq 0$. The operations on M_0 are defined in the obvious way when t is considered infinite. To establish the induction rule for M_0, the author embeds M_0 in the real closed field of fractional power series $\sum_{-p}^{\infty} a_{-i} t^{-i/q}$ (a_i real algebraic), and uses the fact that any non-trivial expression in one free variable in the language considered has only a finite number of zeros. Actually, not only the rule of induction (RIO)

$$[(y)A(0, y) \wedge (x)(y)\{A(x, y) \to A(x', y)\} \cdot \to (x)(y)A(x, y)]$$

for quantifier-free A is valid in M_0, but also the axiom (AIO): $(y)\{A(0, y) \wedge (x)[A(x, y) \to A(x', y)] \cdot \to (x)A(x, y)\}$. In M_0, the following theorems of arithmetic are false: $x^2 \neq 2y^2 \vee x = 0$, $x^p + y^p \neq z^p \vee xyz = 0$ for $p = 3, 4, \cdots$. Thus these results can be stated in the fragment, but not proved. The author also gives alternative axiomatisations by means of special cases of (AIO) (but infinitely many, in contrast to Shoenfield [J. Symbolic Logic **23** (1958), 7–12; MR **20** #5726] for the case of P, $+$ only). More important, he analyzes the usual proofs of the irrationality of $\sqrt{2}$ from two points of view: (a) A rule of inference (PSI), parameter substitution induction, familiar from the literature of primitive recursive arithmetic, is added, where $\vdash A(x, y)$ is inferred from $\vdash A(0, y)$ and $\vdash A[x, t(x', y)] \to A(x', y)$. The elementary argument $a/b = \sqrt{2} \to [a - b < b \wedge (2b - a)/(a - b) = \sqrt{2}]$, fits in here by taking $0 < b \leqq x \to a^2 \neq 2b^2$ for $A(x, a/b)$ with obvious convention, and $t(x', a/b) = (2b \dot{-} a)/(a \dot{-} b)$ (independent of x). (b) Auxiliary function constants and (RIO) applied to formulae in the extended notation, e.g., $[x/y]$ and either $\gcd(x, y)$ or 2^x. It may be remarked that (a) reduces to (b) if one introduces the constant ψ: $\psi(0, x, y) = y$, $\psi(z', x, y) = t[x' \dot{-} z, \psi(z, x, y)]$ and applies (RIO) to $B(u, x, y): x' \geqq u \to A[u, \psi(x' \dot{-} u, x, y)]$. In this particular case, $\psi(z, x, y) = a_z/b_z$, where $a_0/b_0 = a/b$, $a_{m+1}/b_{m+1} = (2b_m \dot{-} a_m)/(a_m \dot{-} b_m)$. To the reviewer's knowledge, this is the first paper where various schemata of the theory of primitive recursive functions are used in an interesting way to systematize (at least a small area of) elementary number theory. It seems to be open whether there are similarly simple (in particular, recursive) non-standard models when one adds division and its recursion equations. Without this fourth basic operation one could hardly be said to be doing number theory.

G. Kreisel (Stanford, Calif.)

Mostowski, Andrzej 5003

Concerning the problem of axiomatizability of the field of real numbers in the weak second order logic.

Essays on the foundations of mathematics, pp. 269–286. Magnes Press, Hebrew Univ., Jerusalem, 1961.

Consider a second-order language T with the relations $a + b = c$, $a \cdot b = c$, $a \leqq b$, between individuals a, b, c and variables V for sets of individuals. A model is called 'weak' second-order if V ranges over all finite (and not arbitrary) sets of individuals. The main observation of the paper is this: If an arithmetic function μ describes (in the obvious sense) a denumerable relational system $\mathfrak{A}_\mu = \langle A, +_{\mathfrak{A}}, \cdot_{\mathfrak{A}}, \leqq_{\mathfrak{A}} \rangle$, the property of (the Gödel number $\ulcorner \Phi \urcorner$ of) the formula Φ: Φ holds in $\mathfrak{A}_\mu$ in the weak second-order sense is hyperarithmetic in μ. The computation is quite straightforward and, of course, parallel to the corresponding work for ω-models, e.g., Gandy-Kreisel-Tait [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **8** (1960), 577–582; MR **28** #2964a]. As a corollary: if a set $\mathscr{A}$ of axioms is analytic, so is the set $\mathrm{Val}_2^w(\mathscr{A})$ of (closed) formulae valid in all denumerable weak second-order models of $\mathscr{A}$. The Skolem-Lowenheim construction applies, and so the validity predicate is Π_1^1 in $\mathscr{A}$. Now, in the (non-denumerable) relational system $\mathscr{F}$ of the field of real numbers, the set of integers is definable in (the language of) T, and hence every analytic predicate. Consequently, there is no analytic set $\mathscr{A}$ such that $\mathrm{Val}_2^w(\mathscr{A})$ consists of just those statements in T which are true in $\mathscr{F}$. The notation and exposition are remarkably agreeable for a paper depending heavily on arithmetization of syntax and semantics.

G. Kreisel (Stanford, Calif.)

Gurevič, Ju. Š. 5004

Elementary properties of ordered Abelian groups. (Russian)

Algebra i Logika Sem. **3** (1964), *no.* 1, 5–39.

This important paper is concerned mainly with the classification by elementary properties of ordered abelian groups. For any relational system K of type σ, let $\Phi(K)$ denote the set of all first-order sentences in the language of σ which are satisfied by K. Let $\Phi_n(K)$ be those sentences in $\Phi(K)$ which are in prenex form and contain at most n quantifiers. The author introduces the concept of an m-chain, where m is any positive integer or ω. Let τ_n be the type corresponding to a single binary relation $x < y$ and $m - 1$ unary relations $|x| = k$ for $1 \leqq k < m$. An m-chain is a model of type τ_m in which the following axioms are satisfied: $x < y$ is anti-symmetric and transitive; the relations $|x| = i$ are mutually exclusive, that is, $\sim(|x| = i \wedge |x| = j)$ holds if $i \neq j$; the relation $x \asymp y$ defined by $\sim(x < y) \wedge \sim(y < x)$ is a congruence with respect to $x < y$ and $|x| = k$ $(1 \leqq k < m)$. With each ordered abelian group G and each n (a positive integer or ω) is associated an m-chain $T_n(G)$ (where m depends on n). The definition of $T_n(G)$ is complicated, and will not be given. The main result of the paper can be stated as follows: let G and H be ordered abelian groups; then there exists a primitive recursive function $k(n)$ such that if $\Phi_{k(n)}(T_n(G)) =$

$\Phi_{k(n)}(T_n(H))$, then $\Phi_n(G)=\Phi_n(H)$. Several corollaries are obtained from this theorem. For example, it follows that $\Phi(G)=\Phi(H)$ if and only if $\Phi(T_\omega(G))=\Phi(T_\omega(H))$. Moreover, the results of Robinson and Zakon [Trans. Amer. Math. Soc. **96** (1960), 222–236; MR **22** #5673] on the elementary equivalence of ordered abelian groups are also obtained as consequences of this main theorem. Finally, it is proved that the theory of ordered abelian groups is decidable. This is accomplished by reducing the decision problem for ordered abelian groups to the decision problem for ordered sets; by a result of Ehrenfeucht this latter theory is decidable. One of the principal tools used in the paper is Ehrenfeucht's game-theoretic method of establishing the equality $\Phi_n(A)=\Phi_n(B)$ for two models A and B of a formal theory [Fund. Math. **49** (1960/61), 129–141; MR **23** #A3666].

R. S. Pierce (Seattle, Wash.)

SET THEORY

See also 4991, 4995, 5009, 5011, 5414, 5415.

Całczyńska-Karłowicz, M. 5005

Theorem on families of finite sets.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 87–89.

The following theorem is proved: Let $\mathfrak{A}$, $\mathfrak{B}$ be two families of finite sets such that $A\cap B\neq\emptyset$ for any $A\in\mathfrak{A}$, $B\in\mathfrak{B}$. Then there exists a countable set N with $A\cap B\cap N\neq\emptyset$ for any $A\in\mathfrak{A}$, $B\in\mathfrak{B}$. An example is given to show that the theorem becomes false when $\mathfrak{A}$ or $\mathfrak{B}$ is allowed to consist of countably infinite sets.

G. Sabidussi (Hamilton, Ont.)

Rotman, B. 5006

Principal sequences and regressive functions.

J. London Math. Soc. **38** (1963), 501–504.

Suppose we are given, for each countable limit ordinal ξ, a sequence of ordinals $f_1(\xi)<f_2(\xi)<\cdots$ converging to ξ. The author proves that, for all but finitely many positive integers i, each function f_i takes $\aleph_1$ values $\aleph_1$ times each; that is, there is a set A_i of cardinal $\aleph_1$ such that $f_i^{-1}(\alpha)$ has cardinal $\aleph_1$ for each $\alpha\in A_i$.

A. H. Stone (Rochester, N.Y.)

Läuchli, H. 5007

The independence of the ordering principle from a restricted axiom of choice.

Fund. Math. **54** (1964), 31–43.

Let T be an axiomatic system of set theory obtained from the Bernays-Gödel system by leaving out the axiom of choice and replacing the axiom of regularity by an axiom of "regularity with respect to a basis set". The Fraenkel permutation method of proving independence of the axiom of choice is applicable to T. Using this method the author proves the following very strong theorem: If T is consistent, then it remains so upon adjunction of the axioms I and II below: (I) There is a set which cannot be ordered; (II) There is a function F whose domain D consists of all sets which can be well-ordered and which correlates with every set x in D a well-ordering of x. (D is called the class of well-orderable sets.)

As a corollary the author obtains the independence of the ordering theorem from the axioms of T and the following additional axioms: (a) There is a choice function for D; (b) If $x\subset D$ and $x\in D$, then $\bigcup x\in D$. Axiom (b) implies that every denumerable union of denumerable sets is itself denumerable.

As usual in the applications of the permutation method, the essential point in the proof is the construction of a suitable group of permutations. The result used by the author and obtained by means of an elaborate group-theoretical proof is the following: Let C be a denumerable set, G a group of permutations of C, $H_G(e)$ [respectively, $K_G(e)$] the group of those f in G which leave e fixed [pointwise fixed] (e and e' denote always finite subsets of C). There exists a group G such that (i) $K_G(e)=H_G(e)$ for each e; (ii) $K_G(e\cap e')$ is generated by $K_G(e)$ and $K_G(e')$ for arbitrary e, e'; (iii) for each e there are three different elements u, v, w and permutations f, g in $K_G(e)$ such that f transforms u in v and v in w, and g transforms w in u and u in v.

A. Mostowski (Warsaw)

COMBINATORIAL ANALYSIS

See also 5441.

Riordan, John 5008

The enumeration of election returns by number of lead positions.

Ann. Math. Statist. **35** (1964), 369–379.

In a ballot candidate A scores n votes and candidate B scores m votes. Denote by α_r and β_r the number of votes for A and B, respectively, among the first r votes counted. Denote by $l_j(n, m; c)$ the number of possible election records in which there are j c-lead positions for A, i.e., in which $\alpha_r>\beta_r+c-1$ holds for j subscripts $r=1, 2, \cdots, m+n$. Let $l_{nm}(x; c)=\sum_k l_k(n, m; c)x^k$. The author finds $l_{nm}(x; 0)$ and $l_{nm}(x; 1)$ explicitly and shows that, for $c>1$, $l_{nm}(x; c)$ can be obtained recursively.

L. Takács (New York)

Rota, Gian-Carlo 5009

The number of partitions of a set.

Amer. Math. Monthly **71** (1964), 498–504.

Let S be a set with n elements. A partition of S is a family of disjoint subsets of S whose union is S. Let B_n denote the number of distinct partitions of S. As the author points out, many problems of enumeration can be interpreted as counting the number of partitions of a finite set. In the present paper a new formula is obtained for B_n. Put $(u)_n=u(u-1)\cdots(u-n+1)$. Let V be the vector space over the reals consisting of all polynomials in the variable u. Let L be the unique linear functional on V such that $L(1)=1$, $L((u)_k)=1$ $(k=1, 2, 3, \cdots)$. It follows from the formula $\sum_\pi (u)_{N(\pi)}=u^n$, where the sum on the left ranges over all partitions of S and $N(\pi)$ denotes the number of blocks in the partition π, that

$$(*)\qquad B_n = L(u^n).$$

The author illustrates the use of (*) by deriving the

familiar recurrence and generating function for B_n. He also obtains the set of orthogonal polynomials

$$h_n(u) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (u)_{n-k}$$

discussed by Touchard [Canad. J. Math. 8 (1956), 305–320; MR 18, 16]. *L. Carlitz* (Durham, N.C.)

Colombo, Umberto 5010

Sulle disposizioni a scacchiera costruite con k campi di Galois. (English, French, Spanish, and German summaries)

Giorn. Ist. Ital. Attuari **26** (1963), 106–117.

Author's summary: "We are given the total number of sets of s mutually orthogonal Latin squares of side p, when $p = p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$ ($p_i^{n_i}$ prime number or power of prime number) and when $s \leq N = p_1^{n_1} - 1$, starting by a given Latin square. The construction is based on the application of the properties of the elements of k Galois fields."

ORDER, LATTICES

See also 4989, 5156, 5888.

Harzheim, Egbert 5011

Beiträge zur Theorie der Ordnungstypen, insbesondere der η_α-Mengen.

Math. Ann. **154** (1964), 116–134.

A cut in a totally ordered set T is a pair (A, B), where A is an initial section of T and $B = T - A$; A or B may be empty. Let $S(T)$ be the set of cuts in T. Totally order $T \cup S(T)$ in the expected way. Define by induction: $T_0 = \emptyset$, $T_{\alpha+1} = T_\alpha \cup S(T_\alpha)$. and, for limit ordinals ρ, $T_\rho = \bigcup_{\theta<\rho} T_\theta$ (with the obvious total ordering on T_ρ). This construction is called the method of successive filling in of cuts. A totally ordered set M is called ω_α-free if M has no subsets of order types ω_α or ω_α^*. Let R_α stand for the set of all sequences of 0's and 1's of length ω_α in which there is a last 1; R_α is ordered according to first differences. The author proves a host of interesting properties of the T_α's and the R_α's, out of which we cite the following small selection. T_{ω_α} has cardinality $\sum_{\beta<\alpha} 2^{\aleph_\beta}$ and is the union of $\aleph_\alpha$-many ω_α-free subsets. For regular ω_α, T_{ω_α} is an η_α-set. For every α, R_α is similar to T_{ω_α}. If ω_α is regular, a totally ordered set M is similar to R_α if and only if M is an η_α-set and M is the union of $\aleph_\alpha$-many ω_α-free subsets. If ω_α is singular, a totally ordered set M is similar to R_α if and only if $M = \bigcup_{\rho<\alpha} M_{\rho+1}$, where each $M_{\rho+1}$ is similar to $R_{\rho+1}$ and, for $\rho' < \rho'' < \alpha$, $M_{\rho'+1}$ is similar to a subset of $M_{\rho''+1}$. *E. Mendelson* (Flushing, N.Y.)

Kogalovskiĭ, S. R. 5012

On a theorem of Frink. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 2 (116), 143–145.

The author generalizes a result of the reviewer which states that a lattice is compact in its interval topology if and only if it is complete. The interval topology of a partially ordered set P results from taking as a sub-base for the closed sets of P the one-sided closed intervals of the form: all $x \geq a$, and all $x \leq b$.

He calls the set P bi-inductive if every chain in P has a least upper bound and a greatest lower bound, and shows that if P is compact in its interval topology, then P is bi-inductive. He shows by a counter-example that the converse is false; not every bi-inductive partially ordered set is compact in its interval topology.

He next shows that every bi-inductive semilattice is complete, and that if all chains of a semilattice S are bounded, then S is bi-inductive if and only if it is complete. Finally, he shows that a semilattice is compact in its interval topology if and only if it is bi-inductive. Since a lattice is bi-inductive if and only if it is complete, this generalizes the theorem of the reviewer.

O. Frink (University Park, Pa.)

Kogalovskiĭ, S. R. 5013

On linearly complete ordered sets. (Russian)

Uspehi Mat. Nauk **19** (1964), *no.* 2 (116), 147–150.

It is well known that a lattice has the fixed-point property if and only if it is complete [A. C. Davis, Pacific J. Math. 5 (1955), 311–319; MR 17, 574]. The author studies conditions on a partially ordered set P, not necessarily a lattice, which are necessary or sufficient for the fixed-point property. This means that every isotone mapping of P into itself has at least one fixed element.

He calls P linearly complete if every maximal chain of P is complete, and shows that linear completeness is necessary but not sufficient for the fixed-point property. For semilattices, however, linear completeness is both necessary and sufficient. He shows that a lattice is complete if and only if it is linearly complete.

E. S. Wolk [Canad. J. Math. 9 (1957), 400–405; MR 19, 243] calls P Dedekind complete if every directed subset has a least upper bound, and every dually directed subset a greatest lower bound. The author shows that P is Dedekind complete if and only if it is bi-inductive, which means every chain has a least upper bound and greatest lower bound. He defines uniform set and shows that a uniform set is linearly complete if and only if it is bi-inductive and hence Dedekind complete. He is then able to derive in another manner some of Wolk's results on partially ordered sets with the fixed-point property.

O. Frink (University Park, Pa.)

Benado, Mihail 5014

Zur abstrakten Begründung der Führertheorie.

Math. Japon. **6** (1960/61), 1–25.

Suppose P is a partially ordered set with relative closure operator: $\bar{y}^u$ is the closure of y relative to u. If $u \geq v$ then f is called the topological conductor of v relative to u if and only if f is the greatest element such that $f \leq v$ and $\bar{f}^u = f$. In a partially ordered set with one operation, written multiplicatively, with $u \geq v$, f is called the algebraic conductor of v relative to u if and only if f is the greatest element such that $f \leq u$ and $uf \leq v$.

The paper presents relationships of these notions to each other, to the extension-contraction correspondence in integral domains, to the Dedekind theory of relatively prime ideals, to the author's previous study of general divisibility relations and of normality, and to modularity.

P. M. Whitman (Providence, R.I.)

Traczyk, T. 5015

An equational definition of a class of Post algebras.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 147–149.

Using his earlier characterization of Post algebras [same Bull. **11** (1963), 3—8; MR **26** #6078] the author gives an equational postulate system, using the functions $\cup$, $\cap$, C, $D_1, \cdots, D_{n-1}$ and the constants $e_0, \cdots, e_{n-1}$ (for the terminology see the paper cited), where D_i is defined by $a = D_1(a)e_1 \cup \cdots \cup D_{n-1}(a)e_{n-1}$ and $C(a)$ is the complement of $D_1(a)$. *G. Grätzer* (University Park, Pa.)

Hermes, Hans 5016

★La teoría de retículos y su aplicación a la lógica matemática [The theory of lattices and its application to mathematical logic].

Conferencias de Matematica, VI. Consejo Superior de Investigaciónes Científicas.

Publicaciones del Instituto de Matemáticas "Jorge Juan", Madrid, 1963. 57 *pp.*

After a brief introduction to the notions of partially ordered set and lattice (modular, distributive, and Boolean), three special topics are studied. First, there is a short section on perspectivities and Galois correspondences. Second, some results are proved about fixed points of lattice endomorphisms, and applications are given to the Schröder-Bernstein Theorem and to a characterization of complete lattices (theorem of A. C. Davis). Third, the propositional calculus is outlined and a completeness theorem is proved for a suitable axiomatization. There is also brief mention of Halmos's polyadic algebras as natural algebraic counterparts of the predicate calculus. The exposition is lucid.

{The monograph has also been published as Publ. Sem. Mat. Zaragoza, Zaragoza, 1962.}

E. Mendelson (Flushing, N.Y.)

Doctor, Hoshang P. 5017

The categories of Boolean lattices, Boolean rings and Boolean spaces.

Canad. Math. Bull. **7** (1964), 245–252.

Stone's theorem [Trans. Amer. Math. Soc. **41** (1937), 375–481] about the equivalence of the theories of Boolean rings and Boolean topological spaces is formulated and proved by the author in terms of the equivalence of categories, in the sense of Grothendieck [Tôhoku Math. J. (2) **9** (1957), 119–221; MR **21** #1328], of Boolean lattices, Boolean spaces, Boolean rings, and their respective homomorphisms or mappings. *P. M. Whitman* (Providence, R.I.)

Goodstein, R. L. 5018

★Boolean algebra.

Pergamon Press, Oxford-London-Paris-Frankfurt; The Macmillan Co., New York, 1963. vii+140 *pp.* $1.95.

After an informal elementary development of the algebra of classes, three axiom systems for Boolean algebra are presented in detail, including proofs of independence and completeness: (1) the usual self-dual axiom system for $\cap$ and $\cup$, with a treatment of canonical forms and of the direct product of a Boolean algebra with itself; (2) axioms for two operations $\cap$ and $'$, with discussion of solutions of equations, of congruences, and of homomorphisms; (3) axiomatization of provability of compound sentences. In the last chapter, Boolean algebra is examined in the context of partially ordered sets and lattices, with some treatment of ideals and Stone's theorem on the representation of Boolean algebras by sets. Finally, there is a brief treatment of Newman algebras, in which the commutative and associative axioms are deleted from (1).

The treatment is generally detailed. Typography is good in a narrow sense, but skimpy in emphasis and in setting off different parts of the material. There are examples and exercises, with solutions for all. The book is a usable introduction for readers with limited background.

P. M. Whitman (Providence, R.I.)

GENERAL MATHEMATICAL SYSTEMS

See also 5015.

Baranovič, T. M. 5019

Free decompositions in the intersection of primitive classes of algebras. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 727–729.

Let us say that a primitive class of universal algebras has a good free decomposition theory if the usual theorems of Schreier-Kuroš type hold: subalgebras of free algebras are free; subalgebras of a free product are free products of a specified kind; any two free decompositions have isomorphic refinements. Suppose that $K_1 = (\Omega_1, \Lambda_1)$ and $K_2 = (\Omega_2, \Lambda_2)$ are two primitive classes of algebras, each with a good free decomposition theory. What can then be said about the "intersection" class $K = (\Omega, \Lambda)$, where $\Omega = \Omega_1 \cup \Omega_2$, $\Lambda = \Lambda_1 \cup \Lambda_2$? In this note the author sketches, without proofs, such a free decomposition theory for K under the assumption that Ω_1 and Ω_2 either do not intersect at all, or intersect in the nullary operator 0 and that then Λ_1 and Λ_2 contain all identities of the form $00 \cdots 0\omega = 0$ for $\omega \in \Omega_1$ or Ω_2. The results are of an axiomatic nature. Seven conditions on K_1 and K_2 are set up which in various combinations ensure that the decomposition theory for K has one desirable feature or another. The results are also quite general and comprise a number of known cases such as groups, non-associative algebras, loops, multi-operator algebras [see A. G. Kuroš, Sibirsk. Mat. Ž. **1** (1960), 62–70; correction, 638; MR **24** #A1278; Acta Sci. Math. (Szeged) **21** (1960), 187–196; MR **24** #A1326] and Ω-loops in the sense of P. J. Higgins [Proc. London Math. Soc. (3) **6** (1956), 366–416; MR **18**, 559].

A more detailed account is held over until the full proofs are available. *K. A. Hirsch* (Madison, Wis.)

Narkiewicz, W. 5020

On a certain class of abstract algebras.

Fund. Math. **54** (1964), 115–124.

A v^{**}-algebra is one in which $a_1, \cdots, a_n$ form an independent sequence if and only if they are C-independent, i.e., a_i does not belong to the subalgebra generated by $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n$ $(i = 1, \cdots, n)$. This generalizes the notion of v^*-algebras which were studied earlier by the author [Fund. Math. **50** (1961/62), 333–340; MR **25** #36; ibid. **52** (1963), 289–290; MR **27** #3575]. Theorem I: If a v^{**}-algebra has a basis, then all bases have the same

number of elements. This result is surprising because all previous results of this kind had to use some form of the exchange axiom, which is avoided in Theorem I. The main result is the following (Theorem II): an algebra $\mathfrak{A}$ is a v^{**}-algebra if and only if Δ_k is a semigroup of transformations $\nabla_k \to \nabla_k$ whenever ∇_k is non-void and $\nabla_1 f(a) = g(a)$ implies $f = g$. In this theorem Δ_k denotes the system of C-independent k-tuples of the algebra $A^{(k)}$ of polynomials in k variables over $\mathfrak{A}$ and ∇_k denotes the system of C-independent k-tuples of $\mathfrak{A}$. This result is an extension of an earlier result of the author for v^*-algebras. However, this does not mean that it was easy to find the conditions characterizing the v^{**}-algebras, nor does it mean that it is possible to carry out the proof along a similar line. The last section considers a few properties (e.g., finiteness) which imply that a v^{**}-algebra is a v^*-algebra.

G. Grätzer (University Park, Pa.)

Karolinskaja, L. N. 5021

Direct decompositions of abstract algebras with distinguished subalgebras. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1960**, *no.* 4 (17), 106–113.

Hashimoto [Osaka Math. J. **9** (1957), 87–112; MR **19**, 935] showed how the direct product representations of abstract algebras may be characterized in terms of certain lattice conditions on the congruences together with the condition of complete permutability of the congruences. In the paper under review the author gives a somewhat more complicated characterization (still relying only upon permutability and lattice conditions parallel to those of Hashimoto) for the case in which the direct product contains a distinguished subalgebra which is itself a direct product of just those factor algebras induced by the congruences acting upon the distinguished subalgebra.

R. M. Baer (Berkeley, Calif.)

THEORY OF NUMBERS

See also 4981, 4984, 5002, 5060, 5062, 5079, 5089, 5189, 5197, 5273, 5300, 5578.

Erdős, Pál; Surányi, János 5022

★Válogatott fejezetek a számelméletből [Selected chapters from number theory].

Tankönyvkiadó Vállalat, Budapest, 1960. 250 *pp.* 12.60 *Ft.*

Cohn, John H. E. 5023

Square Fibonacci numbers, etc.

Fibonacci Quart. **2** (1964), 109–113.

The author proves the old conjecture that $F_1 = F_2 = 1$ and $F_{12} = 144$ are the only perfect squares in the sequence $\{F_n\}$ of Fibonacci numbers. He also shows that $L_1 = 1$ and $L_3 = 4$ are the only perfect squares in the sequence $\{L_n\}$ of Lucas numbers, and he determines all numbers F_n and L_n of the form $2x^2$, x an integer. The methods are elementary. {The main result was obtained independently by others, including the reviewer [Amer. Math. Monthly **71** (1964), 221–222].}

O. Wyler (Albuquerque, N.M.)

Rohrbach, Hans; Weis, Jürgen 5024

Zum finiten Fall des Bertrandschen Postulats.

J. Reine Angew. Math. **214/215** (1964), 432–440.

The authors prove the following result: For each $n \geqq 118$, there exists a prime p such that $n < p \leqq 14n/13$. The proof, which is elementary, is based upon the Chebyshev θ and ψ functions.

E. Cohen (Knoxville, Tenn.)

Gupta, Hansraj 5025

A congruence property of Euler's ϕ-function.

J. London Math. Soc. **39** (1964), 303–306.

Let $a \geqq 2$, $j \geqq 1$ and p prime. The author proves that

$$\phi(a^{p^j} - 1) \equiv 0 \pmod{p^{j(j+1)/2}}.$$

He notes that when $a \equiv 1 \pmod p$ this result can be improved. Indeed, if $a \equiv 1 \pmod{p^u}$, then

$$\phi(a^{p^j} - 1) \equiv 0 \pmod{p^m},$$

where $m \geqq u + j - 1 + j(j+1)/2$.

L. Carlitz (Durham, N.C.)

Ehrhart, Eugène 5026

Nombre de points entiers d'un tétraèdre.

C. R. Acad. Sci. Paris **258** (1964), 3945–3948.

By using the method of an earlier note [same C. R. **256** (1963), 4566–4569; MR **27** #3596a] the author gives a formula for the number of solutions in integers X, Y, Z of $X/a + Y/b + Z/c \leqq 1$; here, a, b, c are integers coprime by pairs.

B. J. Birch (Manchester)

Halberstam, H.; Laxton, R. R. 5027

On perfect difference sets.

Quart. J. Math. Oxford Ser. (2) **14** (1963), 86–90.

A set of integers $k_0, k_1, \cdots, k_m$ is called a perfect difference set (p.d.s.) if in some order the $m^2 + m$ differences $k_i - k_j$ $(i \neq j)$ are congruent to $1, 2, \cdots, m^2 + m \pmod q$ $(q = m^2 + m + 1)$. Using finite projective geometries, J. Singer [Trans. Amer. Math. Soc. **43** (1938), 377–385] has proved that p.d.s.'s exist if $m = p^n$ (p prime). A simpler proof is presented here which is based on the following observation: Let ζ be a generator of the multiplication group $\mathfrak{G}_{3n}$ of the Galois field $GF(p^{3n})$; put $q = p^{2n} + p^n + 1$. Then ζ^{k_1} and ζ^{k_2} are contained in the same coset of the subgroup $\mathfrak{G}_n$ in $\mathfrak{G}_{3n}$ if and only if $k_1 \equiv k_2 \pmod q$. Hence all k arising from the elements of $GF(p^{3n})$ representable in the form $a + b\zeta = \zeta^k$ $(0 \leqq k \leqq p^{3n} - 2;\ a, b \in \mathfrak{G}_n)$ determine a set K of equivalence classes (mod q) of which there are $p^n + 1$ different ones in K. These are shown to represent a p.d.s. Every generator ζ of $\mathfrak{G}_{3n}$ defines such a set, called a ζ-generated p.d.s. Following M. Hall [Proc. Amer. Math. Soc. **7** (1956), 975–986; MR **18**, 560] t is said to be a multiplier of a p.d.s. k_μ $(\mu = 0, 1, \cdots, m)$ if $tk_\mu = k_\nu + s \pmod q$ for some s and for a certain permutation $\mu \to \nu$. Singer [loc. cit.] has conjectured that t is a multiplier of the p.d.s. (k_μ) $(\mu = 0, 1, \cdots, p^n)$ if and only if $t \equiv p^r \pmod q$. This is proved now for ζ-generated p.d.s.'s. Finally the following theorem is mentioned: The p.d.s.'s generated by ζ and ζ^t $((t, q) = 1,\ 1 < t < p^n)$ coincide if and only if $t \equiv p^v$.

H. Schwerdtfeger (Montreal, Que.)

Jordan, J. H. 5028

Pairs of consecutive power residues or non-residues.

Canad. J. Math. **16** (1964), 310–314.

Let k be a positive integer, $p=k+1$ a prime and g a primitive root of p. Let $g^{\operatorname{ind} u}\equiv u \pmod p$.

Recently various authors [the reviewer and E. Lehmer, Proc. Amer. Math. Soc. **13** (1962), 102–106; MR **25** #2025; the reviewer, E. Lehmer and W. Mills, Canad. J. Math. **15** (1963), 172–177; MR **26** #3660] have been concerned with the function $\Lambda(k, m)=\max\{r(k, m, p)\}$, where $r(k, m, p)$ denotes the least r for which

$$(1)\quad \operatorname{Ind}(r)\equiv \operatorname{Ind}(r+1)\equiv\cdots\equiv \operatorname{Ind}(r+m-1)\equiv 0 \pmod{p-1}$$

and the maximum is taken over all primes p for which r exists. $\Lambda(k, 2)$ increases rapidly with k: $\Lambda(2, 2)=9$, $\Lambda(3, 2)=77$, $\Lambda(4, 2)=1224$, $\Lambda(5, 2)=7888$, $\Lambda(6, 2)=202124$, $\Lambda(7, 2)=1649375$.

The author defines another function $\Lambda^*(k, m)$ obtained by deleting the condition of divisibility in (1) so that one asks only for the first appearance of m consecutive integers whose kth power characters are identical. $\Lambda^*(k, 2)$ turns out to be very much smaller. The author has established that $\Lambda^*(2, 2)=3$, $\Lambda^*(3, 2)=8$, $\Lambda^*(4, 2)=20$, $\Lambda^*(5, 2)=44$, $\Lambda^*(6, 2)=80$, $\Lambda^*(7, 2)=343$. The proofs are short enough to be done "by hand", although the last two proofs are omitted to save space. The further results $\Lambda^*(2k, 3)=\Lambda^*(k, 4)=\infty$ are also obtained in the same manner as for Λ. *D. H. Lehmer* (Berkeley, Calif.)

Baker, A. 5029

Rational approximations to certain algebraic numbers.

Proc. London Math. Soc. (3) **14** (1964), 385–398.

Given any irrational number α and any real $k>2$, by Roth's theorem there exists a constant $c=c(\alpha, k)$ such that $|\alpha-p/q|>cq^{-k}$ for all rational numbers p/q. One purpose of this paper is to exhibit a class of algebraic numbers for which c can be made explicit. This class of numbers has the form $\alpha=(u_1a^{1/n}+u_2b^{1/n})/(u_3a^{1/n}+u_4b^{1/n})$, where the integers $a>b$, $n\geqq 3$, u_1, u_2, u_3, u_4 are chosen so that α is irrational, and where $a>(a-b)^\rho(3n)^{2\rho-2}$ with $\rho=\frac{5}{2}(2k-1)(k-2)^{-1}$. For these α an explicit value of c, not set forth in detail here, is obtained as a function of a, b, u_1, u_2, u_3, u_4, n, k. In proving this result the author establishes that with a satisfying the same inequalities, $|ax^n-by^n|\geqq C|x^{n-k}|$ for all integers x and y for a given explicit $C=C(a, b, n, k)$. This generalizes a theorem of C. L. Siegel [Math. Ann. **114** (1937), 57–68]. Furthermore, bounds on $|x|$ and $|y|$ are established for integer solutions x and y of $ax^n-by^n=f(x, y)$ in case f is a polynomial of degree at most $n-3$ with rational coefficients under the assumption $a>(a-b)^{25/2}(3n)^{23}$. The proofs employ the hypergeometric function to get a sequence of rational approximations to $(a/b)^{1/n}$ almost as good as the convergents of the continued fraction expansion but with denominators increasing comparatively slowly.

I. Niven (Eugene, Ore.)

Angel, Myer 5030

Partitions of the natural numbers.

Canad. Math. Bull. **7** (1964), 219–236.

This paper is an extension of work by J. Lambek and L. Moser [Amer. Math. Monthly **61** (1954), 454–458; MR **16**, 17] in the following way. Let $f: f(1), f(2), \cdots$ be a non-decreasing sequence; let $f^+: f^+(1), f^+(2), \cdots$ be the sequence whose nth term $f^+(n)$ is the number of m such that $f(m)<n$. The sequences f and f^+ are said to be inverses. Lambek and Moser have shown that two non-decreasing sequences f and g are inverses if and only if the derived sequences F, G, where $F(n)=f(n)+n$, $G(n)=g(n)+n$, are complementary in the sense that they have no common elements and together exhaust the positive integers. In the author's terminology, complementary sequences form a two-part partition. His extension is to many-part partitions, each part a non-decreasing sequence, no two parts with common elements, and the totality of parts the sequence of positive integers. The development is by means of composite sequences: the composite F_1F_2 has the nth term $F_2(F_1(n))$; $F_1F_2F_3$ has nth term $F_3(F_2(F_1(n)))$, and so on. He first notes that if $\{F_1, F_2\}$, $\{F_3, F_4\}$ are each two-part partitions, then $\{F_1F_3, F_1F_4, F_2F_3, F_3F_4\}$ is a four-part partition as are $\{F_1F_1, F_1F_2, F_2F_1, F_2F_2\}$ and $\{F_3F_3, F_3F_4, F_4F_3, F_4F_4\}$. Thus this kind of multiplication of two-part partitions will yield partitions with 2^k parts. But his main attention is on another kind. Let F^* be the sequence such that $F^*(n)$ is the number of m such that $F(m)\leqq n$. Then the partitions of chief interest are $\{F_1, F_2, \cdots, F_n\}$, where $F_iF_j^*=(F_jF_i^*)^+$ and $F_jF_j^*F_jF_k^*\leqq F_iF_k^*\leqq F_jF_j^*SF_jF_k^*$, S being the operator which replaces a term of a sequence by its successor. The results found are not adapted to brief summary. *J. Riordan* (Murray Hill, N.J.)

Härtter, Erich 5031

Eine Bemerkung über periodische Minimalbasen für die Menge der nichtnegativen ganzen Zahlen.

J. Reine Angew. Math. **214/215** (1964), 395–398.

The definitions of basis and minimal basis of a set of non-negative integers are already to be found in the papers of the author [same J. **196** (1956), 170–204; MR **19**, 122] and Stöhr [ibid. **194** (1955), 40–65, 111–140; MR **17**, 713]. In this paper, the author assumes the two results (Theorems 36 and 37 in his above-mentioned paper) and proves that there are exactly two periodic minimal bases with (Schnirelmann) density $\frac{1}{2}$. *B. K. Ghosh* (Calcutta)

Rotkiewicz, André; Schinzel, André 5032

Sur les nombres pseudopremiers de la forme $ax^2+bxy+cy^2$.

C. R. Acad. Sci. Paris **258** (1964), 3617–3620.

The positive integer n is pseudoprime if n is composite and if n divides 2^n-2. One of the authors has shown that suitable arithmetic progressions contain infinitely many pseudoprimes [Rotkiewicz, same C. R. **257** (1963), 2601–2604]. Now it is shown that, in further imitation of the behavior of the primes, there are infinitely many pseudoprimes represented by a quadratic form $ax^2+bxy+cy^2$ (with natural restrictions on a, b, and c). The authors employ a theorem of Zsigmondy on primitive prime factors of a^n-b^n [Monatsh. Math. **3** (1892), 265–284] and a representation of cyclotomic polynomials given by Schinzel [Proc. Cambridge Philos. Soc. **58** (1962), 555–562; MR **26** #1280].

{Formula (7) should read: $p=ax^2+bxy+cy^2\equiv f \pmod{em}$.} *J. B. Kelly* (Tempe, Ariz.)

Zane, Burke 5033
Uniform distribution modulo m of monomials.
Amer. Math. Monthly **71** (1964), 162–164.

For any polynomial $f(x)$ with integral coefficients say that f is uniformly distributed (mod m), or u.d. mod m, if $f(1), f(2), f(3), \cdots$ are equidistributed in the limit among the residue classes mod m. This amounts to requiring that $f(1), f(2), \cdots, f(m)$ form a complete residue system modulo m. Assuming g.c.d. $(m_1, m_2)=1$ then f is u.d. mod m_1m_2 if and only if f is u.d. mod m_1 and mod m_2. Take f of the special form ax^k with $k>1$. First ax^k is u.d. mod m if and only if m is square-free and ax^k is u.d. mod p for every prime divisor p of m. Define K as the largest divisor of k satisfying $(K, p)=1$. Then ax^k is u.d. mod p if and only if $(a, p)=(K, p-1)=1$.

I. Niven (Eugene, Ore.)

Danzer, L. 5034
Über eine Frage von G. Hanani aus der additiven Zahlentheorie.
J. Reine Angew. Math. **214/215** (1964), 392–394.

Let $\mathfrak{A}$ and $\mathfrak{B}$ denote two infinite sequences of natural numbers, $0 \in \mathfrak{A} \cap \mathfrak{B}$, such that $\mathfrak{A}+\mathfrak{B}$ contains all but finitely many (c_1, say) of the natural numbers. Then if $A(x)$ and $B(x)$ denote the number of terms in $\mathfrak{A} \cap [0; x]$ and $\mathfrak{B} \cap [0; x]$, respectively, it is evident that

$$A(x)B(x) \geqq x+1-c_1.$$

It was conjectured by G. Hanani in an oral communication to P. Erdős [Michigan J. Math. **4** (1957), 291–300; MR **20** #5157; Magyar Tud. Akad. Mat. Kutató Int. Közl. **6** (1961), 221–254] that

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x+1} > 1.$$

It was also conjectured by W. Narkiewicz that if $\mathfrak{A}+\mathfrak{B}$ includes all but finitely many of the natural numbers at least m times, then

$$\limsup_{x\to\infty} \frac{A(x)B(x)}{x+1} \leqq m$$

implies that $\mathfrak{A}$ or $\mathfrak{B}$ is finite [Colloq. Math. **7** (1959/60), 161–165; MR **22** #3722]. The author provides a set $\mathfrak{B}$ and for each m a set $\mathfrak{A}_m$ such that $\mathfrak{B}$ and $\mathfrak{A}_m$ are infinite sets, $\mathfrak{A}_m+\mathfrak{B}$ includes each natural number at least m times, and for each m,

$$\lim_{x\to\infty} \frac{A_m(x)B(x)}{x+1} = m.$$

The author conjectures that if $\mathfrak{A}$ and $\mathfrak{B}$ are two infinite sequences of natural numbers, $0 \in \mathfrak{A} \cap \mathfrak{B}$, such that $\mathfrak{A}+\mathfrak{B}$ contains all but finitely many of the natural numbers, then

$$\liminf_{x\to\infty} [A(x)B(x)-x] = \infty.$$

G. M. Petersen (Swansea)

Ljunggren, W. 5035
On the Diophantine equation $Cx^2+D=y^n$.
Pacific J. Math. **14** (1964), 585–596.

The author extends his results [see Norske Vid. Selsk. Forh. (Trondjhem) **18** (1945), no. 32, 125–128; MR **8**, 136; ibid. **29** (1956), 1–4; MR **17**, 1185] on the number of integral triples x, y, z such that $Cx^2+D=y^z$. Typical of his results is the following. Let D be an odd square-free integer greater than 1. If z is an odd prime, prime to the class number of $Q(\sqrt{-D})$ and $\not\equiv 3 \pmod 8$, then the equation $x^2+4D=y^z$ has no integral solutions with y odd. Furthermore, for a given D, there are but finitely many x, y, and prime z such that $x^2+4D=y^z$, and bounds can be given on the size of y and z after finitely many arithmetical operations. The author's theorems lead to generalizations and extensions of results of the reviewer [Pacific J. Math. **11** (1961), 1063–1076; MR **25** #3005] for the cases $C=1$, $D=7$ and 28. The proofs make use of Skolem's p-adic method as refined by the author. The paper concludes with a short historical review of the general problem.

D. J. Lewis (Ann Arbor, Mich.)

Inkeri, K. 5036
Über die Lösbarkeit einiger Diophantischer Gleichungen.
Ann. Acad. Sci. Fenn. Ser. A I No. 334 (1963), 15 *pp.*

Let l be an odd prime, n a positive integer, and c_1 and c_2 relatively prime positive integers. The author derives a sequence of necessary conditions that the equation

$$c_1(x^{l^n}+y^{l^n}) = c_2z^{l^n}$$

should have integral solutions. Typical of his results is the following: If $n > ((l-1)\log c)/(\log l) - 1$ and $c=\max(c_1, c_2) > 1$, then the equation has no integral solutions with $(l, z)=1$. The methods used are closely related to those used previously in studying Fermat's conjecture.

D. J. Lewis (Ann Arbor, Mich.)

Khinchin, A. Ya. [Hinčin, A. Ja.] 5037
★Continued fractions.
The University of Chicago Press, Chicago, Ill.-London, 1964. xi+95 *pp.* $1.95.

The publishers state that this is the first English translation from the Russian (cf. the German translation in 1956 [*Kettenbrüche*, Teubner, Leipzig, 1956; MR **18**, 274]). It is based on the third edition [Fizmatgiz, Moscow, 1961] of the text which was undertaken after the death of the author. This latter edition was reprinted from the second with only a few brief bibliographical remarks added. Aside from these notes, there is no bibliography. The second Russian edition appeared in 1949 [GITTL, Moscow, 1949; MR **13**, 444] with no essential changes from the first edition in 1935 [ONTI NKTP, Moscow, 1935].

This book is designed as an elementary exposition to fill the gap in the Russian literature. The author confines his attention to continued fractions with positive integral elements. Chapter I consists of the elementary recurrence relations with convergence theorems of regular continued fractions $a_0+1/a_1+1/a_2+\cdots$, where the a_i, $i=1, 2, \cdots$, are positive real numbers and a_0 is an arbitrary real number. Essential formulas needed for applications are given. In Chapter II, the real number system is constructed from continued fractions. Some theorems on the representation of real numbers by regular continued fractions are given, as are applications to the solutions of Diophantine equations, Liouville numbers, square roots of numbers not necessarily perfect squares, and periodic continued fractions. Chapter III is concerned with a presentation of the metric theory of continued fractions.

Here the measures of sets of numbers are investigated. In particular, Gauss's problem in the measure theory of continued fractions and Kuz'min's theorem are described, as well as average values. *E. Frank* (Chicago, Ill.)

Khintchine, A. Ya. [Hinčin, A. Ja.] 5038
★Continued fractions.
Translated by Peter Wynn.
P. Noordhoff, Ltd., Groningen, 1963. iii+101 *pp. Dfl.* 16.25.
This is another translation into English of the third Russian edition [cf. #5037 above].
E. Frank (Chicago, Ill.)

Scott, P. R. 5039
On perfect and extreme forms.
J. Austral. Math. Soc. **4** (1964), 56–77.
A positive quadratic form is said to be perfect if it is uniquely determined by the integral vectors for which it takes its least positive value. The author gives an extensive list of new perfect forms. These are direct sums of forms of type $x_1^2 - x_1x_2 + x_2^2 - x_2x_3 + \cdots + x_{r-1}^2 - x_{r-1}x_r + x_r^2$, and certain linear sections of these forms. Conditions are obtained for such forms to be perfect; it turns out that only in a comparatively small number of cases are the forms not perfect.
The author then shows that certain of these forms are extreme, i.e., that M^n/D is a local minimum, where M is the minimum of the form for integral vectors, D the determinant of the form, and n the number of variables.
J. A. Todd (Cambridge, England)

Ohnari, Setsuo 5040
On the lattice constant of a regular n-gon ($n \equiv 0$ mod 6). (Japanese)
Sûgaku **14** (1962/63), 236–238.
Let φ be a point set on a plane containing the origin 0. A lattice Λ is said to be φ-admissible if $\Lambda \cap \varphi = \{0\}$. The lattice constant $\Delta(\varphi)$ is defined by the infimum of the determinants $d(\Lambda)$ for all φ-admissible lattices Λ. The purpose of the present paper is the computation of the lattice constant of a regular n-gon when $n \equiv 0$ (mod 6). Denote by φ_n the interior of a regular n-gon whose vertices are ω^ν, $0 \leq \nu < n$, where $\omega = \exp(2\pi i \cdot n^{-1})$. Then the result is

$$\Delta(\varphi_n) = \frac{\sqrt{3}}{2}\cos^2\frac{\pi}{n}.$$

The proof is due to several results on the geometry of numbers, and the author explains why the assumption $6|n$ is necessary. *S. Hitotumatu* (Tokyo)

White, G. K. 5041
Lattice tetrahedra.
Canad. J. Math. **16** (1964), 389–396.
Let $\{x\}$ denote the fractional part of x. It is proved that if r_i ($i = 1, 2, 3$) are 3 rational numbers with $r_i + r_j$ not an integer for $i \neq j$, then there exists an integer u for which $1 < \sum_{i=1}^3 \{r_i u\} \leq 1$. From this, the following theorem is deduced. If T is a closed tetrahedron and Λ is a lattice which contains the vertices of T, then the two following conditions are equivalent: (1) The only points of Λ in T other than the vertices lie on a pair of opposite edges of T; (2) There is a pair of parallel lattice planes of Λ through a pair of opposite edges of T such that no points of Λ lie between these planes.
References to related problems are given. {The letters u and w are missing in equation (18) and on p. 395, line 3, respectively.} *I. Danicic* (London)

Schmidt, Wolfgang M. 5042
Über Gitterpunkte auf gewissen Flächen.
Monatsh. Math. **68** (1964), 59–74.
The author shows that under quite mild restrictions there cannot be too many integral points (x, y, z) on a surface $z = f(x, y)$, or, more generally, that there cannot be too many integral points for which the difference $z - f(x, y)$ is very small. Theorem 1 is concerned with integral points for which (*) $1 \leq x \leq A$, $\phi_1(x) \leq y \leq \phi_2(x)$, where $A \geq 2$ is an integer, the $\phi_j(x)$ are differentiable and $\phi_2(x) - \phi_1(x) \leq B$ for some constant B. Suppose that f_y is monotone on each of the curves $y = \phi_j(x)$ and that $|f_y| \leq c$ for some constant c throughout (*). Suppose, further, that f has continuous second derivatives and that the f_{yy} and the Hessian $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ do not vanish in (*). Then the number of pairs (x, y) of integers in (*) for which $f(x, y)$ is an integer is at most

$$150(1+c)^{2/3}A^{5/6}B^{2/3} + 4A.$$

The theorems about the more general problem when $f(x, y)$ is near an integer z are too elaborate to be reproduced here. A special case is that the number of integral solutions (x, y, z) of

$$|\rho y^{\alpha+1}x^{-\alpha} - z| < x^{-\mu}$$

in $\lambda_1 x \leq y \leq \lambda_2 x$, $1 \leq x \leq q$, where $\lambda_1, \lambda_2, \rho, \alpha \neq -1$, $1 < \mu \leq \frac{3}{2}$ are constants, is majorized by a constant multiple of $q^{3/2 + (3/2 - \mu)/7}$, the constant being independent of q. The proofs are technically elementary but very elaborate.
J. W. S. Cassels (Cambridge, England)

Akagawa, Yasumasa 5043
On the construction of Galois extensions. (Japanese)
Sûgaku **14** (1962/63), 209–219.
This is an expository report of the author's work published in Osaka Math. J. **12** (1960), 195–215 [MR **25** #64a].
E. Inaba (Tokyo)

Rodosskiĭ, K. A. 5044
On some trigonometric sums. (Russian)
Mat. Sb. (N.S.) **60 (102)** (1963), 219–234.
Let $f(x) = a_1 x + \cdots + a_k x^k$ be a polynomial of degree ≥ 2 with real coefficients. The author is concerned with the sum $\Sigma_k = \sum_{x=1}^P e^{2\pi i f(x)}$ treated, for example, by I. M. Vinogradoff and L. K. Hua [see the latter's book, *Additive Primzahltheorie*, Teubner, Leipzig, 1959; MR **23** #A1620]. Restricting himself to the case $2 \leq k \leq 11$ and supposing that $|a_r - a/q| \leq 1/q^2$, $(a, q) = 1$, $1 \leq q \leq P^r$, where $2 \leq r \leq k$, he obtains the estimate

$$|\Sigma_k| \leq c(k, \varepsilon)\gamma(q)P^{1-1/\sigma_k+\varepsilon}$$

for arbitrary $\varepsilon > 0$, $c(k, \varepsilon)$ is a constant depending only on k and ε, and

$$\gamma(q) = (Pq^{-1})^{1/\sigma_k} \quad \text{for } 1 \leqq q \leqq P,$$
$$= 1 \quad \text{for } P \leqq q \leqq P^{r-1},$$
$$= (qP^{1-r})^{1/\sigma_k} \quad \text{for } P^{r-1} \leqq q \leqq P^r.$$

and σ_k is given by the table

k	2	3	4	5	6	7	8	9	10	11
σ_k	4	9	20	51	116	247	422	681	1090	1781

S. Chowla (University Park, Pa.)

Lehner, J.; Newman, M. 5045

Weierstrass points of $\Gamma_0(n)$.

Ann. of Math. (2) **79** (1964), 360–368.

The authors consider the subgroup of the modular group denoted by $\Gamma_0(n)$ $((a\tau+b)/(c\tau+d)$, where $c \equiv 0 \bmod n)$. The fundamental domain is regarded as a Riemann manifold S with genus $g(n)$; the authors look for Weierstrass points (namely, points P_0 for which a pole structure exists of type P_0^e for $e \leq g(n)$). The property which is used to determine (some) Weierstrass points is a theorem of B. Schoeneberg [Abh. Math. Sem. Univ. Hamburg **17** (1951), 104–111; MR **13**, 439] which is based on $\mathcal{N}$, the normalizer of G (a discrete group (like $\Gamma_0(n)$) in the Lie group of conformal homeomorphisms of the upper half-plane). Then if $M \in \mathcal{N}$ and p is the period of M (least positive p for which $M^p \in G$) the group $G^* = \sum M^k G$ $(1 \leqq k \leqq p-1)$ is formed and g^* denotes its genus. Finally, if $g^* \neq [g/p]$, then τ, a fixed-point of M, is a Weierstrass point for S. B. Schoeneberg and H. Petersson [H. Petersson, Arch. Math. **2** (1950), 246–250; MR **12**, 394] applied this to the study of $G = \Gamma(n)$, the principal congruence subgroup $((a\tau+b)/(c\tau+d) \equiv \tau \bmod n)$ which is a normal subgroup of the modular group while $\Gamma_0(n)$ is not.

The authors construct the normalizer of $\Gamma_0(n)$ and prove that for n prime $\mathcal{N}/\Gamma_0(n)$ is of order 2. For $\Gamma_0(4n)$ and $\Gamma_0(9n)$ the group G^* is of type $\Gamma_0(2n)$, $\Gamma_0(3n)$; hence a straightforward application of Hecke's formulas yields the theorems that $\tau = 0$ is a Weierstrass point of $\Gamma_0(4n)$ if $g(4n) \geqq 2$ except possibly for $4n = 64$, $4p$, $8p$, $16q$, $4pq$ (primes $p \neq q \equiv -1 \bmod 4$). Likewise for $\Gamma_0(9n)$ if $g(9n) \geqq 2$: $\tau = 0$ is a Weierstrass point except possibly for $9n = 81$, $9p$, $9pq$ (primes $p \neq q \equiv -1 \bmod 3$).

In particular, using the normalizer $T_n(\tau \to -1/n\tau)$ they find that except for finitely many n, the fixed points of T_nA are Weierstrass points of $\Gamma_0(n)$ $(A \in \Gamma_0(n))$. The proof is based on formulas of Fricke connecting $g(n)$ with class number $h(-4n)$. (It seems necessary to know Siegel's result $h \to \infty$.) Thus the authors justify the Weierstrass points $i/\sqrt{23}$, $\frac{1}{2}(i/\sqrt{23}+1)$, etc., for $\Gamma_0(23)$.

The authors finally consider the problem of when $\Gamma_0(n)$ is hyperelliptic. This is true if and only if a $T \in \mathcal{N}$ has period 2. By identifying images of S under T they obtain a Riemann surface of genus 0 for $\Gamma_0^*(n) = \Gamma_0(n) + T\Gamma_0(n)$. This time, using only less advanced estimates on $h(-4q)$ of P. Bateman [Trans. Amer. Math. Soc. **71** (1951), 70–101; MR **13**, 111], they conclude that for prime $q \equiv -1 \bmod 12$ with $g(q) \geqq 2$, $\Gamma_0(q)$ is hyperelliptic exactly when $q = 23, 47, 59, 71$. {In a privately communicated addendum the authors point out that the restriction $q \equiv -1 \bmod 12$ is required so that $\Gamma_0(q)$ is free; otherwise $\mathcal{N}/G$ is isomorphic only to a subgroup of conformal homeomorphisms of S.} Other items in the addenda include the remark that $\Gamma_0(4n)$, $\Gamma_0(9n)$ are free, reference to J. Lewittes [Bull. Amer. Math. Soc. **69** (1963), 578–582; MR **26** #6128], and the remark that $\Gamma_0(37)$ is hyperelliptic but the genus of $\Gamma_0^*(37)$ is 1 since the conformal homeomorphism T need not belong to $\mathcal{N}$. *H. Cohn* (Tucson, Ariz.)

Verma, D. P. 5046

Laurent's expansion of Riemann's zeta-function.

Indian J. Math. **5** (1963), 13–16.

If

$$A_{k,n} = \frac{(\log n)^{k+1}}{k+1} - \sum_{r=1}^{n} \frac{(\log r)^k}{r} \quad (k = 0, 1, 2, \cdots; n = 1, 2, \cdots),$$

then $\lim A_{k,n} = A_k$ $(n \to \infty)$ exists, and

$$\zeta(s) = \frac{1}{s-1} - \sum_{0}^{\infty} (-1)^k A_k (s-1)^k,$$

where the series is shown to converge for $|s-1| < 1$. This expansion of $\zeta(s)$, however, holds on the whole finite plane: the author overlooks the basic fact that $\zeta(s) - 1/(s-1)$ is an entire function of s and, therefore, of $s-1$ also.

Again a method of numerical calculation of the A_k is indicated. *H. Kober* (Birmingham)

Horadam, E. M. 5047

A calculus of convolutions for generalised integers.

Nederl. Akad. Wetensch. Proc. Ser. A **66** = *Indag. Math.* **25** (1963), 695–698.

Let a finite or infinite sequence $\{p\}$ of real numbers $1 < p_1 < p_2 < \cdots$ be given (generalized primes). We form the set $\{l\}$ of all possible p-products $p_1^{v_1} p_2^{v_2} \cdots$, where $v_1, v_2, \cdots$ are non-negative integers of which all but a finite number are 0. Suppose that no two of these "generalized integers" are equal if the v's are different. Then arrange $\{l\}$ as an increasing sequence $1 = l_1 < l_2 < l_3 < \cdots$. {The reviewer remarks that in the case of an infinite sequence $\{p\}$ the possibility of such an ordering for the set $\{l\}$ is equivalent to the condition $p_n \to \infty$ as n increases indefinitely; moreover, this implies $l_n \to \infty$.}

The author considers "arithmetical functions" $f(l)$ defined on the set $\{l\}$ and then introduces the "convolution" of two given arithmetical functions f and g by putting $f(l) * g(l) = \sum_{d\delta = l} f(d) g(\delta)$. Then she develops a calculus quite similar to the one given by the reviewer in the case of ordinary arithmetical functions [Nederl. Akad. Wetensch. Proc. Ser. A **58** (1955), 10–15; MR **16**, 905].

As applications the author proves a number of identities which are used in a following paper to deduce an analogue of Selberg's lemma for the prime number theorem; compare with the next review [#5048].

J. Popken (Amstelveen)

Horadam, E. M. 5048

Selberg's inequality for generalised integers using the calculus of convolutions.

Nederl. Akad. Wetensch. Proc. Ser. A **66** = *Indag. Math.* **25** (1963), 699–704.

Given a finite or infinite sequence $\{p\}$ of real numbers (generalized primes), $1<p_1<p_2<p_3<\cdots$, we form the set of all possible p-products $p_1^{a_1}p_2^{a_2}\cdots$, where $a_1, a_2, \cdots$ are non-negative integers of which all but a finite number are 0. Suppose that no two of these numbers ("generalized integers") are equal (except in the trivial case). Then arrange $\{l\}$ as an increasing sequence $1=l_1<l_2<l_3<\cdots$. Let $\nu(x)=\sum_{l_n\leq x}1$, $\pi(x)=\sum_{p_n\leq x}1$.

Assume $\nu(x)=Cx^{\theta}+o(x^{\theta_1})$, where C, θ, θ_1 are positive constants, $\theta_1<\theta$. Then ("generalized prime number theorem")

$$(1)\qquad \lim_{x\to\infty}\frac{\pi(x)}{x^{\theta}/\log x}=\theta^{-1}.$$

The author applies what she calls "the calculus of convolutions" to obtain an analogue of Selberg's fundamental lemma. From the analogue one obtains the generalized prime number theorem (1) as Bredihin has shown [Mat. Sb. (N.S.) **46** (**88**) (1958), 143–158; MR **21** #87].

S. Chowla (University Park, Pa.)

Wright, E. M. 5049

Direct proof of the basic theorem on multipartite partitions.

Proc. Amer. Math. Soc. **15** (1964), 469–472.

In 1956, the author used generating functions to obtain formulae for the number of partitions of a j-partite number into k parts, distinct or otherwise. In this paper, he gives a direct proof of his results. Use is made of the elementary theory of the symmetric group of permutations on k elements. (For another proof see the reviewer's paper [Proc. Nat. Inst. Sci. India Part A **27** (1961), 579–587; MR **25** #2054].)

H. Gupta (Chandigarh)

May, Warren L. 5050

Binary forms over number fields.

Ann. of Math. (2) **79** (1964), 597–615.

Let K be a number field and let M_K be the set of absolute values of K normalized to agree with the usual absolute values on **Q**. Denote by S any finite subset of M_K containing all the archimedean ones. By an (S, d)-parallelotope, d being a positive integer, is meant a parallelotope P in K such that $P(v)=1$ if $v\notin S$ and $P(v)\in$ (the value group of $v)^d$ if $v\in S$. Denote by I_S the ring of S-integers of K. The object of the paper is a polynomial $F(X, Y)\in I_S[X, Y]$ which is homogeneous of degree d. In the main part of the paper $F(X, Y)$ is subject to the following condition (C): $d>2$, $F(X, 1)$ has no repeated roots and no root lies in K. Under (C) the author considers the number $n(P)$ of pairs $(x, y)\in I_S\times I_S$ such that $F(x, y)\in P$ for an (S, d)-parallelotope P and proves the formula: $n(P)=c\|P\|^{2/d}+O(\|P\|^a)$, where c is a constant depending on the original data K, F and S, a is a positive real number depending on the original data such that $a<2/d$, and

$$\|P\| = \prod_{v\in M_K} P(v)^{[K_v:\mathbf{Q}_v]}.$$

At the end of the paper, the author drops the condition (C) on $F(X, Y)$ and considers a subset T of $K\times K$ stable under the operation by coordinate-wise multiplication of S-units U_S: Denoting by $n'(P)$ the number of pairs $(x, y)\in T$ such that $F(x, y)\in P\cap U_S$ for an $(S, 1)$-parallelotope P, he proves that $n'(P)=c'(\log\|P\|)^{s-1}+O((\log\|P\|)^{s-2})$, where c' is a constant depending on K, F, S and T, and s is the number of elements in S. A condition is given which insures that, for certain T, c' is finite.

T. Ono (Philadelphia, Pa.)

Nanda, V. C. 5051

On the genera of quadratic and hermitian forms over an algebraic number field.

Acta Arith. **8** (1962/63), 431–450.

Let K be an algebraic number field with an involution τ and let k be the fixed field of τ. One can naturally define the notions of hermitian forms of n variables with respect to K/k, class and genus of forms (in case $\tau=1$, "hermitian" is replaced by "quadratic"). The author proves that a genus can be defined by means of a finite set of invariants and also proves the existence of a genus with prescribed invariants (§ 4, Theorem), generalizing the results of Gauss ($\tau=1$, $k=\mathbf{Q}$, $n=2$), Minkowski ($\tau=1$, $k=\mathbf{Q}$), C. L. Siegel ($\tau=1$) and H. Braun ($k=\mathbf{Q}$, K/k: imaginary quadratic). In the proof, the methods of Braun are used [J. Reine Angew. Math. **182** (1940), 32–49; MR **2**, 36; Abh. Math. Sem. Hansischen Univ. **14** (1941), 61–150; MR **3**, 70].

T. Ono (Philadelphia, Pa.)

Kalinka, V. 5052

Generalization of a lemma of L. K. Hua for algebraic numbers. (Russian. Latvian and English summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 1, 149–155.

This article contains a generalization of Hua's lemma [L. K. Hua, Trudy Mat. Inst. Steklov. **22** (1947); MR **10**, 597; errata, MR **11**, 870]. The author's own summary is given as follows.

"Let K be an algebraic number field of degree n with r_1 real conjugates $K^{(1)}, \cdots, K^{(r_1)}$ and r_2 pairs of conjugate complex conjugates $K^{(r_1+l)}$, $K^{(r_1+r_2+l)}$ $(l=1, 2, \cdots, r_2)$, so that $r_1+2r_2=n$. We denote by $\mathfrak{T}$ the set of integers α in K with $0<\alpha^{(i)}<T$ $(i=1, 2, \cdots, r_1)$ and $|\alpha^{(i)}|<T$ $(i=r_1+1, \cdots, r_1+r_2)$, where $\alpha^{(i)}$ is the conjugate of α in $K^{(i)}$, and $T^{2s}>2N(\vartheta)^{1/n}$ (ϑ—the different of K). Let I_{2s} be the number of $2s$-tuples $(\alpha_1, \cdots, \alpha_s, \beta_1, \cdots, \beta_s)$, which satisfy the following condition $\alpha_1^k+\alpha_2^k+\cdots+\alpha_s^k=\beta_1^k+\beta_2^k+\cdots+\beta_s^k$, where α_i, β_i are integers in $\mathfrak{T}$. In the theorem, we prove that $I_{2s}\ll T^{n(2s-k)}$, if $2s\geqq(2^{k-1}+n)nk$."

E. Inaba (Tokyo)

Fogels, E. 5053

On the distribution of prime ideals.

Acta Arith. **7** (1961/62), 255–269.

In 1955 the author proved the existence of an absolute constant c such that for any positive integers x, D, l with $D\geqq 2$ and $(D, l)=1$, there is always at least one prime $p\equiv l \pmod D$ in the interval (x, xD^c), thus extending Linnik's result on the least prime in an arithmetic progression [the author, Dokl. Akad. Nauk SSSR **102** (1955), 455–456; MR **17**, 240]. In the present paper this result is extended to algebraic number fields.

Let K be an algebraic number field of degree n and discriminant Δ, let $\mathfrak{f}$ be an ideal in K, and let $\mathfrak{H}$ be a class of ideals modulo $\mathfrak{f}$. Set $D=|\Delta|\cdot N\mathfrak{f}$ and let $x\geqq 1$. Then it is shown that there is a positive constant c, which depends only on n, such that the interval (x, xD^c) contains at least one prime p that is the norm of an ideal $\mathfrak{p}$ in $\mathfrak{H}$. In the

special case $n=2$, $\mathfrak{f}=[k]$, k a positive rational integer, this result implies the existence of a prime p in the interval (x, xD^c) that is representable by a prescribed primitive quadratic form.

The proof depends on properties of the Hecke L-function that the author established in three previous papers [Acta Arith. **7** (1961/62), 87–106; ibid. **7** (1961/62), 131–147; MR **25** #55; ibid. **7** (1961/62), 225–240].

W. H. Mills (Princeton, N.J.)

Iwasawa, Kenkichi 5054

On a certain analogy between algebraic number fields and function fields. (Japanese)

Sûgaku **15** (1963), 65–67.

In this expository paper, the author explains the motive and ideas of his theory of Γ-extensions of algebraic number fields and its applications.

T. Ono (Philadelphia, Pa.)

Bulota, K. 5055

Some theorems on the density of the zeros of the Hecke ζ-functions. (Russian. Latvian and English summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 1, 29–50.

Let K denote an imaginary quadratic field. Consider the Hecke function

$$Z(s, \chi, N) = \sum_{A\in N} \frac{\chi(A)}{(NA)^s},$$

where $s=\sigma+it$, N is a fixed ideal class of the field K, $\chi(A)$ a Hecke character of the second kind with modulo m and exponent m.

Let $N(\alpha, T, 2T, \chi)$ denote the number of zeros of $Z(s, \chi, N)$ in the strip $\frac{1}{2}<\alpha\leqq\sigma\leqq 1$, $T<|t|\leqq 2T$. Then we have

$$N(\alpha, T, 2T, \chi) = O(T^{4\alpha(1-\alpha)}\log^5 T),$$

$$\sum_{\chi} N(\alpha, T, 2T, \chi) = O((MT)^{4(1-\alpha)+\varepsilon}),$$

$|m|\leqq M$, where $M\leqq cT$.

The results are obtained by the "classical Landau-Carlson-Titchmarsh" method [see also Haselgrove, J. London Math. Soc. **26** (1951), 273–277; MR **13**, 438; Turán, J. Indian Math. Soc. (N.S.) **20** (1956), 17–36; MR **18**, 792]. The author uses his own results in a paper in Litovsk. Mat. Sb. **2** (1962), no. 2, 39–82 [MR **28** #3977].

S. Chowla (University Park, Pa.)

Golod, E. S.; Šafarevič, I. R. 5056

On the class field tower. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 261–272.

The class field tower problem was first proposed by Furtwängler and made well known by Hasse's exposition on class field theory [H. Hasse, Jber. Deutsch. Math. Verein. **35** (1926), 1–55]. The authors of the present article give a decisive answer to this problem. Let k be a finite extension of the rational field R, and p a prime number. We denote by γ the minimal number of generators of the p-Sylow subgroup of the divisor class group and by ρ the minimal number of generators of the unit group. The principal theorem is that, if $\gamma\geqq 3+2(\rho+2)^{1/2}$, then there exists an infinite Galois p-extension of k such that all prime divisors are unramified over k. The simplest example for this is the field

$$R(\sqrt{(-3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19)}) = R(\sqrt{(-4849845)}).$$

The proof can be sketched as follows. Let $\mathfrak{G}$ be the Galois group of the maximal unramified p-extension of k. Then the minimal number d of generators of $\mathfrak{G}$ is equal to γ. If $\mathfrak{F}$ is a free group with d generators, then $\mathfrak{G}$ is isomorphic to a factor group $\mathfrak{F}/\mathfrak{R}$. The minimal number of generators of $\mathfrak{R}$ is to be understood in the topological sense and denoted by r. By a theorem published earlier by Šafarevič [Inst. Hautes Études Sci. Publ. Math. No. 18 (1963), 71–95] we have $r\leqq d+\rho$. On the other hand, we obtain a theorem that the inequality $r>(d-1)^2/4$ holds for every finite p-group, whence it follows that $\mathfrak{G}$ is infinite if $\gamma\geqq 3+2(\rho+2)^{1/2}$. The proof of the latter theorem is carried out by reducing it to a theorem on algebras and by applying the non-commutative theory of Koszul complexes.

E. Inaba (Tokyo)

Riehm, Carl 5057

On the integral representations of quadratic forms over local fields.

Amer. J. Math. **86** (1964), 25–62.

This is an extensive study of the integral representations of quadratic forms over 2-adic fields. Let F be a finite extension of the field of 2-adic numbers, $\mathfrak{o}$ the integers in F, V an n-dimensional vector space over F provided with a symmetric bilinear form $B: V\times V\to F$. A lattice L on V is a finitely generated $\mathfrak{o}$-module in V which contains n independent vectors over F; L admits an $\mathfrak{o}$-base $x_1, \cdots, x_n$. L is said to be unimodular if the matrix $(B(x_i, x_j))$ is unimodular and modular if $(aB(x_i, x_j))$ is unimodular for some $a\in F$. Let L, M be lattices on V. M is said to be represented by L (written $M\to L$) if there is an isometry s such that $s(M)\subseteq L$. The first main theorem gives necessary and sufficient conditions for $M\to L$ when L is unimodular and M is arbitrary. The second main theorem is a special case of the first one and gives a simple condition for L unimodular, M modular: $M\to L\Leftrightarrow B(M, M)\subseteq B(L, L)\subseteq B(M\#, M\#)$, here $M\#=\{x\in V, B(x, M)\subseteq\mathfrak{o}\}$. The author discusses also the representations of lattices on metric spaces with different dimensions by giving conditions when one of the two lattices is unimodular (the third main theorem) and concludes the paper with comments on arbitrary lattices.

T. Ono (Philadelphia, Pa.)

FIELDS AND POLYNOMIALS

See also 5010, 5043, 5053, 5054, 5065, 5121.

Demuškin, S. P. 5058

On 2-extensions of a local field. (Russian)

Sibirsk. Mat. Ž. **4** (1963), 951–955.

For the case $p\neq 2$, the group of the maximal p-extension of a local field has been previously characterized by the author [Dokl. Akad. Nauk SSSR **128** (1959), 657–660; MR **21** #7200; Izv. Akad. Nauk SSSR Ser. Mat. **25** (1961), 329–346; MR **23** #A890]. The proof of Theorem 1 in the latter paper carries over, after a suitable lemma has been established in the present paper, to provide a proof of the corresponding result for the case $p=2$.

R. A. Good (College Park, Md.)

Ljunggren, Wilhelm 5059
On the irreducibility of certain lacunary polynomials.
Norske Vid. Selsk. Forh. (Trondheim) **36** (1963), 159–164.

The author discusses the irreducibility over the field of rationals of the polynomials $f(x)=x^n+\varepsilon_1x^m+\varepsilon_2x^p+r\varepsilon_3$, where n, m, p and r are natural numbers, r a prime, $n>m>p$, and ε_1, ε_2, ε_3 take the values ± 1. The case $r=1$ was completely treated by the author in an earlier paper [Math. Scand. **8** (1960), 65–70; MR **23** #A1627]. For $r>3$ it is easily proved that $f(x)$ is always irreducible; for $r=3$ it is readily seen that $f(x)$ is either irreducible or can be decomposed into two factors, one of which is $x^d\pm 1$ while the other is irreducible—here and in the following, d is the greatest common divisor (n, m, p). The case $r=2$ is more complicated and yields the following theorem: $f(x)$ is either irreducible or has a factor $x^{2d}\pm x^d+1$ and a second, irreducible factor; a necessary condition for reducibility is that $nmp\equiv 0 \pmod 3$. The methods used are similar to those used in the earlier paper [loc. cit.].
H. W. Brinkmann (Swarthmore, Pa.)

Kazandzidis, G. S. 5060
On the cyclotomic polynomial: Morphology, estimates.
Bull. Soc. Math. Grèce (N.S.) **4** (1963), *no.* 1, 50–73.

For a given natural N, the Nth homogeneous cyclotomic polynomial $\phi_N(x, y)$ is defined by $\phi_N(x,y)=\prod(x-\rho y)$, where the product is taken over all the primitive Nth roots ρ of unity. The nonhomogeneous cyclotomic polynomial $\phi_N(x)$ is defined by $\phi_N(x)=\prod_{d\delta=N}(1-x^d)^{\mu(\delta)}=\phi_N(1,x)$, where $\mu(n)$ is the Möbius arithmetical function. The author proves several theorems about $\phi_N(x, y)$ and $\phi_N(x)$ in three sections: (A) 10 theorems, (B) 11 theorems, (C) 16 theorems. E.g., Theorem A3: $\phi_u(x^v, y^v)=\prod_{d|v}\phi_{ud}(x,y)$ when $(u,v)=1$. Theorem B6: If N is not a power of 2, then for any odd prime divisor p of N and for any point $z=re^{\varphi i}$ of the complex plane which is neither zero nor an Nth root of unity, $|(1+r)/(1-r)|^{\bar\phi(N)/\bar\phi(p)}>|\phi_N(z)|/\phi_N(\pm r)>|(1-r)/(1+r)|^{\bar\phi(N)/\bar\phi(p)}$ holds. Here $\bar\varphi(n)=n(1+1/p_1)(1+1/p_2)\cdots(1+1/p_k)$ if $n>1$ and $p_1, p_2, \cdots, p_k$ are the distinct prime divisors of n; $\bar\varphi(1)=1$. Theorem C1: If $z=re^{\varphi i}$, then $|\phi_N(z)|>r^{\varphi(N)/2}|\phi_N(e^{\varphi i})|$, unless $r=1$. Here $\varphi(n)$ is Euler's arithmetical function. At the end the author compares his results with those of W. Klobe [J. Reine Angew. Math. **187** (1949), 68–69; MR **11**, 417] and H. J. Kanold [ibid. **187** (1950), 169–182; MR **12**, 592; ibid. **188** (1950), 129–146; MR **13**, 443; Math. Z. **55** (1952), 284–287; MR **14**, 728].
J. W. Andrushkiw (S. Orange, N.J.)

Inaba, Eizi 5061
Galois extensions associated with generalized Artin-Schreier equations.
Natur. Sci. Rep. Ochanomizu Univ. **14** (1963), 41–44.

Let k be a field containing a finite field P of q elements, and M a nonsingular n-by-n matrix with entries in k. In an earlier paper [same Rep. **13** (1962), no. 2, 1–13; MR **27** #5751] the author associated to M a Galois extension of k by adjoining the entries of the square matrix A satisfying $A^{(q)}=MA$ (where $A^{(q)}$ denotes the matrix whose entries are the qth powers of the entries of A). He showed that the Galois group is a subgroup of $GL(n, P)$. In the present note he sharpens the description of this subgroup in some cases. Let $\mathfrak{o}$ be any P-algebra of n-by-n matrices over P and $G(\mathfrak{o})=\mathfrak{o}\cap GL(n, P)$. If $M\in k\otimes_P\mathfrak{o}$, then the Galois group above is a subgroup of $G(\mathfrak{o})$, and (if the Hilbert irreducibility theorem holds in k) is equal to $G(\mathfrak{o})$ for infinitely many M.
D. Zelinsky (Evanston, Ill.)

Brumer, Armand; Rosen, Michael 5062
Class number and ramification in number fields.
Nagoya Math. J. **23** (1963), 97–101.

The authors show the following theorem. Let K be a finite algebraic number field with absolute degree m, and L be a Galois extension of K with $n=[L:K]$. Then there is an integer $R(m, n)$ depending only on m and n such that h_L is divisible by $h_K\prod e_i/R(m,n)$, where the e_i's are the ramification degrees of the prime ideals of K ramified in L/K, and h_K and h_L denote the class numbers of K and L, respectively.

For its proof, they construct such $R(m, n)$ explicitly. They call it a partial generalization of Leopoldt's result [H. W. Leopoldt, Math. Nachr. **9** (1953), 351–362; MR **15**, 14]. But the $R(m, n)$ given by them does not necessarily divide $h_K\prod e_i$. It is essential in Leopoldt's case that, for a cyclotomic field L over the rational number field Q, the product $\prod e_i$ of the prime natural numbers ramified in L is divisible by $n=[L:Q]$.
K. Masuda (Okayama)

Kolchin, E. R. 5063
Singular solutions of algebraic differential equations and a lemma of Arnold Shapiro.
Topology **3** (1964), *suppl.* 2, 145–155.

The principal result of this important paper considerably broadens the class of partial differential polynomials A in finitely many indeterminates over fields of characteristic zero for which it is known that $(0, \cdots, 0)$ is not a zero of the general component $\mathfrak{P}(A)$ of A, i.e., of that uniquely determined prime component of A which does not contain any separant of A. The author introduces an interesting notion of "domination" of one differential monomial over another and establishes a key lemma which generalizes some results of H. Levi [Trans. Amer. Math. Soc. **51** (1942), 532–568; MR **3**, 264]. This domination lemma, which depends on a generalization of Levi's Low Power Theorem in partial differential algebra [Ann. of Math. (2) **46** (1945), 113–119; MR **7**, 119], leads to the main proposition, a special case of which is the following result: If A has a term which is dominated by every other term of A, then $(0, \cdots, 0)$ is not a zero of $\mathfrak{P}(A)$.

A decisive role in the proof of the domination lemma is played by an unpublished combinatorial lemma by Arnold Shapiro on the redistribution of a system of masses located on the faces of a simplex, relative to another system of masses located at the vertices.
A. Jaeger (Cincinnati, Ohio)

ABSTRACT ALGEBRAIC GEOMETRY

See also 5148, 5448.

Abhyankar, Shreeram 5064
Reduction to multiplicity less than p in a p-cyclic extension of a two dimensional regular local ring (p= characteristic of the residue field).
Math. Ann. **154** (1964), 28–55.

Local uniformization, hence also the resolution of singularities, of an algebraic surface was proved by the author previously [Ann. of Math. (2) **63** (1956), 491–526; MR **17**, 1134; ibid. (2) **78** (1963), 202–203; MR **27** #145]. A simplification of the proof was also given by the author recently [Math. Ann. **153** (1964), 81–96; MR **28** #3037]. One important step of the simplified proof, namely, a reduction step to the case where the multiplicity is less than p, the characteristic of the field, is generalized to the case where the existence of a ground field is not assumed, namely, to p-cyclic extensions of regular local rings of Krull dimension (=altitude=rank) 2, with a slight restriction.
M. Nagata (Kyoto)

Ishida, Makoto 5065
On fields of division points of algebraic function fields of one variable.
Proc. Japan Acad. **39** (1963), 717–720.

Let K be a field of algebraic functions of one variable over an algebraically closed field k of constants, and of genus $g \geqq 2$; let l be an odd prime different from the characteristic of k. The author proves that the extension K/K_l is purely inseparable, where K_l is any field of l-division points of K. (In general, if n is a positive integer, a field of n-division points of K is obtained by adjoining to k a set of elements $\{x_i\}$ from K such that the principal divisors $(x_i)=nD_i$, where the D_i are a complete set of representatives for the divisor classes (degree 0) of orders dividing n in the divisor class group. When n is prime to the characteristic, the number of such classes is n^{2g} [cf. A. Weil, *Variétés abéliennes et courbes algébriques*, Theorems 18 and 19, and Corollary 1 to Theorem 33, Actualités Sci. Indust., No. 1064, Hermann, Paris, 1948; MR **10**, 621].) He remarks that the result remains valid whenever n is divisible by an odd prime different from the characteristic, or by 8, if the characteristic is not 2. He also notes that, for the same prime l, one may have $K=K_l$ for one field K_l of l-division points and $K \neq K_l$ for a second such field K_l'. The techniques of proof follow those of S. Arima [Proc. Japan Acad. **36** (1960), 6–9; MR **22** #4715].
G. B. Seligman (New Haven, Conn.)

Yamada, Hiroshi 5066
On a characterization of fields of moduli.
Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **8**, 133–143 (1964).

The field of moduli of a polarized algebraic variety as defined by Matsusaka [Amer. J. Math. **80** (1958), 45–82; MR **20** #878] is, roughly speaking, the smallest field of definition for an equivalent projective variety. Shimura has an alternate definition for polarized abelian varieties [Ann. of Math. (2) **70** (1959), 101–144; MR **21** #6370], resulting in the characterization of this field as a field of definition for the polarized abelian variety whose elements are left fixed by precisely those automorphisms of the universal domain preserving the polarized abelian variety. The criterion of Shimura is here shown to be valid for an arbitrary variety V, not necessarily abelian, provided V is nonsingular and $G_n(V)/G_a(V)$ has order prime to the field characteristic. Next, given a polarized abelian variety, the dual abelian variety is endowed with a natural polarization, and the duality theorem for abelian varieties is shown to extend to polarized abelian varieties. Finally, the fields of moduli of two dual polarized abelian varieties are the same, up to pure inseparability.
M. Rosenlicht (Berkeley, Calif.)

Kneser, Martin 5067
Erzeugende und Relationen verallgemeinerter Einheitengruppen.
J. Reine Angew. Math. **214/215** (1964), 345–349.

Soit G un groupe algébrique de matrices à coefficients dans un corps de nombres algébriques k et soit S un ensemble fini d'idéaux premiers de k. Pour que le groupe $G_{o(S)}$ des matrices de G dont les éléments n'ont pas de dénominateur en dehors de S soit à engendrement fini, il faut et il suffit que le groupe $G_{k_\mathfrak{p}}$ soit engendré par une partie compacte pour tout $\mathfrak{p}$ dans S. Enoncé analogue pour les relations entre les générateurs. La démonstration s'appuie sur les résultats de Borel [Inst. Hautes Études Sci. Publ. Math. No. 16 (1963), 5–30].
P. Cartier (Strasbourg)

G.-Rodeja F., E. 5068
Irrelevant ideals in the cohomology theory of algebraic varieties. (Spanish)
Rev. Mat. Hisp.-Amer. (4) **24** (1964), 11–15.

The author proves a vanishing theorem which generalizes one of P. Abellanas [Actas Primera Reunión Anual de Mat. Españoles, pp. 158–163, Univ. Madrid, Madrid, 1961; MR **24** #A3164]. Let S^r be an algebraic variety over an infinite field and P its function field. F_i are forms of a fixed degree m, $[F_i]$ the corresponding open components of S^r in the Zariski topology. P_m is the set of all polynomials of P whose degree is $>m$. If there exist $r+1$ forms F_i such that $[F_1] \cup \cdots \cup [F_{r+1}] \cup [\sum t_iF_i] \supset S^r$ ($t_i \in k$, $t_i \neq 0$) and $P_m \subset P(F_1, \cdots, F_{r+1})$, then the rth cohomology group of S^r vanishes.
H. W. Guggenheimer (Minneapolis, Minn.)

LINEAR ALGEBRA

See also 5039, 5079, 5115, 5553.

Taussky, Olga 5069
Matrices C with $C^n \to 0$.
J. Algebra **1** (1964), 5–10.

A theorem of Lyapunov concerning stable matrices (matrices whose characteristic roots have negative real parts) and a theorem of P. Stein concerning matrices C with the property that $C^n \to 0$ are connected by the following theorem. Suppose C satisfies $C^n \to 0$. Let $A=(C+I)^{-1}(C-I)$ and let H be a positive definite hermitian matrix such that $AH+HA^*$ is negative definite. Then $H-CHC^*$ is positive definite. Conversely, let A be a stable matrix and let $C=(I-A)^{-1}(I+A)$; if H is a positive definite hermitian matrix such that $H-CHC^*$ is positive definite, then $AH+HA^*$ is negative definite. It follows from this that if C is such that $C^n \to 0$, then $H-CHC^*=P$ can be solved with positive definite hermitian H for every positive definite hermitian P. A canonical form (under unitary transformations) for matrices C with $C^n \to 0$ is obtained. This canonical form

involves the characteristic roots of H, where $H - CHC^* = I$. The canonical form is used to derive a sufficient condition on matrices K such that if $C^n \to 0$ then $(KC)^n \to 0$.
J. K. Goldhaber (College Park, Md.)

Anderson, T. W.; Das Gupta, S. 5070
Some inequalities on characteristic roots of matrices.
Biometrika **50** (1963), 522–524.
Let $\lambda_i(D)$ denote the ith, in order of size, characteristic root of any positive semi-definite matrix D. Let A be a positive semi-definite matrix, and B and C positive definite matrices, all of the same order p. Then, for any integers $i, j, k = 0, 1, \cdots, p$, such that $j + k \leqq i + 1$,

$$\lambda_i(AB^{-1}) \leqq \lambda_j(AC^{-1}) \cdot \lambda_k(CB^{-1}),$$

$$\lambda_{p-i+1}(AB^{-1}) \geqq \lambda_{p-j+1}(AC^{-1}) \cdot \lambda_{p-k+1}(CB^{-1}).$$

J. Wolfowitz (Ithaca, N.Y.)

Banerjee, K. S. 5071
A note on idempotent matrices.
Ann. Math. Statist. **35** (1964), 880–882.
The theorem proved in this paper and the various published generalizations cited here are essentially contained in Theorem 1, p. 148, of P. R. Halmos's *Finite-dimensional vector spaces* [2nd ed., Van Nostrand, Princeton, N.J., 1958; MR **19**, 725]. *M. Marcus* (Santa Barbara, Calif.)

Sinkhorn, Richard 5072
A relationship between arbitrary positive matrices and doubly stochastic matrices.
Ann. Math. Statist. **35** (1964), 876–879.
The following is the main result of this interesting paper. Theorem: If $A > 0$, then there exists a unique doubly stochastic S for which $D_1AD_2 = S$. The matrices D_1 and D_2 are diagonal with positive main diagonal entries and are uniquely determined to within scalar multiples. As a corollary the author obtains a result announced by the reviewer and M. Newman [Amer. Math. Soc. Notices **8** (1961), 595]. Corollary: If $A > 0$ and A is symmetric, then there exists a diagonal D with positive main diagonal entries such that DAD is doubly stochastic. The author shows that the process of alternately normalizing the row and column sums converges to a positive doubly stochastic matrix. He also shows by example that if A contains zero entries, then the results need no longer hold. Another quite distinct normalization procedure for obtaining the corollary is contained in an announcement by J. E. Maxfield and H. Minc [ibid. **9** (1962), 309].
M. Marcus (Santa Barbara, Calif.)

Carlson, David; Davis, Chandler 5073
A generalization of Cauchy's double alternant.
Canad. Math. Bull. **7** (1964), 273–278.
Let D be a matrix arbitrarily partitioned as D_{pq}; let the (i, j)-element in D_{pq} be $(-1)^{i+j}\binom{i+j-2}{j-1}(x_p + y_q)^{1-i-j}$. Assuming that $x_p + y_q$ never vanishes, det D is evaluated explicitly. The classical case occurs when all the blocks are 1×1 and $D_{pq} = (x_p + y_q)^{-1}$.
O. Taussky-Todd (Pasadena, Calif.)

Charrueau, André 5074
Sur les polynomes caractéristiques de certains produits de matrices.
C. R. Acad. Sci. Paris **258** (1964), 4191–4193.
For a square matrix A over a commutative ring R with an involutory automorphism f it is proposed to compute the coefficients of the characteristic polynomial of $Af(A)$. In the case that R is the field of the complex numbers the result is stated to be important in complex projective geometry. *H. Schwerdtfeger* (Montreal, Que.)

Khan, Nisar A. 5075
Linear relations between higher matrix commutators.
Canad. J. Math. **16** (1964), 315–320.
The author generalizes a linear relation (with scalar coefficients) between the higher commutators of two $n \times n$ matrices A, B, i.e., $B_i = AB_{i-1} - B_{i-1}A$ with $B_1 = AB - BA$ [see Taussky and Wielandt, Proc. Amer. Math. Soc. **13** (1962), 732–735; MR **26** #6191], and partially answers a question raised by those authors. He obtains a relation of lower order under the assumption that the elementary divisors are also known. He does not start off with the case of A having distinct characteristic roots as is done in the cited paper (however, the relation was nevertheless established there for the general case). The elementary divisors of the operator $AX + XB$ have been determined by Givens [Argonne Nat. Lab. Rep. ANL-6546 (1961)]. *O. Taussky-Todd* (Pasadena, Calif.)

Cline, Randall E. 5076
Note on the generalized inverse of the product of matrices.
SIAM Rev. **6** (1964), 57–58.
There is a unique matrix X satisfying $YXY = Y$, $XYX = X$, $(YX)^* = YX$, $(XY)^* = XY$ for a given complex matrix Y. The matrix X is called the generalized inverse of Y and is denoted by Y^+. If A and B are matrices such that AB is defined, then set $B_1 = A^+AB$ and $A_1 = AB_1B_1^+$. The author proves that $(AB)^+ = B_1^+A_1^+$.
M. Marcus (Santa Barbara, Calif.)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 4980, 5014, 5064.

Faith, Carl; Utumi, Yuzo 5077
Intrinsic extensions of rings.
Pacific J. Math. **14** (1964), 505–512.
If R is an extension ring of a ring S, then R is called a left quotient ring of S, $S \leqq R$, if the module ${}_SR$ is an essential extension of the module ${}_SS$; and R is called left-intrinsic over S, $S\nabla_l R$, if $K \cap S \neq 0$ for every nonzero left ideal K of R. If the left-singular ideal S_l^Δ of S is zero, then S has a unique maximal left quotient ring $\hat{S}$. Theorem 1: Let S be a ring such that $S_l^\Delta = 0$ and $\hat{S}$ has no nonzero strongly regular ideal. If R is an extension ring of S, then $S \leqq R$ if and only if $S\nabla_l R$ and for each closed left ideal A of S there exists a left ideal B of R such that $B \cap S = A$. Theorem 2: Let S be a ring such that $S_l^\Delta = 0$, $\hat{S}$ has no nonzero strongly regular ideal, and $S\nabla_r\hat{S}$. If R is an

extension ring of S, then $S \leq R$ if and only if $S\nabla_l R$. These theorems are applied to Goldie prime rings.

R. E. Johnson (Rochester, N.Y.)

Abian, Alexander; McWorter, William A. 5078

On the structure of pre-p-rings.

Amer. Math. Monthly **71** (1964), 155–157.

The authors show that if R is a commutative ring with the identities $px=0$ and $xy^p=x^py$ (p a prime), then R is the direct sum of a p-ring and a nil-ring. (A p-ring is one with the identities $px=0$ and $x^p=x$.)

J. McLaughlin (Ann Arbor, Mich.)

Dade, E. C.; Robinson, D. W.; Taussky, O.; Ward, M. 5079

Divisors of recurrent sequences.

J. Reine Angew. Math. **214/215** (1964), 180–183.

Let R be a commutative ring and I an ideal of R such that R/I is finite. Let $P(X)$ be a monic polynomial with coefficients in R whose coefficients are units modulo I. Theorem 1: Let $u_0, u_1, \cdots$ be any linear recurring sequence in R with characteristic polynomial P, i.e., $u_{n+k}=p_1u_{n+k-1}+\cdots+p_ku_n$, $n\geq 0$, where $P(X)=X^k-p_1X^{k-1}-\cdots-p_k$. Then there is a positive integer r such that if u_0 is in I, so is u_r. This generalizes a well-known fact about Fibonacci sequences. Next, assume $I=Rm$ with m not a zero-divisor in R, and let Q be the total quotient ring of R. With these hypotheses the authors prove Theorem 2: Let A be a square matrix with elements in Q. If $P(A)=0$ and mA has entries in R, then there exists a positive integer r such that A^r has entries in R. This generalises the well-known fact that a rational matrix with integral characteristic polynomial and unit determinant has a power which is integral. Furthermore, it is shown that if $I=Rm$ with m not a zero-divisor, then Theorems 1 and 2 are equivalent. Finally, Theorem 1 is generalized as follows: Let S be a finite set and F a map of the cartesian product of k copies of S into S. Suppose that for every choice of elements $x_1, \cdots, x_k$ of S there is a unique element x_0 such that $x_k=F(x_0, x_1, \cdots, x_{k-1})$; then, Theorem 3: There is a positive integer r such that for every sequence $x_0, x_1, \cdots, x_n, \cdots$ in S with $x_{n+k}=F(x_n, \cdots, x_{n+k-1})$, $n\geq 0$, we have $x_r=x_0$.

A. Rosenberg (London)

Lech, Christer 5080

Inequalities related to certain couples of local rings.

Acta Math. **112** (1964), 69–89.

Let $\mathfrak{p}$ be a prime ideal of a Noetherian ring R. Then $R_\mathfrak{p}$ is a local ring and the Hilbert function $H(\mathfrak{p}; n)=\text{length}_{R_\mathfrak{p}}\, \mathfrak{p}^nR_\mathfrak{p}/\mathfrak{p}^{n+1}R_\mathfrak{p}$ is defined. Starting with $H^{(0)}(\mathfrak{p}; n)=H(\mathfrak{p}; n)$, a sequence of functions $H^{(k+1)}(\mathfrak{p}; n)=\sum_{i=0}^n H^{(k)}(\mathfrak{p}; i)$ is defined. Now the results are the following three theorems.

Theorem 1: Let $\mathfrak{m}$ and $\mathfrak{p}$ be prime ideals of a Noetherian ring R such that $\mathfrak{p}\subset\mathfrak{m}$, height (=rank) of $\mathfrak{m}/\mathfrak{p}$ is one and such that the derived normal ring of $R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}$ is a finite $R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}$-module. Then there is a k such that $H^{(k+1)}(\mathfrak{p}; n)\leq H^{(k)}(\mathfrak{m}; n)$ for all $n=0, 1, 2, \cdots$.

Theorem 2: Let Q be a local ring with maximal ideal $\mathfrak{m}$ and let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal. If $Q/\mathfrak{q}$ contains a field and if $\mathfrak{q}/\mathfrak{q}^2$ is a free $Q/\mathfrak{q}$-module, then the minimum number of generators of $\mathfrak{q}$ is not larger than that of $\mathfrak{m}$.

Theorem 3: Let $\mathfrak{m}$ and $\mathfrak{p}$ be two prime ideals of a Noetherian ring such that $\mathfrak{p}\subset\mathfrak{m}$. Then $H(\mathfrak{p}; 1)+\text{height } \mathfrak{m}/\mathfrak{p} \leq H(\mathfrak{m}, 1)$. In particular, when height $\mathfrak{m}$ = height $\mathfrak{m}/\mathfrak{p}$ + rank $\mathfrak{p}$, then the regularity defect of $\mathfrak{p}$ is not larger than that of $\mathfrak{m}$. Here the regularity defect of a prime ideal $\mathfrak{p}$ of R is defined to be the difference of the minimum number of generators of $\mathfrak{p}R_\mathfrak{p}$ and height $\mathfrak{p}$.

Some parts of the proofs are hard to follow. In particular, the reviewer could not find any proof of Theorem 3 written in this article. Theorem 1 is closely related to a result of the reviewer [Proc. Internat. Sympos. on Algebraic Number Theory (Tokyo & Nikko, 1955), pp. 191–226, Science Council of Japan, Tokyo, 1956; MR **18**, 637].

M. Nagata (Kyoto)

Barbeau, A. 5081

Sur la structure de deux classes d'anneaux.

Canad. Math. Bull. **6** (1963), 373–383.

G. Thierrin [same Bull. **3** (1960), 11–16; MR **22** #1596] determined all right-bipotent rings, i.e., all rings A for which $aA=a^2A$ for each $a\in A$. As a continuation of Thierrin's paper the author investigates rings, all of whose subrings are right-bipotent (completely right-bipotent rings), as well as right-d-bipotent rings, i.e., rings A for which $aA=Aa^2$ for each $a\in A$. The author's main results are: (1) a ring A is a field whose multiplicative group is a torsion group if and only if A is primitive and completely right-bipotent; (2) each right-d-bipotent ring is isomorphic with a subdirect sum of zero rings and/or division rings.

A. Kertész (Debrecen)

Golod, E. S. 5082

On nil-algebras and finitely approximable p-groups. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 273–276.

The first part of this article contains an example of a non-nilpotent nil-algebra with a finite number of generators over an arbitrary field k. This gives a negative answer to Kuroš's problem whether an associative algebraic algebra is always locally finite-dimensional, and also to Levitzki's problem whether an associative nil-algebra is locally nilpotent [A. G. Kuroš, same Izv. **5** (1941), 233–240; MR **3**, 194; J. Levitzki, Bull. Amer. Math. Soc. **51** (1945), 913–919; MR **7**, 237]. In the second part, there is given an example of an infinite finitely-approximable p-group with a finite number of generators. This gives a negative answer to Burnside's problem for groups that have a complete system of linear representations. Both of these examples are based upon the following lemma [see the author and I. R. Šafarevič, #5056 above]. Let R_d be a non-commutative polynomial ring of d variables over a field k, and let I be the ideal generated by an infinite sequence of forms of degree not less than two, where the number of forms with degree i is equal to r_i. We put $r_i\leq s_i$. If the coefficients of the power series

$$(1-dt+\sum_{i=2}^{\infty} s_it^i)^{-1}$$

are all non-negative, then the factor algebra R_d/I is infinite-dimensional.

E. Inaba (Tokyo)

Luh, Jiang 5083
A characterization of regular rings.
Proc. Japan Acad. **39** (1963), 741–742.

A subring M of an arbitrary ring A is called a quasi-ideal of A if $AM \cap MA \subseteq M$. The author proves that the following conditions are equivalent: (i) A is regular; (ii) For every subring M of A, $MAM \subseteq M$ implies $MAM = M$; (iii) For every quasi-ideal M of A, $MAM = M$.
R. McFadden (Belfast)

Kertész, A. 5084
Zur Frage der Spaltbarkeit von Ringen.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 91–93.

The author gives a sufficient condition for a subgroup of the additive group of a ring to be a ring-direct summand. He uses this to prove that (1) in every Artin ring the torsion subgroup is a ring-direct summand (Szász), (2) the underlying additive group of every nilpotent Artin ring is a torsion group (Szele), and (3) if the radical of an Artin ring R is an Artin ring, then R is a ring-direct sum of its torsion subgroup and a semisimple ring.
D. Zelinsky (Evanston, Ill.)

Pareigis, Bodo 5085
Über normale, zentrale, separable Algebren und Amitsur-Kohomologie.
Math. Ann. **154** (1964), 330–340.

If $C \subset K \subset F$ is a tower of fields with F and K Galois over C, Hochschild and Serre produced an exact sequence

$$0 \to H^2(K/C) \to H^2(F/C) \to H^2(F/K)^G \to H^3(K/C) \to H^3(F/C),$$

where, for example, $H^n(K/C)$ denotes the nth cohomology group of the Galois group G of K over C, with coefficients in the multiplicative group of K. Furthermore, they identified $H^2(F/K)$ with the Brauer group $B(F/K)$ of K-algebras split by F, and $H^2(F/K)^G$ with the subgroup $B_G(F/K)$ of "normal algebras"—those to which all C-automorphisms of K are extendable. Amitsur [J. Math. Soc. Japan **14** (1962), 1–25; MR **26** #2481] and Rosenberg and Zelinsky [Osaka Math. J. **14** (1962), 219–240; MR **26** #173] exhibited an exact sequence for a tower of rings (no Galois groups involved) looking exactly like the Hochschild-Serre sequence, except that the cohomology groups denote the cohomology groups of the Amitsur complex and the term $H^2(F/K)^G$ is replaced by a term $H^2(F \otimes K/K)^0$. The present paper shows how to define B_G for rings so that, under the assumption that the known map $H^2(F \otimes K/K) \to B(F \otimes K/K)$ is an isomorphism, this isomorphism carries $H^2(F \otimes K/K)^G$ onto $B_G(F \otimes K/K)$. The author also proves, under suitable normality hypotheses, that $H^2(F \otimes K/K)^G = H^2(F \otimes K/K)^0$, thus identifying the term in the Amitsur exact sequence with a group B_G of normal algebras. Finally, for normal but not necessarily separable field extensions, he shows that the combination $F \otimes K/K$ may be replaced by F/K in all the results above.
D. Zelinsky (Evanston, Ill.)

Murase, Ichiro 5086
On the structure of generalized uniserial rings. II.
Sci. Papers College Gen. Ed. Univ. Tokyo **13** (1963), 131–158.

The major part of this paper is devoted to an analysis of indecomposable generalized uniserial rings for which every minimal left ideal is nilpotent, the structure of all other indecomposable generalized uniserial rings having been thoroughly studied in the first paper of this series [same Papers **13** (1963), 1–22; MR **28** #118]. The author shows that if R is such a ring and the exponent ρ of the radical N of R is not greater than the number n of isomorphism classes of simple R-modules, then eRe is a sfield for any primitive idempotent e of R, and any two such sfields are isomorphic. He then determines completely the structure of a ring satisfying these conditions, presenting it as a vector space over the sfield mentioned above (which is uniquely determined up to isomorphism) with multiplication law described in terms of the Cartan invariants of R.

If $\rho > n$, the situation is more complicated, and the subrings of the form eRe are not sfields. However, the author determines, in a manner similar to that of the preceding case, the structure of such a ring R satisfying the two additional conditions: (a) R is cleft, i.e., R contains a subring S such that $R = S \oplus N$ as abelian groups; (b) Every derivation of a certain sub-sfield of R (whose existence is guaranteed by (a)) is inner.

He then uses his theory to reprove results of Asano and Behrens on uniserial rings (i.e., generalized uniserial rings for which $n = 1$) and a theorem of Nakayama on direct decompositions of modules over generalized uniserial rings. He also discusses the structure of generalized uniserial quasi-Frobenius rings, and proves that any indecomposable generalized uniserial ring satisfying conditions (a) and (b) above is an epimorphic image of such a ring.

Finally, the author determines all minimal faithful modules over certain classes of generalized uniserial rings. He obtains as a consequence the following result, and others of a similar nature. Let S be the algebra of all $n \times n$ matrices over a field F, and T be the subalgebra of S consisting of all upper triangular matrices. If T' is another subalgebra of S, then any F-algebra isomorphism of T onto T' can be extended to an inner automorphism of S.
S. U. Chase (Ithaca, N.Y.)

Djabali, Mahmoud 5087
Étude d'un J-anneau noethérien de dimension 2.
C. R. Acad. Sci. Paris **258** (1964), 5309–5310.

Let A be a left Noetherian ring with zero singular ideal such that no three left ideals have a sum which is their direct sum. If, besides, there exists a principal left ideal I not contained in the radical such that every left ideal contained in I meets the radical, then A has a semisimple left ring of quotients contained in (but not equal to) the injective envelope of A. If the condition on the radical is not satisfied, the conclusion need not hold. No proofs are included.
D. Zelinsky (Evanston, Ill.)

Johnson, R. E. 5088
Distinguished rings of linear transformations.
Trans. Amer. Math. Soc. **111** (1964), 400–412.

A ring R of linear transformations of a vector space M over a division ring D is called distinguished if and only if (1) the lattice J of all R-submodules of M is a distributive sublattice of the lattice L of all subspaces of M, and (2) the set of all linear transformations of M leaving J

invariant is R. A finite distributive sublattice of L containing 0 and M is called an FD-lattice. It is proved (Theorem 2.4) that every FD-lattice is the lattice of submodules of some distinguished ring R. Conversely, if D has characteristic zero, then a finite sublattice of L must be distributive in order to be the lattice of submodules of a ring of linear transformations of M (Theorem 2.6). A main tool is Theorem 1.1 that each non-zero element of an FD-lattice J is a direct union of elements of L, associated with irreducible elements of J.

If J is an FD-lattice, and $N \in L$, then N is called J-distributive if $N \cap (A \cup B) = (N \cap A) \cup (N \cap B)$ for all $A, B \in J$. It is shown (Theorem 3.1) that N is J-distributive if and only if $N = Me$ for some idempotent $e \in R$. All subspaces of M are J-distributive if and only if J is a chain (Theorem 3.3). The reviewer proved [Math. Z. **75** (1960/61), 328–332; MR **23** #A914] that if J is a chain, then R is a Baer ring (each annihilating right or left ideal is generated by an idempotent). The author proves that this is not the only case (J a chain) in which R is a Baer ring, but proves in Theorem 5.3, however, that a more general lattice J rarely has its corresponding ring R, a Baer ring.

Every distinguished ring R is a direct sum $\sum e_i R e_j$, where $1 = \sum_{i=1}^{n} e_i$. Each $e_i R e_j$ is a full ring of linear transformations if $j \leq i$, and is zero if $j = i$. The subring $A = \sum_{j \neq i} e_i R e_j$ is the Jacobson radical of R so that $R/A \cong \sum_{i=1}^{n} e_i R e_i$. The ideal A is nilpotent.

Two distinguished rings are shown to be isomorphic if and only if the obvious isomorphisms hold between their underlying vector spaces.

K. G. Wolfson (New Brunswick, N.J.)

Faddeev, D. K. 5089

On the semigroup of genera in the theory of integral representations. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 475–478.

Let Λ be a **Z**-ring, i.e., a ring with unit which, considered as a **Z**-module, is free on a finite number of generators. The representations of Λ by matrices with coefficients in any ring R containing **Z** (the integers) form a semigroup under formation of direct sums. Let $S, S^p, \tilde{S}, \tilde{S}^p, \bar{S}^p$ be the corresponding semigroups when R is respectively **Z**, the ring of all rationals integral for the prime p, the rational field **Q**, the ring of p-adic integers and the field of p-adic numbers. Then there is an obvious collection of natural maps

$$\begin{array}{ccccc} & & & \tilde{S}^p & \\ & & \varphi_3\nearrow & & \searrow\varphi_6 \\ S \xrightarrow{\varphi_1} G \xrightarrow{\varphi_2} S^p & & & & \bar{S}^p \\ & & \varphi_5\searrow & & \nearrow\varphi_4 \\ & & & \tilde{S} & \end{array}$$

where G is the semigroup of genera, i.e., the quotient of S by the equivalence relation "has the same image in S^p for every p". All the semigroups except S have cancellation [Borevič and the author, Vestnik Leningrad. Univ. **14** (1959), no. 7, 72–87; MR **21** #4968]; $\tilde{S}^p, \tilde{S}, \bar{S}^p$ are free and commutative. Further, the morphisms φ_3, φ_4 are monomorphisms and $\varphi_5, \varphi_5\varphi_2\varphi_1, \varphi_6$ are epimorphisms. Theorem 1 states that

$$\varphi_3 S^p = \varphi_6^{-1}(\varphi_4 \tilde{S}).$$

Theorem 2 describes the semigroup G of genera in terms of the S^p and the φ_5.

If the **Q**-algebra $\tilde{\Lambda} = \Lambda \otimes \mathbf{Q}$ is semisimple and Λ is a maximal **Z**-ring in it, then $G, S^p, \tilde{S}$ are all isomorphic. If, however, Λ is not maximal, then it is stated to be easy to obtain information about G by considering the maximal extensions of Λ. Finally, this is all said to extend to arbitrary Dedekind rings instead of **Z**, with appropriate restrictions about separability.

J. W. S. Cassels (Cambridge, England)

Osofsky, B. L. 5090

Rings all of whose finitely generated modules are injective.

Pacific J. Math. **14** (1964), 645–650.

For a ring R with 1 the following conditions are equivalent: (a) R is an artinian semi-simple ring; (b) every finitely generated unitary right R-module is injective; (c) every cyclic unitary right R-module is injective.

A. Kertész (Debrecen)

Roby, Norbert 5091

Lois polynomes et lois formelles en théorie des modules.

Ann. Sci. École Norm. Sup. (3) **80** (1963), 213–348.

Soient A un anneau commutatif, et M, N deux A-modules. Une loi polynôme sur (M, N) est une loi qui, à chaque A-algèbre commutative R, associe une application R-linéaire $f_R : M \otimes R \to N \otimes R$ de façon compatible avec les homomorphismes $R \to R'$. Une loi polynôme (f_R) est uniquement déterminée par la donnée de f_E, où E est l'algèbre des polynômes en une infinité de variables sur A. L'ensemble $P(M, N)$ des lois polynômes sur (M, N) est un A-module gradué. Étude de la composition des lois polynômes. Parmi les lois polynômes, on trouve les lois linéaires et multilinéaires. Étant donnés une loi polynôme $f = (f_R)$ sur (M, N) et un entier p, la polarisée $P_p(f)$ est la loi polynôme sur (M^p, N) définie par $P_p(f)_R(z_1, \cdots, z_p) = f_R(z_1 + \cdots + z_p)$. Si f est homogène de degré p, $P_p(f)$ est une loi multilinéaire symétrique; alors la correspondance $f \to P_p(f)$ est bijective si $p!$ est inversible dans A. Les chapitres I et II se terminent par une étude des applications quadratiques, des dérivées partielles des lois polynômes, et par une formule de Taylor.

Le chapitre III est une étude détaillée de l'algèbre $\Gamma(M)$ des "puissances divisées" sur le A-module M; liens entre $\Gamma(M)$, l'algèbre symétrique de M, et l'algèbre des tenseurs symétriques sur M. Au chapitre IV on montre que les lois polynômes sur (M, N) correspondent bijectivement et de façon canonique aux applications linéaires $\Gamma(M) \to N$. Applications à l'étude de $\Gamma(M)$ au moyen des lois polynômes: limites inductives, structure de $\Gamma(M)$ quand M est libre, structure de $\Gamma(M/M')$.

Au chapitre V on définit une application diagonale $\Gamma(M) \to \Gamma(M) \otimes \Gamma(M)$, qui fait de $\Gamma(M)$ une co-algèbre. Alors $P(M, A)$ est une algèbre, qui s'injecte canoniquement dans l'algèbre duale de la co-algèbre $\Gamma(M)$.

On passe alors (chapitre V) aux lois formelles. Pour toute A-algèbre commutative R, notons $I(R)$ l'idéal des éléments nilpotents de R; pour tout A-module M, on note $M(R)$ l'image canonique de $M \otimes I(R)$ dans $M \otimes R$. Une loi formelle f sur (M, N) est la donnée, pour chaque A-algèbre commutative R, d'une application linéaire $f_R : M(R) \to N(R)$ de façon compatible avec les homomorphismes $R \to R'$. Toute loi polynôme sans terme constant définit une loi formelle. Les lois formelles sur (M, N)

forment un A-module topologique $F(M, N)$, dont la topologie généralise celle des anneaux de séries formelles et a des propriétés analogues. Composition des lois formelles. Théorème des fonctions implicites et théorème du jacobien.

Au chapitre VII on étudie $F(M, A)$, qui est une algèbre commutative, et le module $F(M, M)$ des "transformations formelles" de M; ce dernier a une structure d'algèbre de Lie. Le chapitre VIII applique les transformations formelles au problème suivant: exprimer un automorphisme de l'algèbre des germes de fonctions analytiques sur $\mathbf{R}^n$ ou $\mathbf{C}^n$ comme une exponentielle de dérivation. Un appendice relie les puissances divisées à la formule de Taylor en caractéristique p. Un autre étudie les logarithmes d'un automorphisme de $\mathbf{C}^n$. *P. Samuel* (Paris)

Tachikawa, Hiroyuki 5092

On dominant dimensions of QF-3 algebras.

Trans. Amer. Math. Soc. **112** (1964), 249–266.

Let B be an algebra over a field k. A B-module M is said to have dominant dimension $\geqq n$ if there exists an exact sequence $0 \to M \to X_1 \to \cdots \to X_n$ with X_i both projective and injective. The supremum of such n is called the dominant dimension of M, written domi $\dim_B M$. In this paper, the author is concerned mainly with domi $\dim_B B$ and domi $\dim_{B^e} B$, where $B^e = B \times B^0$ is the enveloping algebra of B.

Thrall first defined a QF-3 algebra [same Trans. **64** (1948), 173–183; MR **10**, 98] as an algebra B with a faithful projective injective module. This is clearly equivalent to the existence of an exact sequence $0 \to B \to X_1$ with X_1 projective and injective. Since the author is mainly concerned with the case domi $\dim_B B \geqq 1$, he restricts consideration to QF-3 algebras.

If X is the minimal faithful projective injective right module for the algebra B, then X is also a faithful A-module, where A is the B-endomorphism ring of X. Let B' be the A-endomorphism ring of X; it is clear that B can be embedded in B'. One is often interested in showing $B = B'$; B is its own "second centralizer". In the paper under review the author shows that if either domi $\dim_B B > 1$ or domi $\dim_{B^e} B > 1$, then $B = B'$.

He also has several theorems comparing the dominant dimension with injective and projective dimension. For instance, using the above notation, he shows

$$\text{domi dim}_B B \leqq \text{inj dim}_C Y + 1 = \text{proj dim}_A X + 1$$

provided $0 < \text{inj dim}_C Y < \infty$, where Y is the minimal faithful projective injective left B-module and C is its B-endomorphism ring.

There is a section in which he studies dominant dimension in generalized uni-serial algebras. In particular, he obtains results on the dominant dimension of B in the case that the algebra A is generalized uni-serial.

The paper concludes by showing a connection between Tor over A and Ext over B, which is reminiscent of Nakayama's complete cohomology for Frobenius algebras [Osaka Math. J. **9** (1957), 165–187; MR **20** #6449].

J. P. Jans (Seattle, Wash.)

Heller, A.; Reiner, I. 5093

Grothendieck groups of orders in semisimple algebras.

Trans. Amer. Math. Soc. **112** (1964), 344–355.

Notations: R is a Noetherian domain with quotient field F; A an R-algebra which is a finitely generated, torsion-free R-module; $A^* = F \otimes_R A$; $K^0(A)$ the Grothendieck group of the category of finitely generated A-modules (i.e., the abelian group with generators $[M]$, one for each finitely generated A-module M, and relations $[N] = [M] + [P]$ for each exact sequence $0 \to M \to N \to P \to 0$); $K_t^0(A)$ the Grothendieck group of the category of R-torsion, finitely generated A-modules; $K^1(A)$ Bass's analogue of the Whitehead group (the Grothendieck group of the category of all automorphisms of finitely generated A-modules, modulo the further relations $[\lambda\lambda'] = [\lambda] + [\lambda']$ when λ and λ' are automorphisms of the same module).

The authors produce a homomorphism $\Delta: K^1(A^*) \to K_t^0(A)$ which, together with the natural maps, gives an exact sequence

$$K^1(A^*) \xrightarrow{\Delta} K_t^0(A) \xrightarrow{\eta} K^0(A) \xrightarrow{\theta} K^0(A^*) \to 0$$

when A^* is semisimple (but $K^1(A) \to K^1(A^*) \to K_t^0(A)$ is not exact). To define $\Delta([\lambda^*])$ with $[\lambda^*] \in K^1(A^*)$ (so λ^* is an automorphism of an $(F \otimes A)$-module M^*), choose an A-module $M \subset M^*$ such that $FM = M^*$; then $\Delta([\lambda^*]) = [\lambda^* M/(\lambda^* M \cap M)] - [M/(\lambda^* M \cap M)]$ in $K_t^0(A)$.

Note that when A^* is semisimple, $K^0(A^*)$ is the free abelian group generated by the simple A^*-modules, so the last part of this exact sequence splits: $K^0(A) \cong K^0(A^*) \oplus \text{Im } \eta$.

In particular, suppose R is a Dedekind domain of characteristic zero and $A = RG$ is the group ring of a finite group G, and assume that F splits G, i.e., FG is a direct sum of matrix algebras over F. In this case Im η, and hence $K^0(A)$, can be computed explicitly: Let J be the group of fractional R-ideals, n the number of simple A^*-modules, and $J^n = J \times J \times \cdots \times J$. The authors exhibit an automorphism $J^n \to K_t^0(A)$ so that Im η is J^n modulo a subgroup which turns out to be the subgroup generated by $\{(J_1, \cdots, J_n) \mid \text{each } J_i \text{ is a principal ideal}\}$ and

$$\{(P^{q_1}, \cdots, P^{q_n}) \mid \sum q_i d_{ij}^P = 0 \text{ for all } j\},$$

where P ranges over the primes of R and (d_{ij}^P) is the Cartan matrix for the (R/P)-algebra A/PA.

In the case where F does not split G, but is an algebraic number field, $K^0(A)$ is the direct sum of $K^0(A^*)$ and a finite abelian group. *D. Zelinsky* (Evanston, Ill.)

Bokut', L. A. 5094

Letter to the editor. (Russian)

Algebra i Logika Sem. **3** (1964), *no.* 1, 57.

Correction to the author's paper in the same journal [Algebra i Logika Sem. **1** (1962), no. 5, 5–29; MR **27** #2536]; for details see the review in MR **27** #2536.

P. M. Cohn (Chicago, Ill.)

Cohn, P. M. 5095

Free ideal rings.

J. Algebra **1** (1964), 47–69.

Generalizing the concept of principal ideal domain, the author defines a fir [respectively, local fir] as a ring R with unit and no zero divisors, satisfying (i) free right R-modules have unambiguous rank (i.e., all bases have the same cardinal), and (ii) every right ideal [finitely generated right ideal] is a free module. If we change "right" to "left" everywhere in this definition, we get the concept

of a left fir or left local fir. The author conjectures that left firs are the same as firs, and he proves that left local firs are the same as local firs. If R is both a fir and a left fir, then R is a unique factorization domain.

The free product of any family R_λ of local firs over a skew field k is again a local fir. A similar result is true with "local fir" replaced by "fir" provided each unit map $k \to R_\lambda$ has a splitting map $R_\lambda \to k$ whose kernel is a right ideal. These theorems are deduced as corollaries of more general theorems with k a local fir rather than a skew field, together with suitable splitting and freeness hypotheses.

It follows that examples of firs are: every free associative algebra over a commutative field, and the group algebra over a commutative field of any free group.

D. Zelinsky (Evanston, Ill.)

Noronha Galvão, M.ª L. 5096

On a Noether-Krull theory for semi-rings. (Portuguese)

Univ. Lisboa Revista Fac. Ci. A (2) **8** (1960/61), 175–256.

Dans ce travail l'auteur s'occupe de l'extension de la théorie de la représentation de Noether-Krull aux demi-anneaux. Un idéal réticulé est premier [semi-premier] si et seulement s'il est complètement premier [complètement semi-premier]. On introduit les notions d'idéal primal, idéal primal par éléments, idéal primaire, idéal primaire par éléments, μ-composante isolée d'un idéal, μ-composante principale, etc. Dans les demi-anneaux [A. Almeida Costa, même Revista (2) **7** (1959/60), 235–243] tout idéal est l'intersection de ses μ-composantes principales. Dans les demi-anneaux vérifiant la condition de chaîne ascendente, une μ-composante principale est un idéal primal. En partant de l'étude des intersections des idéaux primaux par éléments et primaires par éléments on fait l'étude des intersections des idéaux et des idéaux primaires. On obtient aussi, sous certaines conditions, des théorèmes d'unicité et d'existence des représentations.

E. Lluis (Mexico City)

NON-ASSOCIATIVE ALGEBRA

Kleinfeld, Erwin 5097

Middle nucleus-center in a simple Jordan ring.

J. Algebra **1** (1964), 40–42.

Let R be a simple Jordan ring of characteristic not equal to 2. Let N be its middle nucleus and C its center. The author derives the identity $(n, (n, a, b), (n, x, y)) = 0$ for $n \in N$; $a, b, x, y \in C$, and where $(x, y, z) = (xy)z - x(yz)$. By using this relation and a lemma of Albert [Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 446–447; MR **27** #3679] (for fixed n in N, the additive subgroup B generated by all elements of the form (n, R, R) is an ideal of R), the author generalizes a result of Sandler and the reviewer [Bull. Amer. Math. Soc. **69** (1963), 791–793; MR **27** #5804] to: In a simple Jordan ring of characteristic $\neq 2$ the middle nucleus and center coincide.

R. H. Oehmke (Princeton, N.J.)

Dieudonné, Jean 5098

Sur un théorème de Lazard.

J. Reine Angew. Math. **214/215** (1964), 61–64.

Construction d'une hyperalgèbre non commutative de dimension 8 sur un corps parfait de caractéristique 2, l'algèbre de Lie des éléments primitifs étant de dimension 1.

{D'après Lazard, une série formelle $f(x, y) = x + y + g(x, y)$ (où chaque monôme entrant dans g est de degré total $\geqq 2$) qui vérifie $f(x, f(y, z)) = f(f(x, y), z)$ est symétrique $f(x, y) = f(y, x)$. L'exemple de la série $x + y + x^{p^h} y^{p^k}$ (avec $h > k$) sur un corps de caractéristique $p > 0$ prouve que l'on peut avoir $f(x, f(y, z)) \equiv f(f(x, y), z)$ modulo les puissances p^{h+k}-ièmes des variables sans avoir $f(x, y) \equiv f(y, x)$; on en déduit une hyperalgèbre non commutative de dimension p^{h+k}, d'algèbre de Lie de dimension 1; pour $p = h = 2$ et $k = 1$, on retrouve l'exemple de l'auteur.}

P. Cartier (Strasbourg)

Cohn, P. M. 5099

The embedding of Lie algebras in restricted Lie algebras.

J. London Math. Soc. **39** (1964), 277–287.

Let L be a Lie algebra over a commutative ring Λ with unity of prime characteristic p. It is proved that if $\lambda \to \lambda^p$ is an automorphism of Λ, then L can be embedded in a restricted Lie algebra over Λ. More generally, the conclusion is valid if Λ is without nilpotent elements and if Λ^p is pure in Λ. Examples are given to show that neither of these hypotheses on Λ may be dropped. That L may be embedded in a restricted Lie algebra follows at once if L has an associative embedding; thus the examples cited do not have associative embeddings. A final example shows that (even when $\lambda \to \lambda^p$ is an automorphism of Λ) restrictedness is not sufficient to guarantee an associative embedding.

G. B. Seligman (New Haven, Conn.)

HOMOLOGICAL ALGEBRA

See also 5085, 5092, 5117.

Schmid, Josef 5100

Zu den Reduktionssätzen in der homologischen Theorie der Gruppen.

Arch. Math. **15** (1964), 28–32.

By using, in the category of left **Z**G modules, the relative abelian structure in which exact sequences split over **Z**, the author reproves the reduction theorem of Cartan and Eilenberg [*Homological algebra*, Princeton Univ. Press, Princeton, N.J., 1956; MR **17**, 1040] for cohomology of groups and thence deduces the cup-product reduction theorems using the resolution of Gruenberg [J. London Math. Soc. **35** (1960), 481–494; MR **23** #A3165].

A. Heller (Urbana, Ill.)

Roux, André 5101

Un théorème de plongement des catégories.

C. R. Acad. Sci. Paris **258** (1964), 4646–4647.

Le résultat principal est le suivant. Soit C une catégorie. Il existe un couple (h, P) (où P est une catégorie à limites contravariantes et $h : C \to P$ un foncteur) tel que pour tout foncteur $t : C \to T$ (où T est une catégorie à limites contravariantes), il existe un foncteur $t_\leftarrow : P \to T$ unique à un isomorphisme près, commutant avec les limites contravariantes et tel que $t = t_\leftarrow \cdot h$.

J. R. Isbell (Seattle, Wash.)

Gerstenhaber, Murray 5102

The cohomology structure of an associative ring.

Ann. of Math. (2) **78** (1963), 267–288.

Let A be an associative ring (not necessarily with unit) and P a two-sided S-module. In addition, assume that A and P are modules over a commutative ring S and that all operations of A on A and P are S-homomorphisms. Defining the cohomology groups of the S-algebra A with coefficients in P by the usual cochain formulae, the author is primarily interested in the ring structure of $H^*(A, A)$ and the module structure over $H^*(A, A)$ of $H^*(A, P)$ given by the usual cup product multiplication. Among other things, it is shown that $H^*(A, A)$ is a commutative ring in the sense of graded rings. A bracket product [,] is introduced in $H^*(A, A)$ under which $H^*(A, A)$ becomes a graded Lie ring with $[H^m(A, A), H^n(A, A)] \subset H^{m+n-1}(A, A)$ and such that the bracket operation of $H^1(A, A)$ into itself is the ordinary Poisson bracket of derivations of A into itself. Various other properties of this operation are derived and its role in the author's theory of deformations of algebras indicated. *M. Auslander* (Waltham, Mass.)

MacRae, R. E. 5103

On the homological dimension of certain ideals.

Proc. Amer. Math. Soc. **14** (1963), 746–750.

Let R be a commutative noetherian ring, I an ideal of finite homological dimension in R such that each of the associated prime ideals of I is also an associated prime of a non-zero divisor in R. If R is a local ring, then it is shown that there exist a and b in R such that $I = aR : bR$. If R is a domain, then it is shown that I is projective. The latter result gives another proof of the fact that regular local rings are unique factorization domains.

M. Auslander (Waltham, Mass.)

Uehara, Hiroshi 5104

Homological invariants of local rings.

Nagoya Math. J. **22** (1963), 219–227.

Let R be a commutative noetherian local ring with maximal ideal M generated by $t_1, \cdots, t_n$. Let Λ be the R-algebra $R\langle T_1, \cdots, T_n\rangle$ with $dT_i = t_i$ as defined by Tate [Illinois J. Math. **1** (1957), 14–27; MR **19**, 119]. The author computes for $i = 1, 2, 3, 4$ the Betti numbers $B_i = \dim_K (\mathrm{Tor}_i{}^R (K, K))$, where $K = R/M$, in terms of certain integers deducible from the homology algebra $H(\Lambda)$. These computations are then used to generalize and reprove results already obtained by Eilenberg and Tate. *M. Auslander* (Waltham, Mass.)

GROUP THEORY AND GENERALIZATIONS

See also 5004, 5045, 5082, 5093, 5319, 5374, 5430.

Zieschang, Heiner 5105

Alternierende Produkte in freien Gruppen.

Abh. Math. Sem. Univ. Hamburg **27** (1964), 13–31.

An element of the free group $F(K_1, \cdots, K_n)$ is called an alternating product if it is represented by a reduced word of length $2n$ that contains each K_i and each $K_i{}^{-1}$. Theorem 3: Let θ be a homomorphism of $F(K_1, \cdots, K_n)$ into the free group $F(A_1, \cdots, A_p, B_1, \cdots, B_p)$. If there is an alternating product w such that $\theta(w) = \prod [A_i, B_i]$, then θ is onto. Corollary 3: An endomorphism of $F(A_1, \cdots, A_p, B_1, \cdots, B_p)$ that leaves $\prod [A_i, B_i]$ fixed is an automorphism. Various applications are made to the theory of surface transformations, but the results seem to be already more or less known.

R. H. Fox (Princeton, N.J.)

Petresco, Julian 5106

Algorithmes de décision et de construction dans les groupes libres.

Math. Z. **79** (1962), 32–43.

Let G be a free group on a finite free generating set A; a subset X of G is said to be progressive if $X \cap X^{-1} = \emptyset$ and the A-length $L_A(w)$ of any reduced word w in certain elements x_i of $X \cup X^{-1}$ satisfies $L_A(w) \geqq L_A(x_i)$. Now take a fixed progressive set X and let H be the subgroup generated by X, so that H is free on X. The author gives an algorithm for constructing, for a given reduced word $b \in G$, all words of minimal length in the coset bH, and in particular for deciding whether (a) b itself has minimal length and (b) b lies in H [cf. Nielsen, Math. Scand. **3** (1955), 31–43; MR **17**, 455]. When X is finite, he obtains an algorithm for deciding whether $(G : H)$ is finite; moreover, for any finite set X there is an algorithm for deciding whether X is progressive and one for constructing from X a progressive generating set of H. The proofs depend on an analysis of the usual cancellation arguments. It is noted that the Burnside problem is hereby reduced to a problem of the following type: Given an infinite set X in G, if H is the subgroup generated by X, to decide whether H is of finite rank, and if so, to construct a finite generating set of H. *P. M. Cohn* (Zbl **104**, 243)

Azleckiĭ, S. P. 5107

On certain characteristic subgroups of a finite group. (Russian)

Ukrain. Mat. Ž. **16** (1964), 220–225.

Let $\mathfrak{G}$ be a finite group. Define the $\mathfrak{H}^0$-subgroup of $\mathfrak{G}$ to be the subgroup of $\mathfrak{G}$ generated by one of the following families of subgroups: (i) the centers of all Sylow subgroups of $\mathfrak{G}$; (ii) the kernels of all Sylow subgroups of $\mathfrak{G}$ (the kernel of a group is the intersection of the normalizers of all subgroups); (iii) the commutator subgroups of all Sylow subgroups of $\mathfrak{G}$. The $\mathfrak{H}$-subgroup of $\mathfrak{G}$ is defined correspondingly to be the center of $\mathfrak{G}$, the kernel of $\mathfrak{G}$, or the commutator subgroup of $\mathfrak{G}$. The author investigates conditions which imply the equality of either the $\mathfrak{H}^0$-subgroup with the $\mathfrak{H}$-subgroup, or the equality of $\mathfrak{H}^0$-subgroups defined by different families of subgroups. The results and proofs are fairly elementary. We cite a few of the results which are typical of the article. Theorem 4: If a non-nilpotent group $\mathfrak{G}$ of order n satisfies $\mathfrak{H}(\mathfrak{G}) = \mathfrak{H}^0(\mathfrak{G})$ and $\mathfrak{H}^0$ is defined by (ii), then (1) $n \geqq 216$ if $\mathfrak{H}(\mathfrak{G})$ is Abelian, (2) $n \geqq 432$ if $\mathfrak{H}(\mathfrak{G})$ is Hamiltonian. Theorem 7: Let $\mathfrak{G}$ be a group such that $\mathfrak{H}(\mathfrak{M}) = \mathfrak{H}^0(\mathfrak{M})$ for every proper subgroup $\mathfrak{M}$ of $\mathfrak{G}$. Then $\mathfrak{G}$ is either nilpotent or of type S. (A group is of type S if it is non-nilpotent, but every proper subgroup is nilpotent.)

P. Fong (Berkeley, Calif.)

Oates, Sheila; Powell, M. B. 5108

Identical relations in finite groups.

J. Algebra **1** (1964), 11–39.

The reviewer's question whether the laws of a finite group possess a finite basis [the reviewer, Math. Ann. **114** (1937), 506–525] is answered in the affirmative in this paper. The proof builds on work due to D. C. Cross [reported by G. Higman, Conv. Internaz. di Teoria dei Gruppi Finiti (Firenze, 1960), pp. 93–100, Edizione Cremonese, Rome, 1960; MR **23** #A190]. A finite group is critical if it does not belong to the variety generated by its proper factors (that is, factor groups of subgroups other than the whole group); a variety is a Cross variety if (i) its finitely generated groups are finite, (ii) its laws are finitely based, (iii) the number of critical groups contained in it is finite. Every subvariety of a Cross variety is a Cross variety; so one wants to show that every finite group is contained in some Cross variety. The variety generated by a finite group is generated by the critical factors of that group. Hence it suffices to prove the following key theorem: If $\mathfrak{U}$ is a Cross variety, B a critical group whose proper factors belong to $\mathfrak{U}$, then the variety $\mathfrak{B}$ generated by B and $\mathfrak{U}$ is again a Cross variety. If $\mathfrak{B}^{(n)}$ is the variety of all groups satisfying the n-variable laws of $\mathfrak{B}$, then the laws of $\mathfrak{B}^{(n)}$ are finitely based [the reviewer, loc. cit.]. As $\mathfrak{B} \subseteq \mathfrak{B}^{(n)}$, it suffices to show that n can be so chosen that $\mathfrak{B}^{(n)}$ also satisfies the conditions (i) and (iii) above. The proof of this fact requires different arguments according as the monolith of the (necessarily monolithic) critical group B is non-abelian or abelian. In the former case, (i) is harder to establish; the theory of M-subgroups, that is, subgroups isomorphic to M whose normalizer acts on it as B does on M, is developed specifically for this purpose. In the case of abelian monolith condition (iii) presents the main difficulty. The following invariants of a finite group are introduced: the p-rank of G is the maximum dimension of an absolutely irreducible component of the representation induced by G on any of its chief factors of p-power order; the p-measure of G is the maximum of the p-ranks of the factors of G. Similarly, for a finite non-abelian simple group S, the S-rank of G is the maximum number of factors isomorphic to S in any chief factor of G and the S-measure is the maximum of the S-ranks of the factors of G. The following theorem is the substance of the proof of (iii): The order of a critical group is bounded by a function of its exponent, the maximum of the nilpotency classes of its Sylow subgroups, the maximum of its p-measures and S-measures and the maximum order of its composition factors. {Reviewer's note: Very recently, L. G. Kovács has found a substantial simplification of the proof described above based on the use of a law that bounds the index of the centralizers of the chief factors of a finite group.} *B. H. Neumann* (Canberra)

Sah, Chih-Han 5109

A class of finite groups with abelian 2-Sylow subgroups.

Math. Z. **82** (1963), 335–346.

A finite group G is an AZ-group if G has abelian Sylow 2-groups and cyclic Sylow p-groups for odd primes p. LAZ-groups are $LF(2, p^n)$, $p^n > 3$, $p = 2$ or $p^n \equiv 3$ or 5 mod 8. By an ingenious and intricate way the author classifies AZ-groups. In particular, he proves that simple non-abelian AZ-groups are LAZ-groups, which is a generalization of a theorem of Suzuki [Amer. J. Math. **77** (1955), 657–691, Theorem A; MR **17**, 580]. In some points of the proof the following two theorems (both to appear), which are due to Brauer and Thompson, respectively, are effectively used: (1) Let G have no subgroup of index 2 and have an abelian Sylow 2-group P such that P contains just 3 elements of order 2. Then the order of P is 4. (2) Let G have no subgroup of index 2 and have an abelian Sylow 2-group P such that P has order ≥ 8 and contains an element t of order 2 whose centralizer has the form $\langle t \rangle \times LF(2, p^n)$, $p^n > 3$. Then $p = 3$ and n is odd. *N. Ito* (Nagoya)

Feit, Walter 5110

Groups which have a faithful representation of degree less than $p-1$.

Trans. Amer. Math. Soc. **112** (1964), 287–303.

In a previous paper [Pacific J. Math. **11** (1961), 1257–1262; MR **24** #A3207] the author and Thompson proved the following improvement of a theorem of Blichfeldt: Let G be a finite group which has a faithful representation of degree n over the complex numbers. Let $p > 2n+1$ be a prime number. Then a Sylow p-subgroup of G is abelian and normal in G.

Now in the present paper the author proves a further improvement: Let G be a group of order g which has a faithful representation of degree n over the complex numbers. Let $g = ab$ where $(a, b) = 1$ and $p > n+1$ for every prime p dividing a. Then G contains a normal abelian subgroup A of order a or a/p for some prime p.

The main idea of the proof is rather similar to the previous one. But in this improvement, in particular, the two possible conclusions require (i) a suitable modification of the theory of Brauer and Tuan [Brauer, Amer. J. Math. **64** (1942), 421–440; MR **4**, 2; Tuan, Ann. of Math. (2) **45** (1944), 110–140; MR **5**, 143] for groups whose orders are divisible by a certain prime number only to the first power, and (ii) a more elaborate maneuver of group characters. *N. Ito* (Nagoya)

Hall, P.; Kulatilaka, C. R. 5111

A property of locally finite groups.

J. London Math. Soc. **39** (1964), 235–239.

Theorem I: Every infinite locally finite group contains an infinite abelian subgroup. It is first shown that this is equivalent to Theorem II: Every infinite locally finite group G has at least one element $x \neq 1$ such that the centralizer $C_G(x)$ of x in G is infinite. The proof of this result is elementary but ingenious. Let G be a counter-example to Theorem II. It is first shown that one may assume that G is simple (in addition to satisfying several other conditions). The key step here is to show that G is not residually finite. A clever adaptation of the Brauer-Fowler argument [Ann. of Math. (2) **62** (1955), 565–583; MR **17**, 580] is then used to show that every element in G has odd order. This implies that G is locally solvable [the reviewer and J. G. Thompson, Pacific J. Math. **13** (1963), 775–1029]. The proof is completed by appealing to a theorem of Malcev, which asserts that a simple locally solvable group is finite of prime order.

{This result has also been proved by M. I. Kargapolov [Sibirsk. Mat. Ž. **4** (1963), 232-235; MR **26** #6241].} *W. Feit* (Ithaca, N.Y.)

Kulatilaka, C. R. 5112

Infinite Abelian subgroups of some infinite groups.

J. London Math. Soc. **39** (1964), 240–244.

In a previous paper [see the preceding review #5111] the author and P. Hall showed that every infinite locally finite group G contains an infinite abelian subgroup A. In this paper the author studies conditions on G which ensure that A can be chosen to satisfy certain properties such as subnormality. *W. Feit* (Ithaca, N.Y.)

Kazačkov, B. V. 5113

Letter to the editor. (Russian)

Mat. Sb. (*N.S.*) **63** (**105**) (1964), 646.

As P. A. Gol'dberg [RŽMat **1963** #6A178] and the reviewer [MR **26** #6258] remarked, the author's paper "Conditions for strong group factorizability" [Mat. Sb. (N.S.) **57** (**99**) (1962), 323–332; MR **26** #6258] contained an error. The author points out his mistake, gives the following corrected version of Theorem 8: A countable locally finite group has the Schur-Zassenhaus property, and states that his other results are not affected by the error.

H. Salzmann (Los Angeles, Calif.)

Mennicke, Jens 5114

A note on regular coverings of closed orientable surfaces.

Proc. Glasgow Math. Assoc. **5**, 49–66 (1961).

The group $\mathfrak{f}=\{a,b,c,d\,|\,abcda^{-1}b^{-1}c^{-1}d^{-1}=1\}$ is the fundamental group of F_2, the closed orientable surface of genus 2. In § 2 of this paper the author proves Theorem 1: A finite factor group of the fundamental group $\mathfrak{f}$ cannot have a minimal number e of generators exceeding 4. For $e=2$, every finite 2-generated group occurs as a factor group of $\mathfrak{f}$. For $e=3$ or 4, a necessary condition for a finite group $\mathfrak{G}$ to be a factor group of $\mathfrak{f}$ is that the second factor of the lower central series is at most 2- or 5-generated, respectively.

The author continues the work done in an earlier paper with P. Bergau [Math. Z. **74** (1960), 414–435; MR **27** #1960]. Consider the quaternion algebra $\mathfrak{A}$ over the ring Γ of rational integers with the norm $N=x^2+y^2-3u^2-3v^2$. This algebra consists of all matrices $\begin{pmatrix}x+iy & 3u+3iv\\ u-iv & x-iy\end{pmatrix}$. Consider the group $\mathfrak{U}$ of $+1$-units (with norm $+1$) of $\mathfrak{A}$ and the centre $\mathfrak{Z}=(1,Z)$, $Z=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$. The author proves that the factor group of $\mathfrak{U}$ with respect to the centre $\mathfrak{Z}$ contains $\mathfrak{f}$ as a subgroup of index 2. $\mathfrak{A}$ has as a homomorphic image the algebra $\mathfrak{U}_m$ with the same norm over the residue-class ring Γ_m modulo an arbitrary integer m. The group of $+1$-units of $\mathfrak{A}_m$ contains as a subgroup of index 1 or 2 the homomorphic image of the group of $+1$-units of $\mathfrak{A}_m$. Thus this homomorphism yields a large class of finite factor groups of $\mathfrak{f}$. The structure of these factor groups is investigated. The following is just one of the five theorems proved. Let the group $\mathfrak{P}$ be defined as follows:

$$A^2 = B^2 = C^2 = D^2 = E^2 = F^2 = ABCDEF = 1.$$

$\mathfrak{f}$ is the subgroup of $\mathfrak{P}$ of index 2 which consists of all even products of generators; $\mathfrak{P}$ can be represented as the symmetry group of the hyperbolic tessellation $\{6,4\}$. Theorem: Let $m=3^s z$, $(z,6)=1$. (a) The group $\mathfrak{U}_{3^s}$ of $+1$-units of $\mathfrak{A}_{3^s}$ is an extension of a 2-generated 3-group of order 3^{3s-1} with a factor group elementary abelian of order 4. (b) The group of $+1$-units of $\mathfrak{A}$ is mapped onto the group of $+1$-units of $\mathfrak{A}_m$. (c) The induced factor group of $\mathfrak{P}$ is a factor group of $\mathfrak{U}_m$ with respect to a normal subgroup of order 2. The induced factor group of $\mathfrak{f}$ is of index 2 in the factor group of $\mathfrak{P}$. *W. Moser* (Winnipeg, Man.)

Wonenburger, Maria J. 5115

The automorphisms of $U_n^+(k,f)$ and $PU_n^+(k,f)$. (Spanish summary)

Rev. Mat. Hisp.-Amer. (4) **24** (1964), 52–65.

Let k be a field with an involution $*\neq$ iden, M be an n-dimensional vector space over k, f be a nondegenerate hermitian form on M with respect to $*$, and let $U_n^+(k,f)$ be the special unitary group. The author generalizes a result of J. Dieudonné [Mem. Amer. Math. Soc. No. 2 (1951), XVII, Theorem 26; MR **13**, 531] as follows: For $n\geqq 3$, $n\neq 4$, $\mathrm{ch}(k)\neq 2$, with the possible exception for $U_3^+(\mathbf{F}_9)$, $U_3^+(\mathbf{F}_{25})$, every automorphism of $U_n^+(k,f)$ is of the form $s\to\chi(s)usu^{-1}$, where u is a unitary semisimilitude relative to an automorphism of k commuting with $*$ and χ is a representation of $U_n^+(k,f)$ in its center. This generalization is due to the observation that the Mackey-Rickert characterization of involutions in $U_n^+(k,f)$ can also be applied when $n>6$ is even without any restriction on the index of f. The author also discusses the group $PU_n^+(k,f)$. *T. Ono* (Philadelphia, Pa.)

Wan, Che-hsien [Wan, Cheh-hsian] 5116

A proof of a theorem on automorphisms of linear groups.

Acta Math. Sinica **11** (1961), 380–387 (*Chinese*); *translated as Chinese Math.* **2** (1962), 438–446.

L'auteur donne une nouvelle démonstration du théorème déterminant les automorphismes du groupe unimodulaire $SL_n(K)$ lorsque K est un corps (non nécessairement commutatif) de caractéristique 2 et $n\geqq 2$. Il observe que pour le cas $n=4$, la démonstration donnée par lui-même et L. K. Hua [mêmes Acta **2** (1952), 1–32] contient une erreur dans un des lemmes; sa nouvelle démonstration évite cette erreur, et évite aussi l'utilisation du théorème fondamental de la géométrie projective; il considère pour cela séparément les cas $n=3$ et $n=4$ avant d'aborder le cas général. Dans ses commentaires sur les démonstrations antérieures de ce théorème, l'auteur semble mettre en doute la validité du théorème fondamental de la géométrie projective pour le corps à deux éléments, sous prétexte que dans ce cas une droite projective n'aurait que deux points! (Il n'est pas exclu qu'une erreur aussi grossière provienne d'une faute de traduction.) *J. Dieudonné* (Paris)

Bass, H.; Lazard, M.; Serre, J.-P. 5117

Sous-groupes d'indice fini dans $SL(n, Z)$.

Bull. Amer. Math. Soc. **70** (1964), 385–392.

Les auteurs prouvent que tout sous-groupe d'indice fini de $G(n)=\mathbf{SL}(n,\mathbf{Z})$ est un groupe de congruence pour $n\geqq 3$, c'est-à-dire contient le noyau $G_q(n)$ d'un des homomorphismes canoniques $\mathbf{SL}(n,\mathbf{Z})\to\mathbf{SL}(n,\mathbf{Z}/q\mathbf{Z})$ pour un entier $q\geqq 1$ convenable (il est classique que le résultat correspondant pour $n=2$ est faux). La méthode consiste à comparer les complétés $\hat{G}(n)$ et $A(n)$ de $G(n)$ pour la

topologie des sous-groupes d'indice fini et la topologie des sous-groupes de congruence; on a une suite exacte

$$1 \to C(n) \to \hat{G}(n) \to A(n) \to 1$$

et il s'agit de prouver que le noyau $C(n)=1$. On introduit pour tout entier $q \geqq 1$ le sous-groupe $E_q(n)$ de $G_q(n)$ engendré par les matrices de la forme $1+aE_{ij}$ $(i \neq j)$ où $a \in q\mathbf{Z}$; on prouve que pour $n \geqq 3$, on a $C(n) = \lim_{\overleftarrow{q}} G_q(n)/E_q(n)$, $G_q(n) = E_q(n)G_q(n-1)$ et $(G(n), G_q(n)) \subset E_q(n)$ ((H, H') désignant le groupe engendré par les commutateurs $s^{-1}t^{-1}st$ où $s \in H$, $t \in H'$). De là on déduit d'abord que pour $n \geqq 3$, l'homomorphisme $S: C(n-1) \to C(n)$ est surjectif et que $C(n)$ est contenu dans le centre de $\hat{G}(n)$; il suffit donc de prouver que $C(3)=1$, et cela résultera de ce que le sous-groupe $(S(\hat{G}(2)), C(3))$ est dense dans le sous-groupe fermé $C(3)$ de $\hat{G}(3)$. Par dualité, cela revient à prouver que $H^1(C(2), I)^{A(2)} = 0$, où $I = \mathbf{Q}/\mathbf{Z}$. Or, on a la suite exacte de Hochschild-Serre

$$0 \to H^1(A(2), I) \to H^1(\hat{G}(2), I) \to H^1(C(2), I)^{A(2)} \to H^2(A(2), I)$$

et le reste de la démonstration consiste à prouver que: (1) $H^2(A(2), I)=0$; (2) les deux groupes $H^1(A(2), I)$ et $H^1(\hat{G}(2), I)$ sont des groupes cycliques isomorphes, d'ordre 12. Le second point est assez facile. Pour prouver le premier, on utilise le fait que $A(2)$ est isomorphe au produit des groupes p-adiques $G_p = \mathbf{SL}(2, \mathbf{Z}_p)$; on est ramené à montrer que pour chaque p, on a $H^2(U_p, I_p)=0$, où U_p est un p-groupe de Sylow du groupe des commutateurs de G_p, et $I_p = \mathbf{Q}_p/\mathbf{Z}_p$. Cela résulte de profonds résultats de M. Lazard sur les "pro-p-groupes", non encore publiés. *J. Dieudonné* (Paris)

Kondo, Takeshi 5118

On Gaussian sums attached to the general linear groups over finite fields.

J. Math. Soc. Japan **15** (1963), 244–255.

Gaussian sums $W(\xi, A)$ are attached to every irreducible (complex) representation ξ of the general linear group $\mathrm{GL}(n, q)$ over the finite field $F(q)$ of characteristic p. If t is the trace in $F(q^d)|F(p)$ of the element α of $F(q^d)$ and $e_d[\alpha] = \exp(2\pi it/p)$, then for every A in the total matric ring $M_n(F_q)$ and X ranging over $G(n, q)$ the author defines $W(\xi, A) = \sum_X \xi(X)e_1[\mathrm{tr}(AX)]$. Clearly, for $A \in \mathrm{GL}(n, q)$, $W(\xi, A) = \xi(A)^{-1}w(\xi)\xi(1_n)$, since $W(\xi, 1_n)$ is a scalar matrix. The purpose of this paper is to determine the complex numbers $w(\xi)$. In J. A. Green's paper [Trans. Amer. Math. Soc. **80** (1955), 402–447; MR **17**, 345] the canonical form of a matrix over $\mathrm{GL}(n, q)$ is described by a symbol $(\cdots f^{\nu(f)} \cdots)$ which assigns to each irreducible monic polynomial f of degree d over $F(q)$ (and without 0 roots) a certain partition $\nu(f)$ of the exponent $m = |\nu(f)|$ of f as a factor of the characteristic polynomial. Each class and each irreducible representation ξ of $\mathrm{GL}(n, q)$ is in 1-1 correspondence with such a symbol. Built from certain sums of products of symmetric functions of roots of unity is a character $\pm B^\rho(h)$ from which both the Gaussian sums and character values are derived.

J. S. Frame (E. Lansing, Mich.)

Dixon, John D. 5119

Normal Sylow subgroups of linear groups.

J. Algebra **1** (1964), 70–72.

Recently Feit and Thompson [Pacific J. Math. **11** (1961), 1257–1262; MR **24** #A3207] improved a classical theorem of Blichfeldt as follows: Let G be a finite matric group of degree n over the field of complex numbers. Then a Sylow p-subgroup of G is normal in G whenever $p > 2n+1$.

In the present paper the author proves a partial improvement of the theorem of Feit and Thompson: Let G be an irreducible finite matric group of degree n over the field of complex numbers such that G contains an abelian normal subgroup which is not contained in the center of G. Then a Sylow p-subgroup of G is normal in G whenever $p > n+1$. *N. Ito* (Nagoya)

Dixon, John D. 5120

Complete reducibility of infinite groups.

Canad. J. Math. **16** (1964), 267–274.

Let F be an algebraically closed field, and denote by $\mathfrak{R}$ the class of groups that have a faithful completely reducible representation of finite degree over F. All groups considered are "non-modular", which is defined to mean that no finite factor group has order divisible by the characteristic of F. It is shown (Theorem 1) that a non-modular soluble group of matrices over F is completely reducible if and only if every matrix in it is diagonalizable {the author's statement of this theorem is slightly different and misleading}. This has also been proved, under very similar assumptions, by D. A. Suprunenko and R. I. Tyškevič [Dokl. Akad. Nauk BSSR **4** (1960), 137–139; MR **24** #A1956; Izv. Akad. Nauk SSSR Ser. Mat. **24** (1960), 787–806; MR **25** #124] and D. A. Suprunenko [Dokl. Akad. Nauk BSSR **5** (1961), 321–323; MR **25** #125]. A non-modular soluble group is in $\mathfrak{R}$ if and only if it is a finite extension of an abelian group in $\mathfrak{R}$ (Theorem 2), that is, of a subgroup of a finite direct power of the multiplicative group of F. If a nilpotent group is in $\mathfrak{R}$, then its centre has finite index, bounded by a function of the degree of a faithful irreducible representation and the nilpotency class (Theorem 3a), and hence [J. Wiegold, Proc. Roy. Soc. London Ser. A **238** (1957), 389–401; MR **18**, 716] the order of its derived group is bounded by another such function (Theorem 3b); this is also known [D. A. Suprunenko, *Soluble and nilpotent linear groups*, Amer. Math. Soc., Providence, R.I., 1963; MR **27** #4861b], but the author's proof is quite short {and could have been further shortened by using Wiegold's result referred to above}. A non-modular nilpotent group is in $\mathfrak{R}$ if and only if its centre has finite index and the factor group of its derived group is in $\mathfrak{R}$ (Theorem 4). Three examples are added to show that the assumptions of the theorems cannot be relaxed with impunity.

B. H. Neumann (Canberra)

Menon, P. Kesava [Kesava Menon, P.] 5121

On certain sums connected with Galois fields and their applications to difference sets.

Math. Ann. **154** (1964), 341–364.

Let F be the additive group and G the multiplicative group of $GF(p^{sd})$. A detailed study of the characters of F and of the group ring of F over the field of pth roots of unity leads the author to the following theorem: "If G_m is the subgroup of G formed by its mth powers, where $m = p^d - 1$, then the set of elements $a \in G_m$ such that all solutions of $y^m = a$ can be expressed in the form $y = X - X^p$,

$X \in G$, is a difference set in G_m." The parameters of this difference set are

$$v = \frac{(p^{sd}-1)}{(p^d-1)}, \quad k = \frac{p^{(s-1)d}-1}{(p^d-1)}, \quad \lambda = \frac{(p^{(s-2)d}-1)}{(p^d-1)}.$$

Solutions with the same parameters can be obtained by the well-known method of James Singer from the hyperplanes of a $PG(s-1, p^d)$. It is not clear if these solutions are different from those of the author. At any rate the author's theorem and his study of the group ring of F over the field of pth roots of unity are of interest in themselves. The author also establishes relations between the characters of F and those of G.

H. B. Mann (Columbus, Ohio)

O'Reilly, M. F. 5122

On the modular representation algebra of metacyclic groups.

J. London Math. Soc. **39** (1964), 267–276.

Let KG denote the group algebra of a finite group G over a field of non-zero characteristic p. Consider KG-modules which are finite-dimensional over K, and let $\{M\}$ denote the isomorphism class of the KG-module M. Define addition and multiplication of such symbols by the rules:

$$\{M\}+\{M'\} = \{M \oplus M'\}, \quad \{M\}\cdot\{M'\} = \{M \otimes M'\}.$$

Here, $M \otimes M'$ is the KG-module $M \otimes_K M'$, with the action of G given by $g(m \otimes m') = gm \otimes gm'$, $g \in G$.

J. A. Green [Illinois J. Math. **6** (1962), 607–619; MR **25** #5106] has introduced the modular representation algebra $A(K, G)$, defined as the algebra over the complex field whose generators are the symbols $\{M\}$ ranging over all isomorphism classes of KG-modules, and with addition and multiplication defined as above. He proved that $A(K, G)$ is semisimple if either p does not divide the order of G, or if G is a cyclic p-group.

The present paper generalizes Green's results by proving that $A(K, G)$ is semisimple whenever G has a normal cyclic p-Sylow subgroup P such that the factor group G/P is also cyclic. As is well known, if G has cyclic p-Sylow subgroup, the number of non-isomorphic indecomposable KG-modules is finite. Thus, $A(K, G)$ is a finite-dimensional algebra in this case.

In order to prove the semisimplicity of $A(K, G)$, it suffices to show that for each non-zero element $x \in A(K, G)$, there exists an algebra homomorphism ϕ of $A(K, G)$ into the complex field, such that $\phi(x) \neq 0$. This is established by explicitly giving a full set of indecomposable KG-modules and computing their tensor products. Using the relations thus obtained, the existence of the desired homomorphisms is not too difficult to prove.

It should be mentioned that in a paper to appear in the Illinois J. Math., the author has considerably strengthened the above result. Indeed, he has shown that $A(K, G)$ is semisimple provided only that G has a cyclic p-Sylow subgroup.

I. Reiner (Urbana, Ill.)

Dade, E. C. 5123

Integral systems of imprimitivity.

Math. Ann. **154** (1964), 383–386.

The author considers a finite group G of $n \times n$ matrices as a group of automorphisms of a lattice (i.e., a finite module with respect to the rational integers) of rank n. He calls a decomposition $L = L_1 + \cdots + L_t$ into a direct sum of non-zero sublattices an integral system of imprimitivity for G. If G is a maximal group of automorphisms of L, then a maximal decomposition is determined in a unique way. Let M be the normal subgroup operating trivially on $L/2L$; this is an elementary abelian 2-group. The L_t correspond to the characters χ_t of M in such a way that $\mu l = \chi_t(\mu) l$ for all $\mu \in M$, $l \in L_t$. It is easy to build up the whole group G from the subgroups G_t which leave the L_t fixed. This decomposition was first considered by Minkowski [*Gesammelte Abhandlungen*, Bd. I, p. 203, Teubner, Leipzig, 1911]. If a positive definite Hermitian metric is invariant under G, the L_t are mutually perpendicular and the decomposition coincides with that given by the reviewer [Math. Ann. **125** (1952), 51–55; MR **14**, 851] and M. Kneser [ibid. **127** (1954), 105–106; MR **15**, 780].

M. Eichler (Basel)

Conrad, Paul 5124

Skew tensor products and groups of class two.

Nagoya Math. J. **23** (1963), 15–51.

In this paper the author continues his study of groups of class 2 [Illinois J. Math. **5** (1961), 212–224; MR **26** #5061]. Here he introduces a certain factor group (which is analogous to the exterior algebra) of the tensor product which he calls the skew tensor product. Using the skew tensor product, he obtains a representation of a large class of groups of class 2 which contains every complete group of class 2. He solves the isomorphism problem for this representation, and then uses the representation to get further insight into complete groups of class 2.

M. F. Newman (Canberra)

Chehata, C. G. 5125

Existence theorem for groups.

Proc. Edinburgh Math. Soc. (2) **13** (1962/63), 291–294.

Two different sets of sufficient conditions are given for the existence of a group G^* containing a given group G and inner automorphisms that continue given partial automorphisms (that is, isomorphisms between subgroups) of G in such a way that a prescribed automorphism commutes with all the others. This uses and extends the author's results in Proc. Glasgow Math. Assoc. **1** (1953), 170–181 [MR **15**, 396].

B. H. Neumann (Canberra)

McFadden, R. 5126

Congruence relations on residuated semigroups. II.

J. London Math. Soc. **39** (1964), 150–158.

The author continues his theory of M-congruences of residuated semigroups [same J. **37** (1962), 242–248; MR **25** #5118]. He extends the main result to a residuated semigroup S which need not be a semilattice and which need not contain an identity element. He proves that if $\bigcup X$ exists for the subset X of S, then, for any element a of S,

$$a(\bigcup X) = \bigcup ax, \quad a.\cdot(\bigcup X) = \bigcap (a.\cdot x), \quad x \in X.$$

Let $\mathscr{R}$ be a congruence of S such that each $\mathscr{R}$-class of S contains a greatest element (M-congruence). If every element of S is both a left and a right residual of itself, then

$$\mathscr{R} = \bigcap A_t = \bigcap {}_tA, \quad t \in T,$$

where T is the set of elements t of S which is maximal in its $\mathscr{R}$-classes, and $a\,A_t\,b$ if $t.\cdot a = t.\cdot b$, and $a\,{}_tA\,b$ if $t\cdot.a = t\cdot.b$.

Let S be a commutative residuated semigroup with identity e. G has no proper M-congruences if and only if G is either a partially ordered group or is isomorphic to $\{0, 1\}$, where $0^2 = 01 = 10 = 0$, $1^2 = 1$ and $0 < 1$. In this case, S is integrally closed, i.e., $a \cdot . a = a . \cdot a = e$ for every $a \in S$. In particular, if S is a semilattice semigroup (gerbier), then S is isomorphic to $\{0, 1\}$ as described above or to a subgroup of the multiplicative group R^+ of the positive real numbers.

The interval topology of S is that defined by taking the closed intervals $\{x; a \leqq x \leqq b\}$ of S as a sub-base of closed sets. Let G be a sub-gerbier, with identity e, of the direct product $\prod G_i$, $i \in I$, where each G_i is isomorphic either to $\{0, 1\}$ or to a subgroup of R^+. If there exist members r and s of I such that G_r and G_s are groups, and an element b of G such that $b_r > e_r$ and $b_s < e_s$, then G is not Hausdorff in its interval topology.

E.-A. Behrens (Frankfurt a. M.)

Bastida, Julio Rafael 5127

Groups and homomorphisms associated with a semigroup. I. (Spanish)

Bol. Soc. Mat. Mexicana (2) 8 (1963), 26–45.

In 1951 J. A. Green [Ann. of Math. (2) **54** (1951), 163–172; MR **13**, 100] introduced the $\mathscr{H}$, $\mathscr{L}$, $\mathscr{R}$, $\mathscr{F}$, and $\mathscr{D}$ equivalences in an abstract semigroup. It was shown, among other things, that an $\mathscr{H}$-class H is a group if it contains the product of two of its elements. Later, in 1957, M. P. Schützenberger [C. R. Acad. Sci. Paris **244** (1957), 1994–1996; MR **19**, 249] showed that associated with any $\mathscr{H}$-class H there is a group $\Gamma(H)$ which is a simply transitive group of permutations of H. (Actually, Schützenberger dealt with an $\mathscr{H}$-class of elementary type, as it were. The restriction was later removed.) Using these groups, Schützenberger developed his representation theory. Letting $A \cdot B = \{x | Bx \subseteq A\}$ one may describe $\Gamma(H)$, having adjoined an identity, as $H \cdot H / \mathscr{S}_h$, where $x \equiv y(\mathscr{S}_h) \leftrightarrows hx = hy$ for any fixed $h \in H$. If one desires, the group of a degenerate $\mathscr{H}$-class could simply be defined as the trivial group and $\Gamma(H)$ still taken as above.

Since their introduction, these equivalences and groups have been of singular importance in both abstract and topological semigroups. The Green equivalences are not, in general, congruences. This fact has been the source of considerable difficulty in both theories, particularly the latter. Indeed, there exists the situation, even in a compact connected semigroup with identity where a translate of an $\mathscr{H}$-class properly contains another $\mathscr{H}$-class. This failure of the $\mathscr{H}$-class to behave as a congruence is one of the motivations of the present article. First of all, to make the paper self-contained, the author recalls the necessary preliminary notions such as the Green lemmas. He then shows that $H \cdot H / \mathscr{S}$ is a group if $H \cdot H$ is non-degenerate, no identity being assumed, and takes the second of the above approaches to $\Gamma(H)$. Next it is shown that if $x \equiv y\ (\mathscr{L})$, then $H_x \cdot H_x = H_y \cdot H_y$, and x and y define the same Schützenberger equivalence in the sense that $xp = xq$ if and only if $yp = yq$, where $p, q \in H_x \cdot H_x$. Thus, $\Gamma(H_x) = \Gamma(H_y)$.

The main part of the paper concerns the following situation: Let x, y, z be three elements of S such that $xH_y \cap H_z \neq \emptyset$. Define $A[x, y, z] = (H_z \cdot x) \cap H_y$ and $B[x, y, z] = xH_y \cap H_z$. It follows that these sets are non-vacuous and that $xA[x, y, z] = B[x, y, z]$. Now define the sets $C[x, y, z] = A[x, y, z] \cdot A[x, y, z]$ and $D[x, y, z] = B[x, y, z] \cdot B[x, y, z]$. Using an obvious shortening of notation, it is next shown that $C \subseteq D$. If C is not empty, it is a subsemigroup of $H_y \cdot H_y$ and D is a subsemigroup of $H_z \cdot H_z$. (The hypotheses repeatedly invoked about non-emptiness are fulfilled with an identity or non-triviality of the H-sets in question.) Define now the equivalences $\mathscr{R}'$ as $\mathscr{S}_y$ cut down to C and $\mathscr{O}$ as $\mathscr{S}_z$ cut down to D. It follows that $\mathscr{R}'$ and $\mathscr{O}$ are congruences on C and D, respectively, and finally (with the sets non-vacuous), that $C/\mathscr{R}'$ and $D/\mathscr{O}$ are subgroups of the respective Schützenberger groups. Let these groups be denoted, respectively, by $\Gamma_1[x, y, z]$ and $\Gamma_2[x, y, z]$. With an obvious notation one then has a commutative diagram, all homomorphisms being canonical:

$$\begin{array}{ccc} C & \xrightarrow{\delta} & D \\ \downarrow{\scriptstyle\sigma} & \gamma & \downarrow \\ \Gamma_1 & \rightarrow & \Gamma_2. \end{array}$$

Now if $xH_y \subseteq H_z$, then $\Gamma_1[x, y, z] = \Gamma_y$. If $xH_y \supseteq H_z$, then $\Gamma_2[x, y, z] = \Gamma_z$. If x is left-cancellative upon H_y, then γ is an isomorphism. If S has an identity, $xH_y \subseteq H_z$, and γ is an isomorphism, then x is left-cancellative upon H_y.

Next, let φ be a homomorphism of S onto a semigroup S'. Letting φ_1 be the restriction to C, assuming $C \neq \emptyset$, there is an induced homomorphism φ^1 such that

$$\begin{array}{ccc} C[x, y, z] & \xrightarrow{\varphi_1} & C'[x\varphi, y\varphi, z\varphi] \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ \Gamma_1[x, y, z] & \xrightarrow{\varphi^1} & \Gamma_1'[x\varphi, y\varphi, z\varphi]. \end{array}$$

If S has an identity and φ_2 is the restriction to D, then there is a diagram

$$\begin{array}{ccc} D[x, y, z] & \xrightarrow{\varphi_2} & D'[x\varphi, y\varphi, z\varphi] \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau'} \\ \Gamma_2[x, y, z] & \xrightarrow{\varphi^2} & \Gamma_2'[x\varphi, y\varphi, z\varphi]. \end{array}$$

The rest of the paper is devoted to certain special situations including the commutative case.

R. P. Hunter (University Park, Pa.)

Howie, J. M. 5128

Subsemigroups of amalgamated free products of semigroups. II.

Proc. London Math. Soc. (3) **14** (1964), 537–544.

Terms and definitions used here can be found in the review of Part I [same Proc. (3) **13** (1963), 672–686; MR **27** #3724]. The free product of an amalgam of semigroups S_i with unitary amalgamated subsemigroup U exists and contains each S_i as a unitary subsemigroup; this implies a result proved in Part I. If, for each i, T_i is a unitary subsemigroup of S_i, each intersecting the amalgamation U in the same subsemigroup V of U, the T_i will not in general generate their free product with amalgamation V in the free product of the amalgam of the S_i. This is shown by an example, and a sufficient condition for the T_i to generate their free product with amalgamation V is established.

Hanna Neumann (Canberra)

Bosák, Juraj 5129

General powers in semigroups. (Slovak. Russian and English summaries)

Mat.-Fyz. Časopis Sloven. Akad. Vied **13** (1963), 137–146.

Let S be a semigroup and Q an algebraic system with two binary operations + and $\cdot$. The question the author deals with is to define, in S, powers a^m ($a \in S$) with exponents $m \in Q$ satisfying the usual relations: $a^1 = a$, $a^{m+n} = a^m a^n$, $a^{m \cdot n} = (a^n)^m$. It is assumed that Q is a quasiring, i.e., contains a left identity $1 \in Q$ for the multiplication, and the operations + and $\cdot$ verify the right distributive law. The author states some necessary and sufficient conditions for the existence of powers a^m in the following cases: Q is the set of positive rational numbers, the positive part of a number field, the set of integers, and a field. For example, in the semigroup S it is possible to define powers with integer exponents if and only if S is the union of pairwise disjoint groups. *O. Borůvka* (Brno)

Rédei, Ladislaus 5130

★Theorie der endlich erzeugbaren kommutativen Halbgruppen.

Hamburger Mathematische Einzelschriften, Heft 41. *Physica-Verlag*, *Würzburg*, 1963. 228 *pp*. *DM* 35.00.

Each commutative semigroup generated by n elements is the factor semigroup of the free semigroup with n generators $\varepsilon_1, \cdots, \varepsilon_n$ modulo a suitable congruence relation C. Let R be the affine space over the real numbers with base points associated with $\varepsilon_1, \cdots, \varepsilon_n$. Then the elements of F are representable by the points of R with natural numbers as coordinates and F is imbedded in the module (additively written abelian group) F^0 of the points $\alpha = (a_1, \cdots, a_n)$ with integers a_i as coordinates. F^0 is a lattice if one defines the meet of α and β by $\inf(\alpha, \beta) = (\operatorname{Min}(a_i, b_i); i = 1, \cdots, n)$ and the join by $\sup(\alpha, \beta) = (\operatorname{Max}(a_i, b_i); i = 1, \cdots, n)$.

For a congruence relation C in F the set

$$M_C = \{\alpha - \beta; \alpha C \beta\}$$

is a submodule of F^0, but contrary to the theory of groups, M_C does not determine C completely. Therefore the author supplies M_C by his definition of the kernel function f_C, defined on M_C, with ideals in the semigroup F as values, such that

$$(1) \qquad \alpha C \beta \in \inf(\alpha, \beta) \in f(\alpha - \beta).$$

Such functions are characterized by the following five axioms: (1) f is defined on a submodule M of F^0; (2) $f(\mu)$ is an ideal in F for each $\mu \in M$; (3) $f(0) = F$; (4) $f(-\mu) = f(\mu)$; (5) an axiom equivalent to the transitivity of the relation C_f, defined by (1), namely,

$$(2) \qquad (\mu^+ + f(\mu)) \cap (\nu^+ + f(\nu)) \subseteq \sup(\mu, \nu) + f(\mu - \nu)$$

with $\mu^+ = \sup(0, \mu)$, etc.

The author successfully develops the theory of finitely generated commutative semigroups S by investigating his kernel functions. An F-ideal in F^0 is a set $\mathfrak{A}$ in F^0 bounded from below in the sense of inf, with the property $\mathfrak{A} + F = \mathfrak{A}$. It has a unique finite base $\min \mathfrak{A} = \{\alpha_1, \cdots, \alpha_k\}$, such that $\mathfrak{A}$ is the set-union of the principal ideals $\alpha_i + F$, $i = 1, \cdots, k$. k is the degree of $\mathfrak{A}$. The lattice of the ideals in F is distributive and the ideals in F satisfy the maximal condition. With the ideal $\mathfrak{A}$ in F is associated the congruence relation $C_{\mathfrak{A}}$, defined by

$$(3) \qquad \rho C_{\mathfrak{A}} \sigma \Leftrightarrow F \setminus (\rho + \mathfrak{A}) = F \setminus (\sigma + \mathfrak{A}).$$

(This seems to be equivalent to $\bar{\rho} \mathfrak{R} \bar{\sigma}$ with $\bar{\rho}, \bar{\sigma}$ in the Rees factor semigroup $F/\mathfrak{A}$, and $\mathfrak{R}$ is the congruence relation $\bar{\rho} + \bar{F} = \bar{\sigma} + \bar{F}$ defined by Green [cf. Clifford and Preston, *The algebraic theory of semigroups*, Vol. I, Amer. Math. Soc., Providence, R.I., 1961; MR **24** #A2627].) The factor semigroup $F/C_{\mathfrak{A}}$ is finite. The congruence relations in F are partially ordered by $C_1 \subseteq C_2$ if $\rho C_1 \sigma \Rightarrow \rho C_2 \sigma$ and the kernel functions are partially ordered by $f_1 \subseteq f_2$ if $C_{f_1} \subseteq C_{f_2}$. For each ideal $\mathfrak{A}$ in F, the congruence relation $C_{\mathfrak{A}}$ is maximal in the set of those congruence relations C in which $\mathfrak{A}$ is a class modulo C.

Although M_C does not determine the relation C completely, there is a principal ideal $\zeta + F$ in F such that $\alpha - \beta \in M_C$ implies $\alpha C \beta$. The union of these principal ideals is called the kernel of f. Therefore the principal ideals in F are of great importance for the theory. The range of f consists of but a finite number o_f of ideals, called the order o_f of f. The degree d_f of f is the maximum of the degrees of the ideals in the range of f. The rank r_f of f is the rank of the module M_f. For a submodule M' of M_f the restriction $f' = f | M'$ is a kernel function defined on M'. Therefore the author begins the treatment of the kernel functions "in the small". He gives the kernel functions of the first degree explicitly and the functions of rank 1 by recursion (for all n). This establishes the description of the kernel functions for $n = 2$.

There are conditions for the finiteness of the factor semigroup F/f, i.e., F/C_f, and for the isomorphism $F/f_1 \cong F/f_2$. Each finitely generated commutative semigroup is finitely definable.

E.-A. Behrens (Frankfurt a. M.)

Konguetsof, Léonidas 5131

Construction d'hypergroupes, à partir d'ensembles munis d'opérations partielles. Définition et construction d'hypergroupes à opérateurs.

C. R. Acad. Sci. Paris **258** (1964), 1961–1964.

Let $(E, \cdot)$ be a multiplicative groupoid and let T be a mapping of a subset $E \times E$ into E. Denote the fact that (x, y) belongs to the domain of T by $x \tau y$. The author investigates those systems $(E, \cdot, \tau)$ for which there exists an e in E such that $e \tau x$, $x \tau e$ and $e \cdot x = x \cdot e = e$ for all x in E. For a and b in such a system define $a * b = \{x \in E : (a \cdot b) \tau x\}$. Theorem 1 states that $(E, *)$ is a hypergroup. As a corollary it follows that an "annoïd" A [see the following review #5132] is a hypergroup with respect to $a * b = \{x \in A : ab + x \text{ is defined}\}$. This hypergroup $(A, *)$ can be regarded as a hypergroup with operators in two rather natural ways. *P. F. Conrad* (New Orleans, La.)

Konguetsof, Léonidas 5132

Exemples d'annoïdes. Intégroïdes. Idéaloïdes. Homomorphismes des annoïdes. Modules et algèbres dans un annoïde.

C. R. Acad. Sci. Paris **258** (1964), 3613–3616.

The author continues to announce results about "annoïds" [same C. R. **257** (1963), 21–24; MR **27** #4841; ibid. **257** (1963), 2377–2380; MR **27** #5800]. The title is a good indication of the nature of the eleven theorems and eight propositions that are stated. Two of the examples are as follows. A multiplicative semigroup with a zero is an annoïd if one defines $x + 0 = 0 + x = x$ and $x + x = 0$ for all elements x in the semigroup. The multiplicative semigroup of homogeneous polynomials in n indeterminants over a ring R is an annoïd if addition is restricted to polynomials of the same degree. *P. F. Conrad* (New Orleans, La.)

Tamura, T.; Burnell, D. G. 5133
Extension of groupoids with operators.
Amer. Math. Monthly **71** (1964), 385–391.

Let G (elements $x, y, \cdots$) be a groupoid (i.e., a system in which a closed binary operation $+$ is defined) and suppose that Γ (elements $\alpha, \beta, \cdots$) is a commutative semigroup (multiplicatively written). Let $(\alpha, x)\to\alpha x$ be a mapping of the product set $\Gamma\times G$ into G, such that $x\to\alpha x$ is a 1-1 endomorphism of G satisfying $(\alpha\beta)x=\alpha(\beta x)=(\beta\alpha)x$. {Implicitly it is also assumed that α is uniquely determined by the endomorphism $x\to\alpha x$.} Then the pair G, Γ is said to be a groupoid G with operator semigroup Γ (see also the same authors [Proc. Japan Acad. **38** (1962), 495–498; MR **27** #228]) and is denoted by (G, Γ). The main theorem (Theorem 1) of this paper states: For (G, Γ), there exists $(\bar{G}, \bar{\Gamma})$ having the following properties: (1) G is embedded into $\bar{G}$, (2) Γ and $\bar{\Gamma}$ are isomorphic, (3) each $\bar{\alpha}\in\bar{\Gamma}$ is an extension of (the mapping) $\alpha\in\Gamma$ to $\bar{G}$, (4) $\bar{\alpha}$ is an automorphism of $\bar{G}$, and (5) $(\bar{G}, \bar{\Gamma})$ is the smallest extension of (G, Γ) in a certain natural sense. From Theorem 1 the authors deduce another theorem which was announced recently [loc. cit.]. Applications and examples are given, for instance, G a commutative semigroup, Γ a semigroup of endomorphisms n corresponding to positive integers n such that $n\cdot x=x+\cdots+x$ (n times). {It is claimed that Γ is "isomorphic to" a subsemigroup of the multiplicative semigroup of positive integers, but this should read "a homomorphic image of" (communication of the first author).} *H.-J. Hoehnke* (Berlin)

Radó, F.; Hosszú, M. 5134
Über eine Klasse von ternären Quasigruppen.
Acta Math. Acad. Sci. Hungar. **15** (1964), 29–36.

This paper is a sequel to F. Radó's earlier work [Mathematica (Cluj) **2 (25)** (1960), 325–334; MR **23** #A3771] about the conditions under which a ternary loop coincides with all its principal loop-isotopes that have the same identity element e. Let the ternary loop product of the elements a, b and c be denoted (a, b, c) and let three binary group operations be defined by $x\circ y=(x, e, y)$, $x\nabla y=(x, y, e)$ and $x*y=(e, x, y)$. Then all the principal isotopes with identity e coincide if and only if $(x, y, z)=(x\nabla y)\circ y^{-1}\circ(y*z)=(x\nabla y)*x^{-1}*(x\circ z)=(x\circ z)\nabla z^{-1}\nabla(y*z)$. If the ternary loop Q is continuously differentiable and defined over a real interval, and if again all principal loop-isotopes with the same e coincide, then Q is either a commutative ternary group, or its operation is of the form

$$(x, y, z)=\phi_{-1}\left[\frac{\phi(y)+k\dfrac{\phi(x)-1}{\phi(x)+k}}{\dfrac{1}{\phi(z)}-\dfrac{\phi(x)-1}{\phi(x)+k}}\right].$$

Here ϕ is an arbitrary, invertible, continuously differentiable mapping of Q on a real interval containing the unit point, and k is a real constant distinct from -1. If, on the other hand, local loops with continuous operations over a real interval are considered, such that the operations are defined only in the neighborhood of e, then the required loops are the loop-isotopes of the local quasigroups with the operations of the type

$$(x, y, z)=\frac{x+y+z}{ax+by+cz}.$$

R. Artzy (New Brunswick, N.J.)

McWorter, William A. 5135
On a theorem of Mann.
Amer. Math. Monthly **71** (1964), 285–286.

Generalizing a group-theoretic result of H. B. Mann [Ann. of Math. (2) **43** (1942), 523–527; MR **4**, 35], the author proves the following. Let $(Q, \cdot)$ be a quasigroup and let A and B be two subsets of Q such that not every element of Q has the form ab with $a\in A$ and $b\in B$. Then $|Q|\geqq|A|+|B|$. *R. Artzy* (New Brunswick, N.J.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 5301, 5346, 5405, 5724.

Gładysz, S. 5136
Convex topology in groups and Euclidean spaces.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 1–4.

This paper is complementary to the author's previous paper [Colloq. Math. **9** (1962), 213–221; MR **28** #159]. A convex set in a topological group G is defined to be any set which can be formed as the intersection of left translates of non-empty, closed, maximal semi-groups of G. A group G is said to have a convex topology if, at the identity, there is a base of neighborhoods whose closure is convex. Using results of his previous paper [op. cit.] the author proves the following theorems. Theorem 1: Every topological group with a convex topology is Abelian. Theorem 2: The following two properties are equivalent: (A) G is a locally compact connected topological group with a convex topology; (B) G is isomorphic to a Euclidean space.
F. Hahn (New Haven, Conn.)

Carroll, F. W. 5137
On bounded functions with almost-periodic differences.
Proc. Amer. Math. Soc. **15** (1964), 241–243.

It was proved by R. Doss [same Proc. **12** (1961), 488–489; MR **23** #A3424] that if G is a multiplicative group, if f is a bounded complex-valued function on G, and if for each $h\in G$ the left difference $\Delta_h f$ (defined by $(\Delta_h f)(x)=f(hx)-f(x)$) is right almost-periodic, then f itself is right almost-periodic.

A new proof is given for this theorem, using the theorem that to any topological group G we can find a compact group M and a continuous homomorphism of G onto a dense subgroup G_1 of M such that the right almost-periodic functions on G are exactly those functions on G_1 which can be extended to continuous functions on M [see L. H. Loomis, *An introduction to abstract harmonic analysis*, p. 168, Van Nostrand, Toronto, Ont., 1953; MR **14**, 883]. The author takes the discrete topology on G, and uses a theorem of a previous paper [Trans. Amer. Math. Soc. **102** (1962), 284–292; MR **24** #A3438] to the effect that if f_1 is bounded on G_1, and if all the left differences of f_1 are continuous, then f_1 itself is continuous. *N. G. de Bruijn* (Eindhoven)

Čarin, V. S. 5138
On groups of finite rank. I. (Russian. English summary)
Ukrain. Mat. Ž. **16** (1964), 212–219.

The concept finite special rank (f.s.r.) defined for discrete

groups [A. I. Mal'cev, Mat. Sb. (N.S.) **22 (64)** (1948), 351–352; MR **9**, 493] is carried over to topological groups G: G has f.s.r. k if k is minimal such that every finite set of elements of G generates a subgroup whose closure H has a set of k elements generating a dense subgroup of H. The author works with bicompact groups which are either abelian or are p-groups with no elements of finite order and locally nilpotent (every finite set generates a nilpotent subgroup) or locally solvable [N. N. Mjagkova, Izv. Akad. Nauk SSSR Ser. Mat. **13** (1949), 495–512; MR **11**, 321]. The latter, if they have f.s.r., are shown to be nilpotent (respectively, solvable); the abelian groups have f.s.r. if and only if their necessarily discrete character groups do. Much of the paper follows ideas of Gluškov [V. M. Gluškov, ibid. **20** (1956), 513–546; MR **18**, 280].

L. Zippin (New York)

Boseck, Helmut 5139

Quasiinvariante Masse auf lokal-kompakten Gruppen.

Math. Nachr. **26** (1963), 161–166.

A non-trivial regular Borel measure ν in a locally compact topological group G is called left-quasi-invariant if, for every Borel set E, $\nu(E)=0$ implies $\nu(xE)=0$ for all $x \in G$. Right-quasi-invariance is defined similarly. ν is called two-sided quasi-invariant if it is both left- and right-quasi-invariant and in addition $\nu(E)=0$ implies $\nu(E^{-1})=0$. The following statements are shown to be pairwise equivalent: (1) ν is left-quasi-invariant, (2) ν is right-quasi-invariant, (3) ν is two-sided quasi-invariant, (4) ν is equivalent to a Haar measure μ in G, (5) $\nu(E)=\int_E f\,d\mu$, where f is a positive measurable function on G. In a remark added in proof it is stated that the assumption that ν be regular is redundant.

J. C. Oxtoby (Bryn Mawr, Pa.)

Świerczkowski, S. 5140

Topologies in free algebras.

Proc. London Math. Soc. (3) **14** (1964), 566–576.

In the following an algebra $\mathscr{A}$ is a set A together with a family of operations $\varphi_i: A^{n_i}\to A$, $i \in I$, n_i some natural number for each $i \in I$, and I an arbitrary set. The class of algebraic operations is the smallest class of operations obtained from the φ_i and the projections $(a_1, \cdots, a_n)\to a_j$: $A^n\to A$, $1 \leq j \leq n$, $n=1, \cdots$, by composition. A set $X \subset A$ is said to generate $\mathscr{A}$ if A is the smallest subset of A closed under all algebraic compositions. If $X \subset A$ and if for any algebraic operations $f, g: A^n\to A$, for all mappings $\tau: X\to X$ and for all n-tuples $x=(x_1, \cdots, x_n)$, $y=(y_1, \cdots, y_n) \in X$ we have $f(\tau x)=g(\tau x)$ (where $x=(\tau x_1, \cdots, \tau x_n)$), then X will be called weakly independent. If $\mathscr{A}$ is the free algebra generated by X in a primitive class of algebras, then X is indeed weakly independent (for the concept of a primitive class it may suffice here to mention that the class of rings, commutative rings, Boolean algebras, groups, abelian groups form primitive classes; they all are characterized by polynomial identities involving their operations φ_i). The following theorem is the result of the paper: Let $\mathscr{A}$ be an algebra generated by a subset $X \subset A$ which is weakly independent in $\mathscr{A}$. Suppose that on X a topology is given such that the real-valued functions separate points. Then there is a completely regular topology on A such that (i) all algebraic operations are continuous, (ii) X is a closed subset of A, and (iii) the topology induced on X is finer than the given topology, and both topologies coincide if and only if X is completely regular. The well-organized proof (for which we must refer to the paper) starts with the definition of a basis for the open sets on A employing the real-valued functions and the algebraic operations of $\mathscr{A}$; the separation properties are the most difficult ones to be checked in the sequel. The theorem has the following corollary: Let X be a completely regular space. Then in any primitive class of topological algebras (i.e., algebras in which all operations are continuous) there is an algebra $\mathscr{A}$ on a completely regular space A containing the space X as a closed subspace such that $\mathscr{A}$ is generated by X, that $\mathscr{A}$ is free over X in the given class, and that for every algebra $\mathscr{B}$ in the class every continuous mapping $X\to B$ can be extended to a continuous homomorphism $\mathscr{A}\to\mathscr{B}$. The author mentions previous results along these lines which are due to A. I. Mal'cev [Izv. Akad. Nauk SSSR Ser. Mat. **21** (1957), 171–198; MR **20**, 5249] and, in the case of groups, to A. A. Markov [ibid. **8** (1944), 225–232; MR **7**, 7]; different proofs in the case of groups can also be found in M. I. Graev [ibid. **12** (1948), 279–324; MR **10**, 11], S. Kakutani [Proc. Imp. Acad. Tokyo **20** (1944), 595–598; MR **7**, 240] and B. R. Gelbaum [Proc. Amer. Math. Soc. **12** (1961), 737–743; MR **25** #4025].

K. H. Hofmann (New Orleans, La.)

Štern, A. I. 5141

Completely irreducible representations of the real unimodular group of second order. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 798–801.

In this paper the author classifies all completely irreducible representations (c.i.r.) of the real unimodular group $G=SL(2, R)$ following the method of M. A. Naĭmark as developed in his book [*Linear representations of the Lorentz group* (Russian), see § 15, pp. 199–253, Fizmatgiz, Moscow, 1958; MR **21** #4995] (and applied by him for the similar problem in the case of the complex unimodular group $SL(2; C)$).

Denote by X the group algebra of G, i.e., the space of all indefinitely differentiable functions of compact support on G with convolution as product. Let X_n be the subalgebra of X defined by

$$X_n = \{f \in X : f(u^{-1}gu_1) = C^n(u)\overline{C^n(u_1)}\,f(g)\}, \quad u, u_1 \in U,$$

where $C^n(u_\varphi)=e^{in\varphi}$ if $u=u_\varphi$ is the rotation through φ and U is the maximal compact subgroup of rotations of G. Every representation $g\to T_g$ of G on a Banach space $\mathfrak{H}$ gives rise to a representation of the group algebra $X: f\to T_f=\int_G f(g)T_g\,dg$, $f \in X$. Let $M_n=\{f \in \mathfrak{H}: E^n\xi=\xi\}$ with $E^n=(2\pi)^{-1}\int_U \overline{C^n(u)}T_u\,du$. If $g\to T_g$ is a c.i.r., one obtains a c.i.r. of X_n in the spaces $M_n\neq(0)$; this is one-dimensional since X_n turn out to be commutative. The problem thus depends on finding the form of linear multiplicative functionals in X_n. For this purpose the author proves (Theorem 1) the Paley-Wiener theorem for G, which was also obtained earlier by L. Ehrenpreis and F. Mautner [Trans. Amer. Math. Soc. **84** (1957), 1–55, see Theorems 2.1 and 3.1; MR **18**, 745].

The central result of the paper is Theorem 2: Every c.i.r. of G is defined by a pair of numbers (ρ, j), where ρ is complex, and $j=0, \frac{1}{2}$, and $\rho\neq i(2n+2j+1)$, $n=0, \pm 1, \cdots$. Here (ρ, j) and $(\rho, -j)$ determine one and the same c.i.r., namely $T^{\rho,j}$ in the L^2-space of the maximal nilpotent

subgroup (construction of Gel'fand-Neimark). The pairs $(i(2n+2j+1), j)$ and $(-i(2n+2j+1), j)$, $n=0, \pm 1, \pm 2, \dots$ with $2n+2j+1 \neq 0$ determine three c.i.r., two unitary $D^+_{(|\rho|+1)/2}$ and $D^-_{(|\rho|+1)/2}$ (discrete series of V. Bargmann) and one finite-dimensional representation of dimension $|\rho| = |2n+2j+1|$. The pair $(0, \frac{1}{2})$ determines two inequivalent c.i.r. which are unitary ($D^+_{1/2}$, $D^-_{1/2}$ of V. Bargmann). For irreducible unitary representations of G, see V. Bargmann [Ann. of Math. (2) **48** (1947), 568–640; MR **9**, 133].   *T. S. Bhanu Murthy* (Bangalore)

Yamaguchi, Satoru 5142

On certain zonal spherical functions.

Mem. Fac. Sci. Kyushu Univ. Ser. A **17** (1963), 131–144.

The author considers the hermitian hyperbolic spaces M imbedded as bounded domains in a complex n-space. In the cases $n=1$, $n=2$, formulas for some zonal spherical functions on M are given in terms of such an imbedding. In the case $n=1$ they are expressed in a well-known fashion by means of Legendre functions.

S. Helgason (Cambridge, Mass.)

Berman, D. L. 5143

The theory of linear polynomial operations on topological groups. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 17–19.

The author extends his work on trigonometric polynomials to characters on compact commutative groups.

T. W. Gamelin (Cambridge, Mass.)

Harish-Chandra 5144

Invariant distributions on Lie algebras.

Amer. J. Math. **86** (1964), 271–309.

Dans ce travail, l'auteur commence la longue démonstration du théorème qu'il a annoncé récemment [Bull. Amer. Math. Soc. **69** (1963), 117–123; MR **26** #2545] suivant lequel tout caractère irréductible d'un groupe de Lie semi-simple G, qui, d'après un théorème antérieur de l'auteur, est toujours une distribution, en fait est une fonction localement sommable. La méthode d'attaque consiste à étudier d'abord sur l'algèbre de Lie $\mathfrak{g}$ de G, les distributions (définies sur un ouvert Ω de $\mathfrak{g}$) qui sont invariantes par l'action infinitésimale de $\mathfrak{g}$ (correspondant aux translations de G), et il s'agit de prouver que certaines de ces distributions, satisfaisant à des équations aux dérivées partielles d'un type particulier, sont en fait des fonctions sommables. Ce résultat n'est pas encore démontré dans ce premier mémoire, qui se borne à établir le résultat intermédiaire suivant, qui joue le rôle principal dans le théorème précité. Désignons par ω la forme de Killing sur $\mathfrak{g}$, identifiée à un élément de l'algèbre symétrique de la complexifiée $\mathfrak{g}_c$ de $\mathfrak{g}$; pour tout élément p de cette algèbre symétrique, soit $\partial(p)$ l'opérateur différentiel sur $\mathfrak{g}$ qu'il définit canoniquement; alors, si T est une distribution invariante sur Ω vérifiant une équation de la forme $\sum c_k \partial(\omega^k)T=0$ (où les c_k sont des nombres complexes non tous nuls), et si de plus le support de T est contenu dans $N \cap \Omega$, où N est l'ensemble des $X \in \mathfrak{g}_c$ tels que $\mathrm{ad}(X)$ soit nilpotent, on a nécessairement $T=0$. On peut se ramener au cas où l'équation satisfaite par T est $\partial(\omega)T=0$.

L'idée de la démonstration peut se résumer en les étapes suivantes: (I) En vertu de résultats de Kostant et A. Borel, N est réunion d'un nombre fini d'orbites de G, et si N_q est la réunion de celles de ces orbites qui sont de dimension $\leq q$, N_q est fermé dans N, et pour tout $X_0 \in N_q - N_{q-1}$, l'orbite de X_0 est ouverte dans N_q. Il suffit alors de prouver que si la distribution T vérifie en outre la condition $\mathrm{Supp}(T) \subset N_q \cap \Omega$, alors on a $\mathrm{Supp}(T) \subset N_{q-1} \cap \Omega$. (II) Du fait que ω s'annule identiquement dans N, on déduit que pour tout $X_0 \in N$, il existe un entier $k \geq 0$ tel que $\omega^k T=0$ dans un voisinage de X_0 dans Ω. On fixe alors un $X_0 \in N$, on désigne par q (supposé ≥ 1) la dimension de son orbite et on suppose que $\mathrm{Supp}(T) \subset N_q \cap \Omega$; il suffit de prouver, par récurrence sur k, que la relation $\omega^k T=0$ entraîne que $T=0$ dans un voisinage de X_0. (III) A l'aide du théorème de Jacobson-Morosow, on plonge l'élément nilpotent X_0 dans une sous-algèbre semi-simple $\mathfrak{l}$ de dimension 3, et on se ramène au cas où le centralisateur de $\mathfrak{l}$ dans $\mathfrak{g}$ est réduit à 0. (IV) Soit D le champ de vecteurs sur $\mathfrak{g}$ tel que $D_X=\partial(X)$ pour tout $X \in \mathfrak{g}$; si $T_0=(D+n/2+k-1)T$, on prouve que $\partial(\omega)T_0=0$ et $\omega^{k-1}T_0=0$, donc, par récurrence, $T_0=0$ dans un voisinage de X_0. Il reste à en déduire que $T=0$ dans un voisinage de X_0, ce qui se fait à l'aide d'une étude détaillée de l'opérateur D en relation avec la représentation adjointe de $\mathfrak{l}$ dans $\mathfrak{g}$.   *J. Dieudonné* (Paris)

Hermann, Robert 5145

Compactifications of homogeneous spaces and contractions of Lie groups.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 456–461.

Let G be a connected Lie group and H a closed subgroup. Let ρ be a representation of G on a vector space V, and let W be a subspace of V such that $H=\{g \in G : \rho(g)W \subset W\}$. If $p=\dim W$ and $G^p(V)$ is the Grassmann manifold of p-dimensional subspaces of V, ρ defines in a natural way an equivariant immersion of G/H into the compact space $G^p(V)$. It is stated that if $X \in \mathfrak{g}$, the Lie algebra of G, such that the eigenvalues of $\rho(X)$ are real and simple, then X, considered as a vector field on $G^p(V)$, is the gradient of a smooth function on $G^p(V)$. Some applications to compactifications of symmetric spaces are stated, with emphasis on behaviour of geodesics near the boundary. The definition of contraction of Lie groups is reformulated in terms of the above compactification; application to asymptotic behaviour of spherical functions on symmetric spaces is indicated.

S. Helgason (Cambridge, Mass.)

Moore, Calvin C. 5146

Compactifications of symmetric spaces.

Amer. J. Math. **86** (1964), 201–218.

Let G be a connected semi-simple Lie group with finite center. According to Furstenberg [Ann. of Math. (2) **77** (1963), 335–386; MR **26** #3820], a closed subgroup H of G is called a "boundary subgroup" if G/H is compact and if, for any probability measure μ (i.e., non-negative regular Borel measure with total measure one), we can find a sequence $g_n \in G$ such that $g_n\mu$ tends to a point mass in the weak * topology (this means that there exists a point x_0 in G/H such that we have

$$\int_{G/H} f(x)\, dg_n\mu = \int_{G/H} f(g_n x)\, d\mu \to f(x_0)$$

for all continuous functions f on G/H). The first half of the paper is devoted to an explicit determination of all boundary subgroups of G. Let $\mathfrak{g}_0$ be the Lie algebra of G with an Iwasawa decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{h}^- + \mathfrak{n}_0$, and let F be the restricted fundamental system of $\mathfrak{g}_0$ relative to $\mathfrak{h}^-$ such that $\mathfrak{n}_0$ is generated by eigen-vectors corresponding to positive restricted roots. For a subset E of F, denote by $\mathfrak{g}_0(E)$ the corresponding semi-simple subalgebra of $\mathfrak{g}_0$ and by $\mathfrak{n}_0(E)$ the subalgebra of $\mathfrak{n}_0$ generated by eigen-vectors corresponding to positive restricted roots which are not linear combinations of the fundamental restricted roots in E; then the normalizer $\mathfrak{b}_0(E)$ of $\mathfrak{n}_0(E)$ (in $\mathfrak{g}_0$) is of the form $\mathfrak{b}_0(E) = \mathfrak{g}_0(E) + \mathfrak{l}_0(E)$, where $\mathfrak{l}_0(E)$ is the radical of $\mathfrak{b}_0(E)$ and $\mathfrak{n}_0(E)$ is the derived algebra of $\mathfrak{l}_0(E)$. Let $B(E)$ be the normalizer of $\mathfrak{n}_0(E)$ in G. Then, the author proves that all boundary subgroups (or, what amounts to the same, all closed subgroups containing a minimal boundary subgroup) are conjugate to $B(E)$ ($E \subset F$). Thus, for a semi-simple linear algebraic group, the boundary subgroups are nothing else than the "parabolic subgroups" in the sense of Borel-Godement [R. Godement, Séminaire Bourbaki, 1960/61, 2ième éd., corrigée, Fasc. 1, Exp. 206, Secrétariat mathématique, Paris, 1961; MR **27** #1339; A. Borel, Colloq. Théorie Groupes Algébriques (Bruxelles, 1962), pp. 23–40, Librairie Universitaire, Louvain, 1962; MR **26** #6173]. The second half of the paper deals with the compactification of the symmetric Riemannian space G/K (K being a maximal compact subgroup of G) introduced by Furstenberg [loc. cit.]. Namely, for a boundary subgroup $B(E)$, the homogeneous space $G/B(E)$ carries a unique probability measure μ invariant under K, so that, by means of the map $g \in G \to g\mu$, one can define an equivariant imbedding ϕ_E of G/K into the space of all probability measures on $G/B(E)$. Since the latter space is compact, one can obtain a compactification of G/K by taking the closure $\overline{\phi_E(G/K)}$ of $\phi_E(G/K)$. It is shown that the mapping ϕ_E is injective if and only if E does not contain any component of F. Then, analyzing the structure of $\overline{\phi_E(G/K)}$ in detail for a subset E satisfying this condition, the author finally shows that these compactifications coincide with those introduced earlier by the reviewer [Ann. of Math. (2) **71** (1960), 77–110; MR **22** #9546]. *I. Satake* (Chicago, Ill.)

Moore, Calvin C. 5147

Compactifications of symmetric spaces. II. The Cartan domains.

Amer. J. Math. **86** (1964), 358–378.

This is a continuation of the author's investigation on Furstenberg's compactification of a symmetric Riemannian space of non-compact type G/K [#5146 above; see also Furstenberg, Ann. of Math. (2) **77** (1963), 335–386; MR **26** #3820], and concerns the case where G/K is Hermitian (Cartan's symmetric domain). Such a space has a canonical realization as a bounded domain $\mathscr{D}$ in a complex Euclidean space (Harish-Chandra's imbedding), so that, by taking its closure $\overline{\mathscr{D}}$ in the Euclidean space, one obtains a natural compactification of it. Now the main results of the paper are the following: (1) The restricted fundamental system of G has one of the two forms $\{2\xi_1, \xi_2 - \xi_1, \cdots, \xi_m - \xi_{m-1}\}$ and $\{\xi_1, \xi_2 - \xi_1, \cdots, \xi_m - \xi_{m-1}\}$; (2) The natural compactification $\overline{\mathscr{D}}$ is homeomorphic as a G-space to the compactification $\overline{\phi_E(G/K)}$ corresponding to $E = \{\xi_2 - \xi_1, \cdots, \xi_m - \xi_{m-1}\}$ in the sense of Satake and Furstenberg [Satake, same Ann. **71** (1960), 77–110; MR **22** #9546; Furstenberg, loc. cit.]. These results have been conjectured and actually verified case by case, at least for classical domains, by several mathematicians; but here the author gives a systematic and intrinsic proof. In the first part of the proof for (2), depending essentially on the work of Furstenberg [loc. cit.], it is also shown that, by the above homeomorphism, the Bergman-Šilov boundary of $\mathscr{D}$ corresponds to the subset of point-masses, $G/B(E)$, in the compactification $\overline{\phi_E(G/K)}$. The author also clarifies the relations between the notions of (analytic and metric) boundary components of Pjateckiĭ-Šapiro [*Geometry of classical domains and theory of automorphic functions* (Russian), Fizmatgiz, Moscow, 1961; MR **25** #231] and the construction of $\overline{\phi_E(G/K)}$.

I. Satake (Chicago, Ill.)

Ree, Rimhak 5148

Commutators in semi-simple algebraic groups.

Proc. Amer. Math. Soc. **15** (1964), 457–460.

The author notices that the method of S. Pasiencier and H.-C. Wang proving that every element in a complex semi-simple Lie group is a commutator [same Proc. **13** (1962), 907–913] can be applied to the algebraic case to get "every element in a connected semi-simple algebraic group over an algebraically closed field is a commutator".

T. Ono (Philadelphia, Pa.)

FUNCTIONS OF REAL VARIABLES

See also 4980, 5171, 5203.

Dieudonné, J. 5149

★Fondements de l'analyse moderne.

Traduit de l'anglais par D. Huet. Avant-propos de G. Julia. Cahiers Scientifiques, Fasc. XXVIII.

Gauthier-Villars, Éditeur, Paris, 1963. xviii + 372 *pp.* 60 *F;* $12.50.

This is a translation into French of the author's *Foundations of modern analysis* [Academic Press, New York, 1960; MR **22** #11074].

Drobot, Stefan 5150

★Real numbers.

Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964. vii + 102 *pp.* $3.95.

This short volume treats four aspects of the real number system: (1) historical and axiomatic questions; (2) representations by decimals and by continued fractions; (3) approximations of real numbers by rationals; (4) cardinality and measure. As the author states in the preface, "...the book is not a systematic or complete course...". While many proofs are given, many are not, although adequate references are provided. Thus the material is somewhat descriptive and not everywhere self-contained; this is especially true of the topics labelled (1) and (4) above. But the book is carefully written to provide a wealth of material for a reader who wants an informal but reliable presentation.

I. Niven (Eugene, Ore.)

Tan, Mia-Hung 5151

On convergence and continuity of functions.

Bull. Math. Soc. Nanyang Univ. **1963**, 77–118.

Oeconomidis, Nicolas 5152

Sur les valeurs limites et les nombres dérivés d'une suite de fonctions réelles.

C. R. Acad. Sci. Paris **256** (1963), 1208–1211.

In this sequence of three papers [same C. R. **256** (1963), 3229–3232; MR **29** #199; ibid. **256** (1963), 4346–4349; MR **29** #200] the author establishes a theorem giving a necessary and sufficient condition that if $\{f_n(x)\}$ is a sequence of real functions defined on a set E of real numbers and $f_n(x) \to f(x)$ for $x \in E$, and if $x_0 \in E'$, then $\lim_{n\to\infty} \liminf_{x\to x_0^+} f_n(x) = \liminf_{x\to x_0^+} f(x)$. He also establishes a similar theorem for derivatives.

D. A. Storvick (Minneapolis, Minn.)

Mehdi, M. R. 5153

On convex functions.

J. London Math. Soc. **39** (1964), 321–326.

Let $f(x)$ be a real function defined and convex on (a, b) (i.e., satisfying $f((x+y)/2) \leqq (f(x)+f(y))/2$, $x, y \in (a, b)$). If it is bounded on some Baire set (i.e., some set of the form $G \setminus P \cup R$ with G open and P, R of first category) of second category, then it is continuous on (a, b). The result may be extended to the case of a real convex $f(x)$ defined on an open convex subset of a topological vector space. As a particular corollary, such a function is continuous if it has the Baire property or, in particular, it is Borel measurable. *H. Fast* (Notre Dame, Ind.)

Šmidov, F. I. 5154

A property of a function of two real variables. (Russian)

Dokl. Akad. Nauk BSSR **8** (1964), 145–146.

A result published earlier with a full proof [Izv. Vysš. Učebn. Zaved. Matematika **1963**, no. 2 (33), 155–163, Theorem 4; MR **27** #2587] is stated again without proof.

H. Fast (Notre Dame, Ind.)

Robinson, Raphael M. 5155

On the spans of derivatives of polynomials.

Amer. Math. Monthly **71** (1964), 504–508.

The span of a polynomial of degree n with real roots is defined to be the difference between the largest and smallest roots. The author considers the problem of determining how the span of the kth derivative may be maximized, given the span of the polynomial. After normalizing the problem to monic polynomials with roots in $[-1, 1]$, it is shown that only for polynomials of the form $(x-1)^p \times (x+1)^q$ will the span be maximized.

The problem is trivial when k does not satisfy $k+2 \leqq n \leqq 2k+1$. Otherwise, the choice of p and q above to maximize the span is not certain, although computer evidence indicates that p and q should be taken as nearly equal as possible. Moreover, the cases $n = 2k+1$, $k+2$ and $k+3$ bear this out by ad hoc arguments.

Permitting one of the roots of a polynomial to act as a parameter, the author shows that the span of the kth derivative is always maximized when the varying root is taken to be 1 or -1. This result is then exploited to give the remainder of the proof. The concise arguments employed should prove interesting to specialist and nonspecialist alike. *J. V. Ryff* (Cambridge, Mass.)

MEASURE AND INTEGRATION

See also 5139, 5189, 5301, 5315, 5330, 5462, 5464.

Gaifman, Haim 5156

Concerning measures on Boolean algebras.

Pacific J. Math. **14** (1964), 61–73.

A Boolean algebra B is said to satisfy condition (*) if there exists a sequence of sets $B_n \subset B$ such that $B = \bigcup_1^\infty B_n$ and such that B_n has at most n disjoint elements. It is easily seen that if B admits a finite, finitely additive, strictly positive measure, then it satisfies (*). The main result here is that the converse of this statement is false. By constructing a suitable ideal in the free Boolean algebra on continuum many generators it is shown that the quotient algebra B satisfies (*) but admits no strictly positive measure. The proof of the latter property depends on the notion of intersection number introduced by Kelley [same J. **9** (1959), 1165–1177; MR **21** #7286]. Condition (*) implies the countable chain condition. The converse would imply Souslin's conjecture but has not been disproved. It is shown that Souslin's conjecture is equivalent to each of several statements concerning atomicity or the existence of certain types of measures on a countably additive and countably distributive algebra satisfying the countable chain condition. It remains an open question whether a countably additive and weakly countably distributive algebra that satisfies (*) admits a strictly positive measure.

J. C. Oxtoby (Bryn Mawr, Pa.)

Martin, N. F. G. 5157

A topology for certain measure spaces.

Trans. Amer. Math. Soc. **112** (1964), 1–18.

Let $(X, \mathscr{S}, m)$ be a complete measure space, with $X \in \mathscr{S}$ and $m(X) = 1$. Assume that "$\mathscr{K}$ is a collection of sequences $\{K_n\}$ of sets from $\mathscr{S}$ such that for each $x \in X$ there exists at least one sequence $\{K_n\} \in \mathscr{K}$ such that (i) $x \in K_n$ for each n and (ii) $m(K_n) \to 0$ as $n \to \infty$". For each $E \subset X$ and $x \in X$ define the upper outer density of E at x to be $D^{-*}(E, x) = \sup\{\limsup_n m^*(E \cap K_n)/m(K_n)\}$, this sup taken over all sequences satisfying (i) and (ii) and the quotient being assigned the value zero if $m(K_n) = 0$. The paper is devoted to a study of the family $\mathscr{U} = \{U : D^{-*}(X - U, x) = 0$ for all x in $U\}$, which is shown (Theorem 4.1) to be a topology (called the "density topology") for X.

The inclusion $\mathscr{U} \subset \mathscr{S}$ may fail, but (Corollary 4.4) it holds whenever each E in $\mathscr{S}$ has lower outer density (defined analogously with upper outer density) equal to one at almost all points of E. In this case (4.10) the closed, discrete sets are those of measure zero, so (4.13) $(X, \mathscr{U})$ is a Baire space and m is a category measure for $(X, \mathscr{U})$ in the sense of Oxtoby [J. Reine Angew. Math. **205** (1960/61), 156–170; MR **25** #4054; errata, MR **26**, p. 1543]; (6.3) the measurable functions are those continuous almost everywhere.

Section 7 is restricted "to the reals with Lebesgue

measure and usual metric density", and relates the author's $\mathscr{U}$ to other topologies on this space. Most interesting result: Any bounded $\mathscr{U}$-continuous function is a derivative.

{The level of exposition is very high, and the paper is accessible to any student with a rudimentary knowledge of measure theory and general topology. The reviewer wishes the dependence of $\mathscr{U}$ and $\mathscr{K}$ had been emphasized and clarified by examples. The conditions imposed upon $\mathscr{K}$ seem so lax that certain of the announced results fail: the relation $D(X, x)=1$, for example, used in the proof of 3.5, will not hold if $m(K_n)=0$ whenever $x \in K_n \in \{K_n\} \in \mathscr{K}$. But 3.5 and the theorems dependent upon it will become true if the condition $m(K_n)>0$ is required in addition to (i) and (ii). Another quibble in a different vein is the following: Since the discrete topology on any set is first countable and dominates each topology on the set, the one-line proof of Theorem 7.7 is invalid. In acknowledging by letter the validity of these objections, the author has furnished a correct proof of 7.7.}

W. W. Comfort (Rochester, N.Y.)

Zakon, Elias 5158

A note on regular measures.

Canad. Math. Bull. 7 (1964), 41–44.

A measure m on a σ-field M of subsets of a topological space S is called strongly regular if for any $\varepsilon>0$ and $E \in M$ there exist sets F and G in M such that $F \subset E \subset G$, $m(E-F)<\varepsilon$, $m(G-E)<\varepsilon$, F closed, G open in S. In case S is an "F-space" (i.e., has the property that every closed set is a G_δ), and m is a Borel measure in S such that S is a countable union of open sets of finite measure, it is shown that m is strongly regular, and so is its least complete extension $\overline{m}$.

J. C. Oxtoby (Bryn Mawr, Pa.)

Fishel, B. 5159

A structure related to the theory of integration.

Proc. London Math. Soc. (3) **14** (1964), 415–430.

Author's preface: "Let T be a locally compact space and let $\mathscr{K}$ be the space of continuous functions on T with compact supports. Let $\mu: \mathscr{K} \to R$ be a linear functional such that if $f \geqq 0$, then $\mu(f) \geqq 0$, and if $f_n \downarrow 0$, then $\mu(f_n) \to 0$. Bourbaki extends μ to an outer measure μ^* defined for all functions on T, and uses μ^* to define $L^1(\mu)$, the space of integrable functions modulo null-functions. Here we give an alternative and more direct definition of $L^1(\mu)$. If $\varphi \in \mathscr{K}$ let $\mu_\varphi: \mathscr{K} \to R$ be defined by $\mu_\varphi(f)=\mu(\varphi f)$. We give $\mathscr{K}$ the coarsest topology for which, whenever $S \in \mathscr{S}$, a family of uniformly bounded subsets of $\mathscr{K}$, the family $\{\mu_\varphi\}_{\varphi \in S}$ is equicontinuous. Then $L^1(\mu)$ is the separated completion of the topological vector space $\mathscr{K}$. The above construction can be carried out whenever $\mu: F \to H$ is a linear map, the commutative real algebra F has local units, and the topological real vector space H is locally convex and separated. A suitable choice of $\mathscr{S}$ yields $L^1_{\text{loc}}(\mu)$ as a separated completion of $\mathscr{K}$. If μ is a vector-valued Radon measure we obtain, again as a separated completion of $\mathscr{K}$, a space closely related to the separated space associated with $\mathscr{L}(\mu)$."

J. C. Oxtoby (Bryn Mawr, Pa.)

Kellerer, Hans G. 5160

Masstheoretische Marginalprobleme.

Math. Ann. **153** (1964), 168–198.

Let I be a fixed finite index set, and $\mathfrak{T}$ a fixed family of subsets of I. For each i in I, let K_i be a σ-field of subsets of a set M_i, and denote by K the product σ-field on the Cartesian product $M=\prod_{i \in I} M_i$. If ν is a bounded general measure on K, one defines for each T in $\mathfrak{T}$ a measure ν^T on $K_T=\prod_{i \in T} K_i$ by the requirement $\nu^T(A)=\nu(A \times \prod_{i \in I-T} M_i)$. These ν^T are called the marginal measures of ν. The question arises: Given for each T in $\mathfrak{T}$ a bounded general measure ν_T on K_T, under what conditions are the ν_T the marginal measures of a bounded general measure ν on K? The author shows (Satz 2.2) that the obvious necessary condition $(\nu_{T_1})^{T_1 \cap T_2}=(\nu_{T_2})^{T_1 \cap T_2}$ $(T_1, T_2 \in \mathfrak{T})$ is also sufficient. This he calls the "unrestricted problem".

Next he tackles the more difficult "restricted problem" of finding a bounded general measure ν on K which not only has the given ν_T as marginal measures, but also satisfies $\underline{\nu} \leqq \nu \leqq \bar{\nu}$, where $\underline{\nu}$ and $\bar{\nu}$ are given σ-finite general measures on K. Among other partial solutions to this problem, he obtains the following result (Satz 4.2): Assume that $\underline{\nu} \leqq \bar{\nu}$, and that $\underline{\nu}$ and $\bar{\nu}$ are also bounded; then the following two conditions are equivalent: (a) There exists a solution of the given restricted problem

$$\text{(b)} \qquad \sum_{T \in \mathfrak{T}} \int_{M_T} g_T \, d\nu_T \leqq \int_M \Big(\sum_{T \in \mathfrak{T}} g_T\Big)^+ d\bar{\nu} - \int_M \Big(\sum_{T \in \mathfrak{T}} g_T\Big)^- d\underline{\nu}$$

whenever, for each T in $\mathfrak{T}$, g_T is a bounded K_T-measurable function on M.

J. M. G. Fell (Seattle, Wash.)

Kellerer, Hans G. 5161

Marginalprobleme für Funktionen.

Math. Ann. **154** (1964), 147–156.

In the preceding article [#5160] the author discussed the so-called marginal problem for measures. In the present article he takes up the analogous problem for functions. We retain the notation of the preceding review.

Let M, K, μ be the Cartesian product of the fixed σ-finite measure spaces M_i, K_i, μ_i ($i \in I$, where I is finite), and let $\mathfrak{T}$ be a non-void family of subsets of I. If $f \in L(\mu)$ (the space of μ-summable complex functions) and $T \in \mathfrak{T}$, the function $f^T(x)=\int_{M_{I-T}} f(x, y)\, d\mu^{I-T} y$ $(x \in M_T=\prod_{i \in T} M_i)$ is called a marginal function for f. Conversely, given for each T in $\mathfrak{T}$ a function f_T in $L(\mu^T)$, can a single function f in $L(\mu)$ be found such that the f_T are its marginal functions? The answer is that the obvious necessary condition, namely, that $\int_{M_{T_1-T_2}} f_{T_1}\, d\mu^{T_1-T_2}=\int_{M_{T_2-T_1}} f_{T_2}\, d\mu^{T_2-T_1}$ ($\mu^{T_1 \cap T_2}$-almost everywhere) is also sufficient.

Next the author poses the "restricted problem" for functions: To find an f which will not only have the required marginal functions but will also satisfy $\underline{f} \leqq f \leqq \bar{f}$, where $\underline{f}$ and $\bar{f}$ are given functions. He studies some special cases of this problem, and gives moderately complicated necessary and sufficient conditions for their solvability.

J. M. G. Fell (Seattle, Wash.)

Nikodým, Otton Martin 5162

Summation of fields of numbers in Boolean tribes. Upper and lower summation in the general case. Complete admissibility. Square summability. Equivalence of fields of numbers. IV.

J. Reine Angew. Math. **214/215** (1964), 84–136.

The present paper is the fourth in a series of papers denoted by (I) [Rend. Circ. Mat. Palermo (2) **9** (1960), 169–225; MR **26** #5120], (II), (III) and (IV). (II) and (III) are due to appear in J. Math. Pures Appl. They are devoted to an integration of Burkill-Lebesgue type, which has been introduced by the author in his earlier paper [Rend. Sem. Mat. Univ. Padova **29** (1959), 1–214; MR **22** #4811]. The chapters are numbered throughout from 1 to 12. For notations we refer to the review of (I). In the following the distinguished sequences of complexes being all of the (DARS) type we shall drop (DARS) as a suffix. In (III) (= § 7) were proved several theorems on the summation of a scalar field φ in the case G has no atoms. The paper under review comprising §§ 8–12 is devoted to modifying and proving the theorems of § 7 in the general case when atoms may be available. The condition of admissibility is then reinforced to a condition of complete admissibility referring to the atoms of G. The proofs are not repeated in extenso, only the modifications are indicated. Among the topics dealt with we mention: Boundedness of a sequence $(\varphi(P_n))$ when (P_n) is a distinguished admissible sequence. Upper and lower (finite) summation of real φ. Minimal and maximal distinguished sequences for real φ. Summability. Summation-boundedness of φ on $a \in G$: The scalar field φ is termed summation-bounded whenever for any distinguished sequence (P_n) the sequence $(|\varphi|(P_n))$ is bounded. Implication of the summability of $|\varphi|$ by the summability of φ. Square of a real field φ. μ-square equivalence of real fields φ and ψ on a: $|\varphi-\psi|^2\mu$ is summable on a to 0.

Chr. Y. Pauc (Nantes)

Hayashi, Yoshiaki 5163

A trial production on the integral. IV.

Bull. Univ. Osaka Prefecture Ser. A **12** (1963/64), *no.* 2, 127–138.

Part III appeared in same Bull. **12** (1963), no. 1, 111–126 [MR **28** #180]. A constructive definition of an integral based on transfinite induction is given. This integral, of Cauchy principal value type, was previously defined [ibid. **11** (1962), no. 2, 117–126; MR **27** #2600b].

D. A. Woodward (Argonne, Ill.)

Sinaĭ, Ja. G. 5164a

A weak isomorphism of transformations with invariant measure. (Russian)

Dokl. Akad. Nauk SSSR **147** (1962), 797–800.

Sinaĭ, Ja. G. 5164b

On a weak isomorphism of transformations with invariant measure. (Russian)

Mat. Sb. (*N.S.*) **63** **(105)** (1964), 23–42.

For $i=1, 2$, let M_i be a Lebesgue space with measure μ_i, and let T_i be an automorphism of M_i [cf. V. A. Rohlin, Uspehi Mat. Nauk **15** (1960), no. 4 (94), 3–26; MR **24** #A2002]. T_1 is said to be a homomorphic image of T_2 if there is a homomorphism U of M_2 onto M_1 such that $T_1U=UT_2$. If T_1 and T_2 are such that each is a homomorphic image of the other, they are said to be weakly isomorphic. It is shown that weakly isomorphic automorphisms are spectrally isomorphic and have the same entropy. Weak isomorphism preserves the property of being a K-automorphism [cf. V. A. Rohlin, loc. cit., or V. A. Rohlin and Ja. G. Sinaĭ, Dokl. Akad. Nauk SSSR **141** (1961), 1038–1041; MR **27** #2604]. Weakly isomorphic automorphisms with discrete spectrum are isomorphic.

The main result is Theorem 2: Let T be an ergodic automorphism of a Lebesgue space and let T_1 be a Bernoulli automorphism (i.e., the shift transformation on a space of doubly infinite sequences $x=\{x_n\}$, where each x_n takes countably many values, with product measure). Suppose that, for the entropy of T_1, $h(T_1)\leqq h(T)$. Then T_1 is isomorphic to a factor automorphism of T. Among the corollaries to Theorem 2 are (i) Bernoulli automorphisms with equal entropy are weakly isomorphic, and (ii) any ergodic automorphism has a factor automorphism which is a K-automorphism and has the same entropy.

The proof of Theorem 2, to which the second paper under review is largely devoted, is lengthy and involved. The properties of measurable partitions and entropy, as developed by Kolmogorov [ibid. **119** (1958), 861–864; MR **21** #2035a] are used extensively. An important tool in the proof is an approximation lemma which says (approximately): If ζ and ξ are measurable partitions of M with $\xi\leqq\zeta$ (i.e., ζ is a refinement of ξ modulo null sets), $T\zeta>\zeta$, and $\xi=\{B_i\}_{i=1,\cdots,k}$ is finite, and if $\pi=(\pi_1,\cdots,\pi_k)$ is a finite probability distribution, then (if certain inequalities involving conditional measure and conditional entropy hold) it is possible to find partitions $\xi'=\{B_i'\}$, ζ' with the same properties as ξ, ζ, respectively, for which the conditional measures $\mu(TB_i'|\zeta')$ may be made arbitrarily close to π_i $(i=1, 2, \cdots, k)$.

J. Auslander (College Park, Md.)

Linderholm, Carl E. 5165

If T is incompressible, so is T^n.

Ergodic Theory (*Proc. Internat. Sympos., Tulane Univ., New Orleans, La.*, 1961), *pp.* 191–193. *Academic Press, New York*, 1963.

If f and g are measurable functions on a measure space X, write $f>g$ to mean that $\{x:f(x)\leqq g(x)\}$ is a null set but that $\{x:f(x)>g(x)\}$ is not. Lemma: A measurable transformation T of X into itself is compressible if and only if $fT>f$ for some measurable function f on X. This provides a new proof of the known result stated in the title of the paper.

J. C. Oxtoby (Bryn Mawr, Pa.)

Scoville, Richard 5166

Completely monotone sequences as invariant measures.

Trans. Amer. Math. Soc. **112** (1964), 318–329.

Let $\mathbf{a}=\{a_n\}$ be a completely monotone sequence of real numbers with $\sum_0^\infty a_n=1$. Form a "building" consisting of segments of length a_n, with a_{n+1} lying above a_n. Let T move points one level upward whenever possible, otherwise let T drop the point to the bottom level B_0 and apply a measure-preserving transformation S of B_0. To define S represent B_0 as a building corresponding to $\Delta\mathbf{a}$, its base as a building corresponding to $\Delta^2\mathbf{a}$, and so on. The resulting transformation T_a, which preserves Lebesgue measure, can also be represented as a transformation T acting continuously on the space of increasing sequences of non-negative integers. In the latter representation T is independent of $\mathbf{a}$, but has an invariant measure μ_a determined by $\mathbf{a}$. Any invariant probability

measure for T is of the form μ_a for some a. T is ergodic relative to μ_a only if $a_n = \theta^n(1-\theta)$ for some $0 \leqq \theta < 1$. Some results concerning the spectrum of T in this case are that when $\theta < \frac{1}{2}$, T has no prime roots of unity as eigenvalues, and all powers of T are ergodic. When restricted to the set U of unbounded sequences T is one-to-one and onto. By applying the Kryloff-Bogoliouboff decomposition theorem to the Borel system (T, U) it is possible to obtain the classical representation $a_n = \int_0^1 t^n \, dF_a(t)$, where F_a is an increasing function on $[0, 1]$. An inversion formula for F_a can also be obtained in this way.
J. C. Oxtoby (Bryn Mawr, Pa.)

Révész, P. 5167
A random ergodic theorem and its application in the theory of Markov chains.
Ergodic Theory (Proc. Internat. Sympos., Tulane Univ., New Orleans, La., 1961), *pp.* 217–250. *Academic Press, New York*, 1963.
An extension and elaboration of results first presented by the author in earlier papers [Magyar Tud. Akad. Mat. Kutató Int. Közl. **5** (1960), 375–381; MR **23** #A2892; ibid. **6** (1961), 205–213; MR **26** #2580].
D. Austin (Evanston, Ill.)

Lahiri, B. K. 5168
On the structure of a measurable set.
Indian J. Math. **5** (1963), 51–54.
The author proves two theorems for sets of positive measure in the plane, inspired by A. J. Ward's results on uncountable sets [J. London Math. Soc. **8** (1933), 109–112]. Call a ray through x_0 in direction λ a line of measure for E if, for every δ and η, the part of an angular domain of opening 2η centered on the ray with vertex x_0 that is inside a disk about x_0 of radius δ contains a subset of E of positive measure. A point of E is called special if for each λ either the ray in direction λ or the ray in direction $-\lambda$ through it is not a line of measure for E. Theorem 1: If $m(E) > 0$, then almost every point of E is such that every ray through it is a line of measure. Theorem 2: Let E_2 be the set (of measure 0) of special points of E. Then at all points of E_2 the upper density of E is at most $\frac{1}{2}$.
R. P. Boas, Jr. (Evanston, Ill.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 5045, 5278, 5607.

Goodman, A. W. 5169
On a characterization of analytic functions.
Amer. Math. Monthly **71** (1964), 265–267.
The author proves the following generalization of a theorem of Dzyadyk [Uspehi Mat. Nauk **15** (1960), no. 1 (91), 191–194; MR **22** #2683]. Let R be a given region in the plane and let $u(x, y)$, $v(x, y)$ be continuous real-valued functions with u_x, u_y, v_x, v_y continuous in R. Let R^* contain the image of R under $u = u(x, y)$, $v = v(x, y)$. Finally, let $\phi = \phi(u, v)$ have second-order partial derivatives and satisfy $\phi_u{}^2 + \phi_v{}^2 = 1$ in R^*, and suppose that each level curve for the surface $z = \phi(u, v)$ has only a finite number of horizontal or vertical normals. Then in order that either $u(x, y) + iv(x, y)$ or $u(x, y) - iv(x, y)$ be analytic in R it is necessary and sufficient that, if S is an arbitrary subregion of R, all three of the surfaces $z = u(x, y)$, $z = v(x, y)$, $z = \phi(u, v)$ have the same area over S. The proof is a modification of the one given by Dzyadyk whose result corresponds to the choice for ϕ: $\phi = \sqrt{(u^2 + v^2)}$.
A. Fialkow (Brooklyn, N.Y.)

Baiada, E.; Policarpo, V. 5170
L'esplicitazione di una funzione implicita nel campo complesso.
Atti Sem. Mat. Fis. Univ. Modena **12** (1962/63), 121–128.
Con procedimento topologico gli autori dimostrano il seguente teorema: Siano C_1, C_2 due regioni limitate del piano complesso, e sia $f(x, y)$ una funzione complessa, definita per x appartenente a C_1 e per y appartenente a C_2, la quale sia funzione continua di (x, y). Sia $f(x_0, y_0) = 0$, ove x_0 appartiene a C_1 e y_0 a C_2, e si supponga che esistano quattro numeri reali λ, $\bar{\rho}$, A, α con $0 \leqq \lambda < \pi$, $\bar{\rho} \geqq 0$, $A > 0$, $0 \leqq \alpha < \pi/2$ in modo che per $|\Delta y| \leqq \bar{\rho}$ valgano le disuguaglianze

$$|f(x_0, y_0 + \Delta y)| \geqq A|\Delta y|,$$

$$\lambda - \alpha \leqq \arg f(x_0, y_0 + \Delta y) - \arg \Delta y \leqq \lambda + \alpha.$$

Allora si può determinare un intorno I_{x_0} di x_0 appartenente a C_1 e una funzione complessa $y = y(x)$ definita, continua e univoca in I_{x_0}, i cui valori appartengono a C_2, la quale soddisfa alle condizioni $f(x, y(x)) = 0$ per x appartenente a I_{x_0}; $y(x_0) = y_0$.
S. Cinquini (Pavia)

Myrberg, P. J. 5171
Sur l'itération des polynomes réels quadratiques.
J. Math. Pures Appl. (9) **41** (1962), 339–351.
This summarizes and continues a series of the author's papers on the iteration of the polynomial $P(x) = x^2 + p$, where p is real and x complex [C. R. Acad. Sci. Paris **246** (1958), 3201–3204; MR **20** #89; Ann. Acad. Sci. Fenn. Ser. A I No. 253 (1958); MR **20** #2557; ibid. No. 256 (1958); MR **20** #5876; ibid. No. 259 (1958); MR **21** #7377; ibid. No. 268 (1959); MR **23** #A1808; ibid. No. 336/3 (1963); MR **27** #1552]. There is a discussion of the values of p for which a cycle of zero multiplier (or, more generally, an attractive cycle) exists. The domain of attraction of ∞ [i.e., maximal domain in which $P_n(x) \to \infty$ uniformly, where $P_n(x)$ is the nth iterate of $P(x)$] and its boundary are described. An application is made to the entire functions satisfying a Poincaré functional equation $\Phi(kx) = \Phi^2(x) + p$, where k and p are constant; there is a unique solution for which $\Phi(0) = q = \frac{1}{2} + \sqrt{(\frac{1}{4} - p)}$, $\Phi'(0) = 1$, $k = 2q$; the asymptotic behaviour of this $\Phi(x)$ as $x \to \infty$ in certain angles is derived from the preceding results. There are misprints on p. 344 line 1 and p. 347 line 2: in each case "attractif" should be "répulsif".
I. N. Baker (Zbl **106**, 47)

Lick, D. R. 5172
Sets of non-uniform convergence of Dirichlet series.
J. London Math. Soc. **39** (1964), 333–337.
By G. Brauer's method, the author transforms several theorems on power series into theorems on ordinary Dirichlet series.
G. Piranian (Ann Arbor, Mich.)

Fox, William C. 5173
Erratum to "A new inequality for the Green's function".
Proc. Amer. Math. Soc. **12** (1961), 340–343.

This is a new version of an earlier paper by the author [same Proc. **10** (1959), 562–569; MR **21** #6425], which claims to give a short and elementary proof for Koebe's distortion theorem with the constant $\frac{1}{4}$. Even in this version the Lemma (3.1), on which the proof is based, fails to be correct. Put $r=\frac{3}{4}$, $e=-i$ and $w=i\sqrt{\frac{3}{4}}$, then $w \in J_r$ and

$$\ln\left|\frac{wf(r)^{1/2}+e}{w-e}\right| < 0,$$

which contradicts the inequality given in the Lemma.

A. Pfluger (Zürich)

Krzyż, J. 5174
Some remarks on close-to-convex functions.
Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 25–28.

The author obtains the domain $D(r)$ of variability of $\log f'(r)$ for fixed $r \in (0,1)$ when f ranges over the class of close-to-convex functions in $K=\{z: |z|<1\}$. $D(r)$ is a convex closed domain symmetric with respect to the real axis Ou and the straight line $u=-\log(1-r^2)$. The boundary of $D(r)$ consists of the convex arc

$$W = \log\frac{1-r\exp i\theta_2(\beta)}{[1-r\exp i\theta_1(\beta)]^3}, \qquad 0 \leqq \beta \leqq \pi,$$

where

$$\theta_1(\beta) = \beta - \operatorname{arc}\sin(r\sin\beta),$$

$$\theta_2(\beta) = \pi+\beta+\operatorname{arc}\sin(r\sin\beta),$$

and of its reflection in the real axis. The functions corresponding to the boundary points of $D(r)$ have the form

$$\int_0^z \frac{1-\xi\exp i\theta_2(\beta)}{[1-\xi\exp i\theta_1(\beta)]^3}\,d\xi.$$

It follows from $D(r)$ that if $f(z)$ is close-to-convex, then

$$|\arg f'(z)| \leqq 4\operatorname{arc}\sin|z| \qquad (|z|<1).$$

The maximum occurs for $z=r$ for the function

$$F(z) = \frac{z-z^2\cos\gamma}{(1-ze^{i\gamma})^2}, \qquad \gamma = \tfrac{1}{2}\pi - \operatorname{arc}\sin r = \operatorname{arc}\cos r.$$

The minimum occurs for $z=r$ and $\overline{F(\bar z)}$. For $|z| \leqq 2^{-1/2}$ the bounds obtained are identical with those for the whole class of normalized univalent functions in K.

In his proof the author makes use of the known region of variability of $\log\phi'(r)$ where ϕ ranges over the class of normalized convex functions in K [see A. Marx, Math. Ann. **107** (1932), 40–67] and also of the region of variability of $p(r)$ where p ranges over the class of normalized functions with positive real part in K.

M. S. Robertson (New Brunswick, N.J.)

Ozawa, Mitsuru 5175
Picard's theorem on some Riemann surfaces.
Kōdai Math. Sem. Rep. **15** (1963), 245–256.

Let W be a Riemann surface whose automorphisms form a free abelian group on n generators. If W is an unramified abelian covering surface of a closed Riemann surface and if W has only one end (two ends if $n=1$) satisfying some additional conditions, then W is said to be of class $\mathfrak{G}_n$. Some results concerning $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are given. These lead to conditions for such surfaces to be realizable with finite spherical area.

The author also draws some conclusions related to value distribution theory for functions meromorphic on a surface in $\mathfrak{G}_1$ or $\mathfrak{G}_2$. He shows that these results cannot be derived from L. Sario's general approach [Pacific J. Math. **12** (1962), 1079–1097; MR **27** #286], at least for a certain choice of conformal metric.

B. Rodin (La Jolla, Calif.)

Kuramochi, Zenjiro 5176
Singular points of Riemann surfaces.
J. Fac. Sci. Hokkaido Univ. Ser. I **16**, 80–148 (1962).

A continuation, simplification, and correction of a previous paper of the author [Osaka Math. J. **10** (1958), 83–102; MR **20** #3272]. Several kinds of singular points on the Martin boundary are defined and examined in detail. Behavior of harmonic and analytic functions in the neighborhood of singular points is discussed, and a connection is found with the indivisible sets of Constantinescu and Cornea [Nagoya Math. J. **13** (1958), 169–233; MR **20** #3273].

A. Marden (Minneapolis, Minn.)

Boboc, N.; Mocanu, G. 5177
Sur la notion de métrique harmonique sur une surface riemannienne hyperbolique.
Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **4 (52)** (1961), *no.* 1–2, 3–21 [1963].

A metric is introduced for hyperbolic Riemann surfaces R as follows. If $p, q \in R$, define $\mathscr{S}(p,q)$ to be the family of positive superharmonic functions S on R for which there exists a continuum containing p and q on which $S \geqq 1$. Let $s \to H_s(p,q)$ denote the lower envelope of $\mathscr{S}(p,q)$. For fixed $s \in R$ the function $H_s(p,q)$ is shown to be a continuous metric on $R-\{s\}$. It is claimed that if R is a plane region bounded by a finite number of analytic Jordan curves, then the completion of R in this metric is the closure of the region. Explicit computations are given for $R=\{|z|<1\}$.

In the general case bounds for this metric are obtained in terms of Green's functions and harmonic measures associated with relevant Jordan arcs. Among other results is the following application to a lemma of Koebe. Let $\{p_n\}$, $\{q_n\}$ be sequences of points on R which tend to the ideal boundary, and such that the distance from p_n to q_n is bounded away from 0 in this metric. Let γ_n be an arc from p_n to q_n. If $\psi: R \to R'$ is an analytic mapping of Riemann surfaces and $\psi(\gamma_n) \to p_0 \in R'$, then ψ is constant.

B. Rodin (La Jolla, Calif.)

Springer, George 5178a
Fredholm eigenvalues and quasiconformal mapping.
Bull. Amer. Math. Soc. **69** (1963), 810–811.

Springer, George 5178b
Fredholm eigenvalues and quasiconformal mapping.
Acta Math. **111** (1964), 121–142.

Let D be a plane region of connectivity n bounded by a system of Jordan curves $C_1, C_2, \cdots, C_n$. The lowest nontrivial positive eigenvalue λ associated with $C = \bigcup_{j=1}^n C_j$ is defined as follows. If C is sufficiently smooth, then λ is the lowest eigenvalue of the Neumann-Poincaré integral equation which exceeds 1; for more general C, λ is defined by a certain extremum problem (E) involving Dirichlet integrals.

By means of (E) the author bounds the number λ^* corresponding to C^*, the image of C under a quasiconformal mapping of the plane onto itself. These bounds are illustrated by obtaining useful inequalities for the λ corresponding to specific domains such as confocal elliptic annuli. It is shown that C is quasiconformally equivalent to a system of circles (and also more general canonical curve systems) if and only if $\lambda > 1$. Finally, $\lambda > 1$ if and only if the Ahlfors [Acta Math. **109** (1963), 291–301; MR **27** #4921] cross-ratio condition

$$(\overline{P_1P_2}\cdot\overline{P_3P_4})/(\overline{P_1P_3}\cdot\overline{P_2P_4}) \leqq A < \infty$$

holds for each consecutive set of points P_1, P_2, P_3, P_4 on any C_j. The author's results are, in part, extensions to $n > 1$ of the results in the above-mentioned reference, and the sufficient condition for $\lambda > 1$ when $n = 1$ given by Ahlfors [Pacific J. Math. **2** (1952), 271–280; MR **14**, 182].
E. Reich (Stanford, Calif.)

Yu, Jia-rong [You, Chia-Yung] 5179

On the Borel lines of entire functions defined by Laplace-Stieltjes transforms.

Acta Math. Sinica **13** (1963), 471–484 (*Chinese*); *translated as Chinese Math.* **4** (1964), 511–525.

In this note, the author studies Borel lines of entire functions defined by the Laplace-Stieltjes transform: $F(s) = \int_0^{+\infty} \exp(-sx)\, d\alpha(x)$ $(s = \sigma + it)$ with the uniform convergence abscissa $\sigma_u^F = -\infty$ and linear order $\tau > 0$. Following G. Valiron, he associates with $F(s)$ the Laplace-Stieltjes transform: $f(s) = \int_0^{+\infty} \exp(-sx)\Omega(x)\, d\alpha(x)$ such that (1) $f(s)$ has the uniform convergence abscissa σ_u^f satisfying $0 \leqq \sigma_u^f < +\infty$, (2) $\Omega(x) = \int_{-\infty}^{-1} \exp(-x\sigma)/W(\sigma)\, d\sigma$, where $W(\sigma)$ is a suitably chosen positive function of σ, and (3) $f(s)$ satisfies the functional relation $f(s) = \int_{-\infty}^{-1} F(s+x)/W(x)\, dx$ for $\Re(s) > \sigma_u^f$. From (3) the author can infer relationships between singular points of $f(s)$ and Borel lines of $F(s)$.

Suppose $F(s)$ has linear order $\tau > 0$. If $s_0 = \sigma_0 + it_0$ is a vertex of the horizontal star domain of $f(s)$, then when $0 < \tau < +\infty$, there exists a Borel line $\Im(s) = t'$ of $F(s)$ in the strip: $|t - t_0| < \pi/2\tau$; when $\tau = +\infty$, $F(s)$ has a Borel line $\Im(s) = t_0$. By virtue of this theorem, known theorems about singular points of $f(s)$ can be translated into ones on Borel lines of $F(s)$. Some special cases of these results were already treated in his earlier paper [Ann. Sci. École Norm. Sup. (3) **68** (1951), 65–104; MR **12**, 815].
C. Tanaka (Tokyo)

Kennedy, P. B. 5180

A problem on bounded analytic functions.

Proc. Amer. Math. Soc. **15** (1964), 325–326.

In answer to a problem stated among "Classical Function Theory Problems" [Bull. Amer. Math. Soc. **68** (1962), 21], the author shows that there is a gap series representing a bounded function $f(z)$ in $|z| < 1$ and such that $N(r, 1/f') \sim -\log(1-r)$ $(r \to 1)$. His example is $f(z) = \sum_{n=1}^\infty n^{-2} z^{\lambda_n}$ where (a) $\lambda_{n+1}/\lambda_n \to \infty$, (b) $\log \lambda_{n+1} \sim \log \lambda_n$. He also shows that the condition (b) cannot be dispensed with.
W. H. J. Fuchs (Ithaca, N.Y.)

Kegejan, È. M. 5181

On the radial behavior of functions analytic in a circle. (Russian. Armenian summary)

Akad. Nauk Armjan. SSR Dokl. **37** (1963), 241–247.

Let $\mu(r)$ denote a strictly decreasing function in [0, 1], with $\mu(0) \leqq 1$ and $\mu(1) = 0$; let E denote a closed set of measure 0 on the unit circle; and let $B(\mu, E)$ be the class of nonconstant analytic functions that are bounded by 1 in the unit disk and satisfy the condition

$$|f(re^{i\theta})| \leqq \mu(r) \qquad (e^{i\theta} \in E,\ 0 < r < 1).$$

The following two propositions exemplify the author's results and indicate the spirit of the paper. Theorem 3: If $\mu(r) = \exp(1/(r-1))$, then $B(\mu, E)$ is not empty if and only if the set E is finite. Theorem 4: No matter how slowly $\mu(r)$ approaches 0, there exists a set E of measure 0 for which $B(\mu, E) = 0$. *G. Piranian* (Ann Arbor, Mich.)

Tumarkin, G. C. 5182

Convergent sequences of Blaschke products. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 201–233.

For $k = 1, 2, \cdots$, let b_k denote a Blaschke product with the zeros α_{kj} $(j = 1, 2, \cdots)$, and suppose that $b_k(z) \to f(z)$ in the unit disk D. The author studies relations between the set $\{\alpha_{kj}\}$ and various properties of f. In an earlier paper, he had announced his principal results without proofs [Dokl. Akad. Nauk SSSR **129** (1959), 40–43; MR **21** #7308].

Theorem 1: If $b_k \to f$, then f is a Blaschke product if and only if (1) for each R $(0 < R < 1)$ the number of zeros of b_k in $|z| < R$ is a bounded function of k and (2) the function

$$\Phi(R) = \sup_{k=1,2,\cdots} \sum_{|\alpha_{kj}| > R} (1 - |\alpha_{kj}|)$$

tends to 0 as $R \to 1$.

Theorem 4: Let the jump functions ψ_k be defined by the equation

$$\psi_k(\theta) = -\pi \sum_{0 \leq \arg \alpha_{kj} \leq \theta} (1 - |\alpha_{kj}|^2) \qquad (0 \leqq \theta \leqq 2\pi);$$

write

$$m_k(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\psi_k(\theta),$$

and suppose that $b_k \to f$ and f has no zeros in D. Then $m_k(z) \to f(z)$.

Theorems 1 and 4 lead to a formulation (too long for reproduction), in terms of the distribution of the zeros α_{kj}, of necessary and sufficient conditions for the convergence of a sequence of Blaschke products; they also yield a representation of the limit function as the product of a Blaschke product and a nonvanishing function.

Theorem 11: Suppose that $b_k \to f$. For the convergence in measure of the sequence $\{b_k(e^{i\theta})\}$ of boundary values it is necessary and sufficient that (3) $\limsup_{k\to\infty} \sum (1 - |\alpha_{kj}|) < \infty$ and (4) for all positive ε and δ there exists a set Δ

composed of disjoint intervals $\Delta_1, \cdots, \Delta_n$ on $[0, 2\pi]$, of total length at least $2\pi - \delta$, such that the inequality

$$\sum_{\arg \alpha_{kj} \in \Delta} (1 - |\alpha_{kj}|) < \varepsilon$$

holds for all sufficiently large k.

G. Piranian (Ann Arbor, Mich.)

Tumarkin, G. C. 5183

Conditions for uniform convergence and convergence of boundary values of analytic and meromorphic functions of uniformly bounded characteristic. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 387–417.

From the author's summary: "The methods developed in earlier papers [Dokl. Akad. Nauk SSSR **129** (1959), 40–43; MR **21** #7308; and #5182 above] for proving theorems on sequences of Blaschke products are here applied to establish analogous results for the broader class of sequences of analytic and meromorphic functions. For example, we obtain criteria for the uniform convergence in closed subsets of $|z| < 1$ of sequences with uniformly bounded characteristic."

G. Piranian (Ann Arbor, Mich.)

Zaharjan, V. S. 5184

The radial limit values of a class of functions meromorphic in the circle. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 801–818.

Let $w(z)$ be a function meromorphic in $|z| < 1$. For functions of bounded characteristic the following results are classical: (A) $\lim_{r\to\infty} w(re^{i\theta})$ exists for every θ in $0 \leqq \theta \leqq 2\pi$, except perhaps for a set of θ of measure zero. (B) If $w(z) \not\equiv a$, the set of θ for which $\lim_{r\to 1} w(re^{i\theta}) = a$ is of measure zero. By restricting the condition on $w(z)$, the author obtains sharper results on the exceptional sets. Certain preliminary notions are introduced. The continuous function $H(t) \geqq 0$ defined on $0 < t < \infty$ is said to belong to the class C_H if it satisfies the following four conditions: (1) $H(0) = \infty$, (2) $\lim_{t\to 0} tH(t) = 0$ monotonically, (3) $\int_0 1/tH(t)\,dt < \infty$, (4) $\lim_{x\to 0} (1/xH(x)) \int_0^x H(t)\,dt = c$, where $c \neq 0, \infty$. Let $w(z)$ be meromorphic in $|z| < 1$ and let $A(r)$ denote the spherical area of the region on which $w(z)$ maps the disk $|z| < r < 1$. The function $w(z)$ is said to belong to the class T_H, where $H \in C_H$, if $\int_0^1 A(t)H(1-t)\,dt < \infty$ whenever $\int_0^1 H(1-t)\,dt < \infty$ and $\lim_{r\to 1} A(r) < \infty$ whenever $\int_0^1 H(1-t)\,dt = \infty$. The exceptional sets of θ are described in terms of the notion of convex capacity introduced by K. V. Temko [Dokl. Akad. Nauk SSSR **110** (1956), 943–944; MR **19**, 31]. Finally, without going into detail, exceptional sets Q_H of points a on the Riemann sphere are characterized by certain properties which depend on the function $H(t)$. In particular, these sets Q_H are always of superficial Lebesgue measure zero. The two main theorems, corresponding to the above theorems A and B are as follows: (I) If $w(z) \in T_H$, $\lim_{r\to 1} w(re^{i\theta})$ exists for all θ in $0 \leqq \theta < 2\pi$, except perhaps for a set of θ whose convex capacity relative to the sequence

$$\lambda_n = \sum_{k=n}^{\infty} \frac{1}{kH(1/k)}$$

is zero. (II) If $w(z) \in T_H$, $w(z) \not\equiv a$, and E is the set of θ for which $\lim_{r\to 1} w(re^{i\theta}) = a$, then the convex capacity of E relative to the sequence $\{\lambda_n\}$ is equal to 0, except when $a \in Q_H$.

In the special case that $H(t) = t^{-\alpha}$, $0 < \alpha < 1$, these results reduce to theorems proved by L. Carleson [Thesis, Univ. Uppsala, Uppsala, 1950; MR **11**, 427].

W. Seidel (Detroit, Mich.)

Wang, Hong-sheng [Wang, Hung-shen] 5185

Generalized analytic functions of classes B, H_δ, D, A, and the convergence of sequences of such functions.

Acta Math. Sinica **13** (1963), 531–543 (*Chinese*); *translated as Chinese Math.* **4** (1964), 578–592.

Author's summary: "The contents of this paper are an extension from analytic functions of classes B, H_δ, D, A in the unit circle to generalized analytic functions; and the subsequent application to generalized analytic functions of the uniqueness theorem for functions of class A, of the theorem of F. Riesz for functions of class H_δ and of the theorem of Polubarinova-Kočina for functions of class D. On the basis of the results thus obtained and of the convergence on the boundary of sequences of boundary values of generalized analytic functions, we investigate the uniform convergence of functions of this class within the unit circle. We proceed to apply the theorems of A. Ja. Hinčin and A. Ostrovskiĭ and of G. C. Tumarkin to generalized analytic functions."

Korevaar, J.; Loewner, C. 5186

Approximation on an arc by polynomials with restricted zeros.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 121–128.

If the zeros of a sequence of polynomials are restricted in some manner, then the uniform convergence of the polynomials on a particular subset of the plane to a function not identically zero may imply the uniform convergence on every subset of the plane. Pólya [Rend. Circ. Mat. Palermo **36** (1913), 279–295] proved the following in adding rigor to ideas of Laguerre. If the zeros of the polynomials are real, then the subset can be taken as a disc with center at the origin. The authors show that with the same restriction on the zeros, the subset can be taken as any finite interval of the real axis. More generally, the authors show that if the zeros are restricted to be in the half-plane $\operatorname{Im} z \geqq 0$, then the subset can be taken as any Jordan arc in the half-plane $\operatorname{Im} z \leqq 0$. A counterexample is given to show that the Jordan arc cannot be replaced by an infinite set of points with limit point at the origin. In the counterexample, the points chosen are pure imaginary. It would be of interest to know if there is a counterexample with real points, since this would indicate that the interval in their first quoted theorem is in some sense a natural limit.

J. L. Ullman (Ann Arbor, Mich.)

Govorov, N. V. 5187

On a homogeneous Riemann boundary-value problem with infinite index. (Russian)

Vesci Akad. Navuk BSSR Ser. Fiz.-Tèhn. Navuk **1964**, *no.* 1, 12–17.

The homogeneous Riemann boundary problem ($\phi^+(t) = G(t)\phi^-(t)$) is considered for a domain whose boundary is a simple smooth open contour with initial point t_0 and extending to ∞. The coefficient $G(t)$ is assumed to have

infinite index, i.e., $\lim_t \arg G(t) = +\infty$. Solutions are given but no proofs or derivations are included.

J. F. Heyda (King of Prussia, Pa.)

Ivanov, L. D. 5188

Removable singularities of interior mappings. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1963**, *no.* 1 (32), 81–84.

This paper is devoted to a proof of the following theorem. Suppose K is a perfect nowhere dense subset of a domain G. Suppose $f(z)$ is continuous in G, interior (in the sense of Stoïlow) on $G \setminus K$, and one-to-one on K. Then f is interior on G. This result generalizes certain lemmas of Trohimčuk [for example, in Mat. Sb. (N.S.) **45 (87)** (1958), 233–260; MR **21** #119; Amer. Math. Soc. Transl. (2) **17** (1961), 251–275; MR **23** #A1824].

E. Reich (Stanford, Calif.)

POTENTIAL THEORY

See also 5246, 5249, 5262.

Bateman, P. T. 5189

Sequences of mass distributions on the unit circle which tend to a uniform distribution.

Amer. Math. Monthly **71** (1964), 165–172.

The principal result established in this paper is as follows. Let ϕ be a complex-valued function of a real variable which is continuous, has period 1, and is of bounded variation on (0, 1], so that the curve C: $z=\phi(t)$, $0 \leqq t \leqq 1$, is closed and rectifiable. Let N_1, N_2 be two mass distributions on C, each of total mass 1. For $0<\beta-\alpha\leqq 1$ and $i=1, 2$, let $N_i(\alpha, \beta)$ denote the amount of mass from N_i lying on the part of C specified by the inequality $\alpha<t\leqq\beta$. Write

$$T_i = \sup_{V \leqq N_i} \left| \int_0^1 \phi(t)\, dV(0, t) \right|,$$

where the upper bound is taken over all mass distributions V such that $V(\alpha, \beta) \leqq N_i(\alpha, \beta)$ whenever $0<\beta-\alpha\leqq 1$. Put

$$\Delta = \sup_{\alpha<\beta\leqq\alpha+1} |N_1(\alpha, \beta)-N_2(\alpha, \beta)|$$

and let $L=\sup_\lambda L(\lambda)$, where $L(\lambda)$ denotes the total variation on the unit interval of the function ϕ_λ given by

$$\phi_\lambda(t) = \max\{0, \Re(e^{-2\pi i\lambda}\phi(t))\}.$$

Then $|T_1-T_2| \leqq L\Delta$.

A number of specializations of this theorem are discussed and, in particular, the following result which had been conjectured by the reviewer [same Monthly **69** (1962), 772–775; MR **27** #2473] is proved. Let $\{d_n\}$ be a sequence of positive integers tending to ∞; and suppose that, for any n, D_n is a set of d_n complex numbers of unit modulus. If $0<\beta-\alpha\leqq 1$, denote by $N_n(\alpha, \beta)$ the number of $z \in D_n$ such that $2\pi\alpha < \arg z \leqq 2\pi\beta$. Suppose that

$$\lim_{n\to\infty} \frac{N_n(\alpha, \beta)}{d_n} = \beta-\alpha$$

whenever $0<\beta-\alpha\leqq 1$. Then, writing

$$T_n = \max_{D \subseteq D_n} \left|\sum_{z\in D} z\right|,$$

we have

$$\lim_{n\to\infty} \frac{T_n}{d_n} = \frac{1}{\pi}.$$

L. Mirsky (Sheffield)

Du Plessis, N. 5190

Two counter-examples associated with the Dirichlet problem.

Quart. J. Math. Oxford Ser. (2) **15** (1964), 121–130.

The author gives two examples of exceptional sets for the Dirichlet problem in R^p. The first of these, valid for $p \geqq 3$, generalizes a classical one of Zaremba. It yields a region G whose boundary has two components: S, a sphere of radius $\sqrt{p}$ and center $(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$; and $C^{(p)}$, the p-fold Cartesian product of a one-dimensional Cantor set of parameter γ, $0<\gamma<\frac{1}{2}$. For each $\alpha<p-2$, γ can be chosen so that $C^{(p)}$, which is exceptional for the Dirichlet problem for G, has positive α-capacity.

The second example, which generalizes the Lebesgue "épine", is valid only for $p \geqq 4$. It furnishes a region with a connected boundary that contains an exceptional set of positive α-capacity for each $\alpha<p-3$. The question is raised whether it is possible to construct a region with a connected boundary that contains an exceptional set of positive α-capacity, where $\alpha \in [p-3, p-2)$.

E. C. Schlesinger (New London, Conn.)

Carleson, Lennart 5191

Removable singularities of continuous harmonic functions in R^m.

Math. Scand. **12** (1963), 15–18.

Let D be a domain in the m-dimensional Euclidean space bounded by an outer surface Γ and a compact set E. The set E is said to have β-dimensional measure zero, $0<\beta<m$, if it can be covered by open spheres of radii r_ν such that $\sum_\nu r_\nu{}^\beta$ is arbitrarily small. The set E is called removable for a class of harmonic functions in D if it can be asserted that every function of this class which vanishes on Γ must vanish identically in D. Finally, let H_α be the class of harmonic functions in D which satisfy the Hölder condition $|u(x)-u(x')| \leqq C|x-x'|^\alpha$; $0<\alpha<1$; $x, x' \in D$. The following theorem is proved: E is removable for the class H_α if and only if E has the $(m-2+\alpha)$-dimensional measure zero. *M. Schiffer* (Stanford, Calif.)

Prilepko, A. I. 5192

On the inverse problem for the metaharmonic potential. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 534–537.

A metaharmonic function is a regular solution of the equation $\Delta V-k^2V=0$, and the function $V(x) = \int_T \exp(-k|x-y|)|x-y|^{-1}\,dy$ is called the metaharmonic potential of the body T of density one.

The problem is to determine the body T_1 if its exterior metaharmonic potential is known to be approximately equal, in the sense of some metric, to the metaharmonic potential V of a given body T. It is assumed that the body T is star-shaped with respect to one of its interior points, and that the boundary S of T is such that the functions in its parametric representation are twice differentiable

and that their second derivatives satisfy a Hölder condition with exponent less than one. For a Newtonian potential the problem was solved by V. K. Ivanov [Izv. Akad. Nauk SSSR Ser. Mat. **20** (1956), 793–818; MR **18**, 885].

The author's solution of the problem involves the solution of a non-linear integro-differential equation, and is accomplished by means of Ivanov's method.

D. H. Hyers (Los Angeles, Calif.)

SEVERAL COMPLEX VARIABLES

See also 5147, 5448.

Hitotumatu, Sin 5193

On plurisubharmonic functions. (Japanese)

Sûgaku **11** (1959/60), 163–166.

This is a brief survey of plurisubharmonic functions. After stating the general theory, the author discusses, without proofs, the Hartogs functions and the boundary-value problem of Bremermann with its applications.

K. Noshiro (Nagoya)

Xu, Yi-chao [Hsu, Yi-chao] 5194

On the groups of analytic autohomeomorphisms of bounded positive (m, p)-circular domains.

Acta Math. Sinica **13** (1963), 419–432 (*Chinese*); *translated as Chinese Math.* 4 (1964), 454–469.

From the author's introduction: "Let D be a bounded schlicht domain of the space of two complex variables x, y, including the origin. Let m, p be a pair of coprime positive integers, with $m > p$. The domain D is called the bounded positive (m, p)-circular domain having the origin as center, if the maximal connected fixed component group of analytic autohomeomorphisms at the origin is a one-dimensional real Lie group

$$(1) \qquad x' = xe^{im\theta}, \qquad y' = ye^{ip\theta}, \qquad 0 \leqq \theta < 2\pi.$$

My paper determines all the groups of analytic autohomeomorphisms of bounded positive (m, p)-circular domains. Furthermore, for the cases where the dimensionality of the group exceeds 1, I solve the classification problem underlying the significance of analytic equivalence."

Arai, Hiraku 5195

Holomorphic mappings of normal domains in a space of several complex variables. (Russian)

Mat. Sb. (*N.S.*) **55 (97)** (1961), 481–487.

Spallek, Karlheinz 5196

Einige Untersuchungen über analytische Modulgarben.

Math. Ann. **153** (1964), 428–441.

This paper deals with two theorems of the type of the Osgood-Hartogs theorem (a function of several complex variables is holomorphic if it is holomorphic in each variable separately). In the first one, the author considers a domain G in $\mathbf{C}^n$ with structural sheaf $\mathcal{O}(G)$ and a coherent subsheaf $M \subset m\mathcal{O}(G)$, and constructs to a given analytic subset A of G and the integer q (=0 or 1) the sheaves $M[A, q] \supset M$ consisting of those elements in $m\mathcal{O}(G)$ that for every $y \in A$ satisfy the obvious generalization of the Hartogs hypothesis with respect to a sufficiently large family of q-dimensional analytic sets. $M[A, q]$ is shown to be coherent. The second one deals with families of sections in coherent sheaves of modules which depend holomorphically upon parameters.

H. Röhrl (Minneapolis, Minn.)

Shimizu, Hideo 5197

On traces of Hecke operators.

J. Fac. Sci. Univ. Tokyo Sect. I **10**, 1–19 (1963).

Let $\mathfrak{F}_n$ be the product of n copies of the complex plane minus the real axis, on which the product G of n copies of the group $\mathbf{GL}(2, \mathbf{R})$ of 2×2 invertible real matrices acts in the obvious way. Let Γ be a subgroup of G which is properly discontinuous on F_n, χ a finite-dimensional unitary representation of Γ with kernel of finite index and α an element of G such that $\alpha\Gamma\alpha^{-1} \cap \Gamma$ has finite intersection in Γ and $\alpha\Gamma\alpha^{-1}$. It is assumed that the intersection of Γ with the identity component of G has, on the product of the n upper half-planes, a fundamental domain satisfying condition (F) of a previous paper [Ann. of Math. (2) **77** (1963), 33–71; MR **26** #2641]. One main purpose of the paper is to compute the trace of the Hecke operator $\mathfrak{T}(\Gamma\alpha\Gamma)$ on the space of cusp forms with values in the representation space of χ for an automorphy factor which is a product of n factors of the form $(cz+d)^{-k}|\det \gamma|^{-k/2}$. For this, the author uses the known projector from square integrable automorphic forms to cusp forms [cf. Godement, Séminaire H. Cartan, 1957/1958, Exp. 8, Secrétariat mathématique, Paris, 1958; MR **21** #2750], the trace formula, and evaluates the integrals occurring in the latter. He then considers the case where Γ is the group of units of a maximal order in an indefinite quaternion algebra over a totally real number field and computes the trace of the operator $\mathfrak{T}(\mathfrak{q})$ introduced by Shimura [Ann. of Math. (2) **76** (1962), 237–294].

A. Borel (Princeton, N.J.)

SPECIAL FUNCTIONS

See also 5293, 5306, 5660.

Gradštein, I. S. [Градштейн, И. С.]; Ryžik, I. M. [Рыжик, И. М.] 5198

★Tables of integrals, sums, series and products [Таблицы интегралов, сумм, рядов и произведений].

Fourth edition. Revised with the collaboration of Ju. V. Geronimus and M. Ju. Ceĭtlin.

Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1963. 1100 *pp.* 4.33 *r.*

The third edition [GITTL, Moscow, 1951; MR **14**, 643] of this useful collection has become well known both in its original form and in a German translation [VEB Deutscher Verlag der Wiss., Berlin, 1957; MR **22** #3120]. The new edition is roughly double in size, but has kept the same general arrangement of material.

From the preface: "From the preceding (third) edition have been included without change the parts pertaining to sums, series, products and elementary functions. The remaining parts have been revised. In particular, the

tables of definite integrals of elementary and special functions have been much extended. . . . In general, in the fourth edition the number of special functions considered has been increased in comparison with the third edition." *J. V. Wehausen* (Berkeley, Calif.)

Olsson, Per O. M. 5199

Solutions in a special case of the partial differential equations associated with the Appell function F_2.

Ark. Fys. **25** (1964), 473–480.

Author's summary: "Solutions of the partial differential equations associated with Appell's hypergeometric function $F_2(a, b_1, b_2, c_1, c_2, x, y)$ are given for $a=1$. This choice gives immediately three of the four independent solutions in terms of the hypergeometric functions F_1 or related functions. From a previous investigation [the author, J. Mathematical Phys. **5** (1964), 420–430; MR **28** #3179] analytic continuations of these three solutions are easily obtained. These continuations are all expressed in terms of hypergeometric functions of order two and give the behaviour of the solutions near singular points. The fourth solution is expressed in a series of ${}_3F_2$ functions. Its analytic properties can be studied with the aid of results obtained by Nørlund [Acta Math. **94** (1955), 289–349; MR **17**, 610] who investigated the solutions of the differential equations associated with the functions ${}_pF_{p-1}(x)$."

Chatterjea, S. K. 5200

An integral representation for the product of two generalized Bessel polynomials.

Boll. Un. Mat. Ital. (3) **18** (1963), 377–381.

Put

$$\Phi_n(c, x) = \frac{(c)_n}{n!}\, {}_2F_0[-n, c+n; -; x].$$

The author proves that

$$\frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}\Phi_m(c, x)\Phi_n(d, y) =$$
$$\frac{2^{m+n}}{\pi}\int_0^1\int_{-\pi/2}^{\pi/2} t^{c+m-1}(1-t)^{d+n-1}e^{(m-n)\theta i}\cos^{m+n}\theta$$
$$\times\Phi_{m+n}\left(c+d, \frac{xte^{-\theta i}+y(1-t)e^{\theta i}}{2\cos\theta}\right)d\theta dt \qquad (c>0, d>0),$$

a formula suggested by a double integral representation for $L_n^{(\alpha)}(x)L_n^{(\beta)}(y)$ found by the reviewer [same Boll. (3) **17** (1962), 25–28; MR **25** #1318]. A single integral representation was obtained earlier by Al-Salam [Duke Math. J. **24** (1957), 529–545; MR **19**, 849].

L. Carlitz (Durham, N.C.)

Roberts, J. B. 5201

Some orthogonal functions connected with polynomial identities. II.

Proc. Amer. Math. Soc. **15** (1964), 127–130.

The author continues to generalize the orthogonal step functions given in his earlier paper [same Proc. **11** (1960), 723–730; MR **23** #A100]. These are based on a certain very general polynomial identity and on the digits in the Cantor representation of numbers

$$n = a_0+a_1p_1+\cdots+a_{m-1}p_{m-1} \quad (0 \leqq a_j < n_{j+1}),$$

where n_i are arbitrary integers >1 and $p_j = n_1n_2\cdots n_j$.

D. H. Lehmer (Berkeley, Calif.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 5437, 5466, 5587, 5598, 5599, 5863, 5870, 5877, 5879, 5893.

Rao, P. R. P. 5202

A differential equation.

Amer. Math. Monthly **71** (1964), 530–533.

On traite l'équation différentielle

$$(1) \qquad y'(x) = \sum_{i=0}^{n} f_i(x)y^i(x).$$

La transformation $y=vu-f_{n-1}/nf_n$ réduit (1), sous certaines conditions, à l'équation

$$u' = (f_n^{2-n}U^{1-1/n})(1-C_1u+C_2u^2+\cdots+C_{n-2}u^{n-2}+u^n)$$

séparant les variables. On montre aussi que deux autres types d'équations non-linéaires peuvent être réduits à la forme (1) par les transformations convenables de la variable y seulement. On ne cite aucune bibliographie.

M. Bertolino (Belgrade)

Bagai, O. P. 5203

Evaluation of certain definite integrals by the use of differential equations.

Math. Student **30** (1962), 179–183 (1963).

In this paper the author gives a method for the computation of the integral

$$L_r(a) = 2\int_0^\infty x^{2r+1}\exp(-x^2-ax^{-1})\,dx$$

($\mathrm{Re}\{a\}>0$, r an integer $\geqq 0$) by the aid of differential equations. Only the integral $L_0(a)$ is effectively computed, the corresponding differential equation being $a\,d^3L_0/da^3+2L_0=0$. This equation is solved by the Frobenius method. The result is given in the form of a series.

P. M. Vasić (Belgrade)

Wouk, Arthur 5204

Direct iteration, existence and uniqueness.

Nonlinear Integral Equations (Proc. Advanced Seminar Conducted by Math. Research Center, U.S. Army, Univ. Wisconsin, Madison, Wis., 1963), *pp.* 3–33. *Univ. Wisconsin Press, Madison, Wis.*, 1964.

This paper is largely an exposition and historical survey of questions of existence, uniqueness, and convergence of successive approximations for differential and integral equations, culminating in an abstract uniqueness and iteration theorem for a class of operators on a partially ordered space. This abstract theorem contains most, but not quite all, of the known results on uniqueness and convergence of successive approximations.

F. Brauer (Madison, Wis.)

v. Krzywoblocki, M. Z. 5205
Integral operators in ordinary differential equations.
J. Reine Angew. Math. **214/215** (1964), 137–140.
The author assumes a solution U of the equation $U''-G(x)U'-F(x)U=0$ to be of the form $U(x)=\sum_0^\infty f_n(x)$ in which the functions f_n are determined by quadratures from the recursion formula $f_{n+1}''(x)=G(x)f_n'(x)+F(x)f_n(x)$, $n\geq 0$, $f_0(x)=x$. He also discusses the convergence of the series. *H. A. Antosiewicz* (Los Angeles, Calif.)

Nehari, Zeev 5206
On the zeros of solutions of n-th order linear differential equations.
J. London Math. Soc. **39** (1964), 327–332.
The author proves the following theorem established for $n=2$ by D. London [Pacific J. Math. **12** (1962), 979–991; MR **26** #3958]: Let $p_0(z), p_1(z), \cdots, p_{n-1}(z)$ be analytic functions which are regular in a closed convex region D whose boundary is a piecewise smooth curve C. If

$$\frac{\delta^{n-1}}{(n-1)}\int_C |p_0(w)\,dw| + \sum_{\nu=1}^{n-1}\delta^{n-\nu-1}\int_C |p_\nu(w)\,dw| < 2,$$

where δ is the diameter of D, then no non-trivial solution of the differential equation (1) $y^{(n)}+p_{n-1}(z)y^{(n-1)}+\cdots+p_0(z)y=0$ can have more than $n-1$ zeros in D.

This result is then applied to obtain a non-oscillation criterion for (1). *T. L. Sherman* (Madison, Wis.)

Duffin, R. J. 5207
Chrystal's theorem on differential equation systems.
J. Math. Anal. Appl. **8** (1964), 325–331.
The author gives a new and simple proof of Chrystal's result concerning the number of constants of integration for linear matrix-vector differential systems with constant coefficients. An analogue is given for difference equations. The discussion is restricted to equations of order 3.
W. J. Coles (Madison, Wis.)

Wagner, Norman 5208
Existence theorem for a non-linear boundary value problem in ordinary differential equations.
Contributions to Differential Equations **3** (1964), 325–336.
The title theorem concerns a system of two coupled second-order equations; the linear part is allowed to become singular at one endpoint of the interval. The nonlinearity is of a rather special type, but is such that the boundary-value problem includes a practical and important class of problems of finite axisymmetric bending of thin shells of revolution. In particular, the paper settles the question of existence of solutions for shallow spherical shells under normal pressure, thus completing earlier work of the reviewer [J. Math. and Phys. **38** (1959/60), 209–231; MR **22** #11655].

The existence proof succeeds by introduction of a functional, motivated by the special type of nonlinearity considered, and by reducing the solution of the resulting scalar equation for one unknown function to a minimum problem (M). The existence of solutions of (M) is obtained by methods similar to those employed by Friedrichs and Stoker in nonlinear plate problems [Amer. J. Math. **63** (1941), 839–888; MR **3**, 223].
H. J. Weinitschke (Hamburg)

Skaček, B. Ja. 5209
Asymptotic behaviour of the negative part of the spectrum of one-dimensional differential operators. (Russian)
Approximate methods of solving differential equations, pp. 96–109. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.
Let L be a self-adjoint realization on $L_2(0,\infty)$ of the differential operator $l(y)=(-1)^n y^{(2n)}-q(x)y$. The function $q(x)$ is assumed for large x to be positive and monotone decreasing to zero. Furthermore, $\lim_{x\to\infty} f(x)\ln_j^{k_j-\alpha}x=\lim_{x\to\infty}[f(x)\ln_j^{k_j+\alpha}x]^{-1}=0$, where $\alpha>0$, $k_j>0$ (and $0<k_0<2n$), and $\ln_0 x=x$, $\ln_{j+1}x=\ln(\ln_j x)$. Then the number $m(\varepsilon)$ of eigenvectors to the left of $-\varepsilon$ satisfies the asymptotic formula, as $\varepsilon\downarrow 0$,

$$m(\varepsilon)=\frac{1}{\pi}[1+O(1)]\int_{q(x)\geq\varepsilon}(q-\varepsilon)^{1/2n}\,dx.$$

In particular, when the comparison function is $1/\ln^{k_j}x$ or $1/x^{k_0}$, then $O(1)$ can be refined to $O(\exp[-\tau/\varepsilon])$ or $O(\varepsilon^\tau)$, respectively. The formula for $m(\varepsilon)$ had previously been obtained for $n=1$ under more stringent conditions on $q(x)$ by N. Rosenfeld [Comm. Pure Appl. Math. **13** (1960), 395–405; MR **22** #9656].
Robert McKelvey (Madison, Wis.)

Maksudov, F. G. 5210
On the spectrum of singular non-selfadjoint differential operators of order $2n$. (Russian)
Dokl. Akad. Nauk SSSR **153** (1963), 758–761.
Consider the differential expression

$$l(y)\equiv(-1)^n(P_0(x)y^{(n)})^{(n)}+(-1)^{n-1}(P_1(x)y^{(n-1)})^{(n-1)}+\cdots+P_n(x)y,$$

where the coefficients are complex-valued functions defined on the interval $[0,\infty)$. Under various assumptions about the behavior of the functions $1/P_0(x), P_1(x),\cdots,P_n(x)$, the author uses $l(y)$ and a boundary condition at 0 to define an operator in the Hilbert space $L^2[0,\infty)$, determines the nature of the spectrum of this operator, and shows that its resolvent is an integral operator whose kernel $K(x,t,\lambda)$ is either of Hilbert-Schmidt type or else satisfies the inequalities $\int_0^\infty |K(x,t,\lambda)|^2\,dx<\infty$, $\int_0^\infty |K(x,t,\lambda)|^2\,dt<\infty$. Formulas are given in each case for the asymptotic behavior as $x\to\infty$ of $2n$ linearly independent solutions of the equation $l(y)=\lambda y$.
R. C. Gilbert (Fullerton, Calif.)

Dettman, John W. 5211
The solution of a second order linear differential equation near a regular singular point.
Amer. Math. Monthly **71** (1964), 378–385.
The author suggests the early introduction and development throughout the first course in differential equations of Green's functions and the attendant use of integral equations equivalent to given differential equations. This approach is used to prove an existence-uniqueness theorem for $xy''+p(x)y'+q(x)y=0$, using such notions as continuity, differentiability, uniform convergence, linear operators and the Bessel function I_0.
G. M. Ewing (Norman, Okla.)

Kiyek, Karl-Heinz 5212

★**Zur Theorie der linearen Differentialgleichungssysteme mit einem grossen Parameter.**

Inauguraldissertation zur Erlangung der Doktorwürde der Hohen Naturwissenschaftlichen Fakultät der Julius-Maximilians-Universität Würzburg, Würzburg, 1963. vi + 99 *pp.*

The systems under consideration are of the form $dw/dz = u^r A(z, u)w$, where z is a complex variable, w a complex-valued vector function of z and of the large positive parameter u. The author studies the solution of such systems by means of expressions involving asymptotic series in powers of u^{-1}. This is a well-explored problem, but the author emphasizes an aspect of the theory that has not yet been investigated in any generality for systems, namely, the question as to when such expansions are valid in unbounded regions of the z-plane. Of the existing work on such problems for special differential equations the papers by Olver [see, e.g., Philos. Trans. Roy. Soc. London Ser. A **250** (1958), 479–517; MR **20** #1012] appear to have been particularly useful to the author.

The case that all eigenvalues of $A(z, \infty)$ are distinct is treated in great generality. If multiple eigenvalues occur, particularly if a turning point is present, the results are less complete. One chapter is devoted to the case that $A(z, u)$ has a pole of first order at $z = 0$.

Several known techniques are drawn upon by the author and modified in various ways so as to be applicable in unbounded domains. The conditions that are introduced to get the desired expansions are sometimes rather numerous and their verification in concrete situations may still offer serious difficulties.

Throughout the paper it is assumed that the differential equation as well as the region of validity of the results depend on a second, scalar or vectorial, parameter θ. The role of this parameter has not been quite clear to this reviewer. If θ is interpreted as an angle, it can presumably be used to extend the results from positive u to a complex parameter of the form $ue^{i\theta}$. *W. Wasow* (Madison, Wis.)

Volosov, V. M.; Morgunov, B. I. 5213

Asymptotic behaviour of certain rotary motions. (Russian)

Dokl. Akad. Nauk SSSR **151** (1963), 1260–1263.

In dieser Arbeit wird die Asymptotik von Drehbewegungen mit einem Freiheitsgrad in Abhängigkeit von sich langsam ändernden Parametern untersucht. Die Bewegungsgleichungen des ungestörten Systems haben die Form

$$m(x)\ddot{y} + Q(x, y) = 0, \quad x = \text{const},$$

wobei y eine eindimensionale Koordinate, $m(x)$ die Masse, $x = (x_1, x_2, \cdots, x_n)$ die Gesamtheit der Parameter, $Q(x, y) \equiv \partial V(x, y)/\partial y$ die die Bewegung hervorrufende Kraft ist. $Q(x, y)$ ist periodisch bezüglich y mit der Periode 2π, es ist $\int_0^{2\pi} Q(x, y)\, dy = 0$. Das entsprechende gestörte System wird durch

$$\frac{d}{dt}(m(x)\dot{y}) + Q(x, y) = \varepsilon f(x, y, \dot{y}), \quad \dot{x} = \varepsilon X(x, y, \dot{y})$$

beschrieben. Dabei ist ε ein kleiner Parameter; f und x sind periodisch bezüglich y.

Die Autoren leiten Gleichungen her, die die langsamen Änderungen der Energie $E(\varepsilon, t)$ der gestörten Bewegung und der Parameter x charakterisieren. Die Asymptotik von Schwingungen ähnlicher Systeme wird in den folgenden Arbeiten Volosov's [dieselben Dokl. **106** (1956), 7–10; MR **18**, 41; ibid. **114** (1957), 1149–1152; MR **20** #1822; ibid. **117** (1957), 927–930; MR **20** #1823; ibid. **121** (1958), 959–962; MR **21** #170; ibid. **121** (1958), 22–25; MR **21** #2786] angegeben. In Moiseev [Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 145–158; MR **27** #5978] und Černous'ko [ibid. **3** (1963), 131–144; MR **27** #5986] wird die Asymptotik von Drehbewegungen mit Methoden untersucht, die sich von denen der vordringenden Arbeit unterscheiden.

B. S. Popov (Skopje)

Morgunov, B. I. 5214

Higher approximations in calculating certain rotational motions. (Russian)

Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1963**, *no.* 6, 35–42.

Für die Energie von Drehbewegungen (siehe vorstehendes Referat [#5213] von Morgunov und Volosov) wird mit Hilfe asymptotischer Methoden die zweite Näherung untersucht. Es wird ein für die Anwendungen wichtiger Fall von Bewegungen mit großer Energie betrachtet, und es werden Beispiele aus der Physik eingeführt.

B. S. Popov (Skopje)

Stoljarskaja, N. E. 5215

On the construction of solutions of non-linear systems subject to exterior periodic forces depending explicitly on time. (Russian)

Ukrain. Mat. Ž. **15** (1963), 332–338.

The purpose of the paper is to obtain approximate solutions of the system

$$(1) \qquad \dot{x} = -\omega y + \varepsilon X(\gamma t, x, y), \quad \dot{y} = \omega x + Y(\gamma t, x, y),$$

where X and Y are periodic with respect to γt of period 2π and ε is a small parameter. Solutions are sought in the form

$$z = ae^{i\psi} + \varepsilon u_1(a, \psi, \gamma t) + \varepsilon^2 u_2(a, \psi, \gamma t) + \cdots,$$

where $z = x + iy$, a and ψ are slowly varying functions of time t, and u_k are complex-valued functions of period 2π with respect to γt. Differential equations for a, ψ, $u_k(a, \psi, \gamma t)$ are derived.

As an example, the generalized van der Pol equation is discussed. *C. Olech* (Kraków)

Brown, T. A. 5216

A uniqueness condition for nontrivial periodic solutions to the Lienard equation.

J. Math. Anal. Appl. **8** (1964), 387–391.

The equation considered in this paper is the real differential equation (*) $\ddot{x} + f(x)\dot{x} + q(x)x = 0$. The main hypotheses are, on setting $F(x) = \int_0^x f(\xi)\, d\xi$ and $Q(x) = \int_0^x q(\xi)\xi\, d\xi$, as follows: (i) $q(x) > 0$ $(x \neq 0)$; (ii) there exist real numbers $a < 0$, $b > 0$ such that $F(a) = F(b) = F(0) = 0$ and $xF(x) < 0$ for every other x in the range $a \leqq x \leqq b$; (iii) $f(x) \geqq 0$ for $x < a$ and $x > b$; (iv) $Q(x) \to +\infty$ as $|x| \to \infty$; (v) $Q(a) = Q(b)$. The author shows that under the conditions (i)–(v) there exists a unique (up to translations in t) non-trivial periodic solution to which all other solutions tend. The result extends a result of Levinson and Smith

[Duke Math. J. 9 (1942), 382–403; MR 4, 42]; see particularly Theorem III. The proof of the result is based on some estimates for a certain energy function in the phase space of (*). *J. O. C. Ezeilo* (Ibadan)

Van Vleck, F. S. 5217
A note on the relation between periodic and orthogonal fundamental solutions of linear systems.
Amer. Math. Monthly **71** (1964), 406–408.
Consider the n-vector differential system (*) $dx/dt=Ax$ in which A is a real constant n-by-n matrix. Let $X=X(t)$ be the fundamental solution of (*) and let X^T denote the transpose of X. The fundamental solution is said to be orthogonal if $XX^T=I$ (the unit matrix) for all t. In this short note the author derives a simple characterization of orthogonal fundamental solutions of equations of the type (*). By using this characterization the author further deduces a necessary and sufficient condition for an orthogonal fundamental solution of (*) to be periodic in t.
J. O. C. Ezeilo (Ibadan)

Ivanov, T. F. 5218
On periodic motions of a class of autonomous systems.
Prikl. Mat. Meh. **27** (1963), 1124–1127 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1727–1733.
The author proved in two of his earlier papers [Dokl. Akad. Nauk SSSR **143** (1962), 297–300; MR **25** #5623; Izv. Akad. Nauk SSSR Otd. Tehn. Nauk Meh. Mašinostr. **1962**, no. 5, 129–133] that intermediate and mixed problems can be formed and solved in relation to the equation $\ddot{x}+\dot{x}p(x)+xq(x)=0$ and that the results, so obtained, can lead to all the periodic solutions of this equation. In the present paper, the same method is used to establish three theorems on the existence of such periodic solutions, with the help of two rather elaborately worked examples. The method of the proofs is cursory and based on topological geometry. The convergence of the integrals involved is assumed without mention. {The notation used is not uniform, and there are some misprints}.
B. Dube (Raipur)

Puri, N. N.; Weygandt, C. N. 5219
Second method of Liapunov and Routh's canonical form.
J. Franklin Inst. **276** (1963), 365–384.
The authors establish the well-known equivalence between Liapunov's second method and the Routh-Hurwitz stability criterion for linear differential systems with constant coefficients. In this connection, see the excellent paper by P. C. Parks [Proc. Cambridge Philos. Soc. **58** (1962), 694–702; MR **26** #1580], which is incidentally not quoted by the authors. It should be pointed out that the authors consider only the case when the characteristic roots of the system are distinct, without, however, mentioning it in their statements. The statements about Liapunov's second method starting at the bottom of page 366 and carried over to page 367 are not true without further qualification. Another half-truth appears on page 368 after equation (14). The correct statements may be seen in the book by LaSalle and Lefschetz [*Stability by Liapunov's direct method with applications*, Academic Press, New York, 1961; MR **24** #A2712].
N. P. Bhatia (Cleveland, Ohio)

Puri, N. N.; Weygandt, C. N. 5220
A contribution to the transform calculus used in the synthesis of control systems.
J. Franklin Inst. **277** (1964), 337–348.
The authors present a simple method for finding the Laplace transform of the product of two known time functions, the average value of the product, and the time-weighted average of the product. The method is applicable only in the case that the Laplace transforms of the given time functions are ratios of polynomials.
N. P. Bhatia (Cleveland, Ohio)

Kato, Junji 5221
The asymptotic behavior of the solutions of differential equations on the product space.
Japan J. Math. **32** (1962), 51–85.
The equation discussed here is of the form (1) $dx/dt=F(t,x,y)$, where x is an element of the n-dimensional space E^n, y is an element of a family Y of functions with values in E^m, and t is a real variable over the interval $I=[0,\infty)$. In the first part of this paper, the author proves a variety of converse theorems for (1), concerning the existence of Liapunov functions $V(t,x,y)$ with appropriate properties, under the assumption that (1) has a certain stability behavior with respect to a set $M\subset I\times E^n$. The methods used are closely related to those of T. Yoshizawa [J. Math. Kyoto Univ. **1** (1961/62), 303–323; MR **26** #1577]. The author then applies these results to systems of coupled equations, where the family Y then is chosen, for example, to be a certain family of solutions of a single equation of the system. The theorems are quite general, and include as special cases some theorems concerning the behavior of solutions near families of periodic solutions; see Hale and the reviewer [Arch. Rational Mech. Anal. **6** (1960), 133–170; MR **22** #9670]. For related results, using Liapunov functions in a similar manner, which apply either to ordinary differential equations or to functional differential equations, see Hale [Contributions to Differential Equations **1** (1963), 401–410; MR **26** #6510; ibid. **1** (1963), 411–423; MR **26** #6511].
A. Stokes (Washington, D.C.)

Van, Chzhao-lin [Van, Čžao-lin] 5222
On the converse of Routh's theorem.
Prikl. Mat. Meh. **27** (1963), 890–893 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1354–1360.
Četaev's instability theorem is used to obtain results on the instability of gyroscopically constrained mechanical systems. A result is given, for example, in the case where some of the Poincaré stability coefficients are zero.
J. P. LaSalle (Baltimore, Md.)

Yoshizawa, Taro 5223
Stability of sets and perturbed system.
Funkcial. Ekvac. **5** (1963), 31–69.
Consider the differential equation (1) $dx/dt=F(t,x)$, where $x\in R^n$, the Euclidean n-space, $t\in I$, the non-negative real axis and F is defined and continuous on $I\times R^n$ or on a suitable neighborhood of a set $M\subset I\times R^n$. Call $x(t;x_0,t_0)$ the solution of (1) through (t,x), $M(t)$ the set of points of M with first coordinate t and $d(x,A)$ the distance between x and $A\subset R^n$. The following definition plays an important

role in the paper: M is said to be a stable set of (1) if given arbitrary $t_0 \in I$, $\varepsilon > 0$, $\alpha > 0$, there is a $\delta = \delta(t_0, \varepsilon, \alpha) > 0$ such that if $d(x_0, M(t_0)) < \delta$ and $\|x\| \leqq \alpha$, then

$$d(x(t; x_0, t_0), M(t)) < \varepsilon$$

for all $t \geqq t_0$. Many other definitions patterned on the above are given, and in all there are 25 types of stability that M may possess with respect to (1). Relating these different types of stability and the existence of a Liapounov function for some 33 theorems (and as many corollaries) are given. To state one of them we need two definitions: (i) The solutions of (1) are said to be equi-M-bounded if corresponding to any $\eta > 0$, $\alpha > 0$, $t_0 \in I$, there is a positive number $\beta = \beta(t_0, \eta, \alpha)$, continuous on t_0, such that if $d(x_0, M(t)) \leqq \eta$ and $\|x_0\| \leqq \alpha$, then

$$d(x(t; x_0, t_0), M(t)) \leqq \beta(t_0, \eta, \alpha)$$

for $t \geqq t_0$; (ii) we say that $F(t, x) \in C_0(x)$ if given $\alpha > 0$ then $\|f(t, x) - f(t, x')\| \leqq L(\alpha)\|x - x'\|$ for all $t \in I$ and $\|x\| \leqq \alpha$, $\|x'\| \leqq \alpha$, where $L(\alpha)$ is a constant depending on α. Theorem: Suppose that $F(t, x) \in C_0(x)$, M is a stable set of (1) and the solutions of (1) are equi-M-bounded and approach M as $t \to \infty$; then there exists a continuous Liapounov function $V(t, x) \in C_0$ defined on $I \times R^n$ and such that (a) $V(t, x) = 0$ if $(t, x) \in M$; (b) $a(d(x, M(t))) \leqq V(t, x)$, where $a(r)$ is continuous, positive and increasing, $a(r) \to \infty$ as $r \to \infty$; (c) if $\|x\| \leqq \alpha$, $\|x'\| \leqq \alpha$, and $d(x, M(t)) \leqq \eta$, then $\|V(t, x) - V(t, x')\| \leqq h(\eta) l(\alpha) K(t) \|x - x'\|$, where h, l and K are continuous functions vanishing at the origin;

$$\text{(d)} \quad \lim_{h \to +0} \frac{1}{h}\{V(t+h, x+hF(t, x)) - V(t, x)\} \leqq -cV(t, x), \quad c > 0.$$

M. M. Peixoto (Rio de Janeiro)

Hochstadt, Harry 5224

On the stability of certain second order differential equations.

J. Soc. Indust. Appl. Math. **12** (1964), 58–59.

Ruiz [same J. **11** (1963), 148–158; MR **27** #1659] showed that solutions of the differential equation $y'' + P(t)y' - q^2y = 0$, where $P(t)$ is a periodic function of period 2π and q is a real positive constant, are unstable, that is, there are unbounded solutions. Here the author shows that this result holds for bounded $P(t)$ that are not necessarily periodic. It is also shown that the equation $y'' + P(t)y' + q^2y = 0$, where $P(t)$ is bounded and non-negative, has only bounded solutions. *E. Frank* (Chicago, Ill.)

Jataev, M. 5225

On a critical case of stability. (Russian. Kazak summary)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 3, 3–7.

The author considers systems $x' = y$, $y' = z$, $z' = Z(x, y)$ with $Z(0, 0) = 0$ and examines the stability of the origin for various forms of $Z(x, y)$, such as x^n, $x^n + bx^my$, by constructing appropriate Lyapunov functions. For a more general treatment of the critical case for vector systems $y' = Y(y, z)$, $z' = Az + Z(y, z)$ we refer to Lefschetz [Bol. Soc. Mat. Mexicana (2) **6** (1961), 5–18; MR **24** #A3354].

H. A. Antosiewicz (Los Angeles, Calif.)

Seibert, P. 5226

Prolongations and generalized Liapunov functions. (Russian summary)

Qualitative methods in the theory of non-linear vibrations (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. II,* 1961), *pp.* 332–341. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

The author and J. Auslander investigated stability properties of sets M with respect to an autonomous differential equation in E^n by means of generalized Ljapunov functions (i.e., functions defined in a neighbourhood of M, small in a sufficiently small neighbourhood, bounded away from 0 outside any neighbourhood, and non-increasing along the solutions; under suitable conditions, the last assumption can, as usual, be detected without reference to the solutions themselves). Ordinary (Ljapunov) stability is equivalent to the existence of a generalized Ljapunov function. The authors show that an abstract generalization of Ura's concept of prolongation [T. Ura, Ann. Sci. École Norm. Sup. (3) **70** (1953), 287–360; MR **16**, 247; Funkcial. Ekvac. **2** (1959), 143–200; MR **22** #1727] allows the introduction of a new concept, "absolute stability": M is absolutely stable if it is stable under the "ultimate" prolongation (i.e., under the prolongation transfinitely iterated); and this stability is characterized by the existence of a continuous generalized Ljapunov function. Similarly, total stability (stability under permanent perturbations) is considered—the authors call it "weak stability under perturbations"—and shown to imply absolute stability; it is contrasted with "strong stability under perturbations", which turns out to be equivalent to asymptotic stability. All these stabilities are characterized by the existence of suitable Ljapunov functions. To each type of stability corresponds a type of boundedness: ordinary stability—(equi-) boundedness; absolute stability—absolute boundedness; asymptotic stability—ultimate boundedness; etc. These types of boundedness are characterized by the existence of suitable Ljapunov functions "at infinity".

J. J. Schäffer (Montevideo)

Mihaĭlov, F. A. 5227

Two theorems on the theory of a linear homogeneous differential equation. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 120–126.

The first theorem in question gives the existence of the fundamental system $x^{(1)}(t), \cdots, x^{(n)}(t)$ of complex-valued solutions of an nth-order linear homogeneous differential equation with continuous and real coefficients such that $x^{(i)}(t) \neq 0$ for $t \in I$ and $i = 1, \cdots, n$, where I is a given closed interval.

The second theorem shows the possibility of reducing an nth-order linear homogeneous differential equation to a diagonal system $dz_i/dt = \zeta_i z_i$ $(i = 1, \cdots, n)$ by means of a map of a certain specified type (ζ_i are in general complex-valued functions of t). *C. Olech* (Kraków)

Krasnodembskiĭ, A. M. 5228

Global behaviour of solutions of differential equations of higher order. (Russian)

Ukrain. Mat. Ž. **15** (1963), 205–213.

The author considers a system of equations (1) $dy_i/dx =$

$f_i(x, y_1, \cdots, y_n)$, $i=1, 2, \cdots, n$, where the f_i are continuous and satisfy a uniqueness condition on the region $V: a_i \leqq y_i \leqq b_i, i=1, \cdots, n, -\infty < x < +\infty$, and in addition it is assumed that $0 \leqq f_i(x, y_1, \cdots, y_{i-1}, a_i, y_{i+1}, \cdots, y_n)$, $f_i(x, y_1, \cdots, y_{i-1}, b_i, y_{i+1}, \cdots, y_n) \leqq 0$, $i=1, \cdots, n$. Then it is shown that there exists at least one solution of (1) which remains in V, or, if the f_i are periodic in x, then there exists a periodic solution of (1) in V. Stability results are obtained for the above solutions by making various estimates on the functions $\partial f_i/\partial y_j = f_{ij}$. The estimates are rather complicated, but the essential idea is that $|f_{ii}|$ should dominate $|f_{ij}|, j \neq i$, sufficiently strongly, for each i. The next series of theorems concerns the application of the above results to an equation of the form (2) $y^{(n+1)} = f(x, y, y', \cdots, y^{(n)})$. Here the estimates become quite involved, and while the author does apply his results to a particular equation of the form of (2), for $n=2$, any general application seems unlikely.

A. Stokes (Washington, D.C.)

Moravčík, J. 5229

A remark on the transformation of the solutions of linear differential equations. (Slovak. Russian and German summaries)

Acta Fac. Nat. Univ. Comenian. **6**, 327–334 (1961).

Zwei lineare Differentialgleichungen zweiter Ordnung (a) $y''+a(t)y=0$, (A) $\ddot{Y}+A(T)Y=0$ können bekanntlich stets gegenseitige Transformationen ihrer Integrale y, Y die Form annehmen:

$$y(t) = Y[X(t)]:\sqrt{|X'(t)|}, \quad Y(T) = y[x(T)]:\sqrt{|\dot{x}(T)|}$$

wo X, x Lösungen der nichtlinearen Differentialgleichungen dritter Ordnung

$$\text{(Aa)} \qquad \{X, t\} + A(X)X'^2 = a(t),$$

$$\text{(aA)} \qquad \{x, T\} + a(x)\dot{x}^2 = A(T)$$

[der Referent, Ann. Mat. Pura Appl. (4) **41** (1956), 325–342; MR **20** #1814] sind.

In der vorliegenden Arbeit wird ein analoges Resultat für zwei iterierte lineare Differentialgleichungen vierter Ordnung:

$$\text{(a)} \qquad y^{(\mathrm{iv})} + 10a(t)y'' + 10a'(t)y' + 3[3a^2(t) + a''(t)]y = 0,$$

$$\text{(A)} \quad Y^{(\mathrm{iv})} + 10A(T)\ddot{Y} + 10\dot{A}(T)\dot{Y} + 3[3A^2(T) + \ddot{A}(T)]Y = 0$$

abgeleitet. In diesem Fall lauten die Transformationsformeln so:

$$y(t) = Y[X(t)]:\sqrt{|X'(t)|^3}, \quad Y(T) = y[x(T)]:\sqrt{|\dot{x}(T)|^3};$$

X, x sind wiederum Lösungen nichtlinearer Differentialgleichungen dritter Ordnung von der Form (Aa), (aA).

O. Borůvka (Brno)

Mamrilla, J. 5230

Some properties of the solution of the differential equation $y^{\mathrm{iv}} + 2A(x)y' + [A'(x) + b(x)]y = 0$. (Slovak. Russian and German summaries)

Acta Fac. Nat. Univ. Comenian. **7**, 597–608 (1963).

Es werden einige oszillatorische und asymptotische Eigenschaften von Integralen der linearen Differentialgleichung 4. Ordnung: (a) $y^{(\mathrm{iv})} + 2A(x)y' + [A'(x) + b(x)]y = 0$, $x \in (-\infty, \infty) = j$, A', b stetig, abgeleitet. Ist z.B. $A(x) \leqq 0$, $A'(x) - b(x) \geqq 0$, $x \in j$, $A' - b \not\equiv 0$ in jedem Teilintervall $i \subset j$, so gilt für jedes den Bedingungen $y(x_1) = 0$, $y(x_2) = y'(x_2) = 0$, $x_1 < x_2$ genügende Integral y von

$$\text{(a)} \qquad y(x)y'(x)y''(x) \neq 0$$

für $x > x_2$.

O. Borůvka (Brno)

Jakovlev, M. N. 5231

On the solution of systems of non-linear equations by differentiation with respect to a parameter. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 146–149.

The author notes that by embedding the non-linear system (*) $f(x)=0$, where $f: G \to R^n$ $(G \subseteq R^n)$, in (**) $F(x(\lambda), \lambda) = 0$ $(0 \leqq \lambda \leqq 1)$, where $F(u, 1) = f(u)$ and $x(0) = x_0$ is known, $x^* = x(1)$ satisfying (*) may be obtained by differentiating (**) with respect to the parameter λ and solving the resulting ordinary differential equation for $x(\lambda)$ up to $\lambda = 1$. It is shown that if $f \in C^1$ and the Jacobian J satisfies $(J(x)h, h) \geqq m(r)\|h\|^2$, $\forall x \in S(x_0, r)$ and $m > 0$, with $\int_0^{r^*} m(r)\, dr \geqq \|f(x_0)\|$, where

$$r^* = \sup\{r: \exists J^{-1}(x) \quad \forall x \in S(x_0, r) \subseteq G\},$$

then the differential equations obtained from either (a) $F(x, \lambda) = x - x_0 - \lambda[x - x_0 - f(x)]$ or (b) $F(x, \lambda) = f(x) - (1-\lambda)f(x_0)$ have (with the initial condition $x(0) = x_0$) unique solutions for $\lambda \in [0, 1]$. Applications are noted to Galerkin's method and a finite-difference method for variational problems.

T. I. Seidman (Seattle, Wash.)

Feščenko, S. F.; Škil', N. I. [Škil', M. I.] 5232

An error bound for the asymptotic representation of the solutions of a system of linear differential equations containing a parameter. (Russian)

Ukrain. Mat. Ž. **16** (1964), 132–135.

The authors consider the linear system

$$\text{(1)} \qquad \frac{dx}{dt} = A(\tau, \varepsilon)x + \varepsilon B(\tau, \varepsilon)e^{i\theta},$$

where ε is a small real positive parameter, $A(\tau, \varepsilon)$ is a real n-matrix, and $B(\tau, \varepsilon)$ is a real n-vector. Each of A and B has an expansion in powers of ε for $0 \leqq \tau = \varepsilon t \leqq L$. On this range $A(\tau, 0)$ is assumed to have a multiple eigenvalue $\lambda_1(\tau)$ of constant multiplicity k. It is also assumed that $i\,d\theta/dt = \lambda_1(\tau)$ at isolated points of $0 \leqq \tau \leqq L$. In an earlier paper [same Ž. **14** (1962), 383–392; MR **27** #2682] the second author constructed a formal series solution of (1) in powers of $\mu = \varepsilon^{1/k}$. In the present paper the asymptotic character of these approximations as $\varepsilon \to 0+$ is established. If x_m denotes the sum of the series through terms in μ^m, with initial value the same as that of the solution x, it is shown under appropriate regularity conditions that $\|x_m - x\| = O(\mu^{m+2-2k})$ as $\mu \to 0+$ when $m > 2k-2$.

W. S. Loud (Minneapolis, Minn.)

Ljubarskiĭ, G. Ja.; Rabotnikov, Ju. L. 5233

On the theory of differential equations with random coefficients. (Russian. English summary)

Teor. Verojatnost. i Primenen. **8** (1963), 309–318.

Assume that the equation (1) $\ddot{u}(t) + a_1(t)\dot{u}(t) + a_0(t)u(t) = 0$ has all solutions bounded on $-\infty < t < +\infty$, where a_1, a_0 are real, piecewise continuous and periodic functions of

period 1. The authors consider the related equation (2) $\ddot{u}(t)+a_1(t)\dot{u}(t)+[a_0(t)-\alpha(t)]u=0$, where $\alpha(t)$ is a real random function, and obtain necessary and sufficient conditions that the mean values of $u^2(t)$, $u(t)\dot{u}(t)$, and $\dot{u}^2(t)$ be bounded for all t. Explicit estimates are given, which are quite complicated, but to quote from the author's summary: "The restrictions on $\alpha(t)$ are essentially the following: The correlation length a is much shorter than the period, and $\alpha(t)$, for $-\infty<t<+\infty$, does not exceed the value $\gamma/\sqrt{a}$, where γ is a constant <1." Related references are A. Rosenbloom [Proc. Sympos. Information Networks (New York, 1954), pp. 145–153, Polytechnic Inst. Brooklyn, Brooklyn, N.Y., 1955; MR **17**, 1100], J. C. Samuels and A. C. Eringen [J. Math. and Phys. **38** (1959/60), 83–103; MR **21** #6041], and J. C. Samuels [J. Acoust. Soc. Amer. **32** (1960), 594–601; MR **22** #1945].

A. Stokes (Washington, D.C.)

Khas'minskii, R. Z. [Has'minskiĭ, R. Z.] 5234

The behavior of a self-oscillating system acted upon by slight noise.

Prikl. Mat. Meh. **27** (1963), 683–688 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1035–1044.

The author is concerned with a self-oscillating system with random disturbance $X''+\omega^2X'-\varepsilon f(X,X'')=\mu\xi'(t)$, where $\xi'(t)$, $0\le t<\infty$, is the derivative of a white noise $\xi(t)$, $E(\xi(t))=0$, $E(\xi^2(t))=t$. The pair $Z(t)=(X(t),Y(t))$, $Y(t)=X'(t)$, is a Markov process of diffusion type, whose probability density is the fundamental solution of a parabolic equation. Let $P_\varepsilon(x,y)$ be the stationary probability density of $Z(t)$, and define

$$Q_\varepsilon(r,\varphi) = P_\varepsilon(r\omega^{-1}\sin\varphi, r\cos\varphi).$$

Then using the author's theory on Bogoljubov-Krylov's averaging principle [Teor. Verojatnost. i Primenen. **8** (1963), 3–25; MR **28** #4253], he derives the limiting probability density of $Q_\varepsilon(r,\varphi)$, as $\varepsilon\to0$. Three critical cases are distinguished according as $\sigma\ll1$, $\sigma\gg1$, or $\sigma\sim1$, where $\sigma=\mu/\sqrt{\varepsilon}$. For the van der Pol system $f(x,y)=y(1-x^2)$, if $\sigma\sim1$, the limiting distribution is a truncated normal one. In the case $\sigma\sim1$, $\varepsilon\to0$, the author also computes the limiting stationary mean energy and the effective frequency up to the error $o(\varepsilon)$.

G. Maruyama (Fukuoka)

Bogoljubov, N. N.; Mitropol'skiĭ, Ju. A. 5235

The method of integral manifolds in non-linear mechanics. (Russian. English summary)

Analytic methods in the theory of non-linear vibrations (*Proc. Internat. Sympos. Non-linear Vibrations, Vol. I*, 1961), *pp.* 93–154. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

An English translation of this paper has appeared in Contributions to Differential Equations **2** (1963), 123–196 [MR **26** #6532]. *A. Stokes* (Washington, D.C.)

Lykova, O. B. 5236

A study of the solutions of a system of $n+m$ non-linear differential equations in the neighbourhood of an integral manifold. (Russian. English summary)

Ukrain. Mat. Ž. **16** (1964), 13–30.

A system of $n+m$ non-linear differential equations

$$(1)\quad \begin{aligned} dx/dt &= X(y)x+\varepsilon X^*(t,x,y) \quad (n \text{ equations}),\\ dy/dt &= \varepsilon Y(t,x,y) \quad (m \text{ equations}),\end{aligned}$$

where the vector-functions $X^*(t,x,y)$, $Y(t,x,y)$ are periodic in t of period 2π, satisfy some regularity conditions, $X(y)$ is an $n\times n$ matrix, and, under the assumption that $Y_0(x,y)=\lim_{T\to\infty}(1/T)\int_0^T Y(t,x,y)\,dt$ exists, an auxiliary system (2) $dx/dt=X(y)x$, $dy/dt=\varepsilon Y_0(x,y)$ is considered. It is assumed that (2) has a two-parameter family of solutions of the form (3) $x=0$, $y=y^0(\omega(a)t+\varphi,a)$, where $y^0(\psi,a)$ is periodic in ψ of period 2π, and $\omega(a)$ is Lipschitzian. Some assumptions on the characteristic exponents of the linear systems

$$d\delta x/dt = X(y^0)\delta x,\quad d\delta y/dt = \varepsilon Y_{0y}'(0,y^0)\delta y$$

are also made. Then, the existence and uniqueness of a two-dimensional local integral manifold of the system (1) in a neighbourhood of (3) are proved and its properties are investigated further.

For $X(y)=$ const, similar results are stated by J. Hale [*Oscillations in nonlinear systems*, McGraw-Hill, New York, 1963; MR **27** #401]. *Z. Opial* (Kraków)

Mitropol'skiĭ, Ju. A.; Lykova, O. B. 5237

An integral manifold of non-linear differential equations containing slow and fast motions. (Russian. English summary)

Ukrain. Mat. Ž. **16** (1964), 157–163.

The authors consider the relationship between the solutions of the system

$$(1)\quad \begin{aligned} \dot{x} &= X(y,z)x+\varepsilon X_1(t,x,y,z),\\ \dot{y} &= Y(x,z)y+\varepsilon Y_1(t,x,y,z),\\ \dot{z} &= \varepsilon Z_1(t,x,y,z)\end{aligned}$$

and the system

$$(2)\quad \begin{aligned} \dot{x} &= X(y,z)x,\\ \dot{y} &= Y(x,z)y,\\ \dot{z} &= \varepsilon Z_0(x,y,z),\end{aligned}$$

where $Z_0(x,y,z)=\lim_{T\to\infty}T^{-1}\int_0^T Z_1(t,x,y,z)\,dt$, and x, y, z are vectors. It is assumed that system (2) has an $(s+1)$-parameter family of periodic solutions $x=0$, $y=0$, $z=z^0(\omega(a)t+\varphi,a)$, $z^0(\psi,a)=z^0(\psi+2\pi,a)$, $a=(a_1,\cdots,a_s)$, $\omega(a)>0$, and the characteristic exponents of the linear variational equation have negative real parts except for $s+1$ which are zero. Under various smoothness conditions on the functions involved, the authors prove the existence of an $(s+1)$-dimensional local integral manifold for ε sufficiently small. Related papers: O. B. Lykova [#5236 above], the reviewer and A. P. Stokes [Arch. Rational Mech. Anal. **6** (1960), 133–170; MR **22** #9670].

J. K. Hale (Baltimore, Md.)

Kupka, Ivan 5238

Stabilité des variétés invariantes d'un champ de vecteurs pour les petites perturbations.

C. R. Acad. Sci. Paris **258** (1964), 4197–4200.

If X is a C^k vector field on the manifold M, then it has been known since Poincaré that a periodic solution V will

be preserved up to small isotopies under small perturbations of X, provided the differential induced on a normal hyperplane has no eigenvalues with zero real part. In this paper, a similar result is announced for V an arbitrary compact invariant submanifold. The absolute value of the real part of the eigenvalues is required to exceed a constant $\nu \geqq 0$, depending upon the restriction of X to V and an integer r, $1 \leqq r \leqq k-2$. Then the perturbed invariant manifold is of class C^r. If the absolute value of the real part of some eigenvalue fails to exceed ν, the invariant manifold will disappear for a suitable perturbation. If $k=\infty$, then in general, ν goes to infinity with r. It is stated that ν seems to be closely related to the first variation along the trajectories of X in V.

B. L. Reinhart (College Park, Md.)

Šafieva, D. R. 5239

The critical case of a zero root of countable systems. (Russian. Kazak summary)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 3, 26–34.

The author considers the infinite system of equations

$$\frac{dy}{dt} = F(y, x_1, \cdots),$$

$$\frac{dx_s}{dt} = \sum_{k=1}^{\infty} P_{sk}x_k + \Phi_s(x_1, \cdots, y) \quad (s = 1, 2, \cdots),$$

with Φ_s vanishing to at least second order and

$$F(y, x_1, \cdots) = A(y) + A^{(1)}(y, x_1, \cdots) + \cdots + A^{(m)}(y, x_1, \cdots) + B(y, x_1, \cdots),$$

where $A(y) = F(y, 0, \cdots)$ is a term which vanishes to at least second order, the $A^{(i)}$ are of order i in y, and $B(x_1, \cdots, y) \leqq B|y|^{m+1}$. Using results of K. P. Persidskiĭ [Prikl. Mat. Meh. **14** (1950), 23–44; MR **11**, 520; Izv. Akad. Nauk Kazah. SSR **1951**, no. 62, Ser. Mat. Meh., no. 5, 3–24; MR **14**, 753] on infinite systems, the author proves a stability (instability) theorem of a type done for finite systems by I. G. Malkin [cf. *Theory of stability of motion* (Russian), GITTL, Moscow, 1952; MR **15**, 873; English transl., AEC-tr-3352 (1958), § 91]. A further stability theorem concerns a special case in which $A(y) \equiv 0$.

R. E. L. Turner (Madison, Wis.)

Šafieva, D. R. 5240

The critical case of pairs of purely imaginary roots of countable systems. (Russian. Kazak summary)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 3, 35–38.

Questions of stability are considered for the infinite system

$$\frac{dy_1}{dt} = -\lambda y_2 + F_1(x_1, \cdots, y_1, y_2),$$

$$\frac{dy_2}{dt} = \lambda y_1 + F_2(x_1, \cdots, y_1, y_2),$$

$$\frac{dx_s}{dt} = \sum_{k=1}^{\infty} P_{sk}x_k + \Phi_s(x_1, \cdots, y_1, y_2) \quad (s = 1, 2, \cdots),$$

where the F_i and Φ_s have forms similar to F and Φ_s of the author's earlier paper [#5239]. The introduction of polar coordinates for y_1 and y_2 reduces the problem to that of the previous paper with $F(y, x_1, \cdots)$ periodic in t.

R. E. L. Turner (Madison, Wis.)

Zhang, Bing-gen [Chang, Ping-ken] 5241

Boundedness of solutions of ordinary differential equations of the second order.

Acta Math. Sinica **14** (1964), 128–136 (*Chinese*); *translated as Chinese Math.* **5** (1964), 139–148.

This paper contains a variety of results (ten theorems) about the boundedness of solutions of the equation

$$y'' + A(t)f(y) = 0$$

and of several similar functional equations, including differential-difference equations of the form

$$y'' + A(t)y = f(t, y(t-\tau), y'(t-\tau)),$$

where τ is constant. The proofs of these extensions of well-known theorems [R. E. Bellman, *Stability theory of differential equations*, Ch. 6, McGraw-Hill, New York, 1953; MR **15**, 794] depend mainly on the construction of Ljapunov functions and on variants of Gronwall's inequality.

D. Bushaw (Pullman, Wash.)

Deleanu, A. 5242

Note on differential equations in locally convex spaces.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 677–680.

The author considers the equation (1) $\dot{x} = f(x, t)$, where x is an element of a locally convex linear topological space E. He generalizes previous results of Millionščikov [Dokl. Akad. Nauk SSSR **131** (1960), 510–513; MR **22** #9700], the main generalization consisting of assuming slightly different properties on the function f in (1) concerning the mapping of families of functions on E into compact families. An additional reference is Millionščikov [Mat. Sb. (N.S.) **57** (**99**) (1962), 385–406; MR **27** #6002].

A. Stokes (Washington, D.C.)

Schäffer, Juan Jorge 5243

On Floquet's Theorem in Hilbert spaces.

Bull. Amer. Math. Soc. **70** (1964), 243–245.

The equation considered is (1) $\dot{u} + Au = 0$, where u is an element of a real or complex Hilbert space X, and A is an operator-valued, locally (Bochner) integrable function of the real variable t, periodic of period 1. Denote by U the unique operator-valued solution of (1) that satisfies $U(0) = I$, the identity. (1) has a Floquet representation of order m (m a positive integer) if there exists an operator B such that the function $P(t) = U(t)e^{tB}$ is periodic of period m. The classical theorem for X finite-dimensional states that a Floquet representation of order 1 [or at most order 2] exists if X is complex [real]. It was shown by Massera and Schäffer [Ann. of Math. (2) **69** (1959), 88–104; MR **21** #756] that there need not be any if X is infinite-dimensional. The example given was such that $\int_0^1 \|A(t)\|\,dt$ exceeded π by as little as desired. However, in the above reference, it was also shown (assuming X to be a Banach space) that if $\int_0^1 \|A(t)\|\,dt < \log 4$, then there exists a Floquet representation of order 1. The present paper, for X a Hilbert space, obtains this result assuming $\int_0^1 \|A(t)\|\,dt < \pi$. If $\int_0^1 \|A(t)\|\,dt = \pi$, an example is given

(in two-dimensional real space) to show that a representation of order 1 need not exist. Whether or not a representation of some order exists in this case (when X is infinite-dimensional) is an open question.

A. Stokes (Washington, D.C.)

Panaĭoti, B. N. 5244
Non-linear differential equations with small parameters in a Banach space. (Russian. Azerbaijani summary)
Akad. Nauk Azerbaĭdžan. SSR Trudy Inst. Mat. Meh. **2 (10)** (1963), 39–48.

The equation considered is of the form

$$(1)\qquad \mu\frac{dx}{dt}=f(x,y,t,\alpha),\quad \frac{dy}{dt}=g(x,y,t,\alpha),$$

where μ and t are real, α is a parameter in an m-dimensional space E^m, x is an element of an n-dimensional Banach space, and y belongs to an arbitrary Banach space B. The result desired is to obtain conditions under which a solution of (1) tends to a solution of the system (2) $f(x,y,t,\alpha)=0$, $dy/dt=g(x,y,t,\alpha)$, as $\mu\to 0$. The result obtained is analogous to the standard one for B finite-dimensional, the principal requirements being a stability condition on the equation (3) $dx/dt=f(x,y,t,\alpha)$, uniformly in the variables y,t,α, in a compact subset $P\subset B\times R\times E^m$, and, of course, the possibility of obtaining isolated solutions of the first equation in (2). The requirement that P be compact seems to the reviewer to be a difficult condition to verify if B were not finite-dimensional. No applications are given, or examples of areas in which equations of the form (1), with B infinite-dimensional, could occur.

A. Stokes (Washington, D.C.)

Ibragimov, Š. I. 5245
An approximate solution of the Cauchy problem for the evolution equation with an unbounded operator. (Russian. Azerbaijani summary)
Akad. Nauk Azerbaĭdžan. SSR Dokl. **19** (1963), *no.* 11, 9–14.

Ladyženskaja in an earlier paper [Mat. Sb. (N.S.) **39** **(81)** (1956), 491–524; MR **19**, 279] obtained an existence and uniqueness theorem for generalized solutions of operator equations of the form (1) $dx/dt+A(t)x+B(t)x=f(t)$, $x(0)=x_0$, where x is a mapping of the interval $0\leqq t\leqq L$ into a Hilbert space H, $A(t)$ is a self-adjoint operator with domain independent of t, $B(t)$ is suitably dominated by $A(t)$, and f is a given continuous function of t with values in H, where x_0 is a given element of H. The present paper extends this result to a non-linear equation, where $f(t)$ is replaced in (2) by $f(t,x)$, defined and continuous on $[0,L]\times H$, and Lipschitzian in x uniformly in t. The proof uses finite-difference techniques, also employed by Ladyženskaja. The more important assumptions on A and B are (i) A is positive definite with $A^{-1/2}(0)$ completely continuous and bounded, and (ii) the operator $(A(t)+B(t)+I/h)^{-1}$ exists and is bounded for $h\leqq h_1$, $0\leqq t\leqq L$ (h_1 real).

A. Stokes (Washington, D.C.)

Anger, Gottfried; Wildenhain, Günther 5246
Ein Kapazitätsbegriff für gewöhnliche Differentialgleichungen.
Wiss. Z. Techn. Univ. Dresden **12** (1963), 471–480.

Authors' summary: "Das Ziel der Untersuchungen ist es, gewöhnliche Differentialgleichungen vom Standpunkt der Potentialtheorie aus zu betrachten. Eine gewöhnliche Differentialgleichung der Ordnung m führt dabei auf den Raum der $(m-1)$mal stetig differenzierbaren Funktionen. Ausführlich untersuchen wir die Gleichung $u'=0$. Mit Hilfe der Grundlösung definieren wir das Potential bezüglich eines Maßes und führen einen Kapazitätsbegriff ein. Die Mengen der Kapazität Null sind genau diejenigen Teilmengen des R^1, die keine perfekte Teilmenge enthalten. Anschließend übertragen wir die Ergebnisse auf die gewöhnliche Differentialgleichung $u^{(m)}=0$. Für diese Gleichung erklären wir unter Verwendung der Grundlösung das Potential und die Kapazität. Auf diese Weise ist es möglich, jedem Potential eine auf dem Raum der $(m-1)$mal stetig differenzierbaren Funktionen stetige Linearform zuzuordnen. Die Untersuchungen gelten ganz allgemein für gewöhnliche Differentialgleichungen mit konstanten Koeffizienten."

PARTIAL DIFFERENTIAL EQUATIONS

See also 5190, 5192, 5345, 5403, 5564, 5606, 5660, 5662.

Garabedian, P. R. 5247
★Partial differential equations.
John Wiley & Sons, Inc., New York-London-Sydney, 1964. xii + 672 *pp.* $14.00.

This book is primarily a text for a graduate course in partial differential equations, although the later chapters are devoted to special topics not ordinarily covered in books in this field. As one would expect, the central theme of the presentation is existence and uniqueness theorems. The author has, however, placed the emphasis on constructive procedures, thus sacrificing some generality by imposing rather strong hypotheses in order to simplify the analysis. Proceeding from this point of view the author has made use of an interesting combination of classical and modern analysis in his proofs.

The first three chapters of this book deal with some basic concepts, e.g., the method of power series (which is employed to prove the Cauchy-Kowalewski theorem), a discussion of certain topics in first-order equations, and a classification of partial differential equations of the second order.

In Chapters 4 and 6 the author studies the Cauchy problem in two and more independent variables. The treatment in these two chapters is more or less classical except for the last section of Chapter 6 in which the author deals with the Cauchy problem for hyperbolic equations with analytic coefficients. The main feature of this section is an analytic continuation into the complex domain which permits the use of contour integration and the calculus of residues. Since the author here makes use of the fundamental solution and the parametrix, he introduces and defines these concepts in Chapter 5.

Chapters 7 through 10 deal with elliptic boundary-value problems. Here are presented the standard uniqueness theorems in the Dirichlet and Neumann problems. The Green's, Neumann's and kernel functions are introduced and their properties discussed. Several existence theorems in the Dirichlet problem for the Laplace equation are presented. One is motivated by the kernel function.

Another is based on the Dirichlet principle. Still another is based on the maximum principle, and a final theorem is based on a reduction to a Fredholm integral equation. Some of these theorems were actually proved for slightly more general equations. In order to complete the proof of the theorem based on integral equations, the author devotes one chapter to the Fredholm theory, with special attention given to symmetric kernels and the completeness of eigenfunction expansion.

In Chapter 11 the eigenvalue problem for the vibrating membrane is formulated. The method of Rayleigh-Ritz and the method of symmetrization are treated in some detail. Chapter 12 deals with the formulation of problems for equations of mixed type, with a section on incorrectly posed problems. Chapter 13 is devoted to the formulation of finite-difference analogues for equations of various types. The method of finite differences is used to obtain existence theorems for the heat equation. Chapter 14 is concerned with the derivation of the equations governing the motion of a perfect inviscid fluid, with sections devoted to subsonic, transonic, and supersonic flow, free streamlines and magnetohydrodynamics. Chapter 15 discusses free boundary problems in two dimensions. This chapter deals with questions which have been thoroughly investigated by the author and his colleagues, e.g., the existence of flows with free streamlines. The final chapter is concerned with partial differential equations in the complex domain.

Many of the detailed proofs in this book have been carried through for the case of two independent variables and heavy use has been made of complex variable methods. Generalizations of the results presented by the author are often left as problems. As a consequence the numerous problems appearing at the end of each chapter range from the relatively simple to the extremely difficult.

Because of the author's emphasis on constructive methods for solving problems which are of physical interest, his book will likely be as welcome to the engineer and the physicist as to the mathematician. It might perhaps have been even more valuable if the author had devoted a chapter to parabolic problems.

The author and publisher are to be complimented on the general appearance of the book. The printing is excellent and the text appears to be relatively free of misprints.

L. E. Payne (College Park, Md.)

Volevič, L. R. 5248

A problem in linear programming stemming from differential equations. (Russian)

Uspehi Mat. Nauk **18** (1963), *no.* 3 (111), 155–162.

Every polynomial p in the variables $x_1, \cdots, x_n$ can be represented as the sum of two polynomials $p(x) = \pi p(x) + p'(x)$, where $\pi p(x)$, the principal part of p, is a homogeneous polynomial of degree r and $p'(x)$ is a polynomial of degree $\leqq r-1$. A definition of the principal part πP of an $n \times n$ matrix of such polynomials $p_{ij}(x)$ is proposed as

$$P(x) = \pi P(x) + P'(x),$$

where (a) the (i,j)th element of $\pi P(x)$ either coincides with $\pi p_{ij}(x)$ or is zero; and (b) $\det \pi P = \pi \det P$. This definition is useful, for instance, in the classification of partial differential equations of higher order in n independent variables.

From (a) it follows that the existence and construction of πP hang upon the existence and construction of a certain incidence matrix χ, where $\chi_{ij} = 0$ or 1. The matrix P is called non-degenerate if the degree of $\det P$ coincides with the maximum of the degrees of the individual terms of the sum in the expansion $\det P = \sum \pm p_{1t_1} p_{2t_2} \cdots p_{nt_n}$. If the degree of p_{ij} is a_{ij}, then set up the system of equations in integers:

$$(1) \qquad s_1 + s_2 + \cdots + s_n + t_1 + t_2 + \cdots + t_n = r,$$

$$(2) \qquad a_{ij} \leqq s_i + t_j, \quad i, j = 1, \cdots, n.$$

The main theorem asserts: If P is a non-degenerate matrix, then the system of equations (1), (2) is solvable and

$$\chi_{ij} = 1 \quad \text{if } a_{ij} - s_i - t_j \geqq 0,$$
$$= 0 \quad \text{if } a_{ij} - s_i - t_j < 0,$$

is the incidence matrix defining the principal part πP of P. The result is obtained as a special case of the duality theorem of linear programming. The uniqueness of the solutions of (1) and (2) is investigated.

A. Wouk (Madison, Wis.)

Fulks, W. 5249

An approximate Gauss mean value theorem.

Pacific J. Math. **14** (1964), 513–516.

Let $Lu = \sum_{i,j=1}^n a_{ij} u_{x_i x_j}$, where $a_{ij}(x)$ is symmetric positive semidefinite. Let B be the positive square root of A. Then

$$\int_{|\xi|=r} [u(x + B(x)\xi) - u(x)]\, d\Omega = C_n r^2 Lu(x) + o(r^2),$$

where $d\Omega$ is the element of solid angle in n dimensions and C_n depends only on n.

P. Ungar (New York)

Ovsepjan, S. G. 5250

Approximate solution of the Cauchy problem for certain differential equations with polynomial coefficients. (Russian. Armenian summary)

Akad. Nauk Armjan. SSR Dokl. **38** (1964), 65–69.

The author considers the problem (1) $M(t)u - L(t,x)u = f(t,x)$ with $u|_{t=0} = \varphi_0(x)$,

$$\partial u/\partial t|_{t=0} = \varphi_1(x), \cdots, \partial^{m-1}u/\partial t^{m-1}|_{t=0} = \varphi_{m-1}(x),$$

where $M(t) = \sum_{k=0}^m a_k(t) \partial^k/\partial t^k$, $a_m(t) \neq 0$,

$$L(t,x) = \sum_{k=1}^{s} \sum_{i_0 + |i| = k} b_{i_0,i}(t,x) \partial^k / \partial t^{i_0} \partial x_1^{i_1} \cdots \partial x_n^{i_n},$$

$i_0 < m$, $b_{i_0,i}(t,x) = \sum_{k=1}^{s_{i_0,i}} b_k(t) q_k^{(i)}(x)$. Then if $f(t,x) = \sum_{l=1}^N c_l(t) p_l(x)$, $a_k(t)$, $b_k(t)$, $c_l(t)$ are continuous, $\varphi_0(x), \cdots, \varphi_{m-1}(x)$, $p_l(x)$, and $q_k^{(i)}(x)$ are polynomials with the degree of $q_k^{(i)}(x)$ not greater than $|i| - 1$, it follows that the solution of the above problem can be reduced to solving the Cauchy problem for the ordinary differential equation $M(t)v(t) = h(t)$. As an application the author remarks that if the right side and initial conditions for an equation of the form (1) with M and L as above are approximated by polynomials, then for correct problems one can approximate the solution upon solving the ordinary differential equation above.

R. Carroll (Urbana, Ill.)

Dong, Guang-chang [Tung, Kuang-chang] 5251
Boundary value problems for degenerate elliptic partial differential equations.
Acta Math. Sinica **13** (1963), 94–115 (*Chinese*); *translated as Chinese Math.* **4** (1963), 105–126.

Author's summary: "This paper solves a problem posed by A. V. Bicadze; i.e., a degenerate elliptic equation has a unique solution if the values of the function are prescribed on the nondegenerate boundary and bounded values of its derivative are prescribed on the degenerate boundary."

Agmon, S.; Douglis, A.; Nirenberg, L. 5252
Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II.
Comm. Pure Appl. Math. **17** (1964), 35–92.

In questo lavoro vengono generalizzati ai problemi al contorno per sistemi ellittici di ordine qualunque i fondamentali risultati ottenuti nel caso di una equazione nella parte I [stessi Comm. **12** (1959), 623–727; MR **23** #A2610]; sono infatti dimostrate le maggiorazioni del tipo di Schauder e quelle negli spazi del tipo L_p per una vasta classe di sistemi ellittici lineari e ne sono dedotte alcune interessanti applicazioni quali la regolarizzazione delle soluzioni anche per sistemi di equazioni non lineari.

Nel cap. I viene formulato il problema al contorno. Si consideri il sistema

$$(1)\qquad \sum_{j=1}^{N} l_{ij}(P,\partial)u_j(P) = F_i(P), \qquad i = 1,\cdots,N,$$

dove gli $l_{ij}(P,\partial)=l_{ij}(P;\ \partial/\partial x_1,\cdots,\partial/\partial x_{n+1})$, operatori differenziali lineari, sono polinomi in ∂ a coefficienti a valori complessi dipendenti da P variabile in un dominio D dello spazio euclideo ad $n+1$ dimensioni. L'ordine di tali operatori dipende da due sistemi di pesi interi $s_1,\cdots,s_N$, $t_1,\cdots,t_N$ nel modo seguente: $l_{ij}(P,\xi)$ è un polinomio in ξ di grado $\leqq s_i+t_j$, $i,j=1,\cdots,N$; ovviamente se $s_i+t_j<0$ allora $l_{ij}(P,\partial)=0$. Aggiungendo una opportuna costante ad un sistema di interi e togliendola all'altro si può poi assumere $s_i\leqq 0$ e poichè non tutti gli $l_{ij}(P,\partial)$ sono $\equiv 0$ allora $t_j\geqq 0$. Detta $l_{ij}'(P,\xi)$ la parte di grado s_i+t_j del polinomio $l_{ij}(P,\xi)$ la condizione di ellitticità imposta sul sistema (1) è la seguente: $L(P,\xi)=\det\|l_{ij}'(P,\xi)\|_{i,j=1,\cdots,N}\neq 0$ per ξ reale $\neq 0$. Nel caso $n=1$ viene imposta la seguente condizione supplementare su L: $L(P,\xi)$ è un polinimio in ξ di ordine $\sum_{i=1}^{N}(s_i+t_i)=2m$; per ogni $P\in\dot{D}$ (frontiera di D) se n è la normale a $\dot{D}$ in P e se ξ è un vettore reale $\neq 0$ tangente a $\dot{D}$ in P allora il polinomio $L(P,\xi+\tau n)$, nella variabile complessa τ, ha esattamente m radici $\tau_1^+(P,\xi),\cdots,\tau_m^+(P,\xi)$ con parte immaginaria positiva. Il caso $m=0$ si può facilmente risolvere in maniera esplicita [Douglis e Nirenberg, ibid. **8** (1955), 503–538; MR **17**, 743] e quindi si suppone $m>0$; inoltre si suppone il sistema uniformemente ellittico. Si considerino poi su $\dot{D}$ le condizioni al contorno espresse nella forma seguente:

$$(2)\qquad \sum_{j=1}^{N} B_{hj}(P,\partial)u_j(P) = \varphi_h(P), \qquad h = 1,\cdots,m,$$

con $B_{hj}(P,\partial)$ operatori differenziali lineari a coefficienti a valori complessi dipendenti da P. Gli ordini di tali operatori dipendono da due sistemi di pesi interi $t_1,\cdots,t_N$, $r_1,\cdots,r_m$ nel modo seguente: $B_{hj}(P,\xi)$ è un polinomio in ξ di grado $\leqq r_h+t_j$, $h=1,\cdots,m$, $j=1,\cdots,N$; naturalmente se $r_h+t_j<0$ allora $B_{hj}(P,\partial)=0$. Sia $B_{hj}'(P,\xi)$ la parte di $B_{hj}(P,\xi)$ di grado r_h+t_j. Indicata con $\|L^{jk}(P,\xi+\tau n)\|_{j,k=1,\cdots,N}$ la matrice aggiunta di $\|l_{ij}'(P,\xi+\tau n)\|_{i,j=1,\cdots,N}$, si impone allora sulle condizioni al contorno (2) la seguente condizione complementare: le righe della matrice

$$\|B_{hj}'(P,\xi+\tau n)\|_{\substack{h=1,\cdots,m\\ j=1,\cdots,N}}\cdot\|L^{jk}(P,\xi+\tau n)\|_{j,k=1,\cdots,N},$$

i cui elementi sono considerati come polinomi in τ, devono essere linearmente indipendenti modulo il polinomio, in τ, $\prod_{k=1}^{m}(\tau-\tau_k^+(P,\xi))$. È naturale chiedersi se, dato un sistema ellittico, esistano delle condizioni al contorno che verificano la condizione complementare; per la risposta completa a tale questione gli autori rinviano ad un lavoro non pubblicato di R. Bott; viene solo da essi dimostrato che il problema di Dirichlet per un sistema fortemente ellittico verifica la condizione complementare. Viene dimostrato, sempre nel cap. I, che ogni sistema ellittico può essere rimodellato, con l'aggiunta di nuove variabili, in modo che sia $s_i+t_j\leqq 1$ e che, trasformando di conseguenza le condizioni alla frontiera originarie, la condizione complementare sia ancora verificata.

Il cap. II è dedicato allo studio del sistema (1), (2) con $l_{ij}=l_{ij}'$ ed l_{ij}' a coefficienti costanti e $B_{hj}=B_{hj}'$ e B_{hj}' a coefficienti costanti, D essendo il semispazio $x_{n+1}>0$. Nel n.4 viene costruita, medianti opportuni nuclei di Poisson, una formula esplicita per la soluzione del problema studiato nel caso $F_i=0$, $i=1,\cdots,N$. Tale costruzione è basata sullo studio, fatto nel n.3, del comportamento asintotico delle soluzioni di sistemi di equazioni differenziali ordinarie con condizioni iniziali molto generali. Sempre nel n.4 vengono date alcune maggiorazioni dei nuclei di Poisson che permetteranno nel n.5 di applicare alla soluzione esplicita il teorema di Calderón e Zygmund [Acta Math. **88** (1952), 85–139; MR **14**, 637]. Fondamentale per il seguito è la formula di rappresentazione, ottenuta nel n.6, per le soluzioni del problema non omogeneo. Tale risultato si dimostra usando la formula esplicita ottenuta nel n.4 ed i risultati del n.5 con un ragionamento analogo a quello svolto nella parte I nel caso di una equazione.

Il cap. III è dedicato alla dimostrazione delle maggiorazioni del tipo di Schauder. Tali maggiorazioni vengono dapprima (n.8) ottenute per i problemi considerati nel cap. II, per i quali è stata trovata la formula di rappresentazione, ed infine, con le stesse techniche della parte I, tali maggiorazioni vengono estese al caso generale. Sia l intero $\geqq 0$ e α reale con $0<\alpha<1$; si indica con $C^{l+\alpha}(D)$ lo spazio delle funzioni u continue con le loro derivate fino all'ordine l in $\bar{D}=D\cup\dot{D}$ e inoltre con le derivate di ordine l uniformemente hölderiane di esponente α in D normalizzato da

$$|u|_{l+\alpha}^{D} = \sup_{|h|\leq l}\left(\sup_{x\in\bar{D}}|\partial^h u(x)|\right) + \sup_{|h|=l}\left(\sup_{\substack{x,y\in D\\ x\neq y}}\frac{|\partial^l u(x)-\partial^l u(y)|}{|x-y|^\alpha}\right)$$

intendendo che per ogni $(n+1)$-upla $h=(h_1,\cdots,h_{n+1})$ di interi $h_i\geqq 0$ è $|h|=\sum_{i=1}^{n+1}h_i$ e $\partial^h u=\partial^{|h|}u/\partial x_1^{h_1}\cdots\partial x_{n+1}^{h_{n+1}}$. In modo analogo si definisce lo spazio $C^{l+\alpha}(\dot{D})$ con la norma $|u|_{l+\alpha}^{\dot{D}}$. Si ha allora il Teorema 9.3: sia D un dominio limitato di R^{n+1} di classe $C^{l+\lambda+\alpha}$ con l intero $\geqq l_0=$

$\max(0, r_1, \cdots, r_m)$, α reale con $0<\alpha<1$ e $\lambda = \max(t_1, \cdots, t_N, -s_1, \cdots, -s_N, -r_1, \cdots, -r_m)$. Supponiamo che i coefficienti di l_{ij} siano in $C^{l-s_i+\alpha}(\bar{D})$ e quelli di B_{hj} in $C^{l-r_h+\alpha}(\dot{D})$. Nelle ipotesi fatte su l_{ij} e B_{hj} nel cap. I, sia $u_1, \cdots, u_N$ una soluzione di (1) in D e di (2) su $\dot{D}$ con $F_i \in C^{l-s_i+\alpha}(\bar{D})$ e con $\varphi_h \in C^{l-r_h+\alpha}(\dot{D})$. Se $u_j \in C^{l_0+t_j+\alpha}(\bar{D})$, allora $u_j \in C^{l+t_j+\alpha}(\bar{D})$ e vale la maggiorazione:

$$|u_j|^D_{l+t_j+\alpha} \leq C\left(\sum_{i=1}^N |F_i|^D_{l-s_i+\alpha} + \sum_{h=1}^m |\varphi_h|^{\dot{D}}_{l-r_h+\alpha} + \sum_{k=1}^N |u_k|_0{}^D\right),$$

$$j = 1, \cdots, N,$$

con C costante che non dipende da $u_1, \cdots, u_N, F_1, \cdots, F_N, \varphi_1, \cdots, \varphi_m$. Tale risultato è anche valido sotto opportune condizioni nel caso in cui D sia un dominio illimitato.

Il cap. IV è poi dedicato alle maggiorazioni a priori negli spazi H_{j,L_p}. Per j intero >0, $H_{j,L_p}(D)$ è qui inteso come completamento astratto di $C^\infty(\bar{D})$ rispetto alla norma $\|u\|_{j,L_p} = (\sum_{|h|\leq j}\int_D |\partial^h u|^p\,dx)^{1/p}$, $p>1$; $H_{j-1/p,L_p}(\dot{D})$ è lo spazio delle funzioni φ su $\dot{D}$ che sono "tracce" su $\dot{D}$ di funzioni $v \in H_{j,L_p}(D)$ la norma essendovi definita da $\|\varphi\|_{j-1/p,L_p} = \inf\|v\|_{j,L_p}$ fra tutte le $v \in H_{j,L_p}(D)$ aventi φ come traccia su $\dot{D}$. Tali maggiorazioni sono ottenute sempre a partire dalla formula di rappresentazione stabilita nel n.6, con lo stesso ragionamento della parte I. Il risultato più importante per le maggiorazioni di carattere globale è il seguente Teorema 10.5: sia $l_1 = \max(0, r_1+1, \cdots, r_m+1)$ e sia l un intero $\geq l_1$; sia D un dominio limitato di classe $C^{l+\lambda}$, e supponiamo che i coefficienti di l_{ij} siano in $C^{l-s_i}(\bar{D})$ e quelli di B_{hj} in $C^{l-r_h}(\dot{D})$. Nelle ipotesi fatte su l_{ij} e B_{hj} nel cap. I, sia $u_1, \cdots, u_N$ una soluzione di (1) con $F_i \in H_{l-s_i,L_p}(D)$ e di (2) con $\varphi_h \in H_{l-r_h-1/p,L_p}(\dot{D})$; allora se $u_j \in H_{l_1+t_j,L_p}(D)$ risulta per $j=1, \cdots, N$ $u_j \in H_{l+t_j,L_p}(D)$ e

$$\|u_j\|_{l+t_j,L_p} \leq K\left(\sum_{i=1}^N \|F_i\|_{l-s_i,L_p} + \sum_{h=1}^m \|\varphi_h\|_{l-r_h-1/p,L_p} + \sum_{k=1}^N \|u_k\|_{0,L_p}\right)$$

con K costante indipendente da $u_1, \cdots, u_N, F_1, \cdots, F_N, \varphi_1, \cdots, \varphi_m$. Vengono poi date anche delle maggiorazioni di carattere locale alla frontiera.

Nel cap. V viene dimostrata, con alcuni esempi, la necessità delle ipotesi fatte nel cap. I per avere le maggiorazioni a priori (n. 11) e sono date, nei n. 12, 13, 14, alcune applicazioni dei risultati ottenuti: regolarizzazzione di sistemi non lineari, perturbazione di problemi non lineari, maggiorazioni di Schauder per equazioni semi-lineari. Vengono infine (n. 15) costruiti dei nuclei di Poisson "approssimati" per equazioni a coefficienti variabili.

Come è detto nell'introduzione, alcuni dei risultati di questo lavoro sono stati annunciati da vari autori.

G. Geymonat (Pavia)

Adler, Giorgio [Adler, György] 5253

Principi di massimo relativi alle equazioni di tipo ellittico e parabolico nel caso di condizioni al contorno e di condizioni iniziali rispettivamente non-continue e non-limitate.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **30** (1961), 178–181.

Hirasawa, Yoshikazu 5254

On an estimate for semi-linear elliptic differential equations of the second order.

Kōdai Math. Sem. Rep. **16** (1964), 55–68.

The author considers the equation

$$L(u) \equiv \sum_{i,j=1}^n a_{ij}(x^1, \cdots, x^n)\frac{\partial^2 u}{\partial x^i\,\partial x^j} = f\left(x^1, \cdots, x^n, u, \frac{\partial u}{\partial x^1}, \cdots, \frac{\partial u}{\partial x^n}\right)$$

where the operator L is assumed to satisfy the ellipticity condition with two positive constants $\underline{A} \leq \bar{A}$:

$$\underline{A}|\xi|^2 \leq L(u) \leq \bar{A}|\xi|^2$$

and $\xi = (\xi^1, \cdots, \xi^n)$ is any point of the domain $D \subset E_n$ with boundary $\dot{D}$ and diameter d. $a_{ij}(x^k)$ are assumed to fulfil a Hölder condition with constants H, α. f is defined in a $(2n+1)$-dimensional domain $\{(x, u, p);\ x \in D,\ |u| \leq M,\ |p| < +\infty\}$ with the condition $|f(x, u, p)| \leq B|p|^2 + \Gamma$ where B, Γ, M are positive constants and x, p are vectors. The author proves that if $u(x)$ is a solution of the equation such that $|u(x)| \leq M$, where N is the oscillation of u in D and if $8k(n)BN/\underline{A} < 1$ $(k(n) = (2/(n+1))B(\frac{1}{2}, \frac{1}{2}(n+1))^{-2})$ is fulfilled, then we have

$$\left\{\sum_{i=1}^n \left(\frac{\partial u(a)}{\partial x^i}\right)^2\right\}^{1/2} < C^{(1)}\rho(a)^{-1}\operatorname{osc}\{u(x)\} + C^{(2)}\rho(a),$$

for any point $a \in D$, where $\rho(a) = \operatorname{dist}(a, \dot{D})$ and $C^{(1)}$ and $C^{(2)}$ are positive constants depending only on $\underline{A}$, $\bar{A}$, B, Γ, N, n, d, H and α.

M. Coroi-Nedelcu (Bucharest)

Friedman, Avner 5255

Entire solutions of partial differential equations with constant coefficients.

Duke Math. J. **31** (1964), 235–240.

Let $P(\xi)$ be a non-constant polynomial in $\xi = (\xi_1, \cdots, \xi_n)$ and let $u(x)$ be a solution, in the sense of distributions, of (*) $P(iD_x)u(x) = 0$, where $x = (x_1, \cdots, x_n)$ and $D_x = (\partial/\partial x_1, \cdots, \partial/\partial x_n)$. Then we are concerned with generalizations of the following theorem: If $u \in L^2$ and u satisfies (*), then $u(x) = 0$ almost everywhere. The generalizations are obtained by determining the largest p_0 such that whenever $u(x)$ satisfies (*) and $u \in L^p$ for some $2 \leq p \leq p_0$ then $u(x) = 0$ almost everywhere.

The main theorem of the paper asserts that for a certain class of elliptic operators P, $p_0 = 2n/(n-1)$. It is stated, without proof, that this result can be extended to some types of non-elliptic operators. A number of corollaries of the main theorem are presented of which perhaps the most interesting is that (*) does not hold for $u \in L^p$ if $p > 2$.

P. Cooperman (Teaneck, N.J.)

Ladyženskaja, O. A.; Ural'ceva, N. N. 5256

Hölder continuity of solutions and their derivatives for linear and quasi-linear equations of elliptic and parabolic type. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 1258–1261.

The authors have published several papers [cf., e.g. Uspehi Mat. Nauk. **16** (1961), no. 1 (97), 19–90; MR 2[illegible] #6571] on the question of existence of solutions for linear equations with discontinuous coefficients as well as quasi-linear equations. In the present article they indicate how

some of their results can be proved in a simplified manner by involving the logarithm of solutions into some of the calculations. *A. Friedman* (Evanston, Ill.)

Vaĭnberg, B. R. 5257

Hypo-elliptic equations in the whole space and the principle of limiting absorption. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 20–23.

Consider a hypo-elliptic operator $P(D)=Q(D)R(D)$, where $Q(\xi)$ is a real polynomial, while $R(\xi)$ has no real zeros, and let $P_\varepsilon(\xi)=(Q(\xi)+\varepsilon_1+i\varepsilon_2)R(\xi)$. Making the usual assumption that the real zeros of $P(\xi)$ form a hypersurface of positive curvature, the author constructs a fundamental solution $E_\varepsilon(x)=(2\pi)^{-n}\int\{P_\varepsilon(\xi)\}^{-1}\exp(-ix\xi)\,d\xi$ for $P_\varepsilon(D)$ and states that $E_\varepsilon(x)$ converges weakly towards a fundamental solution $E(x)$ for $P(D)$ when $\varepsilon\to 0$, $\varepsilon_1/\varepsilon_2\geqq c>0$. Further, $E(x)$ coincides with one of the fundamental solutions with known asymptotic behaviour for $|x|\to\infty$, previously studied by the author [same Dokl. **145** (1962), 21–23; MR **25** #2316]. The result implies, under certain "radiation conditions" at infinity, the convergence of the solutions u_ε of $P_\varepsilon(D)u_\varepsilon=f$ (f "finite", i.e., a regular function with compact support) towards a unique solution of $P(D)u=f$. The paper ends with a discussion of the possibility of extending the methods to operators with variable coefficients $P(x,D)=P_0(D)+\lambda P_1(x,D)$, where P_1 is an operator with finite coefficients.

J. Friberg (Göteborg)

Kumano-go, Hitoshi 5258

On an example of non-uniqueness of solutions of the Cauchy problem for the wave equation.

Proc. Japan Acad. **39** (1963), 578–582.

The author considers the equation $Lu=(\Box u+f\,\partial/\partial t+g)u=0$ in (x,y,t)-space, where $\Box=\partial^2/\partial x^2+\partial^2/\partial y^2-\partial^2/\partial t^2$. He constructs real-valued functions u, f, g belonging to a class B (unspecified), satisfying $Lu=0$ and such that u vanishes in $\mathscr{D}=\{(x,y,t);\,x^2+y^2<1\}$. If complex u and g are permitted, then f may be taken to be zero. The construction makes use of a method of Pliś [Comm. Pure Appl. Math. **14** (1961), 599–617; MR **25** #307], using asymptotic expansions of Bessel functions.

M. Schechter (New York)

Morawetz, Cathleen S. 5259

A uniqueness theorem for the relativistic wave equation.

Comm. Pure Appl. Math. **16** (1963), 353–362.

I. E. Segal has shown [Phys. Rev. (2) **109** (1958), 2191–2198; MR **23** #B897] by Fourier transform methods that a solution of the relativistic wave equation $u_{tt}-u_{xx}+\alpha u=0$, $\alpha>0$, which vanishes on the forward light rays $x^2-t^2=0, t\geqq 0$, vanishes identically under certain boundedness hypotheses. This result has been extended by R. Goodman [Thesis, M.I.T., Cambridge, Mass., 1963] to the case $u_{tt}-\Delta u+\alpha u=0$, where there are three space variables x_i, $i=1, 2, 3$, $|x|=r$. The boundedness hypothesis on the solution is that the energy integral $\int(|\Delta u|^2+u_t^2+\alpha u^2)\,du$ is bounded. In this paper the author obtains the above result of Goodman by a procedure that employs only energy integrals and which can be extended to more general equations with non-constant coefficients. *A. K. Aziz* (Washington, D.C.)

Filippov, A. F. 5260

Continuous dependence of the solution on the boundary conditions and on the right-hand side of the equation. (Russian. English summary)

Vestnik Moskov. Univ. Ser. I Mat. Meh. **1963**, *no.* 2, 33–36.

The paper starts with a theorem in functional analysis. If U, Φ are metric spaces and $l\colon U\to\Phi$, or $l(u)=\phi$, a map of U onto Φ, then l^{-1} is continuous if and only if (i) l^{-1} is single-valued; (ii) for every compact set $K\subset\Phi$, the inverse image $l^{-1}(K)$ is a compact subset of the completion $\bar U$ of U; (iii) for any sequence $[u_n]$ with $u_n\in U$, $u_n\to u_0\in\bar U$, $l(u_n)\to\phi_0\subset\Phi$, we have $u_0\in U$, $l(u_0)=\phi_0$. If the role of U is taken by a set of solutions of a differential equation, and the role of Φ is taken by the corresponding initial conditions (with arbitrary metrics in $\bar U$ and Φ), then the theorem above may give statements of continuous dependence of the solutions upon initial values and the right-hand sides. One has only to verify that (1) for any initial condition ϕ the corresponding solution u is unique; (2) for any compact set $K\subset\Phi$ the corresponding set of solutions is compact in $\bar U$; (3) if a sequence of solutions converges in U and the sequence of their initial conditions converges in Φ, then the limit function is a solution (that is, it belongs to U) and satisfies the initial conditions. Examples are given. *L. Cesari* (Ann Arbor, Mich.)

Vinti, Calogero 5261

Un teorema di esistenza per i sistemi di equazioni alle derivate parziali della forma

$$p^{(i)}=f^{(i)}(x,y,z^{(1)},z^{(2)},\cdots,z^{(n)},q^{(i)}).$$

Atti Sem. Mat. Fis. Univ. Modena **12** (1962/63), 33–106.

L'autore prova il seguente teorema: Siano

$$f^{(i)}(x,y,z^{(1)},\cdots,z^{(n)},q^{(i)})\qquad(i=1,\cdots,n)$$

n funzioni definite per $c\leqq x\leqq\bar c$ e per ogni $(n+2)$-pla reale $(y,z^{(1)},\cdots,z^{(n)},q^{(i)})$, le quali nel loro campo di definizione soddisfano alle seguenti ipotesi: (1) ogni $f^{(i)}$ è continua nel complesso delle variabili; (2) ogni $f^{(i)}$ è lipschitziana rispetto a $q^{(i)}$, con costante di Lipschitz indipendente da $x,y,z^{(1)},\cdots,z^{(n)}$; (3) ogni $f^{(i)}$ è limitata; (4) le derivate parziali $f_y^{(i)}$, $f_{z_j}^{(i)}$ $(j=1,\cdots,n)$ di ogni $f^{(i)}$ sono limitate e lipschitziane rispetto a $(y,z^{(1)},\cdots,z^{(n)},q^{(i)})$ con costante di Lipschitz indipendente da x.

Siano $\omega_i(y)$ $(-\infty<y<+\infty;\ i=1,\cdots,n)$ n funzioni continue, aventi la derivata del primo ordine continua e limitata in $(-\infty,+\infty)$, ed esista una funzione $M(y)$ sommabile su ogni intervallo finito, in modo che per $i=1,\cdots,n$ sia

$$\left|\frac{1}{2h}\left\{\frac{d\omega^{(i)}(y+h)}{dy}-\frac{d\omega^{(i)}(y-h)}{dy}\right\}\right|\leqq M(y)$$
$$(-\infty<y<+\infty).$$

Sotto queste ipotesi, fissato comunque un intervallo (a,b), esiste un numero c' con $c<c'\leqq\bar c$, in modo che nel rettangolo $c\leqq x\leqq c'$, $a\leqq y\leqq b$ esiste una n-pla di funzioni lipschitziane $z^{(i)}(x,y)$ $(i=1,\cdots,n)$, la quale soddisfa al sistema $p^{(i)}=f^{(i)}(x,y,z^{(1)},\cdots,z^{(n)},q^{(i)})$ $(i=1,\cdots,n)$ quasi dappertutto, e inoltre alla condizione (di Cauchy) $z^{(i)}(c,y)=\omega^{(i)}(y)$ $(i=1,\cdots,n)$. *S. Cinquini* (Pavia)

Zaidman, Samuel 5262

Quasi-periodicità per l'equazione di Poisson.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 241–245.

Exposé de résultats de l'auteur [Ann. Mat. Pura Appl. (4) **64** (1964), 365–405].

Soit $f(x, t)$ définie pour tout t $[(-\infty < t < +\infty) = J]$ et presque-partout pour $x \in R^n$, mesurable et de carré sommable dans toute bande $R^n \times [a \leqq t \leqq b]$ et telle que:

$$\int_{R^n} |f(x, t)|^2 \, dx < \infty, \lim_{\delta \to 0} \int_{R^n} |f(x, t+\delta) - f(x, t)|^2 \, dx = 0.$$

Soit de plus $\phi(x, t) \in C_0^\infty(R^n \times J)$ (indéfiniment dérivables et à support compact dans $R^n \times J$), alors il existe au moins une fonction $u(x, t)$ mesurable dans $R^n \times J$, de carré sommable dans toute bande et solution de

$$\int_{R^n \times J} u(x, t)\left(\phi_{tt} + \sum_1^n \phi_{x_i x_i}\right) dxdt = \int_{R^n \times J} f(x, t)\phi(x, t) \, dxdt.$$

Lorsque $\int_{R^n} |f(x, t)|^2 \, dx \leqq A^2$ $(0 < A < +\infty, \; -\infty < t < +\infty)$, on dit que f est L^2-bornée, alors si l'équation précédente admet aussi une solution L^2-bornée, les dérivées généralisées u_t et u_{x_i} sont aussi L^2-bornées.

Lorsque f est de plus presque-périodique L^2 en t, alors il en est de même de u, u_t, u_{x_i}; de plus u, u_t, et f ont le même spectre. D'une façon encore plus précise, si f est H^{m-1}-bornée (c'est-à-dire si ses dérivées partielles d'ordre $(m-1)$ sont L^2-bornées), alors u est H^m-presque-périodique, u_t l'est H^{m-1}; si $m-1 > n/2$ ces deux fonctions sont p.p. (de t) dans $C_\infty(R^n)$ (espace des fonctions continues sur R^n et nulles à l'infini avec la convergence uniforme sur R^n).

J. Favard (Paris)

Prouse, Giovanni 5263

Analisi di alcuni classici problemi di propagazione.

Rend. Sem. Mat. Univ. Padova **32** (1962), 338–373.

Sia Ω un aperto limitato e connesso di R^n a frontiera Γ localmente lipschitziana e si consideri in Ω un operatore A differenziale lineare di ordine $2p$ tale che $Au = \sum_{i,j}^{1,s} A_i^*(g_{ij}(x)A_j u)$ dove $g_{ij} \in L^\infty(\Omega)$, $A_i \in \mathscr{L}(\mathscr{D}(\Omega), L^2(\Omega))$ e $A_i \in \mathscr{L}(L^2(\Omega), \mathscr{D}'(\Omega))$, essendo A_i^* l'aggiunto di A_i in quanto elemento di $\mathscr{L}(L^2(\Omega), \mathscr{D}'(\Omega))$ ($\mathscr{D}(\Omega)$ spazio delle funzioni infinitamente differenziabili e a supporto compatto in Ω, $\mathscr{D}'(\Omega)$ spazio delle distribuzioni su Ω). L'autore fa un'analisi interessante ed accurata dei problemi di propagazione (1) $\partial^2 u/\partial t^2 - Au = f(x, t)$, $x \in \Omega$, $t \in R$, $R =]-\infty, +\infty[$, (2) $u(x, 0) = \varphi(x)$, $\partial u(x, 0)/\partial t = \psi(x)$, e (3) $u(x, t)$ verificante opportune condizioni ai limiti di tipo misto su $\Gamma \times R$, dando ad essi un'impostazione debole in certi spazi di Hilbert dedotta da J. L. Lions [*Équations différentielles opérationnelles et problèmes aux limites*, Springer, Berlin, 1961; MR **27** #3935]. Sotto opportune ipotesi su A e sui dati φ, ψ e f vengono ottenuti diversi risultati: nel § 2 viene dato un teorema di esistenza e di unicità per il problema con condizioni ai limiti omogenee, cioè nulle su $\Gamma \times R$, seguendo il noto procedimento di Fourier [cf., ad es., Ladyženskaja, *The mixed problem for a hyperbolic equation* (Russian), GITTL, Moscow, 1953; MR **17**, 160]; nei §§ 3 e 4 sono studiati alcuni casi di condizioni al contorno non omogenee e alcune proprietà di regolarità della soluzione dando un significato più preciso alle condizioni ai limiti; nel § 5 viene dato un teorema di quasi-periodicità per le soluzioni limitate del problema, utilizzando un teorema di L. Amerio [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **28** (1960), 147–152, 322–327, 461–466; MR **22** #6927]; nei §§ 6, 7, 8 vengono poi applicati i risultati ottenuti a diversi casi concreti di classici problemi, tra i quali ad es.: il problema (1), (2), (3) in cui è

$$Au = \sum_{i,j}^{1,m} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) - au, \qquad a_{ij} = a_{ji},$$

A ellittico e le (3) sono date da $u = 0$ su $\Gamma_1 \times R$, $du/dl + ku = 0$ su $\Gamma_2 \times R$, Γ_1 porzione chiusa di Γ, $\Gamma_2 = \Gamma - \Gamma_1$, d/dl derivazione conormale a Γ; e quelli in cui è $Au = \Delta^2 u - bu$ e le (3) sono date da $u = 0$ e $Au + k\,du/d\nu = 0$, ν normale a Γ (problema della piastra appoggiata) oppure $u = 0$, $du/d\nu = 0$ su $\Gamma \times R$ (problema della piastra incastrata).

E. Magenes (Pavia)

Vagabov, A. I. 5264

Correctness conditions for one-dimensional mixed problems for hyperbolic systems. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 1247–1249.

The mixed problem for hyperbolic equations of any order is solved explicitly in the special case where the space variable x is one-dimensional, the coefficients of the equation depend only on x, the (x, t)-domain is rectangular, and the coefficients in the (homogeneous) boundary conditions are constants. There is an additional assumption of "regularity" of the boundary conditions, and some differentiability assumptions on the coefficients of the equations and the non-homogeneous term and on the initial data.

A. Friedman (Evanston, Ill.)

Jeffrey, Alan 5265

The propagation of weak discontinuities in quasi-linear symmetric hyperbolic systems. (German summary)

Z. Angew. Math. Phys. **14** (1963), 301–314.

Author's summary: "Die Abhandlung untersucht die Fortpflanzung kleiner Unstetigkeiten in Systemen von nichtlinearen hyperbolischen Differentialgleichungen. Ein Ausdruck wird abgeleitet, der die Änderung in der Intensität der Unstetigkeit angibt, wenn diese sich entlang eines Strahls des hyperbolischen Gleichungssystems fortbewegt. Schliesslich wird als Beispiel mit Hilfe des angegebenen Verfahrens die Fortpflanzung von Schallwellen behandelt."

J. R. Cannon (Upton, N.Y.)

Ginsberg, F. 5266

On the Cauchy problem for the one-dimensional heat equation.

Math. Comp. **17** (1963), 257–269.

Let $V(x, t)$ satisfy the heat equation in $R = \{(x, t)|0 < x < 1, 0 < t < 1\}$ such that $V(0, t) = \varphi(t)$ and $V_x(0, t) = 0$. If $|V(x, t)| < B$ in R and $\varphi(t)$ is infinitely differentiable in $0 \leqq t \leqq 1$ with $\varphi^{(n)}(0) = \varphi^{(n)}(1) = 0$, then for (x, t) in $R' = \{(x, t)|0 \leqq x \leqq \zeta < 1, 0 < t < 1\}$ there exists a positive constant K such that $|V(x, t)| \leqq K \max_{0 \leqq t \leqq 1} |\varphi(t)|^{1/2}$. A numerical scheme for calculating $V(x, t)$ is discussed along with an example problem.

J. R. Cannon (Upton, N.Y.)

Solonnikov, V. A. 5267

A priori estimates for solutions of second-order equations of parabolic type. (Russian)

Trudy Mat. Inst. Steklov. **70** (1964), 133–212.

The paper consists of two parts. In Part I the author derives inclusion theorems for various function spaces. These spaces are defined by norms which involve fractional derivatives as well as integrations taken over submanifolds. Results of similar nature were previously obtained by several authors. In Part II the author makes use of the results of Part I and derives a priori bounds on solutions of second-order linear parabolic equations. These bounds involve, in case of zero boundary conditions, the estimation of L_p norms of the x-derivatives up to fractional order $2l$ and the t-derivatives up to (fractional) order l of the solution in terms of the L_p norms of the x-derivatives up to order $2l-2$ and the t-derivatives up to order $l-1$ of the non-homogeneous term in the equation. Similar a priori inequalities were previously stated by Slobodeckiĭ [Vestnik Leningrad. Univ. **15** (1960), no. 7, 28–47; MR **22** #2794; Dokl. Akad. Nauk SSSR **123** (1958), 616–619; MR **21** #5061].

A. Friedman (Evanston, Ill.)

Ölçer, N. Y. 5268

On a class of boundary-initial value problems.

Österreich. Ing.-Arch. **18** (1964/65), 104–113.

Let R be a region bounded by a closed surface S. Let M and N be linear partial differential space operators of orders μ and ν, respectively, where $\mu>\nu$, T is a linear differential operator with respect to time t of order γ, and U is a linear differential space operator on S of order less than μ. The boundary-value problem in $f(\mathbf{r}, t)$, $\mathbf{r}$ in R, $t>0$, consists of the differential equation $Mf+Q(\mathbf{r}, t)=TNf$ and conditions $Uf=P(\mathbf{r}, t)$, $\mathbf{r}$ on S, and $\partial^{i-1}f/\partial t^{i-1}=F_i(\mathbf{r})$ when $t=0$ ($i=1, 2, \cdots, \gamma$), where Q, P and F_i are prescribed functions. The eigenfunctions $\chi_m(\mathbf{r})$ of the adjoint of the eigenvalue problem $M\phi_m(\mathbf{r})+\lambda_m N\phi_m(\mathbf{r})=0$, $\mathbf{r}$ in R, $U\phi_m=0$, $\mathbf{r}$ on S, are used to define an integral transform $\bar{f}_m(t)=\int_R fN^*\chi_m\,dR$. The inverse transform $f(\mathbf{r}, t)$ is represented by a series in the functions $\phi_m(\mathbf{r})\bar{f}_m(t)$. The problem in $\bar{f}_m(t)$ is one in ordinary differential equations. Variations of forms of the solution $f(\mathbf{r}, t)$ are noted. The procedure is illustrated with a general problem in heat conduction.

R. V. Churchill (Ann Arbor, Mich.)

Kusano, Takaŝi 5269

On the Cauchy problem for a class of multicomponent diffusion systems.

Proc. Japan Acad. **39** (1963), 634–638.

Let Lu be a second-order linear parabolic operator in the variables $(x, t)=(x_1, \cdots, x_n, t)$ and let f, g be given functions of x, t, u, v. The author considers the Cauchy problem (for u, v): $Lu=f$, $\partial v/\partial t=g$, $u(x, 0)=\varphi(x)$, $v(x, 0)=\psi(x)$, and proves comparison and existence theorems.

A. Friedman (Evanston, Ill.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 5171, 5207.

Jecklin, H. 5270

Über Zusammenhänge zwischen einfachen Funktionalgleichungen und elementaren Mittelwerten. (French, English and Italian summaries)

Mitt. Verein. Schweiz. Versich.-Math. **63** (1963), 19–26.

Die Arbeit beschäftigt sich mit den Potenzmittelflächen der Ordnung p

$$(1)\qquad z=f(x, y)=[(xa^p+yb^p)/(x+y)]^{1/p} \qquad \text{für } p>0,$$

$$(2)\qquad =(a^x b^y)^{1/(x+y)} \qquad \text{für } p=0,$$

wobei die Gewichte x, y veränderliche Grössen, a und b positive Konstanten sind. Ist E eine bestimmte Ebene, so kann man die Flächen (1) bzw. (2) mit E zum Schnitt bringen und eine Schnittkurve erhalten. Der Verfasser hat gezeigt, dass diese Schnittkurven bei gewissen Ebenen E, durch elementaren Funktionalgleichungen von Typus

$$(3)\qquad f[m(x+y)]=m^n[f(x)^{1/n}+f(y)^{1/n}]^n \qquad (m, n>0)$$

charakterisieren können. Die stetigen allgemeinen Lösungen der Funktionalgleichung (3) wurden im Buch von J. Aczél [*Vorlesungen über Funktionalgleichungen und ihre Anwendungen*, Birkhäuser, Basel, 1961; MR **23** #A1959] gegeben.

Z. Daróczy (Debrecen)

Jones, G. Stephen 5271

Fundamental inequalities for discrete and discontinuous functional equations.

J. Soc. Indust. Appl. Math. **12** (1964), 43–57.

The author proves several generalizations of Gronwall's Lemma for discrete and discontinuous problems. His most general result is the following: Let x, f, g, and z be real-valued functions defined on an interval $[a, b]$ and either continuous or of bounded variation. Let g and z be non-negative. Let μ be a non-decreasing functional defined on $[a-\varepsilon, b]$ for some $\varepsilon>0$ which is continuous from the left, and let x, f, g, and z be continuous from the right at all points of discontinuity of μ. If for all t in $[a, b]$

$$x(t) \leqq f(t)+g(t)\int_a^{t-} z(\tau)x(\tau)\,d\mu(\tau),$$

then

$$x(t) \leqq f(t)+g(t)\int_a^{t-} f(\tau)z(\tau)\exp\left(\int_\tau^{t-} g(s)z(s)\,d\mu(s)\right)d\mu(\tau),$$

for all t in $[a, b]$. Let $\zeta_0=a$, let $\{\zeta_i\}$, $i=1, 2, \cdots$, denote the set of discontinuities of μ in $(a, b]$, and for all t in $[a, b]$ let

$$\eta(t)=\mu(t)-\sum_{\zeta_i<t} J(\zeta_i),$$

where $J(t)=\mu(t+)-\mu(t)$. Then for all t in $[a, b]$,

$$x(t) \leqq f(t)+g(t)\int_a^{t-}\left(\prod_{\tau<\zeta_i<t}(1+g(\zeta_i)z(\zeta_i)J(\zeta_i)) \times\exp\left(\int_\tau^t g(s)z(s)\,d\eta(s)\right)z(\tau)f(\tau)\right)d\mu(\tau).$$

Furthermore, if $|f(t)|$ and $g(t)$ are bounded on $[a, b]$ by constants K and c, respectively, then

$$x(t) \leqq K\prod_{a<\zeta_i<t}(1+cz(\zeta_i)J(\zeta_i))\exp\left(c\int_a^t z(s)\,d\eta(s)\right)$$

for all t in $[a, b]$.

The author illustrates the usefulness of his results by using them to prove some theorems on the boundedness and stability of solutions to finite difference equations and Volterra integral equations.

T. A. Brown (Santa Monica, Calif.)

Ghircoiaşiu, N.; Roşcău, H. 5272

L'intégration d'une équation fonctionnelle.

Mathematica (*Cluj*) **4** (**27**) (1962), 21–32.

Dans cet article on considère l'équation fonctionnelle

$$(1)\quad D_n(x,h) = \begin{vmatrix} f(x) & f(x+h) & \cdots & f(x+nh) \\ f(x+h) & f(x+2h) & \cdots & f(x+(n+1)h) \\ \cdot & \cdot & \cdot & \cdot \\ f(x+nh) & f(x+(n+1)h) & \cdots & f(x+2nh) \end{vmatrix} = 0,$$

dans l'ensemble des fonctions f réelles et continues. On démontre l'existence du domaine Ω $(-\infty<x<+\infty;$ $0<h<H)$ dans lequel $D_{n-1}(x,h)\neq 0$. On montre que dans le domaine Ω les équations (1) et

$$a_0(h)f(x)+a_1(h)f(x+h)+\cdots + a_{n-1}(h)f(x+(n-1)h)+a_n(h)f(x+nh) = 0$$

($a_i(h)$ sont des fonctions entières qui dependent de f) sont équivalentes. De là, en utilisant un résultat obtenu par F. Rado [Mathematica (Cluj) **4** (**27**) (1962), 131–143; MR **27** #2667] on a le théorème suivant: L'ensemble des solutions de l'équation fonctionnelle (1) est formé des intégrales de toutes les équations différentielles linéaires et homogènes aux coefficients constants de l'ordre n.

P. M. Vasić (Belgrade)

SEQUENCES, SERIES, SUMMABILITY

See also 5345.

Carlitz, Leonard 5273

A note on power series with integral coefficients.

Boll. Un. Mat. Ital. (3) **19** (1964), 1–3.

The author proves the criterion: Given a sequence of integers $b_1, b_2, \cdots$, the sequence of a_n determined by the system $na_n=\sum_{r=1}^{\infty} b_r a_{n-r}$ is an integral sequence if and only if the b_n satisfy $\sum_{d\delta=n}\mu(d)b_\delta\equiv 0 \pmod n$, where μ is the Möbius function. This result can be reformulated in terms of power series. *E. Cohen* (Knoxville, Tenn.)

Prasad, B. N. 5274

Recent researches on the second theorem of consistency.

Calcutta Math. Soc. Golden Jubilee Commemoration Vol. (1958/59), *Part I, pp.* 225–233. *Calcutta Math. Soc., Calcutta*, 1963.

For definitions of ordinary and absolute Riesz summability, reference may be made to K. Chandrasekharan and S. Minakshisundaram [*Typical means*, Oxford Univ. Press, 1952; MR **14**, 1077].

The theorem due to Riesz, which asserts that $(R,\lambda,\kappa)\subset (R,\lambda,\kappa')$, $\kappa'>\kappa\geqq 0$, is called the "first theorem of consistency" for Riesz summability. Its analogue for absolute summability was given in 1929 by N. Obreschkoff [Math. Z. **30** (1929), 375–386]. Several attempts have been made to answer the more complex question of the relative effectiveness of the methods (R,λ,κ) and (R,μ,κ) or $|R,\lambda,\kappa|$ and $|R,\mu,\kappa|$.

The author confines himself to an exposition of recent researches on the subject of the so-called "second theorems of consistency" which have provided answers to such questions. It may be noted that the monograph *Typical means* [loc. cit.] has given a lucid account of the second theorems of consistency for ordinary Riesz methods. The far-reaching researches of Kuttner in this field, which could not be discussed in this book on account of their then extreme recentness, have now been clearly brought out in the article under review.

The reviewer does not wish to recount again the efforts of various mathematicians which have been so ably presented by the author in this survey article. However, it appears useful to mention some more recent papers. For the latest on the ordinary summability side, reference to *Typical means* is imperative. Besides, the following papers are worth noting. (1) S. M. Mazhar, Proc. Nat. Inst. Sci. India Part A **27** (1961), 11–17 [MR **24** #A2778]; (2) the reviewer, Math. Student **29** (1961), 93–100 [MR **28** #3273]; (3) G. D. Dikshit, Indian J. Math. **1** (1958), no. 1, 33–40 [MR **21** #3695]; (4) ibid. **3** (1961), 7–26 [MR **27** #4007]; (5) Z. U. Ahmad, Rend. Circ. Mat. Palermo (2) **11** (1962), 91–104 [MR **28** #4274]; (6) the reviewer, Math. Student **29** (1961), 101–112 [MR **28** #3274]; (7) Florence M. Mears, Trans. Amer. Math. Soc. **30** (1928), 686–709.

(1) contains a second theorem of consistency for absolute Riesz summability defined along the lines suggested by Flett's famous works, with indices. (2) contains a somewhat simpler presentation of the previous result of Guha for non-integral order. (3), (4), and (5) give extended second theorems of consistency connecting the summability $|R,\mu,\kappa|$ of $\sum a_n$ with the summability $|R,\mu,\kappa|$ of $\sum \varepsilon_n a_n$. (6) contains a necessary-sufficient type of second theorem of consistency for Riesz boundedness, depending mainly on a striking negative result of Kuttner (Lemma 6 of this paper). The memoir (7) contains, inter alia, extensions of Hardy's classical second theorem for double series.

The reviewer takes this opportunity of mentioning that, as stated in *Typical means*, the non-integral case in reference [1] of the article under review [K. Chandrasekharan, J. Indian Math. Soc. (N.S.) **6** (1942), 168–180; MR **5**, 63] requires modification. In references [9] and [11] [the reviewer, Quart. J. Math. Oxford Ser. (2) **5** (1954), 161–168; MR **16**, 351; the author and the reviewer, Math. Ann. **140** (1960), 187–197; MR **22** #5843], the proofs of Lemmas 3 and 5, respectively, have been given under the implicit assumption that $G(\sigma)\in \mathrm{AC}(\delta,\infty)$, although in their statements, which are valid as they stand, BV is mentioned. In applications, this does not mean any material difference. The proofs could, however, be obviously made more precise by an appeal to Theorem 3 of J. B. Tatchell [Proc. London Math. Soc. (3) **3** (1953), 257–266; MR **15**, 118] or Lemma 17 of L. S. Bosanquet [ibid. **11** (1961), 654–690; MR **25** #5352], following the original line of arguments, and replacing $G^{(1)}(\sigma)\,d\sigma$ by $dG(\sigma)$, mutatis mutandis. *T. Pati* (Jabalpur)

Butzer, P. L.; Neuheuzer, H. G. 5275

Sur les conditions taubériennes pour les procédés de Cesàro.

C. R. Acad. Sci. Paris **258** (1964), 4411–4412.

In this interesting paper the authors give four classes of functions, each of which is a Tauberian class for the (C,k) process but not for the $(C,k+1)$ process. Four more classes, each of which is Tauberian for each Cesàro method, but not for the Abel method, are also given.

M. S. Ramanujan (Ann Arbor, Mich.)

Borwein, D.; Matsuoka, Y. 5276

On multiplication of Cesàro summable series.

J. London Math. Soc. **38** (1963), 393–400.

The authors examine the question of the Cesàro summability (ordinary (C, α), strong $[C, \alpha]$ or absolute $|C, \alpha|$) of the Cauchy product of two series $\sum_0^\infty a_n$ and $\sum_0^\infty b_n$ when we have information about the summability of each. They start by giving a natural extension of the definition of $[C, \alpha]$ to cover negative values of α and obtain results which supplement some of Borwein's own results and those of Winn and Boyd (definitions and references are given in the paper under review). One of the results obtained is as follows: If $\mu \geqq 0$ and $\sum_0^\infty a_n$, $\sum_0^\infty b_n$ are summable $[C, -\mu]$ to A, B, respectively, then the Cauchy product is summable $(C, -\mu)$ to AB. *U. C. Guha* (Singapore)

Syrmus, T. [Sõrmus, T.] 5277

Mercer-type theorems for restrained and ordinary convergence. (Russian. Estonian and German summaries)

Tartu Riikl. Ül. Toimetised No. 129 (1962), 274–282.

The author proves some more theorems of Mercerian type for double sequences, including simple statements for quite general summability matrices. He uses and generalizes results by Tedeev [Trudy Stalinirsk. Gos. Ped. Inst. 7 (1959), 237–277]. *L. Schmetterer* (Vienna)

APPROXIMATIONS AND EXPANSIONS

See also 5186, 5307, 5313, 5552.

Walsh, J. L.; Sharma, A. 5278

Least squares and interpolation in roots of unity.

Pacific J. Math. **14** (1964), 727–730.

Let $f(z)$ be analytic on the unit open disk and continuous on its closure. Let $p_n(z)$ be the polynomial of degree at most n found by interpolation to f in the $(n+1)$st roots of unity. The authors prove that $\{p_n(z)\}$ converges to $f(z)$ in the L^2 norm on $C: |z|=1$. The method of proof consists in using a comparison with a point-wise convergent polynomial process on C (known to exist) in conjunction with the fact that the fundamental polynomials of Lagrange interpolation for these nodes are orthogonal on C. Various remarks are then made about consequences of the existence of L^2 norm approximations on C by polynomials in z and $1/z$ to functions $f(z)$ defined and integrable on C. The remarks all pertain to the decomposition $f=f_1+f_2$, where f_1 is of the Hardy-Littlewood class H_2 for $|z|<1$ and f_2 is of the analogous class for $|z|>1$.
J. H. Curtiss (Coral Gables, Fla.)

Walsh, J. L.; Motzkin, T. S. 5279

Best approximators within a linear family on an interval.

Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 1225–1233.

Let a generalized norm be defined by $\|g(x)\| = \int_E \tau(|g(x)|)w(x)\,dx$, where $E=[0, 1]$, $\tau(t)$ is twice continuously differentiable on $[0, \infty)$, $w(x)$ is positive a.e. on E, and $w(x) \in C(E)$, the continuous functions on E. Let P be a linear family of functions, $P \subset C(E)$, and let $p(x) \in C(E)$. If $p(x)$ gives a local minimum to $\|f-p\|$ for every ray in P which ends at p, then $p(x)$ is called a starwise weakly τ-nearest element of P to f. A necessary condition for this is that for every $q \in P$,

$$(*) \quad \int_{E-E_0} q\tau'(|f-p|)\operatorname{sgn}(f-p)w\,dx \leqq \tau'(0)\int_{E_0} |q|w\,dx,$$

where $f-p$ vanishes on $E_0 \subset E$. Additional conditions—which essentially ensure that P is not too large—are found under which (*) is a condition sufficient to ensure that p is a globally τ-nearest element to f. The problem of finding a $w(x)$ to make a given p globally τ-nearest to f is also studied. These results extend previous work of the authors with $\tau(t)=t^p$ [same Proc. **45** (1959), 1523–1528; MR **22** #9773]. *H. O. Pollak* (Murray Hill, N.J.)

Grebenjuk, D. G. [Гребенюк, Д. Г.] 5280

★Polynomials of best approximation with linearly related coefficients [Полиномы наилущего приближения, коэффициенты которых связаны линейными зависимостями].

Akad. Nauk Uzbek. SSR, Inst. Mat. V. I. Romanovskiĭ. *Izdat. Akad. Nauk Uzbek. SSR, Tashkent,* 1960. 239 *pp.* 14.40 *r.*

Golovkin, K. K. 5281

On the approximation of functions in arbitrary norms. (Russian)

Trudy Mat. Inst. Steklov. **70** (1964), 26–37.

Soit $\|\cdot\|$ une norme arbitraire, définie pour une classe des fonctions $u(x)$ sur $-\infty<x<+\infty$ ou bien $x>0$, telles que $\|u(x+z)\| \leqq \|u(x)\|$, $z>0$. Soit $Q_m(x)=(-1)^m[1-(1+x)^m] = \sum_{k=1}^m C_k^m x^k$, $\Delta_h^m u(x)=\sum_1^m C_k^m u(x+h)$, $\zeta(z)$ une fonction indéfiniment différentiable et zéro en dehors de $(0, 1)$, $\omega_m(z)=\sum_1^m k^{-1}\zeta(z/k)$, $u_h^m=h^{-1}\int_0^\infty \omega_m(z/h)u(x+z)\,dz$. En posant

$$I_p[\psi(h)] = \left(\int_0^\infty h^{-1}\psi^p(h)\,dh\right)^{1/p}, \qquad 1 \leqq p < \infty,$$

$$I_\infty[\psi(h)] = \sup_{h>0} \psi(h),$$

l'auteur donne l'estimation du type suivant

$$I[h^{-r}\|u-u_h\|] \leqq CI[h^{-r}\|\Delta_h^m\|], \qquad 0<r<m.$$

On détermine aussi l'allure asymptotique de u_h^m, $h \to 0$. Les résultats sont généralisés pour des fonctions à plusieurs variables. *M. Tomić* (Belgrade)

Golovkin, K. K. 5282

Imbedding theorems for fractional spaces. (Russian)

Trudy Mat. Inst. Steklov. **70** (1964), 38–46.

Les résultats de l'auteur et V. A. Solonnikov [Dokl. Akad. Nauk SSSR **143** (1962), 767–770; MR **25** #444] sont démontrés ici en détail. La méthode de démonstration est fondée sur l'approximation des fonctions par des moyennes du type considéré dans la note analysée ci-dessus [#5281]. Cette méthode est très appropriée pour des problèmes d'immersion relatifs aux espaces fractionnaires de S. M. Nikol'skiĭ [Mat. Sb. (N.S.) **33** (**75**) (1953), 261–326; MR **16**, 453]. *M. Tomić* (Belgrade)

Berg, Lothar 5283

Asymptotische Entwicklungen für Parameterintegrale. II.

Math. Nachr. **27** (1963/64), 133–143.

The author gives a proof of a general asymptotic expansion announced in Part I [same Nachr. **24** (1962), 181–192; MR **26** #6663]. There is an application to Mellin transforms. *R. R. Goldberg* (Evanston, Ill.)

Van der Corput, J. G. [van der Corput, J. G.] 5284

La valeur qu'il faut attribuer à une fonction en un point singulier.

J. Math. Pures Appl. (9) **42** (1963), 351–366.

Consider the function

$$f(w,s) = w\int_0^\infty x^{s-1}(w+x)^{-1}e^{-x}\,dx,$$

where $-\pi+\varepsilon\leqq\arg w\leqq\pi-\varepsilon$ and where s and $\varepsilon>0$ are independent of w, and $\operatorname{Re}(s)>0$. This function possesses the asymptotic expansion $\sum_{h=0}^{\infty}(-1)^h\Gamma(s+h)w^{-h}$. Both the function and the series can be continued into the left half of the complex s-plane, except for the points $s=0, -1, -2, \cdots$.

The purpose of this paper is to show how the function and the series can be assigned "natural" values at these singular points. To obtain these values the author introduces certain "neutrices". It turns out that the values obtained depend on how the neutrix is used as well as on which neutrix is introduced.

Two other examples are given.

T. E. Hull (Vancouver, B.C.)

Berens, Hubert; Butzer, P. L. 5285

On the best approximation for singular integrals by Laplace-transform methods.

Bull. Amer. Math. Soc. **70** (1964), 180–184.

Soit $f(t)$ une fonction intégrable dans $(0, R)$ pour tout $R>0$; on pose

$$J_\rho(t) = \rho\int_0^t f(t-u)k(\rho u)\,du,$$

où $\rho>0$ et où le noyau k possède les propriétés suivantes (P): $k(u)\geqq 0$ $(0\leqq u<+\infty)$; $k\in L(0,\infty)$ et $\int_0^{+\infty}k(u)\,du=1$; alors si $e^{-ct}f(t)\in L_p(0,+\infty)$ $(1\leqq p<\infty)$ pour tout $c>0$, $J_\rho(t)$ existe presque partout et $e^{-ct}J_\rho\in L_p(0,+\infty)$.

Pour f appartenant à l'une de ces classes et pour h localement à variation bornée pour $t\geqq 0$, on considère les transformées de Laplace:

$$\hat f(s) = \int_0^{+\infty}e^{-st}f(t)\,dt,\qquad \check h(s) = \int_0^{+\infty}e^{-st}\,dh(t)$$

$$(s=\sigma+it,\ \operatorname{Re} s=\sigma>0).$$

Sur k on fait de plus l'une des hypothèses suivantes, avec $\operatorname{Re} s>0$:

$$(1)\qquad \lim_{\rho\to\infty}\left(\frac{s}{\rho}\right)^{-\gamma}\left[1-k\left(\frac{s}{\rho}\right)\right] = A$$

existe pour un γ réel $(0<\gamma\leqq 1)$, avec $0<A<+\infty$; (2) il existe une fonction normalisée $Q(u)$, à variation bornée sur $[0,+\infty]$, avec $Q(+\infty)=1$, telle que

$$A^{-1}\left(\frac{s}{\rho}\right)^{-\gamma}\left[1-k\left(\frac{s}{\rho}\right)\right] = \check Q\left(\frac{s}{\rho}\right);$$

(3) il existe une fonction $q\in L(0,+\infty)$, $\int_0^{+\infty}q(u)\,du=1$ et

$$A^{-1}\left(\frac{s}{\rho}\right)^{-\gamma}\left[1-k\left(\frac{s}{\rho}\right)\right] = \hat q\left(\frac{s}{\rho}\right).$$

On a alors les théorèmes d'approximation suivants.

(I) Supposons que $e^{-ct}f$ et $e^{-ct}l\in L(0,+\infty)$, que k satisfasse aux propriétés (P) et à (1); (a) la relation $\|e^{-ct}[\rho^\gamma(f-J_\rho)-l]\|_{L_1(0,+\infty)}=o(1)$ $(\rho\to+\infty)$ entraîne

$$As^\gamma\hat f(s) = \hat l(s)\qquad(\operatorname{Re} s>0)$$

ou

$$Af(t) = \int_0^t\frac{(t-u)^{\gamma-1}}{\Gamma(\gamma)}\,l(u)\,du\qquad\text{(presque partout)};$$

(b) si $\|e^{-ct}(f-J_\rho)\|_{L_1(0,+\infty)}=O(\rho^{-\gamma})$ $(\rho\to+\infty)$, entraîne l'existence d'une fonction $F(t)$ localement à variation bornée pour $t\geqq 0$, avec $\int_0^{+\infty}e^{-ct}|dF|<+\infty$, pour tout $c>0$ et on a:

$$As^\gamma\hat f(s) = \check F(s)\qquad(\operatorname{Re} s>0).$$

(II) Si k satisfait à (P) et à (2), la réciproque de (b) est vraie.

(III) Si k satisfait à (P) et à (3), la condition nécessaire et suffisante pour que $J_\rho(t)$ soit saturée avec l'ordre $O(\rho^{-\gamma})$ pour les fonctions $e^{-ct}f\in L_p(0,+\infty)$ $(1<p<+\infty)$ $(c>0)$, est qu'il existe une fonction $e^{-ct}F\in L_p(0,+\infty)$ $(c>0)$ telle que $As^\gamma\hat f(s)=\hat F(s)$ $(\operatorname{Re} s=0)$.

Une application intéressante est faite à l'équation de la chaleur. *J. Favard* (Paris)

Gapoškin, V. F. 5286

A localization principle and systems of the form $\{\varphi(nx)\}$. (Russian)

Mat. Sb. (N.S.) **63 (105)** (1964), 459–488.

The author has previously investigated expansions in odd functions $\varphi(nx)$ which are "close" to $\sin nx$ [same Sb. (N.S.) **51 (93)** (1960), 239–252; MR **23** #A468], and showed, in particular [Dokl. Akad. Nauk SSSR **144** (1962), 17–18; MR **25** #386], that the principle of localization does not hold in general except when $\varphi(x)=A\sin x$. He now shows that this is still true without any condition of "closeness" at all. He proves a similar thing for systems $\{1, \varphi(nx), \psi(nx)\}$. Here the principle of localization at x_0 means that if f and g are integrable and coincide in a neighborhood of x_0, then the difference of the partial sums of the formal series for f and g converges to 0 at x_0. The author extends the principle of localization to other classes than C and L and to summability. Considering only the T-systems of his first cited paper, he shows that if $x_0\neq p\pi/q$ and $\{\varphi(nx)\}$ admits localization (with convergence) at x_0 in the class of functions whose modulus of continuity is $O\{1/\log(1/\delta)\}$, then $\varphi(x)=A\sin x$. The same conclusion holds if there is localization with Abel summability for the class of bounded functions. A necessary and sufficient condition is given for localization to hold at $x_0=p\pi/q$, improving the condition given in the second cited reference. Finally, the author gives a necessary and sufficient condition for a T-system to be a Riesz basis and some theorems on partial sums of conjugate series.

R. P. Boas, Jr. (Evanston, Ill.)

FOURIER ANALYSIS

See also 5137.

Stečkin, S. B. 5287a

The approximation of periodic functions by Fejér sums. (Russian)

Trudy Mat. Inst. Steklov. **62** (1961), 48–60.

Stečkin, S. B. 5287b

The approximation of periodic functions by Fejér sums.

Amer. Math. Soc. Transl. (2) **28** (1963), 269–282.

Let f be a 2π-periodic function, $E_n(f)$ $(n=1, 2, \cdots)$ its best approximation by trigonometric polynomials of degree $n-1$, $\omega_k(\delta, f)$ its modulus of continuity of order k, $\sigma_n(f)$ its nth Fejér partial sum. The paper contains, for example, the following statements. (i) Let $f \in C$ (C the space of 2π-periodic continuous functions). Then

$$\rho_n(f) \equiv \|f-\sigma_{n-1}(f)\|_C \leqq \frac{B_1}{n} \sum_{\nu=1}^{n} E_\nu(f) \quad (n = 1, 2, \cdots),$$

where B_1 is a constant. A corresponding theorem holds with L_p $(1 \leqq p < \infty)$ instead of C. A corollary of (i) is (ii): Let $f \in C$. Then

$$\rho_n(f) \leqq \frac{B_2(k)}{n} \sum_{\nu=1}^{n} \omega_k\left(\frac{1}{\nu}, f\right) \quad (n = 1, 2, \cdots),$$

where B_2 is a constant depending only on k. (iii) Let $F=\{F_n\}$ $(n=1, 2, \cdots)$ with $F_n \downarrow 0$ and $C(F)$ the class of $f \in C$ for which $E_n(f) \leqq F_n$ $(n=1, 2, \cdots)$. Then

$$n^{-1} \sum_{\nu=1}^{n} F_\nu \leqq \sup \rho_n(f) \leqq \frac{B_3}{n} \sum_{\nu=1}^{n} F_\nu,$$

where the supremum is taken with respect to all $f \in C(F)$.

There are also given estimates for $\rho_n(\tilde{f})$ if $f \in C$ and $\tilde{f}$ denotes the conjugate function of f. (iv) Let $f \in C$ and f' be bounded, then $\rho_n(f) \leqq (M/n)\|\tilde{f}'\|$. (v) Let $f \in C$ and let $t_n(x)$ be a trigonometric polynomial of degree n which fulfills the inequality $\|f-t_n\| \leqq B_4(k)\omega_k(1/n, f)$. Then $\|t_n^{(k)}\| \leqq B_5(k)n^k\omega_k(1/n, f)$. (vi) If $f \in C$ and $\tilde{f} \in C$, then $\rho_n(\tilde{f}) \leqq B_6\{\omega(1/n, f)+E_{n+1}(\tilde{f})\}$. Several corollaries to these theorems are stated. (vii) Let $\omega(\delta)$ be a positive function $(0<\delta \leqq \pi)$, $\omega^{**}(\delta)=\delta \inf_{0<\eta \leqq \delta}\{\eta^{-1} \inf_{\eta \leqq \xi \leqq \pi} \omega(\xi)\}$ $(0<\delta \leqq \pi)$ and $\sum_{n=1}^{\infty} n^{-1}\omega^{**}(n^{-1})<\infty$. Let, furthermore, $H(\omega)$ be the class of $f \in C$ for which $\omega_1(\delta, f) \leqq \omega(\delta)$ $(0<\delta \leqq \pi)$. Then $\sup_{f \in H(\omega)} \rho_n(\tilde{f}) \sim \sum_{\nu=n+1}^{\infty} \nu^{-1}\omega^{**}(\nu^{-1})$.

Finally the author states in two theorems estimates for the deviation of $f \in C$ and $\tilde{f} \in C$ from linear approximating expressions $U_n(f)$, respectively, $U_n(\tilde{f})$. Example: Let k be a natural number and let U_n be a linear approximation method with the following properties: (1) $\|U_n(f)\| \leqq M_0\|f\|$ for every $f \in C$. (2) $\|f-U_n(f)\| \leqq (M_k/n^k)\|f^{(k)}\|$ $(n=1, 2, \cdots)$ for $f \in C$ and $f^{(k)} \in C$. Then $\|f-U_n(f)\| \leqq B(k)(M_0+M_k)\omega_k(1/n, f)$ for any $f \in C$.

G. Goes (Lawrence, Kans.)

Kennedy, P. B. 5288

Note on Fourier series with Hadamard gaps.

J. London Math. Soc. **39** (1964), 115–116.

Let $f(x) \in L(-\pi, \pi)$ and have the Fourier series $\sum_1^\infty (a_k \cos n_k x + b_k \sin n_k x)$, where $\liminf n_{k+1}/n_k > 1$. Suppose also that, for some fixed $x_0 \in [-\pi, \pi]$ and $\alpha > 0$,

$$f(x_0+t)-f(x_0) = O(|t|^\alpha) \quad (t \to 0).$$

Then it is shown that $a_k, b_k = O(1)(\log n_k/n_k)^\alpha$, $k \to \infty$. This theorem is a generalization of the result obtained by the reviewer [same J. **37** (1962), 117–120; MR **24** #A3465].

The very simple proof is based on M. E. Noble's method of approximation [Math. Ann. **128** (1954), 55–62; correction, ibid. **128** (1954), 256; MR **16**, 126].

M. Tomić (Belgrade)

Kadec, M. I. 5289

The exact value of the Paley-Wiener constant. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 1253–1254.

The author proves two theorems concerning Riesz bases in $L^2(-\pi, \pi)$ (i.e., bases $\{x_k\}$ such that each $x=\sum_{k=1}^\infty a_k x_k$ satisfies $A(\sum |a_k|^2)^{1/2} \leqq \|x\| \leqq B(\sum |a_k|^2)^{1/2}$, where $0<A \leqq B<\infty$). Theorem 1: If the sequence of real numbers $\lambda_k=k+\delta_k$ $(k=0, \pm 1, \pm 2, \cdots)$ satisfies $\sup_k |\delta_k|=d<0.25$, then $\{e^{i\lambda_k t}\}_{-\infty}^{\infty}$ is a Riesz basis of $L^2(-\pi, \pi)$. (Paley and Wiener [*Fourier transforms in the complex domain*, Amer. Math. Soc., New York, 1934] have proved this result for $d<1/\pi^2$, Duffin and Eachus [Bull. Amer. Math. Soc. **48** (1942), 850–855; MR **4**, 97] for $d<\ln(2/\pi) \approx 0.22$ and V. D. Golovin [Akad. Nauk Armjan. SSR Dokl. **36** (1963), 65–70; MR **27** #2769] for $d<0.24$, while N. Levinson [Ann. of Math. (2) **37** (1936), 919–936] has proved that the theorem is not valid for $d \geqq 0.25$.) Theorem 2: If $\sup_k |\mathrm{Im}(z_k-ik)|<0.25$ and $\sup_k|\mathrm{Re}\, z_k|<\infty$, then $\{e^{z_k t}\}_{-\infty}^{\infty}$ is a Riesz basis of $L^2(-\pi, \pi)$.

Ivan Singer (Bucharest)

Robertson, M. M. 5290

Integrability of trigonometric series.

Math. Z. **83** (1964), 119–122.

Let $\{\lambda_1, \lambda_2, \cdots\}$ be a sequence of real numbers which can be partitioned into k disjoint sequences $\Lambda_i=\{\lambda_{rk+i}\}$ $(i=1, \cdots, k)$ each of which ultimately increases or decreases to 0. The functions $f(x)=\frac{1}{2}\lambda_0+\sum_1^\infty \lambda_n \cos nx$, $g(x)=\sum_1^\infty \lambda_n \sin nx$ are then said to be "well behaved of order k". It is shown that if f, g are well behaved, then, for $0<\gamma<1$, $f(x)(x-t\pi)^{-\gamma}$, $g(x)(x-t\pi)^{-\gamma} \in L(0, \pi)$ for all t in $[0, 1]$ if and only if $\sum n^{-\gamma}|\lambda_n|$ converges. The method of proof is similar to that of R. P. Boas [Quart. J. Math. Oxford Ser. (2) **3** (1952), 217–221; MR **14**, 867] who considers the case $k=1$. This case is also treated by P. Heywood [ibid. (2) **5** (1954), 71–76; MR **16**, 30].

H. Burkill (Sheffield)

Tobias, T. 5291

On mean-square approximation of linear functionals. (Russian. Estonian and English summaries)

Eesti NSV Tead. Akad. Toimetised Füüs.-Mat. Tehn.-tead. Seer. **13** (1964), 70–82.

The author investigates Wiener measure in a space of continuous functions of n variables. In particular, he establishes the following two theorems. Theorem 1: If $F(x)$ is a Wiener integrable functional on C_n, $x_0(s) \in C_n$, and

$$\frac{d}{ds} x_0(s) = \frac{\partial^n}{\partial s_1 \cdots \partial s_n} x(s_1, \cdots, s_n) \in L_2(S_n),$$

then the translation of a variable $y(s)=x(s)+x_0(s)$ yields

$$\int_{C_n} F(y)\,d_w y = \exp\left\{-\frac{1}{2}\int_{S_n}\left[\frac{d}{ds}x_0(s)\right]^2 ds\right\}$$
$$\times \int_{C_n} F(x+x_0)\exp\left\{-\int_{S_n}\left[\frac{d}{ds}x_0(s)\right]dx(s)\right\}d_w x,$$

where S_n is the unit cube such as $0\leqq s_k\leqq 1$ $(k=1,2,\cdots,n)$ and C_n is a set of continuous functions on S_n such as $x(s_1,\cdots,0,\cdots,s_n)=0$. Theorem 2: Let $\{\alpha_{k_1}(s_1)\cdots\alpha_{k_n}(s_n)\}$ be a complete normalized orthogonal system in $L_2(S_n)$ and put

$$\beta_k(t) = \int_0^{t_1}\cdots\int_0^{t_k}\alpha_{k_1}(s_1)\cdots\alpha_{k_n}(s_n)\,ds,$$

then the Fourier-Hermite development of a linear functional $F(x)$ consists only of linear terms, that is,

$$F(x) = \sum_{(k)=(1)}^{\infty} F(\beta_k)\int_{S_n}\alpha_{k_1}(s_1)\cdots\alpha_{k_n}(s_n)\,dx(s).$$

G. Sunouchi (Sendai)

Alexits, G. 5292

★Konvergenzprobleme der Orthogonalreihen.

Verlag der Ungarischen Akademie der Wissenschaften, Budapest, 1960. 307 *pp.*

Neckermann, L. 5293

Fouriersche Reihenentwicklungen nach einem System orthogonaler Gegenbauerscher Funktionen.

Arch. Math. **15** (1964), 91–107.

The author considers eigenfunctions of the Gegenbauer differential equation with parameter $\nu>-\frac{1}{2}$ and boundary conditions $y(c)=0$, where $-1<c<1$, and $y(x)=O(1)$ for $\nu\geqq\frac{1}{2}$, $y'(x)=O(1)$ when $-\frac{1}{2}<\nu<\frac{1}{2}$ as $x\to 1-$. The case $\nu=\frac{1}{2}$ (Legendre equation) was considered in the author's thesis [Univ. Würzburg, Würzburg, 1962; MR **28** #3170]. The author proves the existence of a system of solutions $C_{\lambda_n}{}^{\nu}(x)$, orthogonal with weight function $(1-x^2)^{\nu-1/2}$; the nth function has n zeros in $(c,1)$; the system is complete with respect to continuous functions. The Fourier series of $f(x)$ converges absolutely and uniformly in $[c,1]$ to $f(x)$ if $f(x)$ is continuous, $f(c)=0$, $f'(x)$ exists in $[c,1)$ and $(1-x^2)^{(\nu+1)/2}f'(x)$ is of bounded variation, if $-\frac{1}{2}<\nu<1$, with conditions on higher derivatives if $\nu\geqq 1$.

R. P. Boas, Jr. (Evanston, Ill.)

Golubov, B. I. 5294

Fourier series of continuous functions relative to a Haar system. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 247–250.

Let $\{\chi_n(t)\}$, $n=1,2,\cdots$, be a Haar system. With $f(t)$ a continuous function on $[0,1]$ let $c_n(f)=\int_0^1 f(t)\chi_n(t)\,dt$ and $E_n(f)=\inf_P\sup_t|f(t)-P(t)|$, where $P(t)$ varies over linear combinations of the first n functions of the system. The author states without proof a number of theorems describing the growth of $c_n(f)$ and $E_n(f)$ when f is continuous, is of Lip α, has variation bounded in L^P mean, is absolutely continuous, and is monotone. The results given are sharp. We cite a few. If $\omega(\delta)\downarrow 0$ and $\delta^{-1}\omega(\delta)\uparrow$ as $\delta\downarrow 0$ and $\sum n^{-1/2}\omega(n^{-1})=\infty$, there exists a continuous function $f(t)$ whose modulus of continuity is $O(\omega(\delta))$ and for which $\sum|c_n(f)|$ diverges. A function $f(t)$ is of Lip α for some $0<\alpha\leqq 1$, if and only if $E_n(f)=O(n^{-\alpha})$ (with $\alpha=1$, this differs from the corresponding result for trigonometric series). For every absolutely continuous $f(t)\not\equiv$ const, $c_n(f)\neq o(n^{-3/2})$, but there exists an absolutely continuous $f_1(t)\not\equiv$ const with $c_n(f_1)=O(n^{-3/2})$ and for every $\varepsilon>0$ there exists an absolutely continuous $f_2(t)$ with $c_n(f_2)\neq o(n^{-1/2-\varepsilon})$. The numbers $c_n(f)$ are non-negative, $n\geqq 2$, if and only if $f(t)$ is non-increasing.

H. J. Landau (Murray Hill, N.J.)

Weiss, M. 5295

On symmetric derivatives in L^p.

Studia Math. **24** (1964), 89–100.

A function $f\in L^p$ has the kth-order L^p-derivative a at x_0 if there exists a polynomial $P(t)$ of degree k with leading coefficient $a/k!$ such that

$$\left\{h^{-1}\int_0^h|f(x_0+t)-P(t)|^p\,dt\right\}^{1/p} = o(h^k).$$

f has the symmetric kth L^p-derivative a at x_0 if $\frac{1}{2}\{f(x_0+t)+(-1)^k f(x_0-t)\}$ has kth L^p-derivative a at $t=0$. Theorem 1: If f has a kth symmetric derivative in L^p at each point of a set E, then at almost all points of E the function has an unsymmetric kth derivative in L^p.

The author proves Theorem 1 for $1\leqq p<\infty$ starting from the known result corresponding to $p=\infty$ [J. Marcinkiewicz and A. Zygmund, Fund. Math. **26** (1936), 1–43]. She uses Theorem 1 to prove a result about trigonometric series

$$(1.1)\qquad \tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+bn\sin nx).$$

Theorem 2: Suppose that the series (1.1) is summable $(C,r-1)$ $(r=1,2,\cdots)$ at each point of a set E of positive measure. Then the series obtained by integrating (1.1) termwise r times converges almost everywhere to a function $F(x)$ which is in every L^p, $p<\infty$, and which has almost everywhere in E an unsymmetric rth derivative in L^p equal to the $(C,r-1)$ sum of the series (1.1).

The proof of Theorem 2, or rather its localised version called Theorem C, involves some clever estimations. {In the course of the proof a_n is used with two different meanings; in § 5 a_n is not the coefficient in (1.1).}

C. S. Herz (Ithaca, N.Y.)

Flor, Peter; Helmberg, Gilbert 5296

Zur definition der Verteilungsfunktion einer reellwertigen Funktion.

J. Reine Angew. Math. **214/215** (1964), 261–267.

Wintner showed that a real-valued almost periodic function f on R has a distribution function in the sense that for all but at most countably many y the limit

$$v(f;y) = \lim_{J\to\infty}\frac{1}{2J}\lambda(\{x:|x|\leqq J, f(x)\leqq y\})$$

exists ($\lambda=$ Lebesgue measure). The proof given by Jessen and Tornehave in their paper [Acta Math. **77** (1945), 137–279; MR **7**, 438] shows that the existence of this distribution function is a consequence of the fact that for every continuous real-valued function g on R the function

$g\circ f$ is again almost periodic and therefore has a mean value

$$M(g\circ f) = \lim_{J\to\infty} \frac{1}{2J}\int_{-J}^{J} g\circ f(x)\,dx.$$

Since these and related properties are possessed by other functional classes (for instance, the weakly almost periodic functions of Eberlein), the authors of the present paper wanted to establish the various basic connections in a general frame, free from superfluous assumptions.

E. Følner (Copenhagen)

Helmberg, Gilbert 5297

Almost periodic functions and Dini's theorem.

Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 173–177.

Let $AP(G)$ denote the space of almost periodic functions on a topological group G. Recently Greve [J. Austral. Math. Soc. **2** (1961/62), 143–146; MR **25** #1407] asserted that a sequence $\{f_n\}$ in $AP(G)$, converging pointwise to zero on G, converges uniformly (so that $Mf_n\to 0$ as well, where M is the usual mean). The author gives a counterexample, and then shows the assertion fails for any non-compact σ-compact maximally almost periodic G; in fact, for $\varepsilon>0$ one can have $Mf_n\geqq 1-\varepsilon$, $0\leqq f_n\leqq 1$.

{Something like σ-compactness is needed for this result even if G is locally compact. For if G is the discrete version $\mathscr{G}_d$ of a compact simple Lie group $\mathscr{G}$, the result fails, for by a well-known result of van der Waerden [Math. Z. **36** (1933), 780–786], the finite-dimensional representations of $\mathscr{G}_d$ are necessarily continuous on $\mathscr{G}$, so that $AP(\mathscr{G}_d)=C(\mathscr{G})$ and Dini's theorem applies directly.}

I. Glicksberg (Seattle, Wash.)

Maravić, Manojlo 5298

Sur certaines relations asymptotiques entre les moyennes sphériques d'ordre supérieur et les moyennes de Riesz des séries de Fourier multiples.

C. R. Acad. Sci. Paris **258** (1964), 4407–4410.

Let $f(Q)=f(x_1,\cdots,x_n)$ be a real, locally integrable function, periodic with period 2π in each of its variables. Let $S_x^{\delta}(Q)$ be the Riesz mean of order δ of the multiple Fourier series of f, and let $f_P(t)$ be the spherical mean of f over the sphere of radius t, center $P=(x_1^0,\cdots,x_n^0)$. The spherical mean of order $s\geqq 0$, $f_{P,s}(t)$, is defined by

$$f_{P,s}(t) = 2B(s,\tfrac12 n)t^{-n-2s+2}\int_0^t (t^2-u^2)^{s-1}u^{n-1}f_P(u)\,du, \qquad s>0,$$

$$= f_P(t), \qquad s=0.$$

The author announces two theorems which state, under certain conditions on the parameters appearing, that if $f_{P,s}(t)$ can be asymptotically approximated in a certain form as $t\to 0+$, then S_x^{δ} can also be asymptotically approximated in a related form, and conversely.

P. G. Rooney (Toronto, Ont.)

Hasegawa, Yoshimitsu 5299

On summabilities of double Fourier series.

Kōdai Math. Sem. Rep. **15** (1963), 226–238.

The author generalizes four theorems on $(C,1)$ and Abel summability of Fourier series and of the conjugate Fourier series to the case of double Fourier and conjugate Fourier series. For example, he proves that if $f(x,y)$ is a continuous function of period 2π in each variable and $|f(x+t,y+s)-f(x,y)|=O(|t|^{\alpha}+|s|^{\beta})$ uniformly in (x,y) as $s,t\to 0$, then for $0<\alpha$, $\beta<1$, $|\sigma_{mn}(x,y)-f(x,y)|=O(m^{-\alpha}+n^{-\beta})$ uniformly in (x,y) as $m,n\to\infty$, where σ_{mn} is the $(C,1,1)$ sum; also, $|f(r,x;R,y)-f(x,y)|=O((1-r)^{\alpha}+(1-R)^{\beta})$ uniformly in (x,y) as $r,R\to 1-0$ where $f(r,x;R,y)$ is the Abel mean. Results are obtained for $\alpha=\beta=1$ and if O is replaced by o.

J. Mitchell (University Park, Pa.)

Mendès France, Michel 5300

Nombres normaux et fonctions pseudo-aléatoires.

Ann. Inst. Fourier (Grenoble) **13** (1963), *fasc.* 2, 91–104.

A pseudo-random function (fonction pseudo-aléatoire) $f(t)$ is a complex integrable function of the real positive variable t such that $\lim_{T\to\infty}(1/T)\int_0^T \overline{f(t)}\,f(t+h)\,dt$ exists and is continuous for $h=0$. The author establishes relations between certain such functions and the theory of normal numbers. He uses, in particular, the multiple correlation

$$\Gamma(k_1,k_2,\cdots,k_p) = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(t+k_1)f(t+k_2)\cdots f(t+k_p)\,dt.$$

A central theorem is the following: Let $f(t)=r_k(x)$ where $k=[t]$ and $r_k(x)=\exp(i\pi[x\cdot 2^k])$, the Rademacher function. A necessary and sufficient condition that the real number x be normal is that $\Gamma(k_1,\cdots,k_p)=0$ for all sequences of integers $k_1,\cdots,k_p$ $(k_1<k_2<\cdots<k_p)$. The well-known result of Borel that almost all real numbers are normal reappears in this context.

H. W. Brinkmann (Swarthmore, Pa.)

Varopoulos, Nicholas Th. 5301

Sur les mesures de Radon d'un groupe localement compact.

C. R. Acad. Sci. Paris **258** (1964), 4896–4899.

Extending the result of a previous note [same C. R. **258** (1964), 3805–3808; MR **28** #4386], the author proves that every non-discrete, locally compact, Abelian group admits Θ-measures.

Edwin Hewitt (Seattle, Wash.)

INTEGRAL TRANSFORMS, OPERATIONAL CALCULUS

See also 5179, 5220, 5268, 5285, 5326, 5335.

Rasof, Bernard 5302

The Laplace transform of the sectionally-continuous derivative of a sectionally-continuous function.

J. Franklin Inst. **277** (1964), 119–127.

The paper is based on a misconception of functions and integrals illustrated by the author's statement that "we do not know what is meant by" the Laplace transform of $f(t)$ when $f(t)=e^t$ $(0<t<1)$ and $f(t)=0$ $(t>1)$, because of the distinct ways of defining f over the two subintervals.

R. V. Churchill (Ann Arbor, Mich.)

Gutiérrez Suárez, Juan José 5303

Characterization of functions representable by integral transforms with a Whittaker kernel. (Spanish)

Rev. Mat. Hisp.-Amer. (4) **24** (1964), 16–51.

Fortsetzung und Schluß der Arbeit unter gleichem Titel in derselben Zeitschrift (4) **23** (1963), 93–129 [MR **28** #431]; ibid. (4) **23** (1963), 139–175 [MR **28** #2411]. In den früheren Kapiteln wurden unter gewissen Bedingungen die asymptotischen Entwicklungen von Funktionen angegeben, die durch Integraltransformationen der Form $f(z)=\int_0^\infty \Phi(tz)\alpha(t)\,dt$ darstellbar sind. Insbesondere wurde die Mittag-Leffler-, die Laplace- und die (verallgemeinerte) Whittaker-Transformation zu Grunde gelegt. In der vorliegenden Note (Kapitel IV der gesamten Arbeit) wird umgekehrt das Problem behandelt, wann eine Funktion mit gegebener asymptotischer Entwicklung durch eine verallgemeinerte Whittaker-Transformation darstellbar ist. *G. Doetsch* (Freiburg)

INTEGRAL EQUATIONS

See also 5192, 5204, 5247, 5332, 5575, 5576, 5577, 5778.

Shinbrot, Marvin 5304

On singular integral operators.

J. Math. Mech. **13** (1964), 395–406.

In a separable Hilbert space $\mathfrak{H}$, let H_+ and H_- be projections satisfying $H_+ + H_- = I$ (the identity), $H_+H_- = H_-H_+ = 0$. Let N be a bounded normal operator with bounded inverse, and P a projection commuting with N. Two equations are studied in $\mathfrak{H}$: (1) $(H_- + NH_+)f = h$; (2) $P(H_- + NH_+)Pf = Ph$. A concrete classical case of these abstract equations is obtained by taking

$$\mathfrak{H} = L^2(-\infty,\infty), \qquad H_\pm = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{f(y)}{y-x\pm i0}\,dy,$$

and taking N and P to be multiplications by functions, N bounded and bounded away from zero, P a characteristic function of a set E. Then (1) is a singular integral equation with Cauchy-type kernel, and (2) restricts (1) to the set E.

The author proves that if, in addition, N is a positive operator, then each equation has a unique solution for every $h\in\mathfrak{H}$. The basic step in the construction of the solutions is the following factorization of N relative to $H_\pm$: $N = C_-C_+$ with $\mathfrak{R}(C_+H_+) = \mathfrak{R}(H_+)$, $\mathfrak{R}(C_-^{-1}H_-) = \mathfrak{R}(H_-)$, where $C_\pm$, $C_\pm^{-1}$ are bounded and $\mathfrak{R}$ denotes the range. This is an extension to the abstract situation of the Wiener-Hopf method. *E. Shamir* (Berkeley, Calif.)

Mel'nik, I. M. 5305

On a singular integro-functional equation. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 100–109.

Let $a(x)$, $b(x)$, $c(x)$, $d(x)$ and $f(x)$ be complex functions of a real variable x and satisfy Hölder conditions for all values of x including the point at infinity. The singular integral equation

$$a(x)\varphi(x)+b(x)\varphi(-x)+\frac{c(x)}{\pi i}\int_{-\infty}^{\infty}\frac{\varphi(\tau)}{\tau-x}\,d\tau + \frac{d(x)}{\pi i}\int_{-\infty}^{\infty}\frac{\varphi(\tau)}{\tau+x}\,d\tau = f(x)$$

is considered. The solution $\varphi(x)$ is sought in the class of functions satisfying Hölder conditions. The given equation is studied for those cases when the solution can be expressed in terms of integrals of the Cauchy type. *H. P. Thielman* (Alexandria, Va.)

Srivastava, K. N. 5306

Dual integral equations with Bessel function and trigonometrical kernels.

Proc. Edinburgh Math. Soc. (2) **13** (1962/63), 351–354.

The author solves the pair of equations

$$\int_0^\infty y^{2k-2}g(y)(xy)^{1/2}J_\nu(xy)\,dy = f(x), \quad 0<x<1,$$

$$\int_0^\infty g(y)(xy)^{1/2}J_\nu(xy)\,dy = 0, \quad x>1,$$

already solved many years ago by Busbridge and Titchmarsh. In the solution under review, the given function $f(x)$, however, must be expressible in a series of Jacobi polynomials and subject to the restriction

$$\int_0^1 x^{\nu+1/2}(1-x^2)^{k-1}f(x)\,dx = 0,$$

and in the solution the unknown function $g(x)$ is expressed as a series of Bessel functions.

The above-mentioned restriction seems severe and may well limit the applications. The main value of the result lies in the fact that it covers some values of ν and k which were not covered by the original Busbridge-Titchmarsh solutions. *J. C. Cooke* (Farnborough)

Lučka, A. Ju. 5307

Approximate solution of linear operator equations in a Banach space by Ju. D. Sokolov's method. (Russian. English summary)

Ukrain. Mat. Ž. **13** (1961), *no.* 1, 39–52.

Let E_1, E_2 be two Banach spaces, and let E_2 have a basis. The problem is to solve the equation

$$(*) \qquad Lx = g+\lambda Bx,$$

where L, B are linear operators from E_1 to E_2, $g\in E_2$, $\lambda\in\mathbf{C}$, and where it is assumed that L has an inverse A such that BA is completely continuous. The author gives the following abstract formulation of a generalization of the method of successive approximations, which in concrete situations was proposed by Ju. D. Sokolov [same Ž. **10** (1958), 419–433; MR **21** #977]: If $\{\varphi_i\}$ is a basis of E_2, determine a sequence $\{f_i\}$ of linear functionals on E_2 such that $f_i(\varphi_j)=\delta_{ij}$. With x_0 arbitrary, construct the sequence $\{x_n\}$ by successively solving

$$x_n = f+\lambda ABx_{n-1}+\lambda\sum_{i=1}^{k}\alpha_{ni}A\varphi_i,$$

where k is a fixed integer, and where $\alpha_{ni}=f_i(B\delta_n)$, $\delta_n = x_n - x_{n-1}$. (The choice $k=0$ yields ordinary successive approximations.) It is shown that if λ belongs to the resolvent set of AB, it is always possible to select k such that the sequence $\{x_n\}$ converges to the solution of (*). There are bounds for the truncation error, and numerical examples. *P. Henrici* (Zürich)

Kurpel', N. S. 5308

Sufficient conditions for the convergence of the method of Ju. D. Sokolov for approximate solution of non-linear integral equations of Hammerstein type. (Russian)

Approximate methods of solving differential equations, pp. 47–53. *Izdat. Akad. Nauk Ukrain. SSR, Kiev,* 1963.

The author studies the Hammerstein equation

$$(1)\qquad u(x) = \phi(x)+\int_a^b K(x,\xi)f[\xi,u(\xi)]\,d\xi$$

(all symbols appearing are real numbers) under the hypotheses of Hilbert-Schmidt K and

$$-\infty < m \leqq \frac{f(x,u_1)-f(x,u_2)}{u_1-u_2} \leqq M < +\infty.$$

The "method of Sokolov" is to solve recursively the Hammerstein equations with a "simpler" kernel

$$(2)\qquad z_n = \phi+(\mathfrak{K}-\mathfrak{K}_s)\mathfrak{f}z_{n-1}+\mathfrak{K}_s\mathfrak{f}z_n$$

and define $u_n=\phi+\mathfrak{K}\mathfrak{f}z_n$. (Here $\mathfrak{f}$ is the Nemytskiĭ operator and $\mathfrak{K}_s$ is a linear operator "close to" $\mathfrak{K}$, e.g., an integral operator with K_s taken as a truncated series for K.)

No prescription for solving (2) is given. Granted this solvability, the author gives sufficient conditions (on the Lipschitz constants of various associated operators) for convergence of u_n to a solution of (1).

G. J. Minty (Ann Arbor, Mich.)

Mangeron, D.; Krivošein, L. E. 5309

Qualitative study of a class of boundary value problems for integro-differential equations and approximate determination of their solutions.

Acta Math. Sinica **13** (1963), 63–67 (*Chinese*); *translated as Chinese Math.* **4** (1963), 70–75.

An expository paper [see the authors, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **9** (1961), 707–712; MR **24** #A2819; C. R. Acad. Sci. Paris **253** (1961), 1190–1192; MR **26** #4143].

Mangeron, D.; Krivošein, L. E. 5310

Some problems concerning the structure of integro-differential equations.

Acta Math. Sinica **14** (1964), 1–6 (*Chinese*); *translated as Chinese Math.* **5** (1964), 1–7.

A continuation of their expository work [#5309 above]. Further references to their joint work in Western journals are given.

McLeod, J. B. 5311

On the scalar transport equation.

Proc. London Math. Soc. (3) **14** (1964), 445–458.

The integro-differential equation

$$\frac{\partial f(x,t)}{\partial t} = -f(x,t)\int_0^\infty f(y,t)\phi(x,y)\,dy + \frac{1}{2}\int_0^x f(y,t)f(x-y,t)\phi(y,x-y)\,dy,$$

with initial conditions of the form $f(x,0)=F(x)$ is studied. $f(x,t)$ is the unknown function, $\phi(x,y)$ and $F(x)$ are given functions, and $\phi(x,y)$ satisfies the two conditions $\phi(x,y)\geqq 0$, $\phi(x,y)=\phi(y,x)$. This is a continuous analogue of the infinite set of differential equations studied by the author [Quart. J. Math. Oxford Ser. (2) **13** (1962), 119–128; MR **25** #3250; ibid. (2) **13** (1962), 193–205; MR **25** #4159; ibid. (2) **13** (1962), 283–284; MR **26** #1524], and he finds analogous results in the present case.

N. P. Bhatia (Cleveland, Ohio)

FUNCTIONAL ANALYSIS

See also 5140, 5149, 5209, 5210, 5260, 5281, 5282, 5289, 5304, 5307, 5421, 5604, 5696.

Garsoux, Julien 5312

★Espaces vectoriels topologiques et distributions.

Avec la collaboration de Daniel Ribbens; Préface de Pierre Houzeau de Lehaie.

Dunod, Paris, 1963. xiii+324 *pp.* 48 *F.*

This book is meant to be a self-contained treatise on topological linear spaces. It is intended for students, and even for theoretical physicists who might desire to obtain a firm group of the rigorous basis of the theory of distributions (in the sense of L. Schwartz). Concerning distributions themselves, there are only a few pages containing essentially just the definitions. These pages are in the middle of the book, following a careful presentation of just that part of the theory on which the definition can be based. The subsequent four chapters continue with the theory of topological linear spaces. The content, notation, and (with one essential exception which is the avowed motive for writing the book) the spirit are essentially those of Bourbaki's *Espaces vectoriels topologiques* [Chapitres I, II, Actualités Sci. Indust., No. 1189, Hermann, Paris, 1953; MR **14**, 880; Chapitres III, IV, V, Actualités Sci. Indust., No. 1229, Hermann, Paris, 1955; MR **17**, 1109; Fascicule de résultats, Actualités Sci. Indust., No. 1230, Hermann, Paris, 1955; MR **17**, 1109]. The difference in spirit is that all the material from the previous books of the Bourbaki series which is actually needed is fitted into the presentation, so that no references to these lower rungs of the Bourbaki ladder are made. A tactical difference is that the author manages to get along with the concept of convergence only of systems $\{x_t\}$ where x_t is a point of the space in question, and the index t runs over some subset of $\mathbf{R}^n$. An appendix, however, presents the concept of filter and shows the relation between the two concepts.

For mathematics students at least, this book should make an excellent textbook, accompanied by the exercises from Bourbaki, together with material on Hilbert space from there or a number of other places. The Bibliography adds just a few sources to those mentioned by Bourbaki—too few in the reviewer's opinion. For example, although there is presented a variant of Zorn's lemma called "Théorème de Tukey [sic]" there is no reference enabling one to discover that this is a Lemma of J. W. Tukey. For economy of reference, a theorem of which S. Mazur might well be proud is called the Hahn-Banach theorem. The publication date of Volume I of Schwartz's book, *Théorie des distributions* [Actualités Sci. Indust., No. 1091, Hermann, Paris, 1950; MR **12**, 31] is 1950 and

not 1957, as the author states in his bibliography. Among other things, he has renamed and presented without reference what Schwartz [op. cit., p. 75] refers to as the Mackey-Arens theorem. *R. Arens* (Los Angeles, Calif.)

Nikol'skiĭ, V. N. 5313

Best approximation by elements of convex sets in normed linear spaces. (Russian)

Kalinin. Gos. Ped. Inst. Učen. Zap. **29** (1963), 85–119.

Let X be a real or complex normed linear space, X^* its conjugate space, and S^* the solid unit sphere in X^*. A subset $\Gamma \subset S^*$ is called fundamental for X if (i) for every $x \in X$ different from 0 there exists an $f \in \Gamma$ such that $|f(x)| = \|x\|$; (ii) Γ is closed in the weak topology on X^* (the weak* topology). An example of a fundamental set is the weak closure of the set of extreme points of S^*. It is observed that there always exist minimal fundamental sets of continuous linear functionals. Calling a set symmetric if it contains, with every f, all multiples εf such that $|\varepsilon| = 1$, the author determines the minimal symmetric fundamental sets for a variety of special spaces.

Now let G be a non-empty convex subset of X, and let Γ be a given fundamental set of continuous linear functionals for X. The following are some of the results proved by the author. Theorem 1: In order that $g_0 \in G$ be a best approximation to $x \in X$, it is necessary and sufficient that for every $g \in G$ there exist an $f \in \Gamma$ such that $|f(x-g_0)| = \|x-g_0\|$ and $\mathrm{Re}\{f(g_0-g)\overline{f(x-g_0)}\} \geqq 0$. Theorem 2: Suppose that $g_0 \in G$ is a best approximation to $x \in X$. In order that g_0 be unique it is necessary and sufficient that for every $g \in G$ different from g_0, and for every $f \in \Gamma$ such that $|f(x-g)| = \|x-g\|$, one have $f(g-g_0) \neq 0$. It is observed further that the best approximation problem has at most one solution for every $x \in X$ and every convex set $G \subset X$ if and only if X is strictly convex, a result proved earlier (for closed subspaces G) by R. A. Hirschfeld [Nieuw Arch. Wisk. (3) **6** (1958), 41–51; MR **20** #4768].

The author uses the above results to reprove and extend various known results of Timan and Remez on best approximation in spaces L_p and C. He finally considers the problem of best approximation by elements of a finite-dimensional subspace G of an arbitrary real X, and obtains both a necessary and a sufficient condition for an element of G to be a best approximation to a given $x \in X$.

J. Korevaar (Madison, Wis.)

Mascart, Henri 5314a

Sur quelques propriétés élémentaires des modules topologiques.

C. R. Acad. Sci. Paris **258** (1964), 1683–1685.

Mascart, Henri 5314b

Sur l'invariance par homothétie du filtre des voisinages de l'origine dans un module topologique.

C. R. Acad. Sci. Paris **258** (1964), 3148–3150.

The author gives some definitions and results on the elementary topological properties of a module E over a topological ring A. The aim is to generalise certain basic ideas and results which are already familiar for topological vector spaces. Bounded subsets and absorbent subsets of E are defined as expected. The definition of a balanced subset of E involves the preliminary choice of an arbitrary subset e of A: $P \subset E$ is said to be balanced if $eP \subset P$. Many of the results are subject to certain restrictions on e, which ensure that its properties resemble sufficiently those of the unit disc in the complex plane. One then has, for example, that the balanced envelope of $P \subset E$ is eP. Another condition sometimes required is the following Axiom VI (in which $\mathfrak{B}$ is the filter of neighbourhoods of 0 in E): If $0 \neq \lambda \in A$, $U \in \mathfrak{B}$, then there exists $V \in \mathfrak{B}$ with $V \subset \lambda U$. This axiom is satisfied in the case of a topological vector space, but is not in general satisfied by a topological ring, regarded as a module over itself. Under this condition (and that mentioned above for e), $\mathfrak{B}$ has a base of closed balanced sets, also a base of open balanced sets. The first note ends with a section on modules of continuous linear maps, and on the properties of E as a module over the ring of continuous A-linear maps of E to itself.

The second note contains results on open, closed, and compact subsets of E, and on metrisable modules, some of which are proved under a weakened version of Axiom VI. (The results on compactness are mostly independent of this axiom.) Denoting the intersection of all $U \in \mathfrak{B}$ by c, assume $c \notin \mathfrak{B}$. Then one has Axiom VI′: If $\lambda \in A$, $\lambda \notin c$, $U \in \mathfrak{B}$, then there exists $V \in \mathfrak{B}$ with $V \subset \lambda U$. A typical result then is: If Axiom VI′ holds, then for any $P \subset E$ and $\lambda \in A$, $\lambda \notin c$, one has $\overline{\lambda P} = \lambda \bar{P}$. The note ends with a theorem giving sufficient conditions for a continuous linear map of one complete metrisable module onto another to be a homeomorphism.

J. H. Williamson (Cambridge, England)

Cartier, P. 5315

Über die Existenz eines Kernes für funktionale Operatoren.

Arch. Math. **15** (1964), 50–57.

Let X and Y be locally compact spaces with Radon measures μ and ν respectively. Grothendieck has shown that any bounded bilinear form B on $\mathscr{L}^1(X, \mu) \times \mathscr{L}^1(Y, \nu)$ is of the form $B(f, g) = \iint K(x, y) f(x) g(y)\, d\mu(x) d\nu(y)$, where K is a member of $\mathscr{L}^\infty(X \times Y, \mu \times \nu)$ [Mem. Amer. Math. Soc. No. 16 (1955); MR **17**, 763]. The present paper contains an elementary proof of this theorem. This is accomplished by repeated application of the following approximation lemma. Let $h^{(j)}$ $(1 \leqq j \leqq m)$ be a family of continuous functions on $X \times Y$, each vanishing off $K \times L$, where K and L are compact subsets of X and Y respectively. For a given $\varepsilon > 0$, there is a finite number of points $x_1, \cdots, x_n$ of K and a finite number of non-negative functions $f_1, \cdots, f_n$ on X with compact supports such that $|h^{(j)}(x, y) - \sum_{i=1}^n f_i(x) h^{(j)}(x_i, y)| \leqq \varepsilon$ for each (x, y) in $X \times Y$ and $j = 1, \cdots, m$.

I. Namioka (Seattle, Wash.)

Nagumo, Mitio 5316

Re-topologization for continuity of a set of operators. (Japanese)

Sûgaku **14** (1962/63), 164–166.

In the present paper, the author investigates the re-topologization of locally convex, bornologic spaces in order that a set of linear operators will be continuous. Also he shows that his theory can be considered a kind of abstract generalization of the Schwartz distribution theory. Moreover, the error in Theorem 4 of an earlier paper [Proc. Japan Acad. **37** (1961), 550–552; MR **26** #593] is corrected.

S. Sakai (Berkeley, Calif.)

Gęba, K. 5317

Algebraic topology methods in the theory of compact fields in Banach spaces.

Fund. Math. **54** (1964), 177–209.

Let E be an infinite-dimensional Banach space, and $X \subset E$ a closed bounded set. The set $\mathscr{F}$ of all compact fields $f: X \to E - \{0\}$ has several important properties: for example [A. Granas, Fund. Math. **48** (1959/60), 189–200; MR **22** #3967], X separates E if and only if $\mathscr{F}$ has more than one homotopy class. In the present paper, the author considers the sets of non-vanishing compact fields that send X into subspaces of finite codimension; he obtains a cohomology theory on the category $\mathfrak{A}$ that has closed bounded pairs (X, A), $A \subset X \subset E$, for objects and compact fields for maps.

Let $E^{-n} \subset E$ be a fixed linear subspace of codimension n, let R_n be the complementary linear subspace, and let $Q \subset E^{-n}$ be a fixed half-ray starting at the origin. Let $\mathscr{L}_n$ be the family of all finite-dimensional linear subspaces L containing $Q + R_n$ and such that $\dim(L \cap E^{-n}) \geqq n+4$, directed by inclusion. Then for any (X, A) in $\mathfrak{A}$ and each $L \in \mathscr{L}_n$, the cohomotopy group $\pi^l(X \cap L, A \cap L)$, where $l = \dim(L \cap E^{-n}) - 1$, is defined, and the author shows that, with suitable connecting homomorphisms, the system $\{\pi^l(X \cap L, A \cap L) | L \in \mathscr{L}_n\}$ is a direct spectrum. The group $\pi^{-n}(X, A)$ is the direct limit of this spectrum; its elements turn out to be in a one-to-one correspondence with the homotopy classes in the set of compact fields $f: (X, A) \to (E^{-n} - \{0\}, E^{-n} - Q)$.

For each sequence $\mathscr{S} = \{E^{-n}, R_n, Q\}$, $n = 0, 1, 2, \cdots$, such that $E^{-n} \subset E^{-n+1}$, $R_n \subset R_{n+1}$, and $Q \subset \bigcap_n E^{-n}$, a coboundary operator $\delta: \pi^{-n}(A) \to \pi^{-n+1}(X, A)$ is defined, and $\Pi = \{\pi^{-n}, f, \delta\}$ is proved to be a cohomology theory on the category $\mathfrak{A}$. If $\bar{\mathscr{S}}$ is any other sequence, the corresponding cohomology theory $\bar{\Pi}$ is almost isomorphic to Π: for any fixed $n \geqq 0$, there is a natural isomorphism $\Omega_n: \bar{\pi}^{-k} \to \pi^{-k}$ defined for all $k \leqq n$. Furthermore, the author proves that $\pi^{-n}(X)$ is naturally isomorphic to the nth S-homotopy group of $E - X$.

Several quick applications are given: for example, if $f: X \approx Y$ is a homeomorphism and a compact field, then $E - X$ and $E - Y$ have the same number of components.

J. Dugundji (Los Angeles, Calif.)

Robertson, A. P.; Robertson, W. J. 5318

★Topological vector spaces.

Cambridge Tracts in Mathematics and Mathematical Physics, No. 53.

Cambridge University Press, New York, 1964. viii + 158 *pp.* $4.95.

This short book, which is completely worthy of its place in the series of Cambridge Tracts, may be used as a text for a first-year graduate course. The chapters are the following: I (Introductory); II Duality and Hahn-Banach theorem; III Topologies on dual spaces and the Mackey-Arens theorem; IV Barrelled spaces and the Banach-Steinhaus theorem; V Inductive and projective limits; VI Completeness and the Closed-Graph theorem; VII Some further topics (Tensor products, extreme points, etc.); VIII Compact linear mappings. There are plenty of examples and a perfectly adequate bibliography. There are no exercises for the student. The terminology and notation agree, to a reasonable extent, with the usage of Bourbaki.

R. Arens (Los Angeles, Calif.)

Choo, Koo Guan 5319

On operators of Banach groups.

Bull. Math. Soc. Nanyang Univ. **1963**, 59–76.

What the author calls a Banach group is a complex Hilbert space. The paper is a slightly confused presentation of the elementary facts about the norms of operators and the relations between projections and subspaces.

P. R. Halmos (Ann Arbor, Mich.)

Raĭkov, D. A. 5320

Free locally convex spaces for uniform spaces. (Russian)

Mat. Sb. (*N.S.*) **63 (105)** (1964), 582–590.

Let X be a uniform space, and let $P(X)$ be the space of all uniformly continuous complex functions on X. By a free locally convex space for X we mean a uniform injection ω of X into a locally convex linear topological space E such that (a) the linear span of $\omega(X)$ is dense in E, and (b) any uniformly continuous map of X into a locally convex linear topological space F extends to a continuous linear map of E into F. First the author proves that an essentially unique free locally convex space exists for any uniform space X, and describes it explicitly: It consists of the free vector space $L(X)$ with X as its set of generators, equipped with the topology of uniform convergence on the class of all equi-uniformly continuous subfamilies of $P(X)$. (Here an element $u = \sum_k \lambda_k x_k$ of $L(X)$ is regarded as a functional on $P(X)$ via the bilinear form $\langle u, f \rangle = \sum_k \lambda_k f(x_k)$ $(f \in P(X))$.) He establishes some further properties of this concept, and relates it to the analogous concept for (non-uniform) topological spaces introduced earlier by A. A. Markov [Dokl. Akad. Nauk SSSR **31** (1941), 299–301; MR **3**, 36]. He also considers a variant of the above definition, in which some distinguished point of X is required to go into the zero element of E. Here too he establishes the existence and uniqueness of E.

J. M. G. Fell (Seattle, Wash.)

Singer, Ivan 5321

Bases and quasi-reflexivity of Banach spaces.

Math. Ann. **153** (1964), 199–209.

A Banach space E is called quasi-reflexive [of order n] if $\operatorname{codim}_{E^{**}} \pi(E) < \infty$ $[\operatorname{codim}_{E^{**}} \pi(E) = n]$, where π denotes the canonical embedding of E into E^{**}. The paper consists of four paragraphs. The first contains definitions of k-boundedly complete and k-shrinking bases (k is a non-negative integer). For $k = 0$ these notions reduce to boundedly complete and shrinking bases in the usual sense. In § 2 the following characterization of quasi-reflexivity for Banach spaces with bases is given: A Banach space E with basis $\{x_i\}$ is quasi-reflexive of order n if and only if there exists an integer k, $0 \leqq k \leqq n$, such that $\{x_i\}$ is $(n-k)$-boundedly complete and k-shrinking. For reflexive Banach spaces this characterization reduces to a theorem of R. C. James [Ann. of Math. (2) **52** (1950), 518–527, Theorem 1; MR **12**, 616]. In § 3 the author proves two general duality theorems concerning k-boundedly complete and k-shrinking bases. In § 4 some characterizations of quasi-reflexivity for general (not necessarily separable) Banach spaces are given.

K. Gęba (Gdańsk)

DeMarr, Ralph E. 5322

Order convergence in linear topological spaces.

Pacific J. Math. **14** (1964), 17–20.

In a partially ordered linear space L, a net $\{x_n, n \in D\}$ is said to order-converge to 0 if there exists a non-empty subset M of L such that M is directed downward (i.e., if x and y are elements of M, then there is an element z in M such that $z \leqq x$ and $z \leqq y$), $\inf M = 0$ and for each y in M there is a $k \in D$ such that $-y \leqq x_n \leqq y$ for all $n \geqq k$. It is proved that a locally convex linear topological space L admits a partial ordering so that the convergence of an arbitrary net to 0 is equivalent to the order-convergence of the net to 0 if and only if L is normable.
I. Namioka (Seattle, Wash.)

Bueckner, Hans F. 5323
Equations in partially ordered spaces.
Nonlinear Integral Equations (Proc. Advanced Seminar Conducted by Math. Research Center, U.S. Army, Univ. Wisconsin, Madison, Wis., 1963), pp. 49–64. Univ. Wisconsin Press, Madison, Wis., 1964.

Expository account (no proofs) of results given in M. A. Krasnosel'skiĭ's book [*Positive solutions of operator equations* (Russian), Fizmatgiz, Moscow, 1962; MR **26** #2862].
D. E. Edmunds (Cardiff)

Luxemburg, W. A. J.; Zaanen, A. C. 5324a
Notes on Banach function spaces. VI, VII.
Nederl. Akad. Wetensch. Proc. Ser. A **66** = *Indag. Math.* **25** (1963), 655–668; 669–681.

Luxemburg, W. A. J.; Zaanen, A. C. 5324b
Notes on Banach function spaces. VIII.
Nederl. Akad. Wetensch. Proc. Ser. A **67** = *Indag. Math.* **26** (1964), 104–119.

The authors of this series of notes make a fresh start with note VI. Notes VI, VII and VIII, which are concerned with abstract Riesz spaces, are essentially independent of the preceding ones. (References to these earlier notes may be found in VI, VIII and V and its review [same Proc. **66** (1963), 496–504; MR **28** #1481].)

Note VI begins with a rather complete presentation of the fundamental facts about Riesz spaces, in other words, vector lattices. The concluding sections discuss integrals and singular functionals on a Riesz space L: Denote by $L^{\sim}$ the Riesz space consisting of the order-bounded linear functionals on L. $\varphi \in L^{\sim}$ is called an integral if whenever $u_1 \geqq u_2 \geqq \cdots$ and $\inf u_n = 0$, then $\varphi(u_n) \to 0$. $\psi \in L^{\sim}$ is called a singular functional if $\inf(|\varphi|, |\psi|) = 0$ for every integral φ.

The nature of the subject matter of note VII will be indicated here by stating some of the definitions and one of the main theorems. Let L_ρ be a Riesz space with a norm ρ such that $\rho(f) \leqq \rho(g)$ whenever $|f| \leqq |g|$. Let L_ρ^* denote the usual dual of the normed linear space L_ρ; then L_ρ^* is a Riesz space, in fact, a subspace of the Riesz space $L_\rho^{\sim}$. Let $L_{\rho,c}^*$ denote the set of integrals in L_ρ^* and $L_{\rho,s}^*$ denote the set of singular functionals in L_ρ^*. Let $L_\rho{}^a$ denote the set of all $f \in L_\rho$ such that $|f| \geqq u_1 \geqq u_2 \geqq \cdots$ and $\inf u_n = 0$ imply $\varphi(u_n) \to 0$ for every $\varphi \in L_\rho^*$. The theorem states that the following conditions are equivalent: (i) $L_{\rho,s}^* = \{\varphi \in L_\rho^* | \varphi(f) = 0 \text{ for all } f \in L_\rho{}^a\}$. (ii) $L_{\rho,s}^*$ is closed in the weak*-topology of L_ρ^*. (iii) $(L_\rho{}^a)^* = L_{\rho,c}^*$, i.e., the mapping of $L_{\rho,c}^*$ onto $(L_\rho{}^a)^*$ defined by restriction is an algebraic and metric isomorphism.

Note VIII is primarily concerned with normal integrals and perfect Riesz spaces: Let L and $L^{\sim}$ be as above; $\varphi \in L^{\sim}$ is called a normal integral if whenever $\{u_\tau\}$ is directed downwards and $\inf u_\tau = 0$, then $\inf|\varphi(u_\tau)| = 0$. We denote by $L_n^{\sim}$ the space of normal integrals on L and embed L in $(L_n^{\sim})_n^{\sim}$ in the obvious way. If $L = (L_n^{\sim})_n^{\sim}$, then we say L is perfect. It is shown that for any L, both $L^{\sim}$ and $L_n^{\sim}$ are perfect.
H. Gordon (Philadelphia, Pa.)

Donoghue, W. F., Jr. 5325
On a theorem of K. Maurin.
Studia Math. **24** (1964), 1–5.

If D is an open set in R^n, then $P_0{}^\alpha(D)$ $(\alpha \geqq 0)$ is the space of functions u which vanish outside of D and have $\|u\|_\alpha{}^2 = \int (1+|\xi|^2)^{\alpha/2} |\hat{u}|^2 \, d\xi < \infty$. With this norm $P_0{}^\alpha(D)$ is a Hilbert space.

The author shows that if D is bounded, then the natural embedding of $P_0{}^\alpha(D)$ in $P_0{}^\beta(D)$ is a Hilbert-Schmidt mapping if and only if $\alpha - \beta > n/2$.

For integral α and β this result had been found by K. Maurin [Math. Scand. **9** (1961), 359–371; MR **25** #4364]. The present proof is quite different from Maurin's. It is based on properties of the Bessel kernel G_α and on the three-lines interpolation theorem of Lions.

If the boundary of D is mildly regular, a similar result holds for the space $P^\alpha(D)$ of restrictions to D of functions u with $\|u\|_\alpha < \infty$.
K. T. Smith (New York)

Rooney, P. G. 5326
A generalization of the Hardy spaces.
Canad. J. Math. **16** (1964), 358–369.

Suppose that $\alpha(x)$ is non-decreasing for $x \geqq c$ and zero for $0 \leqq x < c$, that $f(s)$ is holomorphic for $\operatorname{Re} x > \sigma$ and that $\mu_p(f, x) = (1/2\pi)[\int_{-\infty}^{\infty} |f(x+iy)|^p \, dy]^{1/p}$. The norm is defined by

$$\|f\|_{\alpha,p} = \left[\int_{c+}^{\infty} (\mu_p(f, x/q))^q \, d\alpha(x) + \sup_{x>c} \alpha(c+)(\mu_p(f, x/q))^q\right]^{1/q},$$

where $p^{-1} + q^{-1} = 1$ (σ taking all values $> c/q$).

The space $\mathfrak{H}_p$ is the set of functions satisfying the above properties (finite norm). The spaces $\mathfrak{H}_p$ include as special cases the Hardy spaces $\mathfrak{H}_p(\sigma)$ and generalise the work in a previous paper by the same author [same J. **10** (1958), 421–430; MR **20** #4740].

Several theorems are proved. These in the main show some of the connections between these spaces and the Laplace transform.
J. L. Griffith (Kensington)

Michael, E. 5327
A linear mapping between function spaces.
Proc. Amer. Math. Soc. **15** (1964), 407–409.

Responding to a question of Fred B. Wright, the author shows that if p is an open map from a compact metric space X onto a compact metric space Y, then there exists a continuous linear u from $C(X)$ onto $C(Y)$ such that

$$\inf(f[p^{-1}(y)]) \leqq [u(f)](y) \leqq \sup(f[p^{-1}(y)])$$

for each $f \in C(X)$ and $y \in Y$.

{The interest of the theorem rests in the fact that p is not assumed to admit a continuous cross-section. The rôle normally filled by such a function is here played by a continuous selection for the lower semi-continuous carrier

$\psi: Y \to 2^F$ defined by $\psi(y) = (\text{conv}(p^{-1}(y)))^-$, where F is the (metrizable) completion of the dual space of $C(X)$ in its topology of pointwise convergence on some countable dense subset of $C(X)$. The author has requested that the reviewer draw attention to the fact that the hypothesized map p is taken to be continuous.}

W. W. Comfort (Rochester, N.Y.)

Michael, E. 5328

Three mapping theorems.

Proc. Amer. Math. Soc. **15** (1964), 410–415.

The first theorem associates with each metric space X a Banach space B_X containing X isometrically such that any continuous map on X to a complete locally convex topological linear space admits a linear extension to B_X which is continuous on (X and on) the closed convex hull of every compact subset of X. The second "is the first example of a selection theorem where the range is—at least partially—nonmetrizable". The third theorem is a substantial generalization of the result described in the preceding review [#5327]. {Just as before, the function p is continuous.}

W. W. Comfort (Rochester, N.Y.)

Husain, Taqdir 5329

On completion and completeness of $B(\mathfrak{C})$-spaces.

Math. Ann. **154** (1964), 73–76.

The author continues his study of $B(\mathfrak{C})$-spaces [same Ann. **146** (1962), 413–422; MR **26** #2861], showing that the class of all complete spaces cannot coincide with the class of $B(\mathfrak{C})$-spaces for any $\mathfrak{C}$, and that, under certain conditions, the completion of a $B(\mathfrak{C})$-space is also a $B(\mathfrak{C})$-space.

A. P. Robertson (Glasgow)

Martin, A. D.; Mizel, V. J. 5330

A representation theorem for certain nonlinear functionals.

Arch. Rational Mech. Anal. **15** (1964), 353–367.

Let (T, Σ, μ) be a totally finite measure space, and let B denote the set of essentially bounded real-valued measurable functions on T. If f is a real continuous function on $(-\infty, +\infty)$ with $f(0)=0$, then formula (1) $F(x) = \int_T f(x(t))\, d\mu(t)$ defines a non-linear functional on B with the following properties: (a) $F(x+y) = F(x) + F(y)$ whenever x and y have disjoint support, (b) F is continuous with respect to bounded convergence, and (c) $F(x) = F(y)$ whenever x and y are equimeasurable (i.e., have equal distribution functions). Theorem: If the measure of the atom-free part of T is at least equal to that of any atom, then any functional F on B having properties (a), (b) and (c) can be represented in the form (1). When T is atom-free, the representation is unique. When T is purely atomic, the theorem may fail. Let the atoms have measures $m_i \geqq m_{i+1} > 0$ for all i. Then a necessary condition for the theorem to be valid is that $m_n \leqq 2 \sum_{n+1}^{\infty} m_k$ for infinitely many n. However, this condition is not sufficient, even when the factor 2 is omitted.

J. C. Oxtoby (Bryn Mawr, Pa.)

Gamelin, T. W. 5331

Restrictions of subspaces of $C(X)$.

Trans. Amer. Math. Soc. **112** (1964), 278–286.

A theorem of I. Glicksberg [same Trans. **105** (1962), 415–435] states that the restrictions of the elements of a closed subalgebra A of $C(X)$ to certain closed subsets F of X are closed in $C(F)$. (The condition is that F be an intersection of peak sets.) The argument is now analyzed in its component parts. If A is a vector subspace of $C(X)$, when do the restrictions of the elements of A to F form a closed subspace of $C(F)$? An extension constant is associated with the problem and this number is characterized in terms of induced mappings on adjoint spaces. In this view the main results are strong alternatives in which the extension constant is either zero or not less than one, or (in the logmodular case) either zero or infinite. An extension topology is also considered in which the closed sets are intersections of peak sets.

L. de Branges (Lafayette, Ind.)

Taibleson, Mitchell H. 5332

The preservation of Lipschitz spaces under singular integral operators.

Studia Math. **24** (1964), 107–111.

Let $x = (x_1, \cdots, x_n)$ be a point in Euclidean n-space E_n and for $x \neq 0$ let $x' = x/|x|$ be the projection of x onto the unit sphere Σ. Let $\Omega(x')$ be a complex function on Σ satisfying: (i) $\int_\Sigma \Omega(x')\, dx' = 0$; (ii) there is a non-negative monotone increasing function $\omega(t) \geqq t$, $0 < t \leqq 2$, such that $|\Omega(x') - \Omega(y')| \leqq \omega(|x'-y'|)$ for all $x', y' \in \Sigma$, and such that the Dini condition $\int_0^1 \omega(t) t^{-1}\, dt < \infty$ holds. Let $K(x) = \Omega(x')|x|^{-n}$ be a Calderón-Zygmund kernel. Let T_n be E_n modulo the subgroup of integral lattice points, and set

$$K^*(x) = K(x) + \sum_{k \neq 0} [K(x+k) - K(k)] = K(x) + \bar{K}(x).$$

(Here $k = (k_1, \cdots, k_n)$ is an integral lattice point.) Finally let $K_\varepsilon^*(x) = \chi_\varepsilon(x) K(x) + \bar{K}(x)$, where $\chi_\varepsilon(x) = 1$ if $|x| \geqq \varepsilon$ and is zero otherwise in $-\frac{1}{2} \leqq x_i < \frac{1}{2}$, $i = 1, \cdots, n$, and is transferred in the obvious way to T_n. We set $f_\varepsilon^*(x) = \int_{T_n} f(x-z) K_\varepsilon^*(z)\, dz$ and $f^* = \lim_{\varepsilon \to 0} f_\varepsilon^*$. Let $L_p(T_n)$ and $\|\cdot\|_p$ be defined as usual, and let

$$\|f\|_\alpha{}^p = \|f\|_p + \operatorname*{ess\,sup}_{h \in T_n} \|f(x+h) - f(x)\|_p |h|^{-\alpha}.$$

$\Lambda_\alpha{}^p$ is the set of f for which $\|f\|_\alpha{}^p$ is finite. Let K be as above. The author's main result asserts that the mapping $f \to f^*$ preserves all the classes $\Lambda_\alpha{}^p$. This is of particular interest for $p=1$ and ∞. Specifically, if $f \in \Lambda_\alpha{}^p$, $1 \leqq p \leqq \infty$, $0 < \alpha < 1$, then f_ε^* converges in the L_p norm to a function $f^* \in \Lambda_\alpha{}^p$ and $\|f^*\|_\alpha{}^p \leqq A \|f\|_\alpha{}^p$, where A is independent of f. This strengthens an earlier result of Calderón and Zygmund [same Studia **14** (1954), 249–271; MR **16**, 1017].

I. I. Hirschman, Jr. (St. Louis, Mo.)

Rinehart, R. F.; Wilson, Jack C. 5333

Functions on algebras under homomorphic mappings.

Duke Math. J. **31** (1964), 221–227.

Where $\mathfrak{A}$ and $\mathfrak{B}$ are linear associative finite-dimensional algebras with identity over the real or complex field $\mathfrak{F}$, let $\mathfrak{B}$ be the image of $\mathfrak{A}$ under a homomorphism or anti-homomorphism Ω, and let f be a function on a subset $\mathfrak{T}$ of $\mathfrak{A}$ to $\mathfrak{A}$. This paper concerns conditions on f under which a single-valued function g on a subset of $\mathfrak{B}$ to $\mathfrak{B}$ can be consistently defined by setting $g(\Omega\xi) = \Omega f(\xi)$ for $\xi \in \mathfrak{T}$. It is shown that a sufficient condition for this is that f be

Hausdorff-differentiable, in which case the induced function g will be Hausdorff-differentiable too. It is also shown that another sufficient condition for this is that f be, in a certain sense, the extension to $\mathfrak{A}$ of a function w on the scalar field $\mathfrak{F}$, in which case the induced function g will be, in this same sense, the extension of w to $\mathfrak{B}$.

T. A. Botts (Charlottesville, Va.)

Burov, V. N. 5334

Approximation with constraints in linear normed spaces. II. (Russian. German summary)

Ukrain. Mat. Ž. **15** (1963), 135–144.

This paper is closely related to Part I [same Ž. **15** (1963), 3–12; MR **28** #410]. The author considers the case of the space $C(Q)$ of all real-valued continuous functions on a compact Hausdorff space Q. Let Ω be a closed convex subset of a finite-dimensional subspace of $C(Q)$ and let $f\in C(Q)$. The author gives a necessary and sufficient condition (too complicated to quote here) in order that $g\in\Omega$ be extremal, i.e., $\|f-g\|=\inf_{g'\in\Omega}\|f-g'\|$. The problem of uniqueness of the extremal element and connections between extremal problems in the spaces L_p and C are also studied.

A. Pełczyński (Warsaw)

Semjanistyĭ, V. I. 5335

On some integral transformations in Euclidean space. (Russian)

Dokl. Akad. Nauk SSSR **134** (1960), 536–539.

The problem is to determine a function $f(x)$, defined in euclidean n-space R_n, from a knowledge of its integral over every hyperplane of R_n. Let Ψ be the space of all infinitely differentiable functions on R_n, all of whose derivatives vanish at the origin and which go to zero at infinity more rapidly than any power of r^{-1}. Let Φ be the space of all Fourier transforms of elements of Ψ. An element of the conjugate space of Φ is called a generalized function on R_n. The analogous concept of generalized function is made for the space $\hat{R}_n$ of all hyperplanes in R_n. If f is a generalized function on R_n, $g=P_0\times f$ is the generalized function on $\hat{R}_n$ obtained by integrating f over hyperplanes of R_n. The author now solves for f, given g. The solution depends on an operational calculus for the Laplacian operator and various averaging processes taken over spheres and hyperplanes.

L. de Branges (Lafayette, Ind.)

Temple, G. 5336

The theory of weak functions. I, II.

Proc. Roy. Soc. Ser. A **276** (1963), 149–167; *ibid.* **276** (1963), 168–177.

In the first part the author treats distributions ("weak functions") in one dimension by a method that is very closely related to the method of fundamental sequences of Mikusiński and Sikorski [Mikusiński, Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 589–591; MR **17**, 594; Mikusiński and Sikorski, Rozprawy Mat. **12** (1957); MR **20** #1214; ibid. **25** (1961); MR **23** #A4006] and the reviewer [Nederl. Akad. Wetensch. Proc. Ser. A **58** (1955), 368–378; MR **17**, 63; ibid. **58** (1955), 379–389; MR **17**, 63; ibid. **58** (1955), 483–493; MR **17**, 354; ibid. **58** (1955), 494–503; MR **17**, 354; ibid. **58** (1955), 663–674; MR **17**, 594; also Symposium on Harmonic Analysis and Related Integral Transforms, Dept. of Mathematics, Cornell University, Ithaca, N.Y., 1956 (mimeographed)]. Instead of sequences, the author considers families $\{\varphi(x,\lambda)\}$ depending on a parameter λ that ranges over a set Λ with limit point λ_0. He says that $\{\varphi(x,\lambda)\}$ converges weakly for x on an interval I, as $\lambda\to\lambda_0$, if (i) $\varphi(x,\lambda)$ is continuous for all $x\in I$ and all $\lambda\in\Lambda$; (ii) for some integer $p\geqq 0$, $\varphi(x,\lambda)=D^pf(x,\lambda)$, where D stands for the ordinary derivative d/dx; (iii) $f(x,\lambda)\to f(x)$ uniformly for $x\in I$ as $\lambda\to\lambda_0$. A weak function $\varphi(x)$ is now defined as an equivalence class of weakly convergent families $\{\varphi(x,\lambda)\}$, equivalence meaning that suitable "antiderivatives" $f(x,\lambda)$ have the same limit function $f(x)$. The author briefly discusses derivatives, antiderivatives, products, convolution, division by a power of x, definite integrals and Fourier transforms of weak functions; a more detailed treatment can be found in the papers cited above.

The second part deals with distributions in R^m. "Weak functions" are defined as in the 1-dimensional case, except that the operator $D=d/dx$ is replaced by the spherical Laplacian Δ. The operator Δ is defined by the equation

$$\Delta f(x) = \lim_{R\to 0}\frac{2m}{R^2}\{M(f,R)-f(x)\},$$

where $M(f,R)$ is the average of f over the spherical surface of radius R about x. It is shown that every weak function $\varphi(x)$ can be expressed as $\Delta^m f(x)$, where f is continuous, and that weak functions have derivatives of all orders. Products and convolution are discussed, and the potential due to a weak density function.

J. Korevaar (Madison, Wis.)

Silberstein, J. P. O. 5337

Symmetrisable operators. II. Operators in a Hilbert space $\mathfrak{H}$.

J. Austral. Math. Soc. **4** (1964), 15–30.

This paper is a continuation of Part I under the same main title [same J. **2** (1961/62), 381–402; MR **26** #6789]. It is assumed that A and H are possibly unbounded operators in Hilbert space such that the domains, ranges and null spaces satisfy $\mathfrak{R}_A\subset\mathfrak{D}_H$ (i.e., $\mathfrak{D}_{HA}=\mathfrak{D}_A$) and $\mathfrak{N}_H\subset\mathfrak{N}_A$, H is non-negative definite self-adjoint and A is (left) symmetrizable by H, i.e., HA is self-adjoint. Some of the main results are as follows. Eigenvalues of A (if any) are real, and eigenvectors belonging to different eigenvalues are H-orthogonal. If, in addition, $\mathfrak{D}_A\subset\mathfrak{D}_H$, then all non-zero eigenvalues are simple, and the value 0 is an eigenvalue of multiplicity two at most. The continuous spectrum of A need not be restricted to the real axis, not even when H and A are bounded operators. This result was also obtained by J. Dieudonné [Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), pp. 115–122, Jerusalem Academic Press, Jerusalem, 1961]. If H is bounded and strictly positive definite, then the eigenvalues of A^* are real, and the residual spectrum of A is real. If $\mathfrak{D}_{A^*}\subset\mathfrak{R}_H$, then the residual spectrum of A is empty.

A. C. Zaanen (Pasadena, Calif.)

Silberstein, J. P. O. 5338

Symmetrisable operators. III. Hilbert space operators symmetrisable by bounded operators.

J. Austral. Math. Soc. **4** (1964), 31–48.

This paper is a continuation of Parts I, II under the same

main title [for Part II, see #5337 above]. For the definitions and notations we refer to the preceding review. It is assumed now that H is bounded, and it is shown first that without loss of generality one may then also assume that H is strictly positive definite. The point spectrum and residual spectrum of A are now real (cf. the preceding review). In the special case that $A=BH$ with B bounded and self-adjoint, the continuous spectrum of A is also real, and the residual spectrum is empty. Some conditions, too complicated to reproduce here, are derived in order that a more general symmetrizable A (not necessarily of the form $A=BH$) has these desirable properties.

A. C. Zaanen (Pasadena, Calif.)

Brodskiĭ, V. M. 5339

Eigenvectors of completely continuous linear operators defined in partially ordered non-normed spaces. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 468–471.

Let E be a linear space over the field of reals. Let K be a cone in E, that is, $tK+sK\subset K$ for any $t, s\geqq 0$. (1) Assume that $E=K-K$ and that the only element x such that $x\in K$ and $-x\in K$ is the element zero. (2) Let for any two points $x\neq y$ there exist a linear functional f positive on K (that is, $f(x)\geqq 0$ for $x\in K$) such that $f(x)\neq f(y)$. Define a partial order $x\leqq y$ by the condition: $y-x\in K$, and a convergence $x_n\to x$ by the condition: there exists an element $z\in K$ such that for every $\varepsilon>0$ there exists m such that $-\varepsilon z\leqq x_n-x\leqq \varepsilon z$ if $n>m$. The Cauchy condition is defined similarly. Sequential closedness, completeness, and compactness are defined as usual. A set $M\subset E$ is said to be bounded if there exists an element $z\in K$ such that $-z\leqq x\leqq z$ for all $x\in M$. (3) Assume that the space E is sequentially complete and the cone K is closed. The following theorem is proven. Let a real linear space E with a cone K satisfy the conditions (1)–(3). Let A be a linear operator from E into itself such that $AK\subset K$ and for every bounded set M the image AM is sequentially compact. If there exist elements $u, v\in K$ $(u\neq 0)$ such that (4) $\mu_p u\leqq A^p u\leqq \mu_p v$ $(p=1, 2, \cdots)$ for some positive μ_p, then there exist an element $x\in K$ $(x\neq 0)$ and a number $\eta\geqq(\mu_p)^{1/p}$ $(p=1, 2, \cdots)$ such that $Ax=\eta x$. The condition (4) is satisfied if the following one is satisfied: $\mu u\leqq A^p u\leqq A^p v\leqq \mu v$ $(p=1, 2, \cdots)$ for some positive μ. The proof makes use of Zorn's Lemma and the following fixed-point theorem of Rutman. Let M be a bounded, closed, convex set in a space E satisfying the conditions (1)–(3). If A is a continuous operator mapping the set M into a compact part of M, then there exists a point $x\in M$ such that $Ax=x$ [M. A. Rutman, Mat. Sb. (N.S.) **8 (50)** (1940), 77–96; MR **2**, 104].

W. Bogdanowicz (Washington, D.C.)

Javrjan, V. A. 5340

Some perturbations of self-adjoint operators. (Russian. Armenian summary)

Akad. Nauk Armjan. SSR Dokl. **38** (1964), 3–7.

The perturbation situation is studied in which $A_1=A+V$, where A and A_1 are selfadjoint operators and $R_i^{1/2}VR_i^{1/2}$ is of trace class, $R_z=(A-zI)^{-1}$. There exists a measurable function $\xi(\lambda)$ of real λ such that

$$\mathrm{Sp}\{(A_1-zI)^{-1}-(A-zI)^{-1}\} = -\int_{-\infty}^{+\infty}(\lambda-z)^{-2}|\xi(\lambda)|\,d\lambda$$

whenever z is not real. The function is obtained a.e. by the formula $\xi(\lambda)=\pi^{-1}\lim\arg\Delta(\lambda+i\varepsilon)+c$ as $\varepsilon\searrow 0$, where $\Delta(z)=\det(I+R_z^{1/2}VR_z^{1/2})$ and c is a constant which is independent of λ.

L. de Branges (Lafayette, Ind.)

Cordes, H. O.; Labrousse, J. P. 5341

The invariance of the index in the metric space of closed operators.

J. Math. Mech. **12** (1963), 693–719.

The authors consider the class F of all densely defined operators A in Hilbert space H having the following properties: (a) The range of A is closed; (b) Either A or A^* has a finite-dimensional null-space. For these operators, they put $p(A)=\dim$ null-space $(A)-\dim$ null-space (A^*), and call $p(A)$ the index of A. The metric distance between two densely defined unbounded operators A and B may be taken as $|(I+AA^*)^{-1}-(I+BB^*)^{-1}|$. As the authors show, this metric defines the same topology for bounded operators as the ordinary metric $|A-B|$, and is equivalent to a metric introduced by Kreĭn and Krasnosel'skiĭ [Uspehi Mat. Nauk **2** (1947), no. 3 (19), 60–106; MR **10**, 198] to measure the distance between closed subspaces of a Hilbert space.

Refining results of Kreĭn and Krasnosel'skiĭ [op. cit.], the authors go on to prove: (i) If B is a closed unbounded operator sufficiently close (in the above metric) to $A\in F$, then $B\in F$ and $p(B)=p(A)$; (ii) If $B, A\in F$, and $p(B)=p(A)$, then A and B lie in the same (arcwise) connected component of F; (iii) If $A, B\in F$, and if $p(A)+p(B)$ is not $\infty-\infty$, then $AB\in F$, and $p(AB)=p(A)+p(B)$.

The proof of statement (i) proceeds by relating $p(A)$ to a similar invariant for the bounded operator $A(I+AA^*)^{-1}$. The proof of statement (ii) proceeds by reduction to the case in which A has no null-space, and thence by use of the canonical factorisation of A. The proof of (iii) is direct.

J. T. Schwartz (New York)

Kurepa, Svetozar 5342

A theorem about similarity of operators.

Arch. Math. **14** (1963), 411–414.

Let T be a bounded operator in Hilbert space. In a previous paper [Math. Z. **78** (1962), 285–292; MR **25** #1448] the author showed that if T^n is regular and normal, or if $\exp(T)$ is normal, then T is similar to a normal operator. In the present paper, he applies his earlier results to show that if $\sin(T)$ is self-adjoint and of norm <1, then T must be similar to a self-adjoint operator. The proofs, based upon identities for functions of an operator and on the earlier results, are straightforward.

J. T. Schwartz (New York)

Thorp, B. L. D. 5343

Operators which commute with translations.

J. London Math. Soc. **39** (1964), 359–369.

Let $\mathscr{F}$ denote the space of all complex-valued sequences $x=\{x_n\}_{n\geqq 0}$ and T_k $(k=0, 1, 2, \cdots)$ the translation operator defined by

$$(T_k x)_n = 0 \quad \text{if } 0\leqq n<k,$$
$$= x_{n-k} \text{ if } k\leqq n.$$

A vector subspace D of $\mathscr{F}$ is said to be T-linear if it is not reduced to the zero element, and if both D and $\mathscr{F}\setminus D$ are

closed under all the T_k. A linear operator with domain a T-linear subspace and range in $\mathcal{F}$ is said to be T-linear if it commutes with all the T_k. The Cauchy product (truncated convolution) of $x, y \in \mathcal{F}$ is defined by

$$(x * y)_n = \sum_{i=0}^{n} x_i y_{n-i}.$$

It is shown that any T-linear operator is representable as the Cauchy product with a uniquely determined element of $\mathcal{F}$; this is a discrete analogue of representation theorems given by the reviewer [Pacific J. Math. **7** (1957), 1333–1339; MR **19**, 1183], König and Meixner [Math. Nachr. **19** (1958), 265–322; MR **21** #7410], and Weston [Pacific J. Math. **10** (1960), 1453–1468; MR **23** #A519]. Various properties of T-linear subspaces are given. If M and N are T-linear subspaces, let $T(M, N)$ denote the set of T-linear operators with domain M and range included in N. The set $T(M, N)$ is determined explicitly for various familiar choices of M and N. *R. E. Edwards* (Canberra)

Kužel', A. V. 5344

On non-selfadjoint operators generated by Jacobian matrices. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 1027–1029.

Suppose A' is a linear operator on a Hilbert space defined by a Jacobi matrix with respect to an orthonormal basis $\{e_k\}$. Let $\{P_k(\lambda)\}$ be the corresponding polynomials generated by the matrix, $g_\lambda = \sum P_k(\bar\lambda)e_k$ and $h(\lambda, \mu) = \sum P_k(\lambda)P_k(\mu)$. If $A = A'^*$, it is known that A is either self-adjoint or symmetric with deficiency index $(1, 1)$. In the latter case, let $D(A_\theta)$ be the linear manifold generated by the domain of A and the vector $\theta g_i + g_{-i}$, where θ is a complex number, and on $D(A_\theta)$ define $A_\theta f = A^*f$. A_θ is self-adjoint if and only if $|\theta| = 1$. In any case set $\omega(\lambda, t) = t(\lambda+i)h(\lambda, -i) + (\lambda - i)h(\lambda, i)$, and it turns out that the non-real spectrum of A_θ consists of the non-real zeros of $\omega(\lambda, \theta)$ and the multiplicity of a non-real eigenvalue is one.

If $|\theta| < 1$, then A_θ is dissipative and

$$\prod_{k=1}^{N} \left| \frac{\lambda_k + i}{\bar\lambda_k + i} \right| \geqq |\theta|,$$

where $\{\lambda_k\}_1^N$ $(N \leqq \infty)$ is the non-real spectrum. If A_θ is simple, then the eigenvectors of this operator are dense if and only if the sign of equality holds. If $\tilde h$ corresponds to an operator $\tilde A$ and $h = \tilde h$, then the simple parts of A_θ and $\tilde A_\theta$ are isomorphic. The definition of simple is taken from previous papers of the author [same Dokl. **119** (1958), 868–871; MR **20** #6041; ibid. **125** (1959), 35–37; MR **22** #2905]. The proofs are applications of the methods developed in these papers. *A. Devinatz* (St. Louis, Mo.)

Markus, A. S. 5345

Some tests of completeness of the system of root vectors of a linear operator and the summability of series in that system. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 753–756.

Let A be a completely continuous linear operator in a separable Banach space $\mathfrak{B}$; further, let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \cdots)$ be a system of principal vectors of A in the terminology of V. B. Lidskiĭ [Trudy Moskov. Mat. Obšč. **11** (1962), 3–35; MR **26** #1760]. The author denotes by $E(A)^-$ the closure of the linear span of the system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \cdots)$, and announces necessary conditions for the relations $E(A)^- = \mathfrak{B}$ and $E(A)^- = A(\mathfrak{B})^-$. In case $E(A)^- = \mathfrak{B}$ the system is said to be "complete", and any $f \in \mathfrak{B}$ can be formally expanded in terms of the system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \cdots)$: $f = \sum_{s=1}^{\infty} c_s \mathbf{e}_s$. Lidskiĭ [loc. cit., p. 16] has defined a type of Abel summability of the above series, and the author gives several conditions to insure—besides the completeness of the system—summability in the sense defined by Lidskiĭ; in so doing, he extends a theorem of Lidskiĭ (Theorem 3, loc. cit.; it may be noted that Lidskiĭ's theory is set up in the case where $\mathfrak{B}$ is a Hilbert space). The author concludes with applications to elliptic differential operators in L_p $(p > 1)$; no proofs are given. Among the ten theorems announced in this paper, the following sample result may be singled out. If A is a bounded linear operator on $\mathfrak{B}$, the "numerical range" is defined as the family $W(A)$ of all complex numbers of the form $W_x(Ax)$, where $x \in \mathfrak{B}$, $\|x\| = 1$, and W_x is a bounded linear functional such that $\|W_x\| = \|x\|$ and $W_x(x) = \|x\|^2$. The convex hull of the set $W(A)$ is independent of the choice of W; this has been shown by G. Lumer [Trans. Amer. Math. Soc. **100** (1961), 29–43; MR **24** #A2860], who first introduced the notion of numerical range of an operator in Banach space. Lumer found that $W(A)$ is a subset of the reals if and only if $\|I + i\varepsilon A\| = 1 + o(\varepsilon)$ when $\varepsilon \to 0$ and $-1 < \varepsilon < 1$. The author announces that $W(A)$ is a subset of the reals if and only if $\|(A - \lambda)^{-1}\| \leqq |\text{Im } \lambda|^{-1}$ whenever $\text{Im } \lambda \neq 0$.

G. L. Krabbe (Lafayette, Ind.)

Edwards, R. E. 5346

Convolutions as bilinear and linear operators.

Canad. J. Math. **16** (1964), 275–285.

This paper is concerned with characterizing $f * g$ as a linear or bilinear operation. Let X be a locally compact, not necessarily Abelian group. Let C_c be the set of continuous functions on X with compact support, and let M be the space of all Radon measures on X. Finally, let $\Delta(x)$ be the modular function on X. The author's principal result is the following. If $B = B(f, g)$ is a bilinear operator from $C_c \times C_c$ into M with the properties: B is positive, i.e., $B(f, g) \geqq 0$ if $f \geqq 0$, $g \geqq 0$; for fixed $f \in C_c$, $g \to B(f, g)$ commutes with $R_x(R_x f \cdot (y) = \Delta(x) f(yx^{-1}))$; for fixed $g \in C_c$, $f \to B(f, g)$ commutes with $L_x(L_x f \cdot (y) = f(x^{-1}y))$; for given $f \in C_c$, $B(f, g)$ is the vague limit of finite linear combinations of the $L_x g$ with x in the support of f; then $B(f, g)$ is, apart from a constant multiple, equal to $f * g$. Let X now be a product of lines and circles, and let D be the set of all distributions on X which have compact support and whose Fourier transform is a bounded function on $\hat X$, the dual of X. The author proves that if T is a continuous linear operator mapping C into M which commutes with translations, then $Tf = \mu * f$, where $\mu \in D$. Various other related results are given.

I. I. Hirschman, Jr. (St. Louis, Mo.)

McCarthy, C. A. 5347

On a theorem of Beck and Putnam.

J. London Math. Soc. **39** (1964), 288–290.

The main theorem states that $\varphi(Z) = \varphi(Z^*)$ implies $Z = Z^*$ if φ is a linear transformation on a complex vector space with involution which takes selfadjoint elements into selfadjoint elements and does not have -1 as an eigenvalue. The hypotheses are satisfied by $\varphi: Z \to UZU^*$ in a B^*-algebra with unit if U is a unitary element of the

algebra whose spectrum is contained in an arc of length less than π. This gives a new proof of a theorem of S. K. Berberian [same J. 37 (1962), 403–404; MR 25 #5391], which contains a theorem of W. A. Beck and C. R. Putnam [ibid. 31 (1956), 213–216; MR 17, 1114].

L. de Branges (Lafayette, Ind.)

Boehme, T. K. 5348

Continuity and perfect operators.

J. London Math. Soc. **39** (1964), 355–358.

Let A be a commutative algebra with no non-zero annihilator. A linear operator P on A is called perfect if $P(xy)=xP(y)$ for all x, y in A. Suppose that A has a topology T such that (A, T) is a topological algebra with an approximate identity and (A, T) is a real or complex topological vector space which is "tonnelé". Then every perfect operator is a continuous linear operator on (A, T).

Let D be the convolution algebra of infinitely differentiable functions with compact support on $(-\infty, \infty)$, and furnished with its usual topology. Every perfect operator P on D is of the form $P(f)=Q*f$ for some distribution Q with compact support. *P. Civin* (Eugene, Ore.)

Gleichgewicht, B. 5349

A remark on absolute-valued algebras.

Colloq. Math. **11** (1963), 29–30.

In this paper (as well as in the following two papers [#5350, #5351]) real, not necessarily associative, algebras are considered. A normed algebra is called absolute-valued if its norm $|\ |$ satisfies the condition $|xy|=|x|\cdot|y|$ for all x, y in the algebra. If the algebra has an involution $*$, then it induces a new multiplication $x\circ y=x^*y$. The author proves that in each absolute-valued real algebra with a non-trivial involution, such that $|x^*|=|x|$ for each x, there is an element e such that $x\circ x=|x|^2e$ for each x in the algebra. A consequence of this result is the negative answer to a conjecture of K. Urbanik, claiming that an absolute-valued algebra such that $|x^2+y^2|\geqq|x^2|$ must be isomorphic to the real field.

P. Saworotnow (Washington, D.C.)

Urbanik, K. 5350

Remarks on ordered absolute-valued algebras.

Colloq. Math. **11** (1963), 31–39.

An absolute-valued algebra A is said to be of real character if $x^2+y^2\neq 0$ and $xy+yx\neq 0$ whenever $x\neq 0$, $y\neq 0$. The author shows the existence of an absolute-valued algebra of real character which cannot be ordered and the existence of an infinite-dimensional ordered absolute-valued algebra. He proves that the real field is the only finite-dimensional absolute-valued algebra of real character. Also he establishes a result converse to that proved in the previous paper [#5349]: If the set of all squares in an absolute-valued algebra A is one-dimensional, then the multiplication in A is induced by an involution, i.e., there exists an involution $*$ on A and a multiplication $\circ$ on A such that $xy=x^*\circ y$ for all $x, y\in A$.

P. Saworotnow (Washington, D.C.)

Strzelecki, E. 5351

Algebras under a minimal norm.

Colloq. Math. **11** (1963), 41–52.

The author considers a (non-associative, real, normed) algebra A such that $\|x^2\|=\|x\|^2$ for each $x\in A$ (the norm is minimal). The following results are established. If A is the direct sum of two-sided ideals I_r, $r=1, 2, \cdots, n$, then $\|x\|=\max_r\|x_r\|$, where $x=\sum_{r=1}^n x_r$ and $x_r\in I_r$ for each r. Under an additional assumption the converse is also true: If A, treated as a linear space, is the direct sum of linear subspaces A_1, A_2, $\cdots$, A_n, and $\|\sum_{r=1}^n x_r\|=\max_r\|x_r\|$ $(x_r\in A_r)$, then each A_r is a two-sided ideal in A and A is the direct sum of ideals A_r. If A is associative and finite-dimensional, then A is the direct sum of simple algebras each of which is a division algebra (under a certain assumption each of these division algebras is isomorphic to the real field).

P. Saworotnow (Washington, D.C.)

Gelbaum, B. R. 5352

Banach algebras and their applications.

Amer. Math. Monthly **71** (1964), 248–256.

This article is largely expository. It contains, however, a neat new proof of von Neumann's theorem asserting that, given any countable family F of pairwise commuting normal operators on a Hilbert space, there exists a normal operator A such that every operator in F is a function of A.

J. M. G. Fell (Seattle, Wash.)

Gil de Lamadrid, Jesús 5353

Uniform cross norms and tensor products of Banach algebras.

Bull. Amer. Math. Soc. **69** (1963), 797–803.

In the present paper the author continues the investigations of Gelbaum [Canad. J. Math. **11** (1959), 297–310; MR **21** #2922; Trans. Amer. Math. Soc. **103** (1962), 525–548; MR **25** #2406] concerning tensor products of Banach algebras and the investigations of Gelbaum and the author [Pacific J. Math. **11** (1961), 1281–1286; MR **26** #5394] concerning bases in tensor products. Three questions are examined. From the author's introduction: "One is: Which cross norms α of the algebraic tensor product $A\otimes B$ of two Banach algebras A and B are compatible with multiplication? Compatibility with multiplication means that $\alpha(t_1t_2)\leqq\alpha(t_1)\alpha(t_2)$ for every two tensors $t_1, t_2\in A\otimes B$. The second question is: Is the so-called least cross norm λ, in particular, compatible with multiplication? The third . . . is: Given two Banach spaces E and F, each of which has a Schauder basis, for what cross norms α do the resulting complete tensor products $E\otimes_\alpha F$ have Schauder bases?"

In the present paper the author considers the class of uniform cross norms (α is a uniform cross norm on $E\otimes F$ if it is a cross norm and $\alpha(U\otimes V(t))\leqq\|U\|\ \|V\|\alpha(t)$ for all linear operators $U: E\to E$ and $V: F\to F$ and for $t\in E\otimes F$). If α is a uniform cross norm on $E\otimes F$ and if A and B are subalgebras of $\mathbf{B}(E)$ and $\mathbf{B}(F)$, the algebras of all linear operators on E and F into itself, respectively, then $A\otimes B$ is isomorphic to a subalgebra of $\mathbf{B}(E\otimes_\alpha F)$. The operator norm on $\mathbf{B}(E\otimes_\alpha F)$ induces a norm $\bar{\alpha}$ on $A\otimes B$ which is compatible with multiplication. This fact is used to define the associated cross norm compatible with multiplication to a given uniform cross norm on $A\otimes B$ for arbitrary Banach algebras A and B with units. Let A and B be commutative semisimple Banach algebras of operators and let α be a uniform cross norm such that

the natural embedding $A \otimes_\alpha B \to A \otimes_\lambda B$ is one-to-one. Then the algebra $A \otimes_\alpha^- B$ is semisimple. For other similar results we refer to the paper. No proofs are given except of the fact that the least (weak) cross norm λ is not compatible with convolution in $l \otimes l$. However, λ is compatible with multiplication in $A \otimes B$ if the Gel'fand representation of A is isometric.

Concerning the third question the author announces the following result. Let (x_n) and (y_n) be Schauder bases in the Banach spaces E and F, respectively, and let α be a uniform cross norm on $E \otimes F$. Then the elements $x_n \otimes y_m$ $(n, m = 1, 2, \cdots)$ form, under a certain ordering, a basis for $E \otimes_\alpha F$. *A. Pełczyński* (Warsaw)

Miles, Philip 5354

B^* algebra unit ball extremal points.

Pacific J. Math. **14** (1964), 627–637.

The author investigates extreme points in the unit sphere of a self-adjoint subalgebra (not necessarily uniformly closed) of a B^*-algebra. Main results are as follows. Theorem 1: Let A be a self-adjoint subalgebra of some B^*-algebra B; then x is an extreme point of the unit sphere of A if and only if $(1-x^*x)A(1-xx^*)=\{0\}$. This theorem was obtained by Kadison under the additional assumptions of the existence of the identity in A and the uniform closedness of A [Ann. of Math. (2) **54** (1951), 325–338; MR **13**, 256], and the assumption of the existence of the identity was dropped by the reviewer [Mimeographed Lecture Notes, Yale Univ., New Haven, Conn., 1962]. In Theorem 2, he shows a proof of the reviewer's theorem that a B^*-algebra has the identity if its unit sphere has an extreme point [Pacific J. Math. **6** (1956), 763–773, Appendix; MR **18**, 811], and he also gives a classification of extreme points in the unit sphere of A. Theorem 3: If A is an AW^*-algebra, ϕ a $*$-homomorphism of A into a B^*-algebra, and y an extreme point in the unit sphere of $\phi(A)$, then there is an extreme point x in the unit sphere of A such that $\phi(x)=y$. Theorem 4: Let A, B be AW^*-algebras, with B the image of A under some $*$-homomorphism; then A of (i) type I_n [respectively, (ii) type II_1, (iii) type II_∞, (iv) type III, (v) type I_∞] implies B of (i) type I_n [respectively, (ii) type II_1, (iii) no direct summands of finite types and type I_∞, (iv) type III, (v) no direct summands of finite types].

S. Sakai (Berkeley, Calif.)

Wermer, John 5355

Analytic disks in maximal ideal spaces.

Amer. J. Math. **86** (1964), 161–170.

Let A be a closed subalgebra of $C(X, \mathbf{C})$ where X is compact. Then A shall be called hypo-Dirichlet if there exist functions $f_1, \cdots, f_n$ in A^{-1} (the class of elements in A whose inverses are in A) such that the vector space spanned by the real parts of functions in A and the functions $\log|f_1|, \cdots, \log|f_n|$ is dense in $C(X, \mathbf{R})$. For x, y in X let $\|x-y\| = \sup|f(x)-f(y)|, f \in A$.

The main theorem: Let A be hypo-Dirichlet on its maximal ideal space X. Suppose there are x, y in X such that $0 \neq \|x-y\| < 2$. Then the unit disc $\{|z|<1\}$ can be mapped onto a neighborhood of x in X, such that the elements of A are each holomorphic on the disc. (Thus X is 2-dimensional in a neighborhood of x.) Actually, the author works with an additional hypothesis, but asserts in a footnote that B. O'Neill has shown that this hypothesis may be dropped. The proof also involves a lemma of Gleason [J. Math. Mech. **13** (1964), 125–132; MR **28** #2458]. *R. Arens* (Los Angeles, Calif.)

Zhang, Shang-tai [Chang, Shang-tai] 5356

On several results for certain nonlinear operators.

Acta Math. Sinica **14** (1964), 137–142 (*Chinese*); *translated as Chinese Math.* **5** (1964), 149–155.

From the author's introduction: "In this paper, we use the theory of Orlicz spaces to discuss properties of the Nemyckiĭ and Urysohn operators, and we give some sufficient conditions, and some necessary and sufficient conditions, for these properties. Throughout, measure and integration will be in the sense of Lebesgue."

Zhang, Shang-tai [Chang, Shang-tai] 5357

Continuity and complete continuity of the Uryson operator.

Acta Math. Sinica **13** (1963), 204–215 (*Chinese*); *translated as Chinese Math.* **4** (1963), 222–234.

The author considers the nonlinear integral operator

$$[Tu](x) = \int_G F(x, y; u(y))\, d\mu(y),$$

where $F(x, y; s)$ is assumed real-valued and continuous over $(x, s) \in G \times (-\infty, \infty)$ for almost (μ) all $y \in G$, and where G is bounded closed in euclidean n-space and μ is n-dimensional Lebesgue measure. Under certain additional hypotheses, the continuity and complete continuity of T as an Orlicz space operator are shown.

F. H. Brownell (Seattle, Wash.)

Minty, George J. 5358

On a "monotonicity" method for the solution of nonlinear equations in Banach spaces.

Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 1038–1041.

Let X be a reflexive Banach space. Let Y be its dual. A mapping f defined in $B \subset X$ with values in Y is said to be monotonic if $\mathrm{Re}\langle x_1-x_2, f(x_1)-f(x_2)\rangle \geq 0$, $x_1, x_2 \in B$. It is said to be hemi-continuous (Browder) if for every real half-line T with end-point $x_0 \in B$ and every $z \in X$ the mapping $x \to \langle z, f(x)\rangle$ from $T \cap B$ into Y is continuous. If $D \subset X$, denote by $K(D)$ the closed convex hull of D. If $D_1 \subset X$, we say that B surrounds D_1 densely if for every real line T through $x_0 \in D_1$ there exist on each side of x_0 points in B arbitrarily close to x_0. Theorem: Let f be monotonic and hemi-continuous and let $D \subset K(D) \subset B$, B surrounding $K(D)$ densely, be such that $\mathrm{Re}\langle x, f(x)\rangle \geq 0$, $x \in D$. Then there exists $x \in K(D)$ such that $f(x)=0$. This generalizes results obtained previously for the special case of Hilbert space [the author, Duke Math. J. **29** (1962), 341–346; Browder, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 794–798] and is also related to a result by Vaĭnberg and Kačurovskiĭ [Dokl. Akad. Nauk SSSR **129** (1959), 1199–1202; MR **22** #4930]. A different extension of these results was obtained by Browder [Bull. Amer. Math. Soc. **69** (1963), 862–874; MR **27** #6048]. The proof depends on a reduction to the finite-dimensional case. *J. Peetre* (Lund)

CALCULUS OF VARIATIONS

See also 5863, 5866, 5867, 5870, 5878, 5883, 5884.

ósa, András 5359

Un criterio sufficiente per il minimo assoluto nel caso in cui l'integrale dipende anche dalle derivate di ordine superiore delle funzioni ammissibili.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 130–133.

i tratta di una condizione sufficiente (nell'indirizzo di . Picone), affinchè una funzione $\bar{y}(x)$ renda minimo integrale $\int_a^b f(x, y(x), y'(x), \cdots, y^{(n)}(x))$ in $E(T\times G)$, ove: è costituito dai punti $(x, y_0, y_1, \cdots, y_{n-1})$ soddisfacenti le limitazioni $a\leqq x\leqq b$, $\bar{y}^{(i)}(x)-\rho_i'(x)\leqq y_i\leqq \bar{y}^{(i)}(x)+\rho_i''(x)$ $=0, 1, \cdots, n-1)$ con $\rho_i'(x)$, $\rho_i''(x)$ funzioni definite in $, b)$; G è un insieme di numeri reali y_n; $E(T\times G)$ è la asse delle funzioni $y(x)$ definite e continue in (a, b) isieme con le loro derivate dei primi n ordini e tali che a $y^{(i)}(a)=\bar{y}^{(i)}(a)$, $y^{(i)}(b)=\bar{y}^{(i)}(b)$ $(i=0, 1, \cdots, n-1)$, e che er $a\leqq x\leqq b$ ogni punto $(x, y(x), y'(x), \cdots, y^{(n-1)}(x), y^{(n)}(x))$ ppartenga a $T\times G$. La condizione sufficiente in questione ppone l'esistenza di n funzioni di due variabili $A^i(x, y_{i-1})$ $=1, \cdots, n)$ definite e continue insieme con le loro erivate parziali rispetto a x per $a\leqq x\leqq b$, $\bar{y}^{(i-1)}(x)-$ $_{-1}(x)\leqq y_{i-1}\leqq \bar{y}^{(i-1)}(x)+\rho_{i-1}''(x)$ rispettivamente, in modo ne in ogni punto di $T\times G$ sia verificata la disuguaglianza

$$x, y_0, y_1, \cdots, y_n)-f(x, \bar{y}(x), \bar{y}'(x), \cdots, \bar{y}^{(n)}(x)) - \sum_{i=1}^{n}[A^i(x, y_{i-1})y_i - A^i(x, \bar{y}^{(i-1)}(x))\bar{y}^{(i)}(x)] - \sum_{i=1}^{n}\int_{\bar{y}^{(i-1)}(x)}^{y_{i-1}} A_x^i(x, t)\,dt \geqq 0.$$

S. Cinquini (Pavia)

ósa, András 5360

Sulle funzioni d'invarianza per i problemi del calcolo delle variazioni unidimensionali di secondo ordine.

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **34** (1963), 494–500.

ia T_2 il campo costituito dai punti (x, y_0, y_1, y_2), per i uali è $a\leqq x\leqq b$, $\rho'(x)<y_0<\rho''(x)$ (ove $\rho'(x)<\rho''(x)$ per $\leqq x\leqq b$), mentre y_1 e y_2 possono assumere qualunque alore reale, e sia E la classe delle funzioni $y(x)$ $(a\leqq x\leqq b)$ ontinue insieme con le loro derivate dei primi quattro rdini con $\rho'(x)<y(x)<\rho''(x)$ e si consideri il funzionale

$$J_f[y(x)] = \int_a^b f(x, y(x), y'(x), y''(x))\,dx,$$

ve $f(x, y_0, y_1, y_2)$ è una funzione continua in T_2. Una nzione $v(x, y_0, y_1, y_2)$ continua in T_2 si chiama [M. Picone, tti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. (8) **6** (1962), 281–339; MR **25** #4387] funzione di ivarianza nel caso degli estremi liberi, se, per ogni nzione $y(x)$ di E, $J_v[y(x)]$ ha sempre lo stesso valore; refissati quattro numeri a_0, b_0, a_1, b_1 con $\rho'(a)<a_0<\rho''(a)$, $'(b)<b_0<\rho''(b)$, $v(x, y_0, y_1, y_2)$ si chiama funzione di ivarianza nel caso degli estremi fissi, se $J_v[y(x)]$ ha empre lo stesso valore per ogni funzione $y(x)$ di E oddisfacente alle condizioni $y(a)=a_0$, $y(b)=b_0$; $y'(a)=a_1$, $'(b)=b_1$. L'autore prova che condizione necessaria, ffinchè $f(x, y_0, y_1, y_2)$ sia una funzione di invarianza, è che essa abbia la forma $A(x, y_0, y_1)+B(x, y_0, y_1)y_2$, e dà due condizioni necessarie e sufficienti per l'invarianza (una nel caso degli estremi fissi, l'altra in quello degli estremi liberi), per le quali rinviamo al lavoro in esame.

S. Cinquini (Pavia)

Dubovickiĭ, A. Ja.; Miljutin, A. A. 5361

Extremum problems with certain constraints. (Russian)

Dokl. Akad. Nauk SSSR **149** (1963), 759–762.

This paper outlines a method for handling "classical" and "nonclassical" problems in the calculus of variations. Necessary conditions for extrema are obtained by the first variation, and the method is based upon a separation theorem for convex sets in linear topological spaces. The method is described as follows: $F(w)$ is a given functional on some Banach space W. Relative to some constraints on w one wishes to obtain necessary conditions that w^0 be a minimum of $F(w)$. They distinguish between equality constraints (w lies on some "smooth" manifold), and "inequality" constraints (w lies in the closure of some open set). A variation $\bar{w}$ is said to be excluded if $dF(w^0+\varepsilon\bar{w})/d\varepsilon|_{\varepsilon=0}<0$. A variation $\bar{w}$ is said to be admissible with respect to an inequality constraint if $w^0+\varepsilon\bar{w}$ satisfies the constraint for sufficiently small [positive] ε and for an equality constraint if the line $w^0+t\bar{w}$ is tangent to the constraint manifold. Excluded variations and variations admissible relative to inequality constraints are assumed to form a finite number of convex cones with interior points and variations admissible relative to equality constraints define a subspace. Under very general conditions on F, which are not stated, it is asserted that a necessary condition for w^0 to be a minimum point is that there is no point in the subspace common to the interior of all cones. The basis of the method is the following theorem. Let $\Omega_1, \cdots, \Omega_s$ be a system of convex open cones and let L be a subspace of W. The set Ω_i^* is the set of all linear functionals non-negative on Ω_i and L^* is the set of all linear functionals vanishing on L. The existence of linear functionals $\omega_1, \cdots, \omega_s$, l with $\omega_i\in\Omega_i^*$, $l\in L^*$, not all zero, and satisfying $\omega_1+\cdots+\omega_s+l=0$ is then a necessary and sufficient condition that the intersection of $\Omega_1, \cdots, \Omega_s$, L be the null set.

It is then indicated how the method can be used to derive the Euler equations, Pontryagin's maximum principle, and Bernstein's inequality, as well as to obtain a necessary condition for the optimal solution of the following minimax problem. For the given functional $I(x, u)=\max[\Phi(x, u); t_0\leqq t\leqq t_1]$ find among all solutions $x(t)$, $u(t)$ of the system of equations $dx/dt=f(x, u)$, $x(t_0)=x_1$, $x(t_1)=x_2$ those that minimize $I(x, u)$. The values $u(t)$ belong to some set D of E^r and the time t_1-t_0 is not fixed. The necessary condition for this latter problem is also a maximum principle.

J. P. LaSalle (Baltimore, Md.)

GEOMETRY

See also 5074, 5375.

Pimenov, R. I. 5362

On the foundations of geometry. (Russian)

Dokl. Akad. Nauk SSSR **155** (1964), 44–46.

A system of axioms is constructed for spaces with free mobility, especially for the plane. The initial objects, point and line, and their relations, are introduced with reference to V. F. Kagan [*Foundations of geometry*, Part I (Russian), GITTL, Moscow, 1949; MR **12**, 731; ibid., Part II, 1956; MR **19**, 303]. Then so-called postulated lines are singled out by five axioms (the first is that postulated lines may only move into postulated ones, and into any postulated line). This leads to perpendicularity and the introduction of coordinates with respect to perpendicular axes Ox and Oy; Ox is postulated. Three possibilities exist: a pencil of lines can contain (1) only postulated lines (the homogeneous plane), (2) a unique non-postulated line (the degenerate plane), (3) two or more non-postulated lines (pseudo-plane). Together with the three possibilities indicated by the names Riemann, Euclid and Lobačevskiĭ, there exist nine types of geometry. Their trigonometries can be described with the formulas of spherical trigonometry on the sphere $(\xi j_1)^2+(\eta j_1 j_2)^2+\zeta^2=1$, where ξ, η, ζ are real and j_1, j_2 can take the values 1, e, i ($e^2=0$). A table shows how the nine types are characterized by their coordinate sets in the neighborhood of a point. It is further indicated what the situation is for spaces of more than two dimensions, and what the consequences are for the cinematics of four dimensions, where cosmological theories also play a role. *D. J. Struik* (Utrecht)

Adams, George 5363

★Von dem ätherischen Raume.

Studien und Versuche, Bd. 6.

Verlag Freies Geistesleben, Stuttgart, 1964. 59 *pp.* *DM* 4.80.

This book includes some excellent figures facing pages 16, 17, 32, and 33: a sphere with two tetrahedra in reciprocal positions (one inscribed and one circumscribed), a sphere with an inscribed octahedron and a circumscribed cube, a sphere with two tangent planes and the consequent polar lines, six great circles lying in the "equatorial" planes of a cuboctahedron, circular and parabolic sections of a cone, and Reye's configuration.

H. S. M. Coxeter (Toronto, Ont.)

Steinig, J. 5364

Inequalities concerning the inradius and circumradius of a triangle.

Elem. Math. **18** (1963), 127–131.

Author's summary: "The purpose of this paper is to establish in an elementary manner a number of new inequalities between the inradius and circumradius of a triangle, its area and certain functions of its sides. These inequalities will be strong enough to permit us to deduce without difficulty several well-known results, and to sharpen some inequalities due to other authors."

Schaal, H. 5365

Euklidische und pseudoeuklidische Sätze über Kreis und gleichseitige Hyperbel.

Elem. Math. **19** (1964), 53–56.

Durche eine komplexe Affinität kann ein Kreis $x^2+y^2+2ax+2by+c=0$ in eine gleichseitige Hyperbel $x^{*2}-y^{*2}+2a^*x^*+2b^*y^*+c^*=0$ verwandelt werden und umgekehrt. Die komplexen Ponceletschen absoluten Punkte J_1, J_2 des Kreises werden dabei in die Fernpunkte J_1^*, J_2^* der Hyperbel übergeführt und aus der euklidischen Geometrie der Ebene π des Kreises wird die pseudoeuklidische Geometrie der Ebene π^* der Hyperbel. Sätze der euklidischen Elementargeometrie werden dabei übertragen in Sätze der pseudoeuklidischen Elementargeometrie. Man hat durch dieses Übertragungsverfahren ein einfaches Mittel in der Hand, Theoreme über Kreise, gleichseitige Hyperbeln, Orthogonalität usw. auf bequeme und einleuchtende Art zu beweisen.

Als Beispiel dieser Beweismethode wird die pseudoeuklidische Version verschiedener Sätze der euklidischen elementaren Kreisgeometrie und der Geometrie der gleichseitigen Hyperbel vorgeführt. Man gelangt dabei zu besonders einfachen Beweisen verschiedener bekannter oder neuer Sätze, von denen die beiden folgenden erwähnt seien: (1) Schneiden sich ein Kreis c^* und eine gleichseitige Hyperbel c in drei reellen Punkten P_1, P_2, P_3, so ist ihr vierter Schnittpunkt P der zum Höhenschnittpunkt Q des Dreiecks (P_1, P_2, P_3) diametrale Hyperbelpunkt [Brianchon, Poncelet]. (2) Sind P, Q diametrale Punkte einer gleichseitigen Hyperbel c, so schneidet der Kreis c^* um Q durch P die Hyperbel c in drei weiteren reellen Punkten P_1, P_2, P_3, die ein gleichseitiges Dreieck bilden [Brocard]. *K. Strubecker* (Karlsruhe)

Nicolaï, Abraham Herman 5366

★Geometry of one-dimensional aggregates.

Proefschrift ter Verkrijging van de Graad van Doctor in de Wiskunde en Natuurwetenschappen aan de Rijksuniversiteit te Groningen.

n.v. Dijkstra's Drukkerij v/h Boekdrukkerij Gebroeders Hoitsema, Groningen, 1964. vii + 63 *pp.*

A normal rational scroll C_s^u in projective space S_{u+s-1} of $u+s-1$ dimensions may be defined as the locus with equations

$$\left\|\begin{matrix} x_{11} & x_{12} & \cdots & x_{1u} \\ x_{21} & x_{22} & \cdots & x_{2u}\end{matrix}\right\| = 0,$$

where each x_{hk} is a linear form in the coordinates in S_{u+s-1}.

The author takes as his basic scroll Segre's C_{n+1}^{n+1}, given by the parametric equations

$$\rho z_{ij} = x_i y_j \qquad (i=0,\cdots,n;\ j=0,1),$$

and develops its properties by considering the correspondence on the one hand between ranges of points and pencils of primes (hyperplanes) in S_n and on the other between points and primes in S_{2n+1}. On this basis he discusses the structure of C_s^u and its projections.

Using the correspondence, the author is able also to derive several projective configurations, for example, the set of $n+1$ associated lines in S_n, and the poristic sets of simplexes inscribed in one scroll and circumscribed about another, and to extend them.

Reading would have been easier if the author had backed up his synthetic arguments with more detailed algebraic treatment. *T. G. Room* (Sydney)

Denniston, R. H. F. 5367

On H. Simpson's six-conic theorem.

J. Austral. Math. Soc. **3** (1963), 454–455.

Let $P_1, \cdots, P_6$ be six points of a real projective plane,

not all on a conic and no three on a line. Denote by (1), ⋯, (6) the conics through the quintuplets obtained from them by suppressing $P_1, \cdots, P_6$, respectively. Then, in the pencil of centre P_1 there are five pairs of an involution, given by P_1P_i and the tangent in P_1 at (i) ($i=2, \cdots, 6$); P_1 is described as "elliptic" or "hyperbolic", according to the type of this involution. Moreover, P_1 is called an "in-point" or an "out-point" according as it is internal or external with respect to (1). Finally, a line such as P_1P_2 is described as "touched" when there are (two) real conics through P_3, P_4, P_5, P_6 touching it.

In this paper a number of results connecting the above qualifications of points and lines are given, with only some hints about proofs. The latter are of the continuity type used, e.g., by H. Simpson [Proc. Amer. Math. Soc. **12** (1961), 931; MR **24** #A2877]. An alternative, more immediate (but less elementary) way consists in referring to the real cubic surface represented by the ∞^3 linear system of plane cubics through $P_1, \cdots, P_6$, and applying some known results to it [cf. the reviewer, ***The non-singular cubic surfaces***, p. 48 (i), Oxford Univ. Press, Oxford, 1942; MR **4**, 254]. *B. Segre* (Rome)

Skopec, Z. A. 5368

A homology transformation of two second-order curves into two circles. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 2 (39), 139–143.

Let there be given two non-degenerate real curves of second order in the projectively extended euclidean plane, in homogeneous coordinates. It is required to find a central projection (homology) reducing them into two circles. There are certain obvious restrictions concerning the common points of the given conics. If two are conjugate imaginary, let p be the real line through these two points. Then there are two one-parameter families of homologies transforming the conics into circles, and if M_1, M_2 are the centres of these homologies, then they are symmetric to p and the intersection $\overline{M_1M_2} \cap p$ coincides with the intersection of p with the diameters of the given curves. *H. Schwerdtfeger* (Montreal, Que.)

Strommer, J. 5369

Konstruktionen allein mit dem Zirkel in der hyperbolischen Ebene.

J. Reine Angew. Math. **214/215** (1964), 192–200.

The well-known result of Euclides Danicus in euclidean geometry is synthetically shown to hold also in hyperbolic geometry. Observing that all straightedge and compass constructions can be reduced to the following three: To find the intersections of (1) two straight lines; (2) a straight line and a circle; (3) two circles, it is clear that for the construction with the compass alone only the first two problems have to be dealt with. The proof is prepared: (a) By solving ten basic problems: to find the symmetric point of a point P with respect to a given line; to construct the perpendicular straight line at the midpoint of a segment; to duplicate a given segment; problem (2) with the restriction that the line does not pass through the centre of the circle; to find an inner point of a given segment; to draw a parallel through a given point to a given line; etc. (b) By the introduction of the inversion with respect to a circle K about a point O. The definition is based on the notion of "Randbild" of a plane E' in the plane E and thus makes use of a construction in hyperbolic space [cf. H. Liebmann, Ber. Verh. Sächs. Ges. Wiss. Leipzig Math.-Phys. Kl. II **54** (1902), 244–260, in particular, pp. 250–260]. A circle F about the non-euclidean centre O of K touching a horocycle perpendicular to K is called "Fluchtkreis". The inversion is seen to have the usual properties with the following refinement: The image of a circle C is an ordinary circle, a horocycle, a hypercycle or a straight line if C avoids, touches, intersects or intersects orthogonally (respectively) the Fluchtkreis. (c) By giving a compass construction of the inverse point M' to a point M with respect to a circle K if M lies outside the Fluchtkreis. For two intersecting lines AB and CD it is now possible to determine a point A' so that the (unknown) intersection $AB \cap CD$ coincides with the midpoint of the segment AA'. All this leads up to the construction of the bisectrix of an angle, whence a solution of (2) in the restricted case is derived. Finally, also problem (1) is reduced to the previously treated questions. *H. Schwerdtfeger* (Montreal, Que.)

Jasinskaja, E. U. 5370

Metric invariants of equations of quadrics and pairs of planes in semi-non-Euclidean spaces. (Russian)

Trudy Sem. Vektor. Tenzor. Anal. **12** (1963), 315–337.

In a previous paper [Dokl. Akad. Nauk SSSR **137** (1961), 1327–1330; MR **23** #A3496] the author investigated general properties of distances between points, of angles between hyperplanes, of motions and of conformal transformations in semi-non-Euclidean spaces of general type. In the present paper he studies metric invariants of pairs of planes of any dimension and metric invariants of equations of quadrics in these spaces. These theories of metric invariants are the generalizations of those in Euclidean spaces (where metric invariants are lengths of general perpendiculars and stationary angles and so on) and in non-Euclidean spaces (where lengths of general perpendiculars are metric invariants and so on). *A. Kawaguchi* (Sapporo)

Dembowski, Peter 5371

Eine Kennzeichnung der endlichen affinen Räume.

Arch. Math. **15** (1964), 146–154.

Ein Blockplan ist ein Paar, bestehend aus einer endlichen Menge, deren Elemente Punkte genannt werden, und einer Menge von Punktmengen, den Blöcken, das die folgenden Bedingungen erfüllt: Alle Blöcke enthalten gleichviel Punkte; je zwei Punkte sind in derselben Anzahl von Blöcken gemeinsam enthalten; jeder Block hat mit der Komplementärmenge jedes anderen Blockes mindestens zwei Punkte gemeinsam. Zu je zwei Punkten P, Q wird als Gerade der Durchschnitt aller die Punkte P, Q enthaltenden Blöcke gebildet. Unter einer Parallelität in einem Blockplan sei eine Äquivalenzrelation in der Menge der Blöcke verstanden, bei der durch jeden Punkt zu jedem Block genau ein paralleler Block geht. Alle Äquivalenzklassen einer Parallelität enthalten dieselbe Anzahl von Blöcken. Ein Blockplan ist genau dann isomorph zum Blockplan der Punkte und Hyperebenen eines endlichen affinen Raumes, wenn er eine Parallelität besitzt, bei der jede Äquivalenzklasse ebenso viel Elemente enthält wie eine Gerade. Diese Bedingung für die

Parallelität erweist sich noch als gleichwertig mit der Aussage: Jede Gerade trifft jeden Block, zu dem es durch sie keinen parallelen Block gibt. *G. Pickert* (Giessen)

Shrikhande, S. S. 5372

A note on finite Euclidean plane over $GF(2^n)$.

J. Indian Statist. Assoc. **1** (1963), 48–49.

Qvist [Ann. Acad. Sci. Fenn. Ser A I Math.-Phys. No. 134 (1952); MR **14**, 1008] has shown that in a projective plane of even order n all the tangents of an $(n+1)$-arc (i.e., a set of $n+1$ points no three of which are collinear) pass through one point. In particular, this is true for the tangents of a conic in the plane over $F=GF(2^n)$. The author gives an alternative proof of this last fact, using the linear metrization by Archbold [Mathematika **7** (1960), 145–148; MR **23** #A558] of the affine plane over F.

P. Dembowski (Frankfurt a.M.)

Sandler, Reuben 5373

On subplanes of free planes.

Canad. J. Math. **16** (1964), 379–385.

Let π^k denote the free projective plane of rank $k+6$ (freely generated by k points on one line and two other points off that line), and let S_m^* be the set (ordered by inclusion) of all subplanes of rank $\leqq m$ [subplanes of free planes are free, cf. Kopeĭkina, Izv. Akad. Nauk SSSR Ser. Mat. **9** (1945), 495–526; MR **8**, 167] of π^k. The main result is that S_m satisfies the ascending chain condition. Also it is shown that if ρ is a subplane of rank n which is contained in no subplane of rank $<n$, then every descending chain of subplanes of rank n containing ρ is finite. The author remarks that, due to a result of the reviewer, these results hold also if k is infinite.

P. Dembowski (Frankfurt a.M.)

Roth, R. 5374

Flag-transitive planes of even order.

Proc. Amer. Math. Soc. **15** (1964), 485–490.

Supplementing some results of D. G. Higman and the reviewer [Illinois J. Math. **5** (1961), 382–397; MR **24** #A1069], the author shows that if G is a flag-transitive group of collineations on a finite projective plane and if the order, n, of the plane is such that either $n+1$ or n^2+n+1 is a prime, then either G is doubly transitive on the points of the plane or G is regular on the flags of the plane.

J. McLaughlin (Ann Arbor, Mich.)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 5040, 5042, 5836.

Kotzig, Anton 5375

On the theory of Euler polyhedra. (Russian. English summary)

Mat.-Fyz. Časopis Sloven. Akad. Vied **13** (1963), 20–31.

Let P be Euler's polyhedron and denote one of its vertices [edges] by v [h]. Consider the set V [H] of the faces of P incident with v [h] and let $\Delta(v)$ [$\sigma(h)$] stand for the sum of the numbers of the edges of P which are incident with the elements of V [H]. Euler's polyhedron P is said to be of the nth degree if each of its vertices is incident exactly with n edges ($n=3,4,5$). The author states several theorems concerning the values of the functions $\Delta(v)$ and $\sigma(h)$. For example: If P is of the third degree without triangles, one has, for an appropriate h, $\sigma(h)\leqq 11$. There are Euler polyhedra P of the third degree without triangles such that $\sigma(h)\geqq 11$ for each h of P. Given a positive integer M, there are Euler polyhedra P of the fourth degree with $\Delta(v)>M$ for every vertex v of P.

O. Borůvka (Brno)

Grünbaum, Branko 5376

Common secants for families of polyhedra.

Arch. Math. **15** (1964), 76–80.

Definitions: (1) A convex cone C with vertex at the origin is called the associated cone of a convex polyhedron $P\subset E^n$ with respect to the vertex v of P if $v+C$ is the union of all half-lines with endpoint v which contain at least one point of P different from v. A polyhedron P' is related to a polyhedron P provided each associated cone of P' is an intersection of associated cones of P. A family $\mathscr{P}$ of polyhedra is related to P if each member of $\mathscr{P}$ is related to P. (2) A family $\mathscr{P}$ of polyhedra in E^n is said to have property S if there exists a hyperplane intersecting every member of P; the family $\mathscr{P}$ is said to have property $S(k)$ if every k-membered subfamily of $\mathscr{P}$ has property S. With these definitions the author proves the following theorems: (1) For families $\mathscr{P}$ related to a centrally symmetric convex polyhedron $P\subset E^n$ with $2p$ vertices, $S(p(n+1))$ implies S. (2) For every positive integer k there exists a $t=t(k,n)$ such that for families $\mathscr{P}$ related to a convex polyhedron $P\subset E^n$ with k vertices, $S(t)$ implies S; moreover, we may take $t(k,n)\leqq\binom{k}{2}(n+1)$. (3) Let a centrally symmetric convex body $K\subset E^2$ have the property that $S(t)$ implies S for families of positive homothets of K, where t depends on K only; then K is a polygon.

L. A. Santaló (Buenos Aires)

Venkov, B. A.; Venkov, B. B. 5377

On the automorphisms of a convex cone. (Russian. English summary)

Vestnik Leningrad. Univ. **17** (1962), *no.* 7, 42–57.

Let K be an open convex cone with vertex 0 in Euclidean space R^n. An automorphism of K is a non-singular matrix S such that Sx and $S^{-1}x$ belong to K for each $x\in K$. A group G of automorphisms of the cone K satisfies the compactness condition if each element of G has determinant ± 1 and there exists a closed conical set K_1 (i.e., $tx\in K_1$ for $t\geqq 0$ and $x\in K_1$) with vertex 0 and contained in $K\cup\{0\}$ such that each point of K is equivalent, relative to G, to a point of K_1 (i.e., for each $x\in K$, there exists an $S\in G$ such that $Sx\in K_1$). The main results obtained are the following. If G is any group of automorphisms of K satisfying the compactness condition, then any boundary point of K is equivalent to a point with arbitrarily small coordinates. If K admits an abelian group (with finite index) of automorphisms satisfying the compactness condition, then K is a simplicial polyhedral angle (i.e., K is the set of all non-negative linear combinations of a fixed linearly independent set of n points).

J. G. Ceder (Goleta, Calif.)

Barthel, Woldemar; Bettinger, Werner 5378

Bemerkungen zum isoperimetrischen Problem.

Arch. Math. **14** (1963), 424–429.

For compact sets A, B in E^n define the exterior and interior Minkowski areas of A relative to B by

$$F_+(A, B) = \liminf_{\rho\to 0+} \rho^{-1}(|A+\rho B|-|A|),$$

$$F_-(A, B) = \liminf_{\rho\to 0+} \rho^{-1}(|A|-|A-\rho B|).$$

Let B^H be the convex hull of B, then for convex A with $|A|>0$, or when A is a proper polyhedron, $F_+(A, B^H)=F_+(A, B)=F_-(A, B)=F_-(A, B^H)$. This and the usual Minkowski inequalities yield the following generalization of the latter: Denote by A_0 the set of those points of A every neighborhood of which intersects A in a set of positive measure. If A_0 is convex or is a proper polyhedron, then $F_+(A, B)\geqq n|A|^{(n-1)/n}|B^H|^{1/n}$, $F_-(A, B)\geqq n|A|^{(n-1)/n}|B^H|^{1/n}$. In the first relation equality holds for $A=A_0$ and $|B^H|>0$ only when A is homothetic to B^H. (This result, for star-shaped B, goes back to the reviewer [Amer. J. Math. **71** (1949), 743–762; MR **11**, 200].) In the second relation equality holds for $|A|>0$ and $|B^H|>0$ only when A_0 is homothetic to B^H.

H. Busemann (Los Angeles, Calif.)

Imre, Margit 5379

Kreislagerungen auf Flächen konstanter Krümmung.

Acta Math. Acad. Sci. Hungar. **15** (1964), 115–121.

The author proves a theorem which is an extension to the elliptic and hyperbolic cases of a theorem by Fejes Tóth [*Lagerungen in der Ebene, auf der Kugel und im Raum*, III, § 8, Springer, Berlin, 1953; MR **15**, 248]. The theorem is as follows. Let V be a union of n surfaces of regular spherical, euclidean or hyperbolic triangular mosaics (Dreikantmosaiks), r and R being the radii of the inscribing and circumscribing circles. We consider n circles with radius ρ, where $r\leqq\rho\leqq R$. Then the measure of a covering of the n circles with respect to V takes its greatest value when the circles are each concentric with the mosaic surfaces. The proof is analogous to the one by Fejes Tóth, but devices are necessary in the details.

M. Kurita (Nagoya)

Molnár, József 5380

Collocazioni di cerchi sulla superficie di curvatura costante.

Celebrazioni Archimedee del Sec. XX (Siracusa, 1961), *Vol. I, Parte II, pp.* 61–72. *Edizioni "Oderisi", Gubbio*, 1962.

The paper studies the arrangement of circles on surfaces S_2 of constant curvature (in particular, the sphere and the euclidean and hyperbolic planes) in connection with the problem of determining the maximum radii of n circles whose interiors have, pairwise, no common points. The density δ of a family $\{C_i\}$ of circles of S_2 with respect to a domain of S_2 of area D is the sum of the areas $C_i\cap D$ divided by D (i.e., $\delta=(\sum C_i\cap D)/D$). Density relative to the euclidean or hyperbolic planes is defined by a limiting process. The theorems proved give upper bounds for δ, and figures are supplied showing the regular and (for certain values of n) irregular lattices formed by the centers of the circles in extreme cases.

L. M. Blumenthal (Columbia, Mo.)

Fejes Tóth, Gábor 5381

Über die Blockierungszahl einer Kreispackung.

Elem. Math. **19** (1964), 49–53.

Because of the small parking area available, the tenants of an apartment block must sometimes park their automobiles in such a way that some cars block others. In such a case, some tenants must leave their car keys with the parking lot attendant so that any owner can get his car unblocked. This practical problem motivates the author to consider the similar problem for packing of circles and convex regions in the plane. More precisely, for any positive integer n, let $B(n)$ (the blocking number) be the smallest integer such that in any packing of n unit circles, any circle of the packing can be unblocked by shifting (along straight lines) at most $B(n)$ of the other circles. The author exhibits a particular packing of unit circles (whose centres are the vertices of a half-regular tessellation (3, 3, 4, 3, 4)) which proves that $B(4)\geqq 1$, $B(6)\geqq 2$, $B(10)\geqq 3$, $B(13)\geqq 4$, $B(18)\geqq 5$, $B(23)\geqq 6$, $B(29)\geqq 7$.

Let R be any positive number, and n any positive integer. The author exhibits a configuration of n circles, each of radius $\leqq R$, with blocking number $[(n-2)/4]$. Finally, a simple argument (using results in a paper by L. Fejes Tóth and A. Heppes [Compositio Math. **15** (1963), 119–126; MR **28** #4435] is used to show that in any packing of n (not necessarily congruent) convex regions, the blocking number is $\leqq[(n-2)/2]$.

W. Moser (Winnipeg, Man.)

Gilbert, E. N. 5382

Randomly packed and solidly packed spheres.

Canad. J. Math. **16** (1964), 286–298.

The packings considered use spheres of different radii. In a packing of D-dimensional space, let $f(R)$ denote the volume fraction occupied by spheres of radius $\geqq R$; let $m(R)$ be the number of spheres per unit volume which have radius $\geqq R$. The main problem investigated is that of finding what kinds of functions $f(R)$ and $m(R)$ can be achieved by actual packings. The simplest method is to fill a container with large spheres, then use the largest spheres possible to fill up as much as one can of the remaining space, and so on. An IBM computer was used to extend the empirical results in the case of circles in the plane; this led to the approximate formulas $m(R)\approx 0.07R^{-1.3}$, $f(R)\approx 1-0.41R^{0.7}$, which the author compares to the bound

$$f(R) \leqq \{\pi p^2+2(1-p^2)\arcsin p(1+p)^{-1}\}/\{2p(1+2p)^{1/2}\}$$

(where $p=R/B$) conjectured by Fejes Tóth and Molnár [Math. Nachr. **18** (1958), 235–243; MR **20** #2669] and proved by Florian [Rend. Circ. Mat. Palermo (2) **9** (1960), 300–312; MR **27** #5171]. The author proves the slightly weaker theorem: When non-overlapping circles are packed in the plane, the circles of radius $\geqq R$ leave uncovered a fraction

$$1-f(R) \geqq (2\sqrt{3}-\pi)m(R)R^2 > 0.32251m(R)R^2$$

of the plane area.

In D-dimensional space, two random processes for packing unequal spheres are achieved as follows. Consider $(D+1)$-space with Cartesian co-ordinates $(x_1, x_2, \cdots, x_D, t)$ satisfying $|x_i|<\infty$ $(i=1, \cdots, D)$, $0\leqq t<\infty$. Select a random pattern of points from this space by a Poisson process of density α per unit $(D+1)$-dimensional volume.

Each selected point represents a sphere, with centre $(x_1, \cdots, x_D)$, which one tries to add to the packing at time t. The radius of the sphere to be tried at time t is the function $R(t)$. The first criterion rejects a trial sphere only if it overlaps a sphere which was added to the packing at an earlier time. The second criterion rejects a sphere if it overlaps another trial sphere with a smaller value of t; in particular, a sphere is rejected if it overlaps an earlier trial sphere which itself was rejected. Thus for a given Poisson pattern of points the second packing is only a subset of the first packing. The author investigates these two packings in great detail and compares his inequalities with those obtained empirically by G. D. Scott [Nature **188** (1960), 908–909], the known lattice packing in 2, 3, 4, 5 dimensions, and the upper bound of lattice packings obtained by H. F. Blichfeldt [Math. Ann. **101** (1929), 605–608]. *W. Moser* (Winnipeg, Man.)

DIFFERENTIAL GEOMETRY

See also 5146, 5147, 5379, 5447, 5784, 5792, 5797, 5798.

Goodman, A. W. 5383

A partial differential equation and parallel plane curves.

Amer. Math. Monthly **71** (1964), 257–264.

It is shown that if $\phi(x, y)$ is a solution of $\phi_x^2 + \phi_y^2 = 1$ in a given region R, then $z = \phi(x, y)$ is a ruled surface generated by a set of lines each of which makes an angle of $\pi/4$ with the xy-plane. If the region R is not too complicated, the intersection of these lines with the plane is a smooth curve C such that the projection of each ruling on the xy-plane is normal to C and $\phi(x, y)$ is the directed distance of (x, y) from C. These ideas lead the author to a technical definition of distance surface which is valid over any plane region R. He proves the theorem: Let ϕ_{xx}, ϕ_{xy}, ϕ_{yy} be continuous in R. Then $\phi(x, y)$ is a solution of $\phi_x^2 + \phi_y^2 = 1$ if and only if $z = \phi(x, y)$ is a distance surface. A related question deals with curves $C(q)$ lying in R and parallel to C at a distance q. The family $C(q)$ covers R in a simple manner if each curve is a simple curve and no two curves intersect. The author proves the result: Let m be the minimum value of the radius of curvature along C, and suppose that the length of C is less than $m\pi/2$. Then the region covered by the family $C(q)$ with $|q| < m$ is covered in a simple manner by that family.

A. Fialkow (Brooklyn, N.Y.)

Amur, K. S. 5384

On some invariants associated with a rectilinear congruence.

J. Indian Math. Soc. (*N.S.*) **27** (1963), 33–44.

Among the theorems proved we find one on rectilinear congruences for which the middle surface coincides with the focal surfaces. Then the focal surfaces meet the developables of the congruence in the asymptotic lines. The converse theorem also holds. A necessary and sufficient condition that a congruence be one of Guichard is that the geodesic torsion of a ray with respect to a focal surface be zero. The notation is based on that of Weatherburn [*Differential geometry of three dimensions*, Vol. II, Cambridge Univ. Press, London, 1930].

D. J. Struik (Utrecht)

Özbek, A. Riza 5385

Über Weingarten'sche Isothermflächen, auf denen die Kurven festen Krümmungsmasses eine zueinander geodätisch parallele Kurvenschar bilden. (Turkish summary)

İstanbul Tek. Üniv. Bül. **14** (1961), 31–44.

Die Kurven festen Krümmungsmasses K auf einer Fläche konstanter mittlerer Krümmung H sind dann und nur dann geodätisch parallel wenn die Fläche auf eine Rotationsfläche abwickelbar ist. Dafür ist auch notwendig und hinreichend dass diese Kurven mit den beiden Scharen von Minimallinien ein Sechseckgewebe bilden. Auch formen solche Kurven mit den Scharen von Krümmungslinien ein Sechseckgewebe, aber dieser Satz ist nicht umkehrbar. Es werden Flächen konstanter H angegeben, auf denen die Kurven $K = \text{const}$ mit den Krümmungslinien ein Sechseckgewebe bilden ohne zu einander parallel zu sein. Ein Hinweis über die Invarianten-bestimmung solcher Flächen wird gegeben. Der Fall $H = 0$ wurde vom Autor in dasselbe Bül. **13** (1960), 19–43 [MR **24** #A485] behandelt. *D. J. Struik* (Utrecht)

Ermolaeva, È. N. 5386

On the curvature of curves on a smooth surface at points where the second derivatives fail to exist. (Russian)

Ukrain. Mat. Ž. **16** (1964), 89–93.

When a curve passes through a point P of such a surface, we can define as its curvature k_ν the limit of $2d/h^\nu$, $1 < \nu \leqq 2$, where h and d are the distances from a point φ of the curve to the tangent line at P and to P, respectively [M. Ja. Vygodskiĭ, *Differential geometry* (Russian), GITTL, Moscow, 1949; MR **12**, 127]. Analogous to such a curvature of order ν can also be defined an osculating plane α of order ν, where h now signifies the distance of φ to α. The surface now considered is given by $z = (ax^2 + 2bxy + cy^2)^{\nu/2}$ $(1 < \nu \leqq 2)$, called a paraboloid of order ν. It is shown that for all curves passing through its summit P, having a common tangent and a common osculating plane of order ν, there exists at P a curvature of order ν. Between the curvatures of these curves there exists a relation analogous to that of Meusnier. The paper concludes with an example showing how a smooth surface having an osculating paraboloid of order ν can be obtained.

D. J. Struik (Utrecht)

Cypkin, M. E. 5387

An expression for the measures of curvature and torsion for an unusual ruled surface. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 144–152.

This paper deals with special ruled surfaces, in particular the surfaces formed by the principal normals or the binormals of a curve with constant torsion.

H. Busemann (Los Angeles, Calif.)

Diskant, V. I. 5388

On the minimal number of vertices of a knotted curve. (Russian)

Sibirsk. Mat. Ž. **5** (1964), 234–235.

A procedure is developed by which it is possible to construct knotted space curves having no more than two vertices

(here $dk/ds=0$ when k is the first curvature). An example is constructed by starting from a limaçon-like plane curve $\mathbf{p}(\sigma)$ with one loop composed of arcs of astroids and circles with the symmetry axis along $x=\sqrt{2}$ and tangent to the x-axis, taking the involute $\mathbf{r}_1(\sigma)$ of length L, taking the curve $\mathbf{r}_2$ symmetrical with respect to the axis Ox, and considering at last the curve $\mathbf{r}(s)=\mathbf{r}_1(\sigma)$ for $0\leq s\leq L$ and $=\mathbf{r}_2(\sigma_1)$ for $L\leq s\leq 2L$; here $\sigma=\sigma(s)$ where s is the arc length of $\mathbf{r}(s)$, and $\sigma_1=\sigma(2L-s)$. Then the curve $\mathbf{R}(s)=\mathbf{r}(s)+z(s)\mathbf{k}$, where $z(s)=(s-a)^4(s-b)^4$ when s lies within the segments which build up the curve $\mathbf{R}$, and $z=0$ outside of them. The vector $\mathbf{k}$, not defined, seems to lie along the z-axis. *D. J. Struik* (Utrecht)

Lane, N. D.; Singh, K. D.; Scherk, P. 5389

Monotony of the osculating circles of arcs of cyclic order three.

Canad. Math. Bull. **7** (1964), 265–271.

It is known that if a differentiable curve has a monotone increasing curvature, then the circles of curvature at distinct points have no points in common. The authors give conformal proofs of two related results concerning an arc A_3 of cyclic order three; that is, one which no circle meets more than three times: (1) Two general osculating circles at distinct points of A_3 have no points in common. (2) All but a countable number of points of A_3 are strongly conformally differentiable. The proofs depend upon theorems derived by Lane and Scherk in earlier papers [Canad. J. Math. **5** (1953), 512–518; MR **15**, 251; Trans. Amer. Math. Soc. **81** (1956), 358–378; MR **18**, 64], which also contain definitions of the terms used here. *A. Fialkow* (Brooklyn, N.Y.)

Nitsche, Johannes C. C. 5390

A necessary criterion for the existence of certain minimal surfaces.

J. Math. Mech. **13** (1964), 659–666.

The following problem is considered. Let Γ_1, Γ_2 be Jordan curves lying in parallel planes in E^3. When does there exist a doubly connected minimal surface bounded by these curves? Intuitively one would expect no such surface to exist if Γ_1 were fixed and if Γ_2 were moved too far away, either laterally or vertically. A theorem is proved which confirms this, and gives a quantitative expression for the relative magnitudes. *R. Osserman* (Stanford, Calif.)

Godeaux, Lucien 5391

Surfaces associées à une suite de Laplace terminée. II.

Bull. Soc. Roy. Sci. Liège **32** (1963), 811–813.

A continuation of the note which appeared in same Bull. **32** (1963), 597–601 [MR **28** #1529].

Ivlev, E. T. 5392

On a pair of ruled surfaces in three-dimensional projective space. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 3–10.

The author constructs the canonical frame of an arbitrary pair of ruled surfaces in three-dimensional projective space. He gives the geometrical characteristics of the elements of the frame, and explains the geometrical sense of the invariants defining the pair of ruled surfaces. The author also considers special classes of pairs of ruled surfaces and gives some other applications of the canonical frame. The author uses S. P. Finikov's notations and terminology. *W. Wrona* (Warsaw)

Ivlev, E. T. 5393

The canonical frame of a parabolic pair of complexes in P_3. (Russian)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. **161** (1962), 42–53.

The author constructs the canonical frame of a parabolic pair of complexes in three-dimensional projective space. He gives the geometrical characteristics of the elements of the frame and explains the geometrical sense of the corresponding invariants. Then he examines some special classes of parabolic pairs of complexes. The author uses S. P. Finikov's notations and terminology. *W. Wrona* (Warsaw)

Kovancov, N. I. 5394

The theory of complexes in a bi-axial space. (Russian)

Izv. Vysš. Učebn. Zaved. Matematika **1964**, *no.* 1 (38), 56–68.

The author dwells mainly on the theory of complexes in bi-axial spaces of hyperbolic and elliptic type, i.e., on spaces with real or conjugate-imaginary basic straight lines. He considers the first differential neighbourhood of the ray. The notion of normal correlation on the ray appears to be fundamental. This correlation is defined as the projective correspondence between the points of the ray and the planes tangent to the cones of the complex with the vertices in these points. The consideration of the normal correlation makes it possible to introduce many notions generalizing the corresponding notions of Euclidean geometry. *W. Wrona* (Warsaw)

Verbickiĭ, L. L. 5395

Three-dimensional hypersurfaces realizing a conformally Euclidean metric. (Russian)

Trudy Sem. Vektor. Tenzor. Anal. **12** (1963), 339–354.

When developing the analytic method used by A. Finzi [Atti Ist. Veneto Sci. Lett. Arti (8) **5** (**62**) (1902/03), 1049–1062], the author finds a geometric construction of special line varieties V_3 (exactly one principal curvature of which is identically zero) lying in Euclidean space E_4 and realizing a conformally Euclidean metric.

{Note: The result may be regarded as a partial constructive supplement of the well-known existence theorem for V_n $(n>2)$ to be conformally Euclidean [cf. J. A. Schouten, *Ricci-calculus*, 2nd ed., p. 309, Springer, Berlin, 1954; MR **16**, 521].} *A. Urban* (Prague)

de Barros, Constantino M. 5396

Espaces infinitésimaux; théorie générale.

C. R. Acad. Sci. Paris **258** (1964), 3624–3627.

This paper contains an abstract algebraic system generalizing many differential geometric theories, e.g., vector fields on differentiable manifolds and Lie derivatives defined by them on various modules, differential forms, connections, etc. For K and F a commutative ring and

a commutative K-algebra, respectively, a (K, F)-linear structure on a set X is defined by a K-module structure on X and a K-homomorphism π_L of F into $\mathrm{End}_K(X)$. A (K, F)-Cartan space on X is, in addition, a K-Lie algebra structure on X and a map $\pi_C: X \to \mathrm{End}_K(F)$ satisfying certain axioms. An example would be where X is the Lie algebra of vector fields on a manifold M, K denotes the real numbers, F is the algebra of real smooth functions on M. A pseudo-infinitesimal space V with respect to X is given by a (K, F)-linear structure on V and a K-linear function $\pi_I: X \to \mathrm{End}_K(V)$ such that if $\pi_I(X)v = Xv$, then $X(fv) = (Xf)v + f(Xv)$, where $f \in F$. For example, V could be the p-forms on M, $\pi_I(X)$ the Lie derivative. An exterior derivative and curvature are defined for any X-infinitesimal space. *H. H. Johnson* (Seattle, Wash.)

Flanders, Harley 5397

★Differential forms with applications to the physical sciences.

Academic Press, New York-London, 1963. xiii + 203 *pp.* \$7.50.

The author states in the preface that, though the book is aimed primarily at engineers and physical scientists, it could also serve as an introduction to modern differential geometry for graduate mathematics students. The reviewer feels that the author has accomplished both of these purposes remarkably well. The book tends to encourage in the reader an intuitive understanding for the subject of differential forms by the use of numerous examples and illustrations from geometry and the physical sciences. Furthermore, he has presented the theoretical material with sufficient precision and rigour to make it highly useful as an introductory book for mathematics students.

If one were to use the book for a text on differential geometry one would include Chapters I (Introduction), II (Exterior algebra), III (The exterior derivative), V (Manifolds and integration), VI (Applications in Euclidean space), VIII (Applications to differential geometry), IX (Applications to group theory). Two additional chapters, VII (Applications to differential equations) and X (Applications to physics), which can be read after Chapters II, III, and V, may be substituted for certain others in a course given for science and engineering students.

In the chapter on manifolds and integration a quick introduction is made to tangential spaces, differential forms, Euclidean simplices, chains and boundaries. Stokes's theorem is proved by a specific computation over a $(p+1)$-simplex using the definition for exterior differentiation.

De Rham's first and second theorems are presented with some discussion of their meanings; however, a proof is not given. In the chapter on differential geometry much of the material usually covered in a classical course in differential geometry or tensor calculus is introduced in the notation of differential forms. For instance, topics such as parallel displacement, Gaussian curvature, geodesics, the principal curvatures, the Riemann curvature tensor, Christoffel symbols, the Bianchi identity, torsion, and affine connections are discussed. A short discussion of harmonic integrals is also provided.

The chapter on differential equations begins with harmonic functions of n variables and a derivation of the various Green's formulae using differential forms. From these the author also obtains Gauss's mean-value theorem, the maximum (minimum) principle, and Liouville's theorem. The heat equation, Frobenius's integration theorem, systems of differential equations, and Lie's third theorem are also discussed.

In the last chapter on applications to physics the author uses the notations of differential forms to discuss the Hamiltonian theory, relative integral invariants, Poisson and Lagrange brackets, and contact transformations. Some applications are also given to integration over r-chains transversal to the trajectories; for instance, a proof of the Liouville theorem concerning integration over regions in phase space, and some remarks concerning fluid dynamics. *R. P. Gilbert* (College Park, Md.)

Okumura, Masafumi 5398

Certain almost contact hypersurfaces in Euclidean spaces.

Kōdai Math. Sem. Rep. **16** (1964), 44–54.

The main result of the paper states that only E^{2n-1}, S^{2n-1} and $E^r \times S^{2n-r-1}$ can carry a normal (ϕ, ξ, η) structure induced by the euclidean structure of E^{2n} (for definitions, see S. Sasaki and Y. Hatakeyama [Tôhoku Math. J. (2) **13** (1961), 281–294; MR **25** #1513]). On the way, the author develops the tensor analysis of these structures quite a bit further.

H. W. Guggenheimer (Minneapolis, Minn.)

Lehmann-Lejeune, Josiane 5399

Sur l'intégrabilité de certaines G-structures.

C. R. Acad. Sci. Paris **258** (1964), 5326–5329.

It is shown that a nilpotent transformation field on a differentiable manifold with constant canonical form $\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$, I an identity matrix, is integrable if the structure tensor vanishes. *R. J. Crittenden* (Providence, R.I.)

Solodovnikov, A. S. 5400

Geometric description of all possible representations of a Riemannian metric in Levi-Civita form. (Russian)

Trudy Sem. Vektor. Tenzor. Anal. **12** (1963), 131–173.

In a recent paper [Dokl. Akad. Nauk SSSR **141** (1961), 322–325; MR **26** #714] the author formulated the problem and gave the effective geometric construction of all possible representations of Riemannian metric in Levi-Civita form, i.e., in the form

$$(*) \qquad ds^2 = \sum_{\alpha=1}^{p} \prod\nolimits_\beta' |f_\beta - f_\alpha| \, ds_\alpha^2$$

for the most interesting case $K = \mathrm{const} < 0$ only. ($p > 1$ denotes the number of groups (x^{i_α}), $\alpha = 1, \cdots, p$, of the coordinates x^i, $i = 1, \cdots, n$; ds^2 is a positive definite metric form depending on x^i only; f_α is a function of x^{i_α} if the group (x^{i_α}) consists of one coordinate only, $f_\alpha = \mathrm{const}$ in the other cases; $\prod_\beta' |f_\beta - f_\alpha| = \prod_{\beta=1,\beta\neq\alpha}^{p} |f_\beta - f_\alpha|$; K is the curvature of the form $d\bar{s}^2 = \sum_{\alpha=1}^{p} \prod_\beta' |f_\beta - f_\alpha| \, dy_\alpha^2$ associated with the form (*).)

The present paper gives a complete solution of the problem for $K = \mathrm{const}$. All the coordinate systems for which ds^2 has the form (*) are constructed geometrically.

A. Urban (Prague)

Kručkovič, G. I. 5401

On a class of Riemannian spaces. (Russian)

Trudy Sem. Vektor. Tenzor. Anal. **11** (1961), 103–128; *erratum, ibid.* **12** (1963), 95.

A class of Riemannian spaces, called semi-reducible, is considered. These are Riemannian spaces whose metrics, in appropriate coordinates, can be expressed in the form $ds^2 = ds_0{}^2 + \sigma(x^1, \cdots, x^q)ds_1{}^2$, where $ds_0{}^2 = g_{ij}(x^k)dx^i dx^j$ $(i, j, k = 1, 2, \cdots, q)$ is called the principal part and $ds_1{}^2 = g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta$ $(\alpha, \beta, \gamma = q+1, \cdots, n)$ the additional part. This class includes many important known Riemannian spaces.

The author gives necessary and sufficient conditions for a Riemannian space to be semi-reducible. The conditions are given in a geometrical (Theorem 1) and in a tensorial (Theorem 2) form. The latter shows the existence of a symmetric tensor A_{ab} not proportional to g_{ab} and a gradient $u_{,a}$ satisfying the conditions

$$A_{ab,c} = -\tfrac{1}{2}(u_{,a}A_{bc} + u_{,b}A_{ac})$$

and $A_{ac}A_b{}^c = A_{ab}$.

Then the author investigates some special kinds of semi-reducible spaces. It is shown that the semi-reducible spaces with a one-dimensional principal part of the metrics coincide with those for which a non-constant solution of the equation $f_{,ab} = \varphi g_{ab}$ exists for some function φ. For $V_0(K)$ spaces, i.e., spaces with metrics of form $ds^2 = ds_0{}^2 + \sigma ds_1{}^2$ such that the "adjoint" metric $d\overset{*}{s}{}^2 = ds_0{}^2 + \sigma dy^2$ is of constant curvature K, and only for those spaces, the equation $f_{,ab} = -(Kf+L)g_{ab}$ (K, L constant) has a non-constant solution. The number of functionally independent solutions coincides with the dimension of the metrics $ds_0{}^2$ (Theorem 4). The principal part of a semi-reducible space is $(n-1)$-dimensional if and only if there exists a non-isotropic vector field A_a and a gradient $u_{,a}$ such that $A_{a,b} = \tfrac{1}{2}(A_a u_{,b} - A_b u_{,a})$ (Theorem 5).

In the last section the author restricts himself to spaces with positive definite metrics and investigates the question of uniqueness of the representation of the metrics in semi-reducible form. The decomposition $ds^2 = ds_0{}^2 + \sigma ds_1{}^2$ is unique under the assumption that the principal part $ds_0{}^2$ is not semi-reducible.

Every other decomposition arises from here by decomposing the additional metrics (Theorem 6). A decomposition $ds^2 = ds_0{}^2(x^i) + \sigma_1(x^i)ds_1{}^2 + \cdots + \sigma_t(x^i)ds_t{}^2$ with non-semi-reducible $ds_0{}^2$ of dimension > 1 and non-constant ratios σ_p/σ_r is also unique up to the order of the parts $ds_1{}^2, \cdots, ds_t{}^2$, a change of coordinates within any of the parts and a substitution of $c\sigma_p$ instead of σ_p and $(1/c)ds_p{}^2$ instead of $ds_p{}^2$ (Theorem 7). *A. Goetz* (Wrocław)

Tabata, Morio 5402

Conformal transformations of Riemannian spaces. (Japanese)

Sûgaku **14** (1962/63), 152–164.

An expository paper on recent results on the subject of the title. In particular, it explains the background for the following conjecture: Let M be a compact Riemannian manifold of dimension $\geqq 3$ with constant scalar curvature $k > 0$. If M admits an infinitesimal conformal transformation which is not an infinitesimal isometry, then is M isometric to the sphere of radius $\sqrt{k}$? *K. Nomizu* (Providence, R.I.)

Milnor, J. 5403

Eigenvalues of the Laplace operator on certain manifolds.

Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 542.

In this note it is shown that there exist two Riemannian flat tori of dimension 16 which are not globally isometric (though affinely equivalent), and yet whose Laplacian for exterior forms has the same sequence of eigenvalues. Their defining lattices in R^{16} were discovered by Witt. *J. Eells* (Cambridge, England)

Avez, André 5404

Espaces harmoniques compacts.

C. R. Acad. Sci. Paris **258** (1964), 2727–2729.

The author proves a conjecture of Lichnerowicz that a compact completely harmonic riemannian manifold V_n (positive definite metric) is locally symmetric.

The proof makes use of results of Allamigeon [same C. R. **252** (1961), 1093–1095; MR **24** #A2348] that, if $x_0 \in V_n$, the geodesics which issue from x_0 are all closed and all have the same length $2L$; moreover, all geodesic arcs of length less than L contain no conjugate points. It follows that the symmetry map $S_{x_0}: x \rightsquigarrow y$, where $\exp_{x_0}{}^{-1} x = -\exp_{x_0}{}^{-1} y$, is globally defined over V_n.

The map S_{x_0} induces a new metric $\bar{g}$ from the given harmonic metric g by means of the relation $\bar{g}(x) = g(S_y x)$. Let Δ, $\bar{\Delta}$ denote the Laplacian operators corresponding to g, $\bar{g}$. Then it is proved that $(\bar{\Delta} - \Delta)$ annihilates all eigenfunctions of the elliptic operator Δ. A result of Kolmogoroff [Math. Ann. **108** (1933), 149–160] shows that this implies that $(\bar{\Delta} - \Delta)$ annihilates all C^2-functions defined over V_n, and this leads to the result $g = \bar{g}$.

{Reviewer's note: It would be extremely interesting if one could prove the result "$(\bar{\Delta} - \Delta)$ annihilates all C^2-functions over V_n" without making use of the assumption of compactness. This would imply that a locally harmonic riemannian manifold with positive definite metric is locally symmetric, which was the original form of Lichnerowicz's conjecture.} *T. J. Willmore* (Liverpool)

Wolf, Joseph A. 5405

Curvature in nilpotent Lie groups.

Proc. Amer. Math. Soc. **15** (1964), 271–274.

The author proves the following theorem: Let M be a Riemannian manifold admitting a transitive, connected, noncommutative nilpotent Lie group of isometries. If $m \in M$, there exist two-dimensional subspaces S, T of the tangent space M_m such that the sectional curvatures satisfy $K(S) < 0 < K(T)$. *S. Helgason* (Cambridge, Mass.)

Hermann, Robert 5406

An incomplete compact homogeneous Lorentz metric.

J. Math. Mech. **13** (1964), 497–501.

The author shows that there exists a compact, homogeneous Lorentzian manifold M which is incomplete in the sense that the geodesics on M are not in general indefinitely extendable. The example is the orthogonal group SO(3) with a suitable left-invariant Lorentzian structure. *S. Helgason* (Cambridge, Mass.)

Takamatsu, Kichiro 5407

On a decomposition of an almost-analytic vector in a K-space with constant scalar curvature.

Tôhoku Math. J. (2) **16** (1964), 72–80.

Let M be an almost Hermitian manifold with complex structure F_i^h and Hermitian metric g_{ji}. If these satisfy (i) $\nabla_j F_i^h = 0$, (ii) $\nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0$, and (iii) $\nabla_j F_i^h + \nabla_i F_j^h = 0$, where ∇_j denotes the covariant differentiation with respect to g_{ji}, M is called (i) a Kähler, (ii) an almost Kähler, (iii) an almost Tachibana manifold, respectively.

If a vector field v^h in an [almost] complex manifold satisfies $\underset{v}{£} F_i^h = 0$, v^h is called a contravariant [almost] analytic vector field, where $\underset{v}{£}$ denotes the Lie differentiation with respect to v^h.

Y. Matsushima [Nagoya Math. J. **11** (1957), 145–150; MR **20** #995] proved the theorem that in a compact Kähler-Einstein space, any contravariant analytic vector field v^h is uniquely decomposed in the form

$$v^h = p^h + F_i^h q^i,$$

where p^h and q^h are both Killing vector fields.

A. Lichnerowicz [*Géométrie des groupes de transformations*, Dunod, Paris, 1958; MR **23** #A1329] generalised this result to the case of a compact Kähler manifold with constant scalar curvature. S. Sawaki [Math. Ann. **146** (1962), 279–286; MR **25** #538] generalised the same result to the case of a compact almost Tachibana-Einstein manifold.

In the paper under review the author generalises the same result to the case of a compact almost Tachibana manifold with constant scalar curvature.

K. Yano (Tokyo)

Adler, Alfred W. 5408

Classifying spaces for Kähler metrics. II. Universal Laplacians and a theorem of Chern.

Math. Ann. **154** (1964), 257–266.

In a previous paper [same Ann. **152** (1963), 164–184; MR **28** #1569] the author constructed a real-valued 2-form Ω on a riemannian manifold $B^+_{U(n)}$ which induces in a natural way the fundamental forms of all Kähler metrics on all Kähler manifolds of real dimension $2n$. He now exhibits a differential operator on $B^+_{U(n)}$ which induces the Laplace operator on all Kähler manifolds of real dimension $2n$. All the results apply more generally to the G-manifolds defined by Chern [*Algebraic geometry and topology*, pp. 103–121, Princeton Univ. Press, Princeton, N.J., 1957; MR **19**, 314]. Here G is a closed subgroup of $O(d)$, and the Kähler geometry corresponds to the case $d=2n$, $G=U(n)$. As an application the author presents a new proof of a theorem of Chern about the Laplace operator on a G-manifold. *R. L. E. Schwarzenberger* (Liverpool)

Moalla, Fatma 5409

Espaces de Finsler complets à courbure de Ricci positive.

C. R. Acad. Sci. Paris **258** (1964), 2734–2737.

Following the theory of Finsler spaces as developed by É. Cartan [*Les espaces métriques fondés sur la notion d'aire*, Actualités Sci. Indust., No. 79, Hermann, Paris, 1934], the author generalises to these spaces theorems previously obtained by Myers [Duke Math. J. **8** (1941), 401–404; MR **3**, 18] for Riemannian manifolds. In particular, the following results are proved. Let F be an n-dimensional Finsler space for which the Ricci curvature at every point and in every direction is not less than the constant value $e^2 > 0$. Then there exists no minimal geodesic arc of length $\geq \pi(n-1)^{1/2}e^{-1}$. In particular, if F is complete, then F is closed and has diameter $\leq \pi(n-1)^{1/2}e^{-1}$.

T. J. Willmore (Liverpool)

Losik, M. V. 5410

Kawaguchi spaces associated with Klein spaces. (Russian)

Trudy Sem. Vektor. Tenzor. Anal. **12** (1963), 213–237.

The theory of Kawaguchi spaces, which are a natural generalization of Finsler spaces as well as of Klein spaces, was created by the reviewer [Rend. Circolo Mat. Palermo **56** (1932), 245–276; Proc. Imp. Acad. Japan **13** (1937), 237–240]. A Klein space means an n-dimensional differentiable manifold with a given finite r-parametric Lie group (transitive or intransitive) of differentiable transformations which define automorphisms of the space. It is well known that in a Klein space we may introduce the invariant measuring of arc length of curves by various methods (i.e., by adopting one of the integral invariants), so that this Klein space turns into a Kawaguchi space. Such a Klein space defines an integral class of Kawaguchi spaces in exactly the same way as a Euclidean space considered as metric space defines an integral class of Riemannian spaces. The main purpose of the present paper is to find the similar connection between such classes of Kawaguchi spaces and Klein spaces as that between those of Finsler spaces, particularly their important special case—Riemannian spaces, and Euclidean spaces, i.e., to attempt to associate certain classes of Kawaguchi spaces with Klein spaces, in exactly the same way as one associates Riemannian spaces with Euclidean spaces. In this paper it is shown that in every Klein space and in any space with curvature corresponding to it we can construct an invariant Kawaguchi metric which completely characterizes this space. There are shown also the simplest characteristic Kawaguchi metrics and the corresponding classes of Kawaguchi spaces connected with equi-affine, symplectic and affine spaces, and there is a discussion of the problem on the characteristic of such Kawaguchi spaces possessing a maximal group of motions. Some of these results were published earlier in a paper by the present author [Dokl. Akad. Nauk SSSR **134** (1960), 1299–1302; MR **23** #A2855]. In conclusion, there is introduced the concept of bending of surfaces in Kawaguchi spaces and there are obtained some results concerning bending of surfaces in a symplectic space.

A. Kawaguchi (Sapporo)

Nickerson, H. K. 5411

On differential operators and connections.

Trans. Amer. Math. Soc. **99** (1961), 509–539.

In this paper the author defines derivations on the differential forms with values in a vector bundle and classifies them. Let B be a vector bundle over a manifold M and ${}^sB = B \times \cdots \times B$, the s-times tensor product. (In particular, ${}^0B$ is the trivial line bundle.) Let ${}^sS^p$ be the sheaf of germs of p-forms on M with values in ${}^sB$. Then $\mathscr{S} = \{{}^sS^p\}$ is a doubly graded algebra in an obvious manner. A derivation of $\mathscr{S}$ of degree q is a linear mapping D such that (1) $D({}^sS^p) \subset {}^sS^{p+q}$ and (2) $D(\Phi\Psi) = (D\Phi)\Psi + (-1)^{pq}\Phi(D\Psi)$ for $\Phi \in {}^*S^p$, $\Psi \in {}^*S^*$. Let $\mathscr{D}^q$ be the sheaf of germs of derivations of $\mathscr{S}$ of degree q. Obviously, $\mathscr{D}^q$ forms a sheaf of graded Lie algebras. Let B^{q+1} denote

the sheaf of germs of $(q+1)$-forms with values in the tangent bundle $T(M)$. Let P be the principal bundle associated with B with structure group G. Let Σ^q denote the sheaf of germs of q-forms with values in the vector bundle $T(P)/G$ over M. The main result states that $\mathscr{D}^q$ is isomorphic with $\Sigma^q + B^{q+1}$, the isomorphism $(\sigma, W) \in \Sigma^q + B^{q+1} \rightarrow D \in \mathscr{D}^q$ being given by $D\Phi = \sigma\Phi + \sigma \barwedge W$, where the action of σ on Φ is induced by the natural mapping $P \times F \rightarrow B$ (F is the fibre of B) and $\barwedge$ is the tensor product followed by an obvious contraction.

There have appeared a few papers dealing with similar problems since the appearance of this paper, whose review here was unfortunately delayed; see, in particular, C. J. Henrich [Trans. Amer. Math. Soc. **109** (1963), 411–419; MR **27** #5195].

S. Kobayashi (Berkeley, Calif.)

GENERAL TOPOLOGY

See also 5005, 5017, 5114, 5157, 5188, 5320, 5411, 5841.

Duda, R. 5412

On biconnected sets with dispersion points.

Rozprawy Mat. **37** (1964), 60 *pp.*

This memoir is composed of four chapters. While the first chapter is given over to definitions and simple lemmas (whose proofs in many cases could have been left to the reader), the notion of a "relative quasicomponent" of a point set seems to be new and useful. Suppose that p is a point of the closure of a subset A of a topological space X. By the p-quasicomponent of A relative to X is meant the set of all points x such that x belongs to every closed subset of X which contains p and whose common part with $p+A$ is relatively open. This point set is denoted by $Qc_p(A, X)$. Whenever X is compact, $Qc_p(A, X)$ is a continuum. In general, one observes that the p-quasicomponent of X relative to X is the p-quasicomponent of X. Relative quasicomponents are used in the proof of the following theorem: Suppose that A is a subset of X (the Cartesian product of the Cantor set and the closed unit interval) such that if the open set U contains a point of A then there are $\mathfrak{c}$ components T of X such that $T \cap U \cap A$ is of power $\mathfrak{c}$; if $\dim A = 1$ except on a nowhere dense subset, then A contains $\mathfrak{c}$ quasicomponents each of power $\mathfrak{c}$. {The author poses the problem: Can the last hypothesis be weakened to "$\dim A = 1$"? The answer to this is "no". Let A consist of all points of some one component of X and the "irrational" points of all others.}

Chapters II and III contain several theorems dealing with connected (usually separable and metric) spaces P such that for some point a, $P-a$ is totally disconnected (i.e., a is a dispersion point of P). The author shows that if M is a finite subset of $P-a$, then $P-M$ is connected. There are several theorems dealing with the number (and the dimension) of quasicomponents in $P-a$. It is of considerable interest that if S is a connected space, there exists a space P (as above) such that S is the continuous image of P.

The main results are the examples contained in Chapter IV. It is true that there exist spaces P such that every quasicomponent of $P-a$ is degenerate and it is almost true [Corollary 9.4] that every space P contains such a subspace. Nevertheless, the author constructs (using the continuum hypothesis) a space P such that any subset of $P-a$ whose homeomorphic image (even in some other space) can be made connected by the addition of a single point must contain $\mathfrak{c}$ quasicomponents of power $\mathfrak{c}$. This is Example 3. On the other hand, there are spaces P such that $P-a$ has only countably many distinct quasicomponents and (1) [Example 2] each of them is nowhere dense in P or (2) [Example 1] none of them is nowhere dense in P. Example 4 is a constructive justification for the theorem (in Chapter III) which states (among other things) that if n is a positive integer, there exists a space P of dimension n which is not the union of $\aleph_0$ disjoint closed subsets.

F. Burton Jones (Riverside, Calif.)

Levine, Norman 5413

Inherited properties of functions in quotient spaces.

Amer. Math. Monthly **70** (1963), 969–973.

Let X and X^* be topological spaces with equivalence relations R and R^*, respectively. Let $f\colon X \rightarrow X^*$ have the property that $x\,R\,y$ implies $f(x)\,R^*f(y)$. Then f induces a mapping f^* of the quotient space X/R into X^*/R^*. If, in addition, $x\,R'\,y$ implies $f(x)\,R^{*\prime}f(y)$, where R' and $R^{*\prime}$ denote the negations of R and R^*, f is termed 1-1(R, R^*). Question: Under what conditions is a specified property of f inherited by f^*? Sample results: (1) "continuity" is always inherited, but not "sequential continuity"; (2) "homeomorphism" is inherited if and only if f is 1-1(R, R^*). In each of the following cases the property in question is inherited under the condition stated, but not in general: (3) "open (closed)" provided f is 1-1(R, R^*) and onto; (4) "compact preserving" provided X is compact and X/R is Hausdorff. Other properties considered are "connected preserving" and "separation preserving".

J. C. Oxtoby (Bryn Mawr, Pa.)

Frolík, Zdeněk 5414

On the descriptive theory of sets. (Russian summary)

Czechoslovak Math. J. **13** (88) (1963), 335–359.

A space, according to Choquet, is said to be analytic if it is completely regular and is a continuous image of some $K_{\sigma\delta}$ subset of some space. The concepts of an analytical structure (i.e., "a determining system with a nucleus, satisfying a certain condition of completeness") and a complete sequence of countable coverings are introduced and studied in some detail. A sequence of coverings $\{\mathscr{M}_n\}$ of P is complete if $\bigcap \overline{\mathscr{B}} \neq \emptyset$ whenever $\mathscr{B}$ is any family of subsets of P having the finite intersection property and such that for each n, some $B \in \mathscr{B}$ is contained in an $M \in \mathscr{M}_n$. A Hausdorff (regular) space admitting such a complete sequence of (closed) coverings is called a B-space (a space from E). Among other things, it is proved that a completely regular space belongs to E if and only if it is an $F_{\sigma\delta}$ in its Stone-Čech compactification. From this it follows that analytic spaces are exactly those completely regular spaces which are continuous images of B-spaces. It is shown that B-spaces have many properties similar to those of Borel subsets of separable, complete metrizable spaces. Moreover, analytic spaces are characterized as those completely regular spaces admitting an analytical structure. And regular spaces which are continuous images of the irrationals (i.e., spaces "analytic in

the classical sense" or "les espaces de Souslin") are characterized in terms of an analytical structure.

J. G. Ceder (Goleta, Calif.)

Frolík, Zdeněk 5415

On bianalytic spaces. (Russian summary)

Czechoslovak Math. J. **13** (88) (1963), 561–573.

Continuing his previous work [#5414] the author introduces and studies the notion of a bi-analytic set. A space X is bi-analytic if both X and $\beta(X)-X$ are analytic. For a family $\mathscr{M}$, let $\mathscr{B}(\mathscr{M})$ denote the smallest family containing $\mathscr{M}$ which is closed under countable intersections and countable unions. For a set X, let $F(X)$ and $Z(X)$ denote the closed subsets and the zero-sets of X, respectively. Among the main results obtained are the following: A space is bi-analytic if and only if $X \in \mathscr{B}(Z(Y))$, where Y is any space for which $\tilde{X}=Y$. A metrizable space X is bi-analytic if and only if X is separable and $X \in \mathscr{B}(F(Y))$ for any separable metric space Y containing X. A metrizable space is bi-analytic if and only if there exists a complete sequence $\{\mathscr{M}_n\}$ of countable disjoint coverings of X such that all the sets from $\bigcup_{n=1}^{\infty} \mathscr{M}_n$ are analytic.

J. G. Ceder (Goleta, Calif.)

Chen, Chuan-Chong 5416

A brief survey of the Tychonoff topology with a detailed proof of the Tychonoff theorem.

Bull. Math. Soc. Nanyang Univ. **1963**, 38–46.

The proof of Tychonoff's theorem given in this note is essentially the Bourbaki proof.

O. Wyler (Albuquerque, N.M.)

Jongmans, F. 5417

Incidents de frontière.

Bull. Soc. Roy. Sci. Liège **32** (1963), 814–822.

The author obtains some formulas involving unions, intersections, interiors, closures, and frontiers.

E. Michael (Seattle, Wash.)

Hayashi, Eiichi 5418

On λ-topology. (Japanese)

Sûgaku **14** (1962/63), 167–168.

Continuing a previous paper [Sûgaku **11** (1959), 99–100], the author studies the λ-topology of a given topological space R. For example, R is λ-Hausdorff if and only if R is Hausdorff. If R is λ-regular (λ-normal), then R is regular (or normal), but not vice versa.

K. Nomizu (Providence, R.I.)

Tamano, Hisahiro 5419

Note on paracompactness.

J. Math. Kyoto Univ. **3** (1963), 137–143.

The author introduces the notion of a "linearly locally finite" family of subsets of a space; this is an indexed family $\{S_\alpha | \alpha \in A\}$ with a linearly ordered index set A such that, for each $\alpha \in A$, the subfamily $\{S_\lambda | \lambda \leqq \alpha\}$ is locally finite. Thus every σ-locally finite family is linearly locally finite (but not conversely). All the spaces considered are to be completely regular and T_1. Theorems: The union of a linearly locally finite system of open sets with paracompact closures is paracompact. A space is paracompact if and only if every open covering has a linearly locally finite open refinement. If X is paracompact and M metrizable, $X \times M$ is paracompact if and only if it is countably paracompact. These results are deduced from a new criterion for a subspace of a space to be paracompact; the proofs also use the author's theorem about $X \times \beta X$ [Pacific J. Math. **10** (1960), 1043–1047; MR **23** #A2186]. Finally, another condition making $X \times Y$ paracompact is stated without proof. {On p. 142, line 8, "maximal" presumably means "maximal, for a previously given k".}

A. H. Stone (Rochester, N.Y.)

Leader, Solomon 5420

On products of proximity spaces.

Math. Ann. **154** (1964), 185–194.

The author studies closure operations on the subsets of a set which are distributive but not necessarily idempotent; that is, $c(A \cup B)=cA \cup cB$, but not always $ccA=cA$. A set with such a closure operator is called a c-space. He defines for c-spaces such generalized topological notions as compactness, continuity of mappings, and cartesian products, and proves some theorems about them.

By a distributive relation, or d-relation α over a set he means a binary relation such that $A\alpha(B \cup C)$ if and only if $A\alpha B$ or $A\alpha C$. A d-relation α determines a closure operator given by $cA=\{x:\{x\}\alpha A\}$. Conversely, a closure operator c determines at least one d-relation given by $A\alpha B$ if and only if A meets cB. The d-relations are generalizations of proximity relations.

A space with a d-relation is called a d-space. In a proper d-space related sets are non-empty, and sets that intersect are always related. The author extends to d-spaces the notion of a cluster of sets previously studied by him [Fund. Math. **47** (1959), 205–213; MR **22** #2978]. Clusters are analogous to ultrafilters; a proper d-space is called cluster-compact if every cluster contains a singleton. He shows that a product of proper d-spaces is cluster-compact if and only if every factor space is cluster-compact.

Finally he shows that a product of proximity spaces is a proximity space, and the Smirnov compactification of the product is homeomorphic to the topological product of the Smirnov compactifications of the factor spaces. This result is obtained here as a special case of a more general theorem. It may also be verified directly.

O. Frink (University Park, Pa.)

Michael, E. 5421

A short proof of the Arens-Eells embedding theorem.

Proc. Amer. Math. Soc. **15** (1964), 415–416.

The theorem of Arens and Eells [Pacific J. Math. **6** (1956), 397–403; MR **18**, 406] referred to in the title states that every Hausdorff uniform space (in particular, every metric space) can be embedded homeomorphically (isometrically) as a closed subset of a locally convex linear space. The author shows here that every metric space X can be embedded isometrically as a closed, linearly independent subset of a normed linear space E of all real-valued functions f defined on the metric space $Y=X \cup \{p\}$ (with metric d), $p \notin X$, such that $f(p)=0$ and $|f(x)-f(y)| \leqq Kd(x,y)$, $x, y \in Y$, for some $K \geqq 0$, where the smallest such K is the norm of f.

T. Husain (Ottawa, Ont.)

Krishnan, V. S. 5422

An additive, asymmetric, semi-uniform spaces and semi-groups.

J. Madras Univ. B **32** (1962), 175–198 (1963).

A generalized uniform structure on a set S is a family of subsets (U_i), $i \in I$, of the product set $S \times S$, subject to the condition that each U_i contains the diagonal of $S \times S$. The generalized uniform structure is additive if for each pair U_i, U_j, there exists a U_k which is a subset of $U_i \cap U_j$. It is feebly symmetric if for each U_i there exists a U_j such that $U_j^{-1} \subset U_i$. It is feebly transitive if for each U_i there exists a U_k such that if (x, y) and (y, z) are in U_k, then (x, z) is in U_i.

An additive semi-uniform topological space is a space whose topology is determined by an additive, feebly transitive uniform structure. The first part of the paper defines completeness for additive semi-uniform spaces, and also gives a construction for the completion of such spaces. The last section of the paper deals with a class of ordered semigroups with a uniform structure. Theorems are obtained concerning extensions and complete extensions of such semigroups. *E. Duda* (Coral Gables, Fla.)

Gähler, Siegfried 5423

2-metrische Räume und ihre topologische Struktur.

Math. Nachr. **26** (1963), 115–148.

Let R be a point set and σ a real-valued function defined on $R \times R \times R$. The author calls σ a 2-metric if it satisfies the following conditions: (1a) if the cardinality of R is greater than 2, then for each distinct pair a and b there is at least one point c in R so that $\sigma(a, b, c) \neq 0$ (he actually does not allow R to consist of exactly two points but this restriction is somewhat awkward and seems unnecessary); (1b) $\sigma(a, b, c) = 0$ if two of the coordinates of the triple (a, b, c) are identical; (2) $\sigma(a, b, c) = \sigma(a, c, b) = \sigma(b, c, a)$; and (3) $\sigma(a, b, c) \leqq \sigma(a, b, d) + \sigma(a, d, c) + \sigma(d, b, c)$.

The most natural example of such a function is the euclidean 2-metric on E^m for $m > 1$. If a, b and c are collinear, then $\sigma(a, b, c) = 0$; if not, then $\sigma(a, b, c)$ represents the area of the triangle with vertices a, b and c. In view of this, condition (3) is aptly dubbed the tetrahedral inequality.

If α is a positive real, then the α-neighborhood of the pair (a, b) with respect to σ is the set $\{c | c \in R$ and $\sigma(a, b, c) < \alpha\}$ which is denoted by $U_\alpha(a, b)$. The collection of all such sets $U_\alpha(a, b)$ forms a sub-basis for a topology called the natural 2-metric topology induced by σ. A space is termed 2-metrizable if it has a 2-metric whose induced topology is equivalent to the original topology.

K. Menger introduced the analogous n-metric for euclidean m-space in Chapter III of Math. Ann. **100** (1928), 75–163; he also discussed abstract m-metrics which, however, did not necessarily have the author's property (1a). The author investigates the topological structure of such spaces, which Menger only touched on in the euclidean case; apparently for the sake of applications, he limits himself to 1-metrics—ordinary metrics—and 2-metrics.

Some features of the theory resemble those of metric spaces. For example, by Theorems 7 and 8, a 2-metric space is a regular Hausdorff space. In Theorem 19 the euclidean metric and 2-metric are seen to induce the same topology, while in Theorem 20 it is proved that every metric space is 2-metrizable. On the other hand, an example is given in § 17 of a 2-metric space that does not satisfy the first axiom of countability and hence is not metrizable.

The author closes with a remark that these results are to be applied in another paper on "linear 2-normed spaces". *R. H. Rosen* (Princeton, N.J.)

Gray, William J. 5424

On the metrizability of invertible spaces.

Amer. Math. Monthly **71** (1964), 533–534.

The author shows that an invertible space is either metrizable or nowhere locally metrizable.

J. R. Isbell (Seattle, Wash.)

Vollrath, Hans-Jochachim 5425
[Vollrath, Hans-Joachim]

★Grundzüge einer Theorie der Ω-metrischen Räume.

Von der Fakultät für Mathematik und Physik der Technischen Hochschule Darmstadt zur Erlangung der Würde eines Doktors der Naturwissenschaften (Dr. rer. nat.) genehmigte Dissertation, Darmstadt, 1963. ii + 29 *pp.*

The author generalizes the notion of metric space by considering distance functions satisfying the usual conditions, but taking values in the ring of real Ω-numbers. The Ω-numbers were introduced and studied by C. Schmieden and D. Laugwitz [Math. Z. **69** (1958), 1–39; MR **20** #2404]. They consist of arbitrary sequences of rational numbers.

An Ω-metric space has a topology which is Hausdorff, and is definable in terms of ε-neighborhoods or of Moore-Smith convergence of nets. It is natural to use nets defined over the directed set of positive integral Ω-numbers as a generalization of sequences. The author defines completeness in terms of Cauchy nets, and proves that every Ω-metric space can be densely imbedded in a complete space. He also obtains conditions for an Ω-metric space to be compact. *O. Frink* (University Park, Pa.)

Borsuk, Karol 5426

Bericht über einige neue Ergebnisse und Probleme aus dem Gebiet der anschaulichen Topologie.

Jber. Deutsch. Math.-Verein. **66** (1963/64), *Abt.* 1, 94–106.

A map $f: X \to Y$ is an r-map if there exists $g: Y \to X$ such that fg is the identity on Y. If such an f exists, Y is said to be r-dominated by X. Two spaces r-dominated by one another are called r-equivalent. This notion generalizes that of retracts. In this survey-type presentation, some problems concerning r-dominations are formulated. Also discussed is the status of some problems the author previously posed [same Jber. **60** (1958), Abt. 1, 101–114; MR **19**, 1186]. They are mainly about AR and ANR sets. Samples follow.

Is every n-dimensional compactum homeomorphic to a subset of an $(n+1)$-dimensional AR set? This has been solved by Bothe [Fund. Math. **52** (1963), 209–224; MR **27** #1953] who proves a much stronger result: There exists an $(n+1)$-dimensional AR set containing every compactum of dimension at most n.

Does every n-dimensional ANR set contain at least one

$(n-1)$-dimensional ANR set? This remains unsolved. In this connection, however, Bing and the author have constructed [ibid. **54** (1964), 159–175; MR **28** #4520] a 3-dimensional AR set containing no disk.

K. W. Kwun (Tallahassee, Fla.)

Din' N'e T'ong 5427

Preclosed mappings and a theorem of A. D. Taĭmanov. (Russian)

Dokl. Akad. Nauk SSSR **152** (1963), 525–528.

A Hausdorff extension cY of a Hausdorff space Y is perfect if for all closed subsets A, B of Y such that $A \cup B = Y$, the closure of $A \cap B$ in cY is the intersection of the closures of A and B in cY. A mapping f from X onto Y is preclosed if for every $y \in Y$ and for every neighborhood O of the set $f^{-1}(y)$ there is a set H such that $f^{-1}(y) \subseteq H \subseteq O$ and $f(H)$ is open; f is perfect if it is a closed mapping and the inverse image of every point is compact. Every preclosed mapping is quasi-compact in the sense of G. T. Whyburn [Duke Math. J. **17** (1950), 69–74; MR **11**, 194]; if f is continuous from X onto a Hausdorff space Y every point y of which has a denumerable fundamental system $\{O_k y\}$ of neighborhoods such that $y_k \to y$ whenever $y_k \in O_k y$, then f is preclosed if and only if f is quasi-compact. Let aX and cY be Hausdorff extensions of X and Y respectively, and let f be a continuous mapping from X onto Y admitting a continuous extension f_{ac} from aX onto cY. Extending a theorem of Taĭmanov, the author shows that if f is preclosed and monotone (the inverse image of every point is connected), if f_{ac} is perfect, and if cY is a perfect extension of Y, then f_{ac} is monotone. Also, if aX is a perfect extension of X and if f_{ac} is closed and monotone, then cY is a perfect extension of Y.

S. Warner (Durham, N.C.)

Guay, M. D.; Hocking, J. G.; Kronk, H. V. 5428

Local near-homogeneity.

Amer. Math. Monthly **70** (1963), 827–833.

A space S is defined to be near-homogeneous at a point $p \in S$ if, for each neighborhood U of p and each point $x \in S$, there exists a homeomorphism h of S onto itself so that $h(x) \in U$. A space S is defined to be continuously near-homogeneous if, for each point $x \in S$ and each open set U in S, there is an isotopy h_t of S onto itself such that $h_1(x) \in U$. The authors present numerous simple theorems about the set of all points at which a space is near-homogeneous (continuously near-homogeneous). Their main result about continuously near-homogeneous spaces is the following theorem. A nondegenerate locally connected continuum in the plane is a simple closed curve if it is continuously near-homogeneous. Some of the methods and theorems are similar to some work by Doyle and Hocking on continuously invertible spaces [Pacific J. Math. **12** (1962), 499–503; MR **26** #4323].

C. E. Burgess (Salt Lake City, Utah)

Doyle, P. H. 5429

A sufficient condition that an arc in S^n be cellular.

Pacific J. Math. **14** (1964), 501–503.

If an arc A in the n-sphere S^n is such that $S^n - A$ is not an open n-cell, then it has a subarc B such that no small n-cell whose boundary meets B in just one point can contain either end point of B in its interior.

R. H. Fox (Princeton, N.J.)

Doyle, P. H.; Warne, R. J. 5430

Some properties of groupoids.

Amer. Math. Monthly **70** (1963), 1051–1057.

"Groupoid" means a set together with a closed binary operation, written multiplicatively. In Section 1 the authors discuss some properties of a groupoid G in terms of subgroupoids A ($AA \subseteq A$), antigroupoids B ($BB \subseteq G - B$, the complement of B in G), expansive sets C (defined as non-void subsets which are not subgroupoids and not antigroupoids), and what they call partial ideals. Section 2 concerns topological groupoids (defined to be Hausdorff topological spaces such that the multiplication is continuous in both variables; see also the second author [Publ. Math. Debrecen **8** (1961), 143–146; MR **24** #A538]). A result of A. D. Wallace [An. Acad. Brasil. Ci. **25** (1953), 335–336; MR **15**, 854] on the topological closure of an algebraic subgroup of a topological semigroup G is seen to remain valid also for a topological groupoid G. Some of the "associative-like" properties of g-threads (cf. the second author, loc. cit.) induced by the topology are mentioned.

H.-J. Hoehnke (Berlin)

Anderson, L. W.; Hunter, R. P. 5431

The $\mathscr{H}$-equivalence in a compact semigroup. II.

J. Austral. Math. Soc. **3** (1963), 288–293.

The authors continue their earlier study [Bull. Soc. Math. Belg. **14** (1962), 274–296; MR **27** #1920] by first setting forth conditions under which the sets $(x.\cdot x) \cap (Se \times G)$ and $(H.\cdot x) \cap Se$ are topologically isomorphic, where S is a semigroup, H is an $\mathscr{H}$-class in S, $x \in H$, and G is a subgroup of S with identity e. It is shown that these conditions are also sufficient to give $(H.\cdot x) \cap Se = (H.\cdot x)e = (x.\cdot x)G$.

Next, using the topologized Schützenberger group and transformation group theory, it is shown that if S is a compact semigroup with identity and group kernel K, and if $a \in S$ such that $kH_a = K$ for some $k \in K$, then the existence of a local cross-section in the kernel of the induced homomorphism $\varphi_a : \mathscr{S}G_a \to K$ implies a fibre bundle structure on Sa, with base space K, fibre $(e.\cdot e) \cap Sa$, and projection mapping $\lambda_e | Sa$, where $\lambda_e(sa) = esa$. Moreover, if Sa is connected, then the fibre of the bundle is also connected. In particular, if $\mathscr{S}G_a$ is finite-dimensional, then the existence of the local cross-section is assured. By combining this result with one from the earlier paper, the following noteworthy corollary is obtained. Let S be a compact connected semigroup with group kernel K such that $S = Se$. If there exist x, y in S such that $xH_y = K$, then K is a group orbit of some maximal subgroup H_f, and the core contains a non-degenerate continuum at e, the identity of K. Moreover, if S is n-dimensional and K is $(n-1)$-dimensional, then there exists a standard thread from f, the identity of H_f, to e.

If S is any finite-dimensional compact semigroup, $x \in S$, then certain subgroups of $H_x.\cdot x$ satisfy the above hypotheses and therefore have the structure of fibre bundles.

The analogy between the behavior of an $\mathscr{H}$-class and that of a group is further strengthened by a theorem including the following result: If S is a compact semigroup

with identity, and if $f: S \to T$ is a continuous epimorphism, then $f|H$ cannot raise dimension, where H is any $\mathscr{H}$-class of S.

The paper concludes with the theorem: Let S be a compact semigroup with identity such that for each $x \in S$ there is an integer n ($=n(x)$) such that $x^n \in E$. Then the natural map of S to $S/\mathscr{H}$ is light. It is a corollary that if S is also connected, then S is acyclic.

D. R. Brown (Knoxville, Tenn.)

Kneser, Hellmuth 5432

Eine nichtkompakte zusammenhängende Fläche ohne Fluchtweg.

Ann. Acad. Sci. Fenn. Ser. A I No. 336/14 (1963), 9 *pp.*

Let R^+ be the set of the non-negative real numbers and B the long line with the initial point but without the end point. By a surface one understands a space locally homeomorphic to 2-space (but not necessarily metrizable). A "Fluchtweg" in a surface F with $R^+(B)$ as domain is a continuous map f of $R^+(B)$ into F such that for each compact set $K \subset F$, there exists $t_0 \in R^+$ ($t_0 \in B$) such that for no $t \geqq t_0$, $f(t) \in K$. A non-compact connected surface without "Fluchtweg" is constructed. A partial motivation for considering such paths appears to be that the existence or non-existence of such paths sometimes distinguishes topological types of given surfaces.

K. W. Kwun (Tallahassee, Fla.)

Gillman, David S. 5433

Sequentially 1-ULC tori.

Trans. Amer. Math. Soc. **111** (1964), 449–456.

A sequence of tori $\{T_1, T_2, \cdots\}$ is called sequentially 1-ULC if for each positive ε there is a positive δ and integer N such that if $n > N$ and α is a simple closed curve on T_n of diameter less than δ that bounds a disk on T_n, then α bounds a disk on T_n of diameter less than ε.

Let A be an arc in three-space, E^3. Theorem 1: If (1) A lies on a 2-sphere S in E^3, (2) A lies on a simple closed curve J which is the intersection of a nested sequence of tori plus their interiors, and (3) the sequence of tori is sequentially 1-ULC, then A is tame. That a tame arc satisfies the above condition for appropriately chosen S, J, and sequence $\{T_1, T_2, \cdots\}$ of tori is obvious. As a corollary the author observes that the "Bing sling" [Bing, J. Math. Pures Appl. (9) **35** (1956), 337–343; MR **18**, 407] contains no subarc that lies on a disk. Thus this curve has neither property P nor property Q defined by the reviewer [Bull. Amer. Math. Soc. **63** (1957), 293–305; MR **19**, 568] at any point. The method of proof is to show that by modifying the given S appropriately, S is locally tame at each interior point of A.

If P_A is the subset of A consisting of those points at which A pierces some disk, then tameness of an arc may be described as follows. Theorem 2: If A is an arc in E^3 such that (1) A lies on a 2-sphere S in E^3, (2) A lies on a simple closed curve J that is the intersection of a sequence of nested tori plus their interiors, and (3) P_A is dense in A, then A is tame.

Using this theorem and a construction based on an example of Bing [Duke Math. J. **28** (1961), 1–15; MR **23** #A630], the author gives an example of an arc that pierces no disk but itself lies on a disk. This curve, then, has property P (referred to above) at no point, but has property Q at each point.

O. G. Harrold (Knoxville, Tenn.)

Bean, Ralph J. 5434

Disks in E^3. I. Subsets of disks having neighborhoods lying on 2-spheres.

Trans. Amer. Math. Soc. **112** (1964), 206–213.

Theorem 3: Let D be a disk in E^3 such that the set W of all its wild points is contained in the interior of D. If there is a connected open neighborhood U of W in D that lies on a 2-sphere, then the entire disk D itself lies on a 2-sphere. It is also shown, with the help of a modification of a striking example due to Bing, that D always lies on a 2-sphere if W consists of one point, but need not lie on a 2-sphere if W consists of two points.

R. H. Fox (Princeton, N.J.)

Bing, R. H. 5435

Retractions onto spheres.

Amer. Math. Monthly **71** (1964), 481–484.

By elementary means the author proves that the union of a 2-sphere with its exterior in E^3 admits a retraction onto the 2-sphere. Also proved are the following. Theorem 3: Let X be an $(n-2)$-connected ANR imbedded as a closed subset of a compact n-manifold M, then for each point $p \in M - X$ there is a retraction of $U + X - \{p\}$ onto X, where U is the component of $M - X$ containing p. Theorem 4: The union of an $(n-1)$-sphere with its interior in E^n is an AR.

L. Neuwirth (Princeton, N.J.)

Haddock, A. G. 5436

Rotation groups under monotone transformations.

Fund. Math. **53** (1963/64), 173–175.

This note generalizes some results of L. Whyburn [Fund. Math. **28** (1937), 124–130] on homeomorphisms of a space into itself to comparable assertions about monotone transformations. The term "rotation group" is used in a technical set-theoretic sense and has no contact with Lie groups.

E. Dyer (Houston, Tex.)

Auslander, Joseph 5437

Generalized recurrence in dynamical systems.

Contributions to Differential Equations **3** (1964), 65–74.

Let $(x, t) \to xt$ be a continuous flow on a locally compact separable metric space X, $\mathscr{V}$ the class of continuous maps $f: X \to E$ for which $f(xt) \leqq f(x)$ for all $x \in X$ and $t > 0$, and denote by $\mathscr{R}$ the set of points $x \in X$ such that $f(xt) = f(x)$ for all $f \in \mathscr{V}$ and $t \geqq 0$. The author gives various characterizations of the set $\mathscr{R}$ which he calls the generalized recurrent set; one is in terms of the theory of prolongations [see, e.g., the author and P. Seibert, Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics, pp. 454–462, Academic Press, New York, 1963; MR **27** #416], the other may be stated as follows. There is an $f \in \mathscr{V}$ such that $x \in \mathscr{R}$ if and only if f is constant on the orbit of x and if $x \notin \mathscr{R}$, then $f(xt) < f(x)$ for $t > 0$. He also shows that if the flow is parallelizable, then $\mathscr{R} = \emptyset$ and if $\mathscr{R} = \emptyset$, then the flow is completely unstable, but that the converses in general do not hold. The author also proves,

among others, the following stability theorem. Let M be a compact, positively invariant set which is absolutely stable [loc. cit.]. Then M is asymptotically stable if and only if there is a neighborhood U of M for which $(U-M)\cap \mathscr{H}=\emptyset$. *H. A. Antosiewicz* (Los Angeles, Calif.)

Gottschalk, Walter H. 5438
Minimal sets occur maximally.
Trans. New York Acad. Sci. (2) **26** (1963/64), 348–353.

The present note provides some examples of minimal sets in symbolic flows. The reviewer indicates his regret that the paper does not contain more background material, although suitable references are certainly provided.

A symbolic flow is an action of the additive group of integers on the Cantor discontinuum which describe geodesic flows over certain surfaces of constant negative curvature. A flow (X, T) is minimal if X is a compact Hausdorff space and the closure of each orbit $\overline{xT}$ is all of X. The author refers to existence theorems for minimal flows. The basic theorem is that if (Y, T) is a flow on a compact Hausdorff space, then $X=\overline{xT}$, an orbit closure, is a minimal set if and only if x is almost periodic.

The author proceeds to give examples of almost periodic points in some symbolic flows. The first example is the following. Consider the sequence (a) 0, 1, 10, 11, 100, 101, 110, 111, $\cdots$ of non-negative integers in binary notation. Take the sums of the digits of members of (a) to obtain (b) 0 1 1 2 1 2 2 3$\cdots$ which is determined by $0\to 0\,1$, $1\to 12$, $2\to 23$, $\cdots$ on infinitely many symbols starting with 0. Now reducing (b) modulo any integer $n\geqq 2$ will produce a non-periodic, almost periodic sequence.
P. E. Conner (Charlottesville, Va.)

ALGEBRAIC TOPOLOGY

See also 5317, 5445, 5446, 5828, 5829.

Bedrosian, S. D. 5439
Generating formulas for the number of trees in a graph.
J. Franklin Inst. **277** (1964), 313–326.

A graph without loops is called full if each pair of vertices is joined by the same number $n\geqq 1$ of edges. Formulae are given for the number of spanning trees in certain graphs derivable from full ones by deleting edge-sets of specified kinds. *W. T. Tutte* (Waterloo, Ont.)

Dirac, G. A. 5440
On the structure of 5- and 6-chromatic abstract graphs.
J. Reine Angew. Math. **214/215** (1964), 43–52.

Suppose the vertex a is adjacent to each vertex of a cycle of length 4. Any refinement of such a graph is called a pyramid with apex a. It is shown that if (a, b) is any edge in a critical 5-chromatic graph, then a is the apex of a pyramid in the graph which contains (a, b) and has certain additional properties. It is also shown that any connected 6-chromatic graph can either be contracted into a complete 6-graph or else the graph obtained from it by removing a suitable set of at least 11 edges can be contracted into a graph containing a complete 6-graph with one edge missing. *J. W. Moon* (London)

Ore, Oystein 5441
Arc coverings of graphs.
Ann. Mat. Pura Appl. (4) **55** (1961), 315–321.

The graphs which the author considers are finite, unoriented and loopless. Let G be a graph, label the vertices of G by $a_0, a_1, \cdots, a_m$, and let (a_i, a_j) denote the edge between a_i and a_j in G. $A=(a_0, a_1)(a_1, a_2)\cdots(a_{n-1}, a_n)$ is called an arc of length n where no vertex a_i appears more than once in it. An arc is a circuit if $a_0=a_n$. A family of arcs $\{A_i\}$ of G is said to form an arc covering of G when they are disjoint and each vertex in G lies on one of them, where A_i is permitted to be a single vertex. An arc covering is maximal when it contains the greatest possible number of edges.

The main result seems to be Theorem 2.1: When a maximal arc covering of G contains $k\geqq 2$ arcs then $k\leqq m-\rho(t)-p(t')$, where m is the number of vertices in G, t and t' are two vertices in G not connected by an edge, and $\rho(t)$ is the number of edges in G having t as an endpoint. The consequences of the theorem give various sufficient conditions for a graph G to have a Hamilton arc (the arc including all vertices of G), and to have a Hamilton circuit. Also, see the author's papers [Amer. Math. Monthly **67** (1960), 55; MR **22** #9454; J. Math. Pures Appl. (9) **42** (1963), 21–27; MR **26** #4336].
C. Y. Chao (Pittsburgh, Pa.)

Brown, Edgar H., Jr.; Peterson, Franklin P. 5442
Algebraic bordism groups.
Ann. of Math. (2) **79** (1964), 616–622.

In this paper the authors introduce and study algebraic analogues of bordism and cobordism groups. The first algebraic notion to appear (§ 2) is that of a Poincaré algebra. This is so defined that if V is an n-manifold, then $H^*(V; Z_2)$ is a Poincaré algebra. The given structure of a Poincaré algebra H^* includes a "fundamental class" in $\mathrm{Hom}(H^n; Z_2)$; it also includes operations from the mod 2 Steenrod algebra A.

The authors next introduce a notion of "cobordism". A Poincaré algebra H is a "boundary" if there exist two further algebraic objects H', H'' with specified properties. The definitions are so written that if W is a manifold with boundary V, and $H=H^*(V; Z_2)$, then we may take $H'=H^*(W; Z_2)$, $H''=H^*(W, V; Z_2)$. It is proved (Theorem 2.2) that a Poincaré algebra H is a boundary if and only if all its "Stiefel-Whitney classes" vanish. Thus the "algebraic cobordism" group ${}^aN_n$, defined by classifying Poincaré algebras in the obvious way, is isomorphic to the usual cobordism group N_n.

Next (§ 3) the authors introduce bordism groups. The usual bordism groups $N_*(K)$ depend on a space K, and reduce to the usual cobordism groups N_* if one takes K to be a point. Generalising their previous work, the authors introduce "algebraic bordism groups" ${}^aN_n(X)$; these depend on an algebraic object X, of such a nature that one can substitute $X=H^*(K; Z_2)$; and they reduce to the "algebraic cobordism groups" ${}^aN_n$ if one takes X to be Z_2.

The authors calculate these groups by proving (Theorem 3.2) that $\mathrm{Hom}({}^aN_*(X); Z_2)$ is naturally isomorphic to $H^*(BO; Z_2)\otimes_A X$. They deduce (Corollary 3.3) that if K ranges over a suitable class of spaces, then the algebraic bordism group ${}^aN_n(H^*(K; Z_2))$ is naturally isomorphic to the usual bordism group $N_n(K)$.
J. F. Adams (Manchester)

Mann, L. N. 5443

Actions of elementary p-groups on $S^n \times S^m$.

Michigan Math. J. **11** (1964), 47–51.

An elementary p-group of rank k is a k-fold direct product of the cyclic group Z_p with itself (p a prime). A group acts effectively on a space if and only if every group element different from the identity moves at least one point in the space. Now P. A. Smith showed that if an elementary p-group of rank k acts effectively on the n-sphere S^n, then $k \leq (n+1)/2$ if $p \neq 2$ and $k \leqq n+1$ if $p=2$.

The author of this paper considers the problem of which elementary p-groups can act effectively on the product $S^n \times S^m$. The major result is the answer $k \leqq [(n+1)/2] + [(m+1)/2]$, $p \neq 2$, and $k \leqq n+m+2$ if $p=2$. Actually a more general result is proved. Namely, if X is an orientable cohomology n-manifold mod p with $\sum_0^n \beta_i(X) \leqq 4$, where

$$\beta_i(X) = \dim H^i(X; Z_p)$$

and if an elementary p-group of rank k is effective on X, then $k \leqq (n+2)/2$, $p \neq 2$, and $k \leqq n+2$, $p=2$. It turns out that induction over n will do the argument since being a generalized cohomology manifold mod p is a property inherited by the fixed-point set of every subgroup of a p-group.

The question of what happens for an arbitrary product of spheres is clearly implicit, but there is at this point no information available beyond the twofold product.

P. E. Conner (Charlottesville, Va.)

Ginsburg, Michael 5444

On the homology of fiber spaces.

Proc. Amer. Math. Soc. **15** (1964), 423–431.

In this paper some results are obtained concerning the homology of a fibre space under the hypothesis that the Lusternik-Schnirelmann category of the base space is k. When $k=2$, a generalization of the Wang sequence is obtained. Also, if $\operatorname{cat} X \leqq 2$, the additive structure of $H_*(\Omega X)$ is computed and the Pontryagin ring structure is partially determined. For arbitrary k, a spectral sequence is obtained which relates the homology groups of the total space, the fibre and the loop space of the base and for which the differentials d_r are zero for $r \geqq k$.

The method consists of combining the following two facts. If $F \to E \to B$ is a fibration, the singular chains of E are chain equivalent to a twisted tensor product $\bar{B}(C(\Omega B)) \otimes C(F)$, where C and $\bar{B}$ denote the singular chains and the bar construction, respectively [the reviewer, Ann. of Math. (2) **69** (1959), 223–246; MR **21** #4423]. If X is 1-connected and $\operatorname{cat}(X) \leqq k$, then the identity map of $\bar{B}(C(\Omega X))$ onto itself is chain equivalent to a chain map which factors through $\bar{B}^{k-1}(C(\Omega(X)))$. This follows from a result of the author [ibid. (2) **77** (1963), 538–551; MR **26** #6976]. *E. H. Brown* (Waltham, Mass.)

TOPOLOGY AND GEOMETRY OF DIFFERENTIABLE MANIFOLDS

See also 5142, 5146, 5147, 5238, 5403, 5408, 5442.

Novikov, S. P. 5445

Homotopically equivalent smooth manifolds. I. (Russian)

Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 365–474.

One of the main objects of this paper is to give a condition for diffeomorphism between simply connected differentiable manifolds of the same homotopy type and such that any two of them admit a homotopy equivalence taking the tangent bundle of one into that of the other, the dimensions being greater than or equal to 5.

Let $\{M_i^n\}$ be a class of manifolds of dimension $n \geqq 5$ as just described. Pick $M \in \{M_i^n\}$ and let T_N be the Thom space corresponding to the normal bundle of M suitably embedded in R^{n+N} (Euclidean space) for sufficiently large N. It is shown that T_N has a fundamental cycle $[T_N]$ in the image of the Hurewicz homomorphism $H: \pi_{n+N}(T_N) \to H_{n+N}(T_N)$. Let $A(M) = H^{-1}([T_N])$. It is shown that (1) the group $\pi(M, SO_N)$ operates transitively on $A(M)$, and (2) the group $\pi^+(M, M)$ of homotopy classes of orientation-preserving diffeomorphisms of M on itself preserving the tangent bundle operates on the orbit set $A(M)/\pi(M, SO_N)$.

With the aid of these results the following is proved. Classification theorem: There is a map

$$F: \{M_i^n\} \to (A(M)/\pi(M, SO_N))/\pi^+(M, M)$$

such that (a) $F(M_1) = F(M_2)$ if and only if M_1 is either diffeomorphic to M_2 or to $M_2 \# \tilde{S}$, where $\tilde{S}$ is a Milnor sphere, and (b) if $\dim M \neq 4k+2$, F is onto.

The general technique leading to this result and others in the paper is a systematic investigation of the relation between maps of manifolds and maps of the corresponding Thom spaces.

The following is a brief sketch of the contents of the paper. The first chapter starts with a description of spherical modifications, here called the Morse construction, and a description of the construction of the Thom space $T(M_n)$ of a manifold M_n in R^{n+N}. In particular, the definition of the fundamental cycle $[T_N(M)]$ is given and the construction of the set $A(M) = H^{-1}[T_N]$ mentioned above. Let $\alpha \in A(M_n)$. Then there is a map $f_\alpha: S^{n+N} \to T_N$ representing α such that $f_\alpha^{-1}(M) = M_\alpha$ is a differentiable submanifold of S^{n+N} and f_α relates the normal bundle of M_α in S^{n+N} to that of M in T_N. Also let $\bar{A}(M)$ be the set of α in $A(M)$ such that, in addition, M and M_α are homotopy equivalent. This, of course, is the set of special interest in this article.

It is shown that, in addition, f_α can be chosen so that it induces isomorphisms on the homology groups for dimensions $< [n/2]$. For dimension $[n/2]$ two cases are possible. If $n = 4k$ or $2k+1$, f_α also induces isomorphism at dimension $[n/2]$, and in this case M and M_α are homotopy equivalent. If $n = 4k+2$, the kernel of $f_{\alpha*}$ on dimension $2k+1$ is $Z+Z$. The argument here is carried out by setting up systems of generators for the kernels of the homomorphisms in question and then killing them by spherical modifications. In the second case ($n \neq 4k+2$) there is an invariant whose vanishing gives the condition for the possibility of killing the kernel of $f_{\alpha*}$ at dimension $2k+1$. This part of the discussion gives the essential argument leading to part (b) of the above classification theorem.

Take $\alpha \in \bar{A}(M)$ and assume that f_α is chosen so that it is a homotopy equivalence of M and M_α. If $M_{\alpha 1}$ and $M_{\alpha 2}$, along with the corresponding $f_{\alpha 1}$ and $f_{\alpha 2}$, satisfy this condition, then either $M_{\alpha 1} = M_{\alpha 2}$ or $M_{\alpha 1} = M_{\alpha 2} \# \tilde{S}$, where $\tilde{S}$ is a Milnor sphere, according as n is even or odd. This leads to part (a) of the classification theorem.

To complete the formulation of the classification theorem, the group actions involved must be described. The action of $\pi(M, SO_N)$ on $A(M)$ is generated as follows.

$\alpha \in \bar{A}(M)$ is represented by a map $f_\alpha : S^{N+n} \to T_N(M)$. This gives a map of the normal bundle of M_α to the normal bundle of M. An element of $\pi(M, SO_N)$ gives a twisting of this map. When extended to S^{n+N} a new element of $\bar{A}(M)$ is obtained. The action of $\pi^+(M, M)$ on $A(M)/\pi(M, SO_N)$ arises from the fact that $\pi^+(M, M)$ acts on the set of homotopy classes of maps of M_α into M. In this section also relations between maps of M into itself and maps of $T_N(M)$ into itself are worked out.

In the second chapter some properties of Thom spaces of bundles over complexes are obtained as preliminaries for later results. In particular the Thom spaces for the various skeletons of a cell decomposition of a manifold are set up, and in terms of these, obstructions are constructed to the diffeomorphism of two manifolds M_α, M_β, where α, β are in $\bar{A}(M)$. Next, some elementary operations are introduced for the changing of the differential structure on a manifold without changing its combinatorial structure, in some suitable triangulation. The first consists in forming the connected sum with a Milnor sphere. The second consists in removing a solid torus and replacing it with a new identification of the boundaries. The latter appears as a special case of sticking together manifolds along cycles carried by disjoint unions of spheres. Also the relation between change of differential structure and the action of a spherical modification h is studied. If the structure on M is changed, the obstruction is constructed to the extension of this change to $h(M)$ and the trace of h. This is done by studying the Thom spaces.

The third chapter consists of applications and examples. The first application is to products of spheres. It is shown that if $n-k \not\equiv 2\ (4)$ then any products $\tilde{S}^k \times \tilde{S}_1^{n-k}$ and $\tilde{S}^k \times \tilde{S}_2^{n-k}$, where $\tilde{S}^k \in \theta^k(\pi)$ and $\tilde{S}_i^{n-k} \in \theta^{n-k}(\pi)$, are diffeomorphic modulo a point. Also it is shown that there are combinatorially non-equivalent simply connected π-manifolds of the homotopy type of $S^2 \times S^6$.

A study is made of the relation between a manifold M and $M \# \tilde{S}$ for a Milnor sphere $\tilde{S}$. The following are two of the results obtained: (1) There is a manifold M^n for $n = 9$ or 10 such that $w_2(M^n) \neq 0$, and there is a Milnor sphere $\tilde{S}^n \in \theta^n(\pi)$ such that $M^n = M^n \# \tilde{S}^n$; (2) There is a manifold M^{13} such that $p_1(M^{13}) \not\equiv 0\ (3)$ and $M^{13} = M^{13} \# \tilde{S}^{13}$ for all $\tilde{S}^{13} \in \theta^{13}(\pi)$.

Independent derivations are given for some results of Browder [Colloq. on Algebraic Topology, 1962, Lectures, Mat. Inst., Aarhus Univ., Aarhus, 1962; MR **26** #3565] on the conditions for a bundle over a manifold M to be the normal bundle of a manifold of the same homotopy type as M embedded in some Euclidean space.

The paper concludes with announcements of results to be dealt with in later papers, namely: (1) results on the non-invariance of Pontryagin numbers with respect to homotopy type; (2) the transfer of the results of this paper to the combinatorial case; (3) the study of the effect of adding Milnor spheres to a manifold; (4) the study of the embedding of Milnor spheres in Euclidean spaces.

A. H. Wallace (Bloomington, Ind.)

Holmann, Harald 5446

★Vorlesung über Faserbündel.

Aschendorffsche Verlagsbuchhandlung, Münster, 1962. v + 200 *pp.* *DM* 21.00.

Many of the basic results on fibre bundles still hold when other definitions are substituted for that given by Steenrod [*Topology of fibre bundles*, Princeton Univ. Press, Princeton, N.J., 1951; MR **12**, 522]. The new definition may be more general to include fibre spaces which are not locally trivial or which have no structural group; it may be more special, as for fibre bundles in the categories of differentiable, real-analytic or complex-analytic spaces and maps or algebraic fibre spaces defined using the Zariski topology over an arbitrary field.

Grothendieck showed in 1955 how to state all these definitions in a single formalism which is easy to understand, easy to apply, and independent of the choice of category [*A general theory of fibre spaces with structure sheaf*, Univ. Kansas, Lawrence, Kan., 1st ed., 1955; 2nd ed., 1958]. The present lectures give an exposition of the basic definitions and results from this point of view.

The first four chapters consist largely of definitions. Chapter I, Fibre spaces: a fibre space is a triple (B, π, X) consisting of spaces B, X and a surjection $\pi: B \to X$. Chapter II, Fibre bundles: a fibre bundle with typical fibre F is a fibre space (B, π, X) which is locally isomorphic to $(X \times F, p, X)$, where $p: X \times F \to X$ is the product projection. Chapter III, Sheaves: a sheaf is a fibre space (B, π, X) for which π is a local homeomorphism; a sheaf of groups (G, q, X) operates on a fibre space (B^0, p, X) if there is a fibre-preserving operation $G \times_X B^0 \to B^0$. Chapter IV, Fibre spaces with structure sheaf: a fibre space (B, π, X) is of type B^0 with G as structure sheaf if there exists an open cover $\{U_i\}$ of X such that (i) there are isomorphisms $h_i: B | U_i \to B^0 | U_i$, and (ii) there is a section g_{ij} of G over $U_i \cap U_j$ such that $h_j h_i^{-1}(b) = g_{ij}(p(b)) \cdot b$ for all $b \in p^{-1}(U_i \cap U_j)$. As an application of these definitions the author establishes the classification of fibre spaces with structure sheaf G by elements of the first cohomology group of X with coefficients in G.

The next two chapters apply especially to the differentiable case, and could be usefully supplemented by a reading of the exposition by Dold [Ann. of Math. (2) 78 (1963), 223–255; MR **27** #5264]. They include the following topics. Chapter V, Fibre bundles with structure group: study of projection $\pi: B \to B/H$, where H is a subgroup of a topological group B; differentiable approximations to continuous sections of differentiable fibre bundles. Chapter VI, Reduction of structure sheaf and structure group: homotopic homomorphisms of fibre bundles over a paracompact space X; extension of sections of fibre bundles over a paracompact space X.

Chapter VII, Vector bundles: direct sum, tensor product, dual, and exterior powers of vector bundles are all defined functorially, and applications cover the tangent bundle of a differentiable manifold and reduction of the structure group to subgroups of the general linear group. With a few exceptions the theorems proved are independent of the choice of category. For more non-trivial results it is necessary to make additional assumptions about the base spaces involved. Examples are the use of paracompactness by Milnor [*Notes on characteristic classes*, Princeton Univ. Press, Princeton, N.J., 1957; *Differential topology*, Princeton Univ. Press, Princeton, N.J., 1958]; of holomorphic completeness by Grauert [Math. Ann. **135** (1958), 263–273; MR **20** #4661]; and of compactness and completeness by Atiyah [Trans. Amer. Math. Soc. **85** (1957), 181–207; MR **19**, 172; Bull. Soc. Math. France **84** (1956), 307–317; MR **19**, 172].

R. L. E. Schwarzenberger (Liverpool)

Griffiths, Phillip A. 5447

On the differential geometry of homogeneous vector bundles.

Trans. Amer. Math. Soc. **109** (1963), 1–34.

Recall that a Wang C-space is a homogeneous complex manifold $X=E/F=A/B$, where E, F are complex Lie groups, E is semi-simple and A a maximal compact (real) subgroup of E. A homogeneous principal G bundle $P\to X$ is one on which E acts commuting with the action of G. The author first computes the A-invariant complex connexions in P. An oversimplification of this computation is that, since $A\times G$ acts transitively, P lifts back to the trivial bundle over A and invariant connexions are defined by the right kind of linear mappings from the Lie algebra of A to that of G; the curvature measures its departure from being a homomorphism. The difficulties occur in the conditions for complex compatibility and lead the author to work in the complexification of the real Lie algebras. In the same terms, he defines the complex torsion of an affine connection in X and, in the case where X has A-invariant Hermitian structure, he computes the unique A-invariant connexion which preserves the inner product. This leads to results about Kähler C-spaces and new proofs of some known results. The tangent bundles to X can be considered as the homogeneous vector bundle $E\times_F \mathfrak{e}/\mathfrak{f}$, where $\mathfrak{e}$, $\mathfrak{f}$ are the Lie algebras of E, F and where F acts adjointly. The author's canonical complex affine connexion is then defined by the linear mapping $\tilde{\chi}:\mathfrak{e}\to \mathrm{gl}(n, \mathbf{C})$, where $\tilde{\chi}(f)=\mathrm{ad}\, f$ and $\tilde{\chi}$ is zero on the $\mathfrak{b}$-reductive complement of $\mathfrak{f}$. This connexion goes over to a connexion in a general homogeneous vector bundle and its curvature is a computable Dolbeault image of the Atiyah class. There are many applications of the techniques. It is proved, for example, that a homogeneous vector bundle admits a holomorphic connexion if and only if it is trivial. Section VIII contains some of the ground work for the author's description of the cohomology in the sheaf of germs of holomorphic sections of a homogeneous vector bundle [Acta Math. **110** (1963), 115–155; MR **26** #6993].

H. B. Shutrick (Manchester)

Shioda, Tetsuji 5448

On algebraic varieties uniformizable by bounded domains.

Proc. Japan Acad. **39** (1963), 617–619.

Let D be a bounded domain in complex Euclidean space and Γ a discontinuous group of analytic automorphisms of D which acts without fixed points, such that the quotient $V=\Gamma/D$ is a projective algebraic variety. The following theorems are proved: (a) V is a minimal model in the sense of algebraic geometry, i.e., any meromorphic mapping from a complex manifold into V is holomorphic; (b) The group of automorphisms of V is isomorphic to the group of automorphisms of the function field of V. Further, this group is finite if D is simply connected.

Previous, less general, work along these lines has been done by Hawley [Ann. of Math. (2) **52** (1950), 637–641; MR **12**, 603], Sampson [Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 895–898; MR **14**, 633], and Igusa [Amer. J. Math. **76** (1954), 669–678; MR **16**, 172]. The proofs given here are remarkably simple, using only general geometric ideas.

R. Hermann (Evanston, Ill.)

PROBABILITY

See also 4980, 5008, 5164a–b, 5167, 5233, 5234, 5291, 5501, 5714, 5852.

Fréchet, Maurice 5449

Une remarque d'ordre historique sur le classement des probabilités nulles (ou de la raréfaction).

C. R. Acad. Sci. Paris **258** (1964), 4877–4878.

Author's summary: "Cournot, sans donner de définitions aussi précises que celles d'Émile Borel, a eu la prescience de la possibilité d'une hiérarchisation des probabilités nulles."

Fréchet, Maurice 5450

Les probabilités nulles et la raréfaction.

Ann. Sci. École Norm. Sup. (3) **80** (1963), 139–172.

E. Borel a été le premier mathématicien à établir une échelle entre probabilités nulles. A cet effet il a introduit la notion de raréfaction d'un ensemble de mesure nulle.

Le present mémoire se décompose en deux parties. La première est un exposé de l'ensemble des raisons qui ont amené Borel à la notion de raréfaction. La deuxième, la plus intéressante, est destinée à des généralisations diverses de cette notion.

A. Fuchs (Strasbourg)

Calot, Gérard 5451

★Cours de calcul des probabilités.

Collection Statistique et Programmes Économiques, Vol. 3.

Dunod, Paris, 1964. xv+256 *pp.* 32 *F.*

This is an elementary text for a course originally designed for economists, given between descriptive and mathematical statistics. It covers the usual ground of combinatorics, algebra of sets, random variables and their numerical characteristics (including conditional expectation and generating function), a good number of useful special distributions, stochastic convergences and random numbers. The presentation of combinatorics and the discussion of functions of random variables are good features. The definition of an (absolutely) continuous random variable as one whose range of variation has the power of the continuum could be misleading, and the definition of "total" independence of more than two events is too vague. The many diagrams and examples, added to explicit style and clear printing, should make this book useful for self-study and review at the indicated level.

K. L. Chung (Stanford, Calif.)

Polljak, Ju. G. 5452

On the comparison of two probabilities. (Russian. English summary)

Teor. Verojatnost. i Primenen. **8** (1963), 195–196.

Author's summary: "A new method of experiment for comparison of probabilities is discussed."

Morris, K. W. 5453

A note on direct and inverse binomial sampling.

Biometrika **50** (1963), 544–545.

It is shown that the probability of at least k successes in n Bernoulli trials equals the probability that the kth success

appears for the first time in at most n trials. This result is used in discussion of a simple problem of statistical inference. *J. Riordan* (Murray Hill, N.J.)

Sazonov, V. V. 5454
On a higher-dimensional central limit theorem. (Russian. Latvian and English summaries)
Litovsk. Mat. Sb. **3** (1963), *no.* 1, 219–224.
Author's summary: "Let $\{\xi_k\}$, $k=1, 2, \cdots$ be a sequence of independent uniformly bounded random vectors taking their values in R^m. Let $\{A_n\}$, $n=1, 2, \cdots$ denote a sequence of linear transformations and $\{a_n\}$, $n=1, 2, \cdots$ be a sequence of vectors. Then the distributions of $A_n \sum_{k=1}^n \xi_k + a_n$ converge for some $\{A_n\}$ and $\{a_n\}$ to a nondegenerate normal distribution if and only if $\mathbf{D}(t, \sum_{k=1}^n \xi_k) \to \infty$ when $n \to \infty$ for every $t \in R^m$."

Mitalauskas, A. 5455
An asymptotic expansion for independent random variables in the case of a stable limiting distribution. (Russian. Latvian and German summaries)
Litovsk. Mat. Sb. **3** (1963), *no.* 1, 189–193.
Author's summary: "In dieser Note ist die asymptotische Entwicklung für die charakteristische Funktion (ch. F.) von normierten Summen der unabhängigen Zufallsgrößen erhalten, wenn die Zufallsgrößen unidentisch verteilt sind, und die ch. F. der normierten Summe gegen die ch. F. des stabilen Gesetzes mit $\alpha \neq 1$ strebt. Die Entwicklung ist nach die Analogen der Liapunovschen Brüchen durchführt. Man beweist, daß die Koeffizienten bei diesen Analogen in Bezug auf n gleichmäßig beschränkt sind."

Orazov, G. 5456
Limit theorems for a random number of random terms. (Russian)
Izv. Akad. Nauk Turkmen. SSR Ser. Fiz.-Tehn. Him. Geol. Nauk **1964**, *no.* 1, 24–29.
The Berry-Esseen estimate for the remainder term in the central limit theorem is extended to the distribution of $\xi_0+\xi_1+\cdots+\xi_\nu$, where $\xi_0, \xi_1, \cdots$ are independent and identically distributed with finite third moments and ν is independent of $\{\xi_j\}$ and has a Poisson or binomial distribution. *W. Hoeffding* (Chapel Hill, N.C.)

Rosenblatt-Roth, M. 5457
Some theorems concerning the strong law of large numbers for non-homogeneous Markov chains.
Ann. Math. Statist. **35** (1964), 566–576.
A sequence of random variables $\{\xi_n\}$ is said to satisfy the strong law of large numbers if there exists a sequence of numbers $\{d_n\}$ such that $\lim_{n\to\infty} [(1/n) \sum_{i=1}^n \xi_i - d_n] = 0$ a.e., and the normal strong law of large numbers if it is possible to choose $d_n = (1/n) \sum_{i=1}^n E\xi_i$ in the above definition. The author considers sequences $\{\xi_n\}$ which form a Markov chain, and gives several sufficient conditions for the normal strong law in terms of the so-called ergodic coefficients of the transition functions. Unfortunately, this concept is not well known, at least in this country, and while references are given, this paper would be considerably improved if the definitions and basic results had been included.
The author also proves a necessary condition for the strong law. *J. R. Blum* (Albuquerque, N.M.)

Survila, P. 5458
On a local limit theorem for density functions. (Russian. Latvian and German summaries)
Litovsk. Mat. Sb. **3** (1963), *no.* 1, 225–236.
Let $\xi_1, \xi_2, \cdots$ be independent random variables, $E\xi_k=0$, $E\xi_k^2=\sigma_k^2$, and let each ξ_k have a probability density $p_k(x)$ bounded by a constant C_k. Let $\bar{p}_n(x)$ denote the probability density of

$$\bar{S}_n = (\xi_1+\cdots+\xi_n)/(\sigma_1^2+\cdots+\sigma_n^2)^{1/2},$$
$$\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2).$$

Theorem 1 states that $\sup_x|\bar{p}_n(x)-\phi(x)|\to 0$ under fairly simple conditions; they are satisfied, for instance, if $\bar{S}_n$ obeys the central limit theorem and $\sigma_k^2 C_k^2$ is bounded. Theorem 2 gives a sufficient condition for $|x|^m|\bar{p}_n(x)-\phi(x)|\to 0$ as $n\to\infty$, where $0\le m\le 2$.
W. Hoeffding (Chapel Hill, N.C.)

Cramér, Harald 5459
Model building with the aid of stochastic processes. (French summary)
Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 3–30.
A survey of the most commonly used stochastic models, with 58 references. *D. G. Kendall* (Cambridge, England)

Zítek, F. 5460
Convergence des suites de processus stochastiques. Applications à la théorie des files d'attente. (English summary)
Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 329–333.
Let $X(t)$ be a stochastic process, taking non-negative integral values, with independent increments such that $P(X(0)=0)=1$, $EX(t)\le EX(T)<\infty$, $T>t$. The process is said to be regular if $M(t)=EX(t)$ is continuous and said to be singular if $M(t)$ is constant except in a set of at most denumerable points, given in advance, which are called singularity points. Let $X^{(n)}(t)$, $0<t<1$, be a sequence of singular processes whose singularity points are jk_n^{-1}, $j=1, 2, \cdots, k_n$, with $k_n\to\infty$. The conditions that the limit process $X(t)$ of $X^{(n)}(t)$ exists in the sense that $\lim_{n\to\infty} P(X^{(n)}(t)=k)=P(X(t)=k)$ for all $0<t<1$ and $k=0, 1, 2, \cdots$, and the condition that $X(t)$ is regular, are directly obtained from the general results of E. G. Kimme [Trans. Amer. Math. Soc. **84** (1957), 208–229; MR **18**, 770]. The author gives a sufficient condition which guarantees the regularity of the limit process $X(t)$ and also gives a condition which assures the absolute continuity of $M(t)$. Also he shows that if a regular process or a process with absolutely continuous $M(t)$ is given, then there exists a sequence of singular processes satisfying respective conditions given by him, which converges to the given process. *T. Kawata* (Washington, D.C.)

Pitcher, Tom S. 5461
On adding independent stochastic processes.
Ann. Math. Statist. **35** (1964), 872–873.

For a stochastic process x on an interval T, let P_x be its canonical measure on path space Ω. Let $M_x=\{f\in\Omega: P_{x+f}\prec P_x\}$. Let x, y now be separable and independent processes. Let P and Q be the canonical measures on $\Omega\times\Omega$ associated with the vector processes (x, y) and $(x+y, y)$, respectively. Then it is shown that $Q\prec P$ if and only if $P_y(M_x)=1$. In the latter case, $dQ/dP(\cdot, g)=dP_{x+g}/dP_x$, for P_x a.e., g in Ω. *J. Feldman* (Berkeley, Calif.)

Nedoma, Jiří 5462

Über die Ergodizität und r-Ergodizität stationärer Wahrscheinlichkeitsmasse.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 90–97 (1963).

Let (X, Π) be a given measurable space. Let T together with its inverse be a one-to-one measurable transformation of X into itself with T^n defined by $T^n(x)=T^{n-1}(T(x))$ for $x\in X$, $T^0(x)=x$ and $T^{-n}=(T^n)^{-1}$. The probability measure μ defined on the Borel field Π is said to be periodic with period r if r is the smallest integer such that $\mu(E)=\mu(T^r(E))$ for all $E\in\Pi$. If $r=1$, the probability measure μ is said to be stationary. The set $E\in\Pi$ is said to be r-invariant if $E=T^r(E)$ and is said to be (r,μ)-indecomposable if $\mu(E)>0$, $E=T^r(E)$, and for every r-invariant measurable subset $F\subset E$, $\mu(F)>0$ implies $\mu(F)=\mu(E)$. The probability measure μ is r-ergodic if it is periodic with period r and the set X is (r,μ)-indecomposable. If $r=1$, the simpler terms invariant sets, μ-indecomposable sets and ergodic probability measures are used. If μ is a periodic probability measure with period r, one defines $\mu^{[r]}(E)=(1/r)\sum_{i=0}^{r-1}\mu(T^i(E))$ for all $E\in\Pi$. The main result of the paper is the following theorem. Let r be a natural number, $r\geqq 1$. Let μ be an ergodic probability measure, then there exists an r-ergodic probability measure π such that $\mu=\pi^{[r]}$. Two preliminary lemmas and proofs are too involved to detail here.

For related work see Jacobs [Trans. 2nd Prague Conf. Information Theory, pp. 231–249, Publ. House Czechoslovak Acad. Sci., Prague, 1960; MR **24** #B2502] and Winkelbauer [ibid., pp. 685–831; MR **23** #B2093].

S. Kullback (Washington, D.C.)

Pop-Stojanović, Zoran 5463

Sur les propriétés des champs aléatoires continus et homogènes.

C. R. Acad. Sci. Paris **257** (1963), 3286–3289.

Author's summary: "Le sujet de cette Note est l'étude des champs aléatoires, continus et homogènes donnés dans R^2, se présentant comme des notions analogues au processus aléatoire complexe stationnaire dans le sens général, et qui furent introduits dans les travaux de Yaglom, Ito et Tschang-Tse-Pay."

M. Rosenblatt (Providence, R.I.)

Mahowald, Mark 5464

On the measurability of stochastic processes.

Illinois J. Math. **8** (1964), 312–315.

Roughly speaking, the author proves that if a stochastic process $x_t(s)$ has a standard modification $y_t(s)$ which is jointly measurable in t and s, then the Kakutani version (i.e., the function space version) is jointly measurable in "time and space" variables. This solves, in some sense, the problem raised by Doob [Bull. Amer. Math. Soc. **53** (1947), 15–30; MR **8**, 472] of the joint measurability of the Kakutani version, reducing it to the problem of determining whether or not the original process has a measurable modification.

D. L. Hanson (Columbia, Mo.)

Nevel'son, M. B. 5465

The behaviour of an invariant measure of a diffusion process with small diffusion on a circle. (Russian. French summary)

Teor. Verojatnost. i Primenen. **9** (1964), 139–146.

We are given a "dynamical system" $dx=B(x)\,dt$ on $0\leqq x\leqq 1$, and associated with it the diffusion equation $u_t=B(x)u_x+\varepsilon u_{xx}$. The problem is to study the limiting behavior of the invariant measure of the associated diffusion process when the diffusion constant $\varepsilon\to 0$. In the first theorem it is shown, under the condition $B(x)>0$, that the density $P_\varepsilon(x)$ of the invariant measure tends uniformly to $[B(x)]^{-1}$, the invariant measure of the dynamical system. The following two theorems concern the case when $B(x)>0$ except for a finite number of zeros x_i, i.e., when the dynamical system does not have a stable equilibrium position. Now the limiting invariant measure $\lim_\varepsilon \int_V P_\varepsilon(dx)$ is concentrated at those among the points x_i where B has a zero derivative of the highest order. Finally, when the dynamical system has stable equilibrium positions, it is shown that the limiting invariant measure is concentrated in several of these points. The situation here is more complicated than in the analogous classical results of Pontryagin et al. (1933) and S. Bernstein (1933) where B was assumed to be the gradient of a potential. *F. L. Spitzer* (Ithaca, N.Y.)

Mil'man, V. D.; Myškis, A. D. 5466

Random shocks in linear dynamical systems. (Russian)

Approximate methods of solving differential equations, pp. 64–81. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

Let $t_0<t_1<t_2\to\infty$ be a sequence, $A=\|a_{ij}\|_1{}^p$ a matrix of real numbers and ξ^0, $\Delta_i\xi$, $i=0, 1, \cdots$, mutually independent p-dimensional random vectors. The system of equations

$$(1)\qquad \frac{d\xi}{dt}=A\xi+\sum_{t_i<t}\Delta_i\xi\delta(t-t_i),\qquad \xi(t_0)=\xi^0,$$

is called a linear dynamical system with random shocks. It is supposed that all characteristic values of A have negative real parts. If ζ_1, ζ_2 are two random vectors with distribution functions F_1 and F_2, the authors set

$$\rho(\zeta_1,\zeta_2)=\sup_u\int_{|x-u|\leq 1}|F_1(x)-F_2(x)|\,dx,$$
$$\rho'(\zeta_1,\zeta_2)=\operatorname{Var}|F_1-F_2|.$$

For sequences of vectors with uniformly bounded second moments, the convergence in the metric ρ is equivalent with weak convergence of distributions. Let

$$\frac{d\eta}{dt}=A\eta+\sum_{t_i<t}\Delta_i\eta\delta(t-t_i),\qquad \eta(t_0)=\eta^0,$$

be a dynamical system of the same type as (1). If the second

moments of $\Delta_i\xi$, $\Delta_i\eta$ are uniformly bounded, $\inf(t_{i+1}-t_i)>0$ and $\lim_{n\to\infty}\rho(\Delta_n\xi, \Delta_n\eta)=0$, then $\lim_{t\to\infty}\rho(\xi(t), \eta(t))=0$. If, moreover, $\lim_{n\to\infty}\rho'(\Delta_n\xi, \Delta_n\eta)=0$, then we may write

$$\xi(t) = \xi_1(t)+\xi_2(t),\ \eta(t) = \eta_1(t)+\eta_2(t),$$

with

$$\lim_{t\to\infty}\rho'(\xi_1(t), \eta_1(t)) = 0,\ \lim_{t\to\infty}(E|\xi_2(t)|^2+E|\eta_2(t)|^2) = 0.$$

Petr Mandl (Prague)

Kalmykov, G. I. 5467

On the reversibility of Feller processes. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 251–254.

A stationary Markov process whose state space is an open linear interval I is considered. The process is of Feller type, that is, the class of continuous functions on I with finite limits at the endpoints is supposed invariant under the usual semigroup. There is a stationary density $p(x)$ and a transition density $p(t, x, y)$. If $p(x)p(t, x, y)$ is symmetric in (x, y), the process is "reversible". Theorem 1: If each endpoint of I is inaccessible or reflecting and if the backward differential equation is satisfied, then the forward one is also and the process is reversible. Theorem 2: If both backward and forward equations are satisfied, the process is reversible and each boundary point is either inaccessible or reflecting. *J. L. Doob* (Urbana, Ill.)

Kalmykov, G. I. 5468

Correlation functions of a Gaussian Markov process. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 495–498.

Let $\{X_1(t), X_2(t)\}$ be a bivariate stationary Gaussian Markov process with zero means and unit variances. The author finds all the possible matrices $\|r_{ij}(t)\|$ $(i, j=1, 2)$ such that $r_{ij}(t)=E\{X_i(s)X_j(s+t)\}$ for some such process. *L. Weiss* (Ithaca, N.Y.)

Aleškjavičene, A. 5469

A sharpening of limit theorems for homogeneous Markov chains. (Russian. Latvian and German summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 1, 9–20.

Suppose $\xi(t)$ to be a stationary Markov chain with state space Ω, F being a σ-algebra of subsets of Ω. The n-step transition functions $p^{(n)}(\omega, A)$ are assumed to satisfy the condition that for some integer k and for some $\delta<1$, $\sup|p^{(k)}(\omega, A)-p^{(k)}(\omega', A)|=\delta<1$, the sup being over all ω, ω' in Ω and all $A\in F$. This implies the existence of an invariant probability measure which is taken as the measure of $\xi(0)$. Given an arbitrary F-measurable real-valued function X on Ω, let $X_n=X(\xi_n)$, $n=0, 1, \cdots$. The author finds conditions under which the sequence of dependent random variables X_n belongs to the domain of attraction of a stable law of index $0<\alpha<2$, $\alpha\neq 1$. These conditions involve moments of the signed measure $F(x)-V(x)$ (F is the distribution of X_1 and V the appropriate stable distribution) and also moments of multivariate distributions corresponding to partial sums of the X_n. Not only the precise conditions but also the precise error estimates obtained are too complicated to reproduce here. The results extend recent work by Siraždinov and Nagaev who obtained, in the same setting, sufficient conditions and error estimates for the convergence of the partial sums $X_1+\cdots+X_n$ to the normal law. *F. L. Spitzer* (Ithaca, N.Y.)

Khas'minskii, R. Z. [Has'minskiĭ, R. Z.] 5470

On the stability of the trajectory of Markov processes.

Prikl. Mat. Meh. **26** (1962), 1025–1032 (*Russian*); *translated as J. Appl. Math. Mech.* **26** (1963), 1554–1565.

The author discusses sufficient conditions for stability of the solution of a stochastic differential equation $dX/dt=b(X(t, w), t)+\sigma(X(t, w), t)\dot\xi(t, w)$ where $b(0, t)\equiv 0$ and $\sigma(0, t)\equiv 0$. A trajectory $X(t)\equiv 0$ is said to be stable for $t\geq t_0$ if $\varepsilon>0$ and $s\geq t_0$ imply $\lim_{x\to 0}P_{s,x}\{\sup_{t>s}|X(t)|>\varepsilon\}=0$.

Author's summary: "A study is made of the problem of stability under different assumptions (than those considered by Kac and Krasovskiĭ [Prikl. Mat. Meh. **24** (1960), 809–823]). Only continuous Markov processes of the diffusion type are considered. A necessary and sufficient condition for the stability of such processes is found which is analogous to Liapunov's second method. The relation between the stability of a system of ordinary equations and the stability of stochastic systems obtained from the former by the addition of diffusion is also investigated." *D. Austin* (Evanston, Ill.)

Vorob'ev, N. N. 5471

On families of random transitions. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 53–71.

Author's summary: "It is important for game-theoretical purposes to extend the notion of Markov transition probabilities to the case of consistent families of measures. To the families of random transitions is applicable the notion of consistency and if so, that of extensibility.

"Theorem: The regularity of complex $\mathfrak{K}$ is the necessary and sufficient condition of extensibility of any consistent family of random transitions connected with complex $\mathfrak{K}$.

"Some of the random transitions and their families can be found as limits of sequences of conditional probabilities obtained from consistent families of probability measures. The set of such families of random transitions is described and a criterion whether a family of random transitions belongs to this set is given." *G. Marinescu* (Bucharest)

Krickeberg, Klaus; Pauc, Christian 5472

Martingales et dérivation.

Bull. Soc. Math. France **91** (1963), 455–543.

This paper contains a detailed general approach to martingale theory for directed index sets. Let θ, the parameter set, be an arbitrary directed set. A Boolean σ-algebra $\mathfrak{B}$ is given, together with a "stochastic base" $\{\mathfrak{B}_\tau, \tau\in\theta\}$, where $\mathfrak{B}_\tau$ is a sub-σ-algebra of $\mathfrak{B}$. All algebras have the same zero and unit element and $\mathfrak{B}$ is monotone increasing. The stochastic base is fixed throughout. If, for each τ in θ, ϕ_τ is a σ-additive extended real-valued function on $\mathfrak{B}_\tau$, $\Phi=\{\phi_\tau\}$ is a "premartingale". If $\phi_{\tau_1}\leq\phi_{\tau_2}$ on $\mathfrak{B}_{\tau_1}$ whenever τ_1 precedes τ_2, Φ is a "submartingale", or "martingale" if there is equality. This formulation of martingale theory in terms of set functions goes back to Bochner [Ann. of Math. (2) **62** (1955), 162–169; MR **17**, 167], whose parameter sets were also only partially ordered. A martingale defines an additive function (equal

to ϕ_τ on $\mathfrak{B}_\tau$) on the algebra $\mathfrak{A}=\bigcup_\tau\mathfrak{B}_\tau$. Notions like that of σ-additivity, absolute continuity and so on, applicable to this set function on $\mathfrak{A}$, can be translated into corresponding notions for the generating martingale, but independent definitions are also given. Thus a martingale of bounded variation has a unique decomposition into the sum of a σ-additive martingale and a purely finitely additive one. In the positive case, the first component is the greatest σ-additive martingale dominated by the given one. A submartingale Φ defines a martingale Φ^m, where $\phi_\tau{}^m=\lim_{\tau'}\phi_{\tau'}$ on $\mathfrak{B}_\tau$, under a certain finiteness hypothesis, and the martingale thus obtained is the smallest one dominating Φ.

If a premartingale is given by $\phi_\tau(A)=\int_A f_\tau\,d\mu$, where f_τ is measurable on $\mathfrak{B}_\tau$ and μ is strictly positive σ-finite and σ-additive on each $\mathfrak{B}_\tau$ and $\mathfrak{B}$, the family of integrands is a "stochastic process", an "integrating representation of Φ". If Φ is a martingale of bounded variation generated in this way, Φ has a unique decomposition into the sum of a purely singular (relative to μ) martingale and an absolutely continuous one Φ_{ac}, where $\phi_{ac\,\tau}(A)=\int_A f_\omega\,d\mu$ and $f_\omega=$ stoch $\lim_\tau f_\tau$. Martingale convergence theorems for the integrating representations are proved under various hypotheses. For example, if Φ is a martingale with integrating representation $\{f_\tau\}$, if each f_τ is in a given Orlicz space and if the family of norms of these functions is bounded, there is stochastic convergence of the process to an element f_ω in the Orlicz space and $\phi_\tau(A)=\int_A f_\omega\,d\mu$.

In the application to derivation each $\mathfrak{B}_\tau$ is usually generated by a partition, and the order relation is fineness of the partition. Both stochastic and essential convergence of generalized function sequences are used and Vitali-type covering hypotheses play an essential role. Throughout the paper the completeness of the approach leads to numerous improvements in old results and to new ones. For earlier work on martingale theory with partially ordered index sets see Krickeberg [a series of papers beginning with Trans. Amer. Math. Soc. **83** (1956), 313–337; MR **19**, 947], Helms [ibid. **87** (1958), 439–446; MR **20** #1350], and Chow [ibid. **97** (1960), 254–285; MR **22** #12565]. *J. L. Doob* (Urbana, Ill.)

Paz, A. 5473

Graph-theoretic and algebraic characterizations of some Markov processes.

Israel J. Math. **1** (1963), 169–180.

Author's summary: "An algebraic decidable condition for a stationary Markov chain to consist of a single ergodic set, and a graph-theoretic decidable condition for a stationary Markov chain to consist of a single ergodic noncyclic set are formulated. In the third part of the paper a graph-theoretic condition for a non-stationary Markov chain to have the weakly ergodic property is given." *J. L. Snell* (Hanover, N.H.)

Ney, P. E. 5474

Generalized branching processes. I. Existence and uniqueness theorems.

Illinois J. Math. **8** (1964), 316–331.

The system which is investigated in this paper may be described as follows. At time 0 we have a parent particle of energy x_0 which at time T splits into N new particles of energies, $x_1, \cdots, x_N$, respectively. Each of these new particles then behaves as if it were a parent particle, the behavior being independent of any other existing particle. In this paper the author investigates the total energy $x(t)$ of all particles in existence at time t, and the closely related quantity $Y(t)$ of the total energy of the system up to time t. Heuristic considerations lead to an integral equation which is satisfied by the Laplace-Stieltjes transform of $x(t)$. Rather than trying to justify this integral equation directly, the author starts with the equation and then, using analytical arguments, shows there is a unique probabilistically significant solution. Criteria are given for moments to exist, and the equations satisfied by these moments when they do exist are derived. *S. C. Port* (Santa Monica, Calif.)

Ney, P. E. 5475

Generalized branching processes. II. Asymptotic theory.

Illinois J. Math. **8** (1964), 332–350.

In this paper the author continues his investigation of the quantity $x(t)$ for the generalized branching process described above [#5474]. Using similar techniques to those used in the investigation of age-dependent branching processes it is shown that the moments for $x(t)$ satisfy renewal equations from which their asymptotic behavior may be readily derived. It is also shown that under certain conditions, $x(t)/ex(t)$ converges in the mean square to a random variable w. Some properties of w are also given. In the case of exponential generation times this convergence result is improved to convergence with probability one by the observation that $\{x(t)/ex(t)\}$ is in this case a martingale. *S. C. Port* (Santa Monica, Calif.)

Okonjo, Chukuka 5476

★Über stationäre Null-Eins-Prozesse.

Inaugural-Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Universität zu Köln, Cologne, 1962. v+50 *pp.*

A gap process is a stationary process $\{y_n, n\geqq 0\}$ of zeros and ones. If the ones occur at the integers $n_1, n_2, \cdots$, then $\lambda_t=n_t-n_{t-1}$ is called the length of the tth gap. The author begins with a discussion of gap processes and the related sequences of gaps, referring to work of Akaike [Ann. Inst. Statist. Math. **8** (1956), 87–94; MR **19**, 327; ibid. **10** (1959), 233–259; MR **21** #4488] and Wever [Math. Z. **71** (1959), 283–288; MR **21** #7570]. A gap process is viewed also as a functional of a Markov chain $\{x_n\}$ with states $U_0, U_1, \cdots$: $y_n=f(x_n)$, where $f(U_0)=1$, $f(U_i)=0$ for $i\neq 0$. More generally, a Markov gap process is defined as a zero-one-valued functional of a stationary Markov chain. The sequence of gaps is then a functional of a Markov chain. If the gap process has bounded gaps which themselves form a Markov chain, it has an absolutely continuous spectral function whose spectral density is rational in $\exp 2\pi i\lambda$. *H. D. Brunk* (Columbia, Mo.)

Kerstan, Johannes; Matthes, Klaus 5477

Stationäre zufällige Punktfolgen. II.

Jber. Deutsch. Math.-Verein. **66** (1963/64), *Abt.* 1, 106–118.

The theory of labelled random point sequences (l.r.p.s.) was developed in Part I of the paper [Matthes, same Jber. **66** (1963/64), Abt. 1, 66–79; MR **28** #3479]. The present

part deals with infinitely divisible stationary l.r.p.s. An l.r.p.s. $[M_K, \mathfrak{M}_{\mathfrak{K}}, P]$ is called infinitely divisible if it may be written as a sum of independent stationary l.r.p.s. with arbitrary small probability of occurrence of events in a bounded time interval. Let $I_1, \cdots, I_n$ be intervals, $L_1, \cdots, L_n \in \mathfrak{K}$, and let p denote the distribution of the random vector

$$X = (\mathrm{Card}\{\Phi \cap I_1 \times L_1\}, \cdots, \mathrm{Card}\{\Phi \cap I_n \times L_n\}).$$

Then p is infinitely divisible and

$$p = e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} q^m,$$

where q is a distribution on non-negative integer-valued vectors of R_n. The authors establish the existence of a measure $\bar{P}$ on $\mathfrak{M}_{\mathfrak{K}}$ such that

$$\bar{P}(X = (m_1, \cdots, m_n)) = \lambda q((m_1, \cdots, m_n)),$$

for any choice of $I_1, \cdots, I_n$ and $L_1, \cdots, L_n$. Infinitely divisible l.r.p.s. are classified to be singular if $\bar{P}(\mathrm{Card}\ \Phi < \infty) = 0$ and to be regular if $\bar{P}(\mathrm{Card}\ \Phi = \infty) = 0$. The properties of these two types of l.r.p.s. are elucidated by some theorems. *Petr Mandl* (Prague)

Kesten, H.; Spitzer, F. 5478

Ratio theorems for random walks. I.

J. Analyse Math. **11** (1963), 285–322.

Consider a completely general homogeneous random walk on a rectangular lattice R. The probability of an event E concerning paths with initial point x is denoted by $P_x(E)$ and most of the results relate to the limit of $P_x(E_1)/P_0(E_2)$ as the length of walk increases. The methods are essentially elementary in the spirit of recent work of Spitzer [J. Math. Mech. **11** (1962), 593–614; MR **25** #2655].

If Ω is any subset of the lattice, $P_x(n, r, \Omega)$ denotes the probability that a walk of length n starting from x visits Ω r times. The principal theorem proved is that $\lim_{n\to\infty} P_x(n, r, \Omega)/P_0(n, 0, \{0\})$ exists for every bounded Ω for every aperiodic random walk. Moreover, in case the set Ω consists of one point only and the walk is persistent, then the limit is unity for every $r \geqq 1$. This represents a significant generalization of previous results of Hunt [Trans. Amer. Math. Soc. **81** (1956), 264–293; MR **18**, 54] and Kac [ibid. **84** (1957), 459–471; MR **19**, 184].

In the course of the development of the main theorem many beautiful and interesting other results are proved incidentally. For example, let r_n denote the probability that a walk starting from the origin does not return there during the first n steps. Then in one dimension, $\sup(r_n/r_{2n}) < \infty$; in higher space, $\lim_{n\to\infty} (r_n/r_{2n}) = 1$. *J. Gillis* (Rehovot)

Dudley, R. M. 5479

Pathological topologies and random walks on abelian groups.

Proc. Amer. Math. Soc. **15** (1964), 231–238.

Generalizing an earlier result [same Proc. **13** (1962), 447–450; MR **25** #4578] concerning the existence of a recurrent random walk on discrete abelian groups, the author proves Theorem 1: A locally compact abelian group G has a recurrent random walk if and only if (a) for every open subgroup H of G, G/H is countable, and (b) G does not have a discrete subgroup free abelian on three generators. The assertion of Theorem 2, that any countable abelian group may be discretely imbedded in a metrizable abelian group with a dense cyclic subgroup, implies in particular that "locally compact" cannot be replaced by "complete separable metric" in Theorem 1. An amusing corollary of Theorem 2: There is a topology for the abelian group Z_4 for which it has a recurrent random walk, and in which there is a subgroup free on three generators with discrete relative topology. [A random walk on a group G is, of course, called recurrent if every open $U \subset G$ is visited infinitely often with probability one.] *F. L. Spitzer* (Ithaca, N.Y.)

Keilson, Julian 5480

The homogeneous random walk on the half-line and the Hilbert problem. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 279–291.

The author discusses certain Markov processes (with discrete or continuous time) on a half-line, which lead to problems familiar in queueing theory. The analysis is based on ideas from the theory of Wiener and Hopf: the factorization of characteristic functions in the complex domain leads to solutions under weaker conditions than in previous work of Smith and Lindley. *F. L. Spitzer* (Ithaca, N.Y.)

McKean, H. P., Jr. 5481

Excursions of a non-singular diffusion.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 1 (1962/63), 230–239.

Let $x(t)$ be a non-singular diffusion process on the half-line $[0, +\infty)$, with a reflecting barrier at the origin, whose precise description is given in Section 2 of the present paper. The random t-set of roots of $x(t) = 0$ is denoted by $\mathfrak{Z}$. The Lebesgue measure of $\mathfrak{Z}$ is assumed to be 0 with probability one. Let $\mathfrak{Z}_1$, $\mathfrak{Z}_2$, etc., be the open intervals (components) of the complement of $\mathfrak{Z}$, labelled in a certain way. The excursions of $x(t)$ are the stochastic processes defined by $\mathfrak{e}_1(t) = x(t + \inf \mathfrak{Z}_1)$ for $0 \leqq t \leqq |\mathfrak{Z}_1|$ = the length of $\mathfrak{Z}_1$, $\mathfrak{e}_2(t) = x(t + \inf \mathfrak{Z}_2)$ for $0 \leqq t \leqq |\mathfrak{Z}_2|$, etc. As a special case, the excursions of a reflecting Brownian motion on $[0, +\infty)$ were investigated by Lévy [*Processus stochastiques et mouvement brownien*, pp. 225–237, Gauthier-Villars, Paris, 1948; MR **10**, 551] and Itô and McKean [*Diffusion processes and their sample paths*, Chapter 2, Sections 9–11, Springer, Berlin, 1964 (to appear)].

In the present paper it is shown how one can construct the excursions of $x(t)$ conditional on $\mathfrak{Z}$, using only the absorbing process $x_0(t)$ derived from $x(t)$ (i.e., the process stopped at $\mathfrak{m}_0 = \inf\{t > 0, x(t) = 0\}$). The author derives the idea from the following observation for Brownian excursions. Let $r_-(t)$, $r_+(t)$ be two independent copies of the Bessel motion (i.e., the radial part of a 3-dimensional Brownian motion starting at the origin). Consider the motion defined by $r_-(t)$ for $0 \leqq t \leqq |\mathfrak{Z}_1|/2$ and by $r_+(|\mathfrak{Z}_1| - t)$ for $|\mathfrak{Z}_1|/2 \leqq t \leqq |\mathfrak{Z}_1|$, tied so as to have $r_\pm(|\mathfrak{Z}_1|/2) = a$. Itô and McKean found that the above tied motion is identical in law to the Brownian excursion $\mathfrak{e}_1(t)$ conditional on $\mathfrak{Z}$ and $\mathfrak{e}_1(|\mathfrak{Z}_1|/2) = a$. The Bessel motion, however, is nothing but the Brownian "h-path" process corresponding to the minimal harmonic function $h(b) = b$. {Brownian h-path processes were introduced by Doob [Bull. Soc. Math.

France **85** (1957), 431–458; MR **22** #A844] in the study of Martin boundaries of a Green space.} Moreover, by a simple computation, one gets (*) $\lim_{s\to\infty} P(B|\mathfrak{m}_0>s)=\hat{P}_a(B)$, where $P_a(\hat{P}_a)$ is the probability law of the reflecting Brownian motion paths (the Bessel motion paths) with starting point a and B, an event of the past before t for some finite t. {Note that each side of (*) is determined by the absorbing Brownian motion alone.} The author shows that a similar argument still works for the general case. More precisely, a non-singular diffusion process (for short, "dot motion") satisfying (*) is obtained as an h-path process (modified with mass creation if h is not harmonic) of the absorbing process $x_0(t)$ corresponding to a specified function h, superharmonic (not harmonic in general) relative to $x_0(t)$, and the dot motion plays the same role as the Bessel motion for Brownian excursions to construct the excursions of $x(t)$ conditional on $\mathfrak{Z}$. Finally the author gives an application to the behavior of Brownian motion paths in higher dimensions. {There are several misprints: The constant coefficients in (3b), p. 230, (5), p. 231 and (6), p. 238 should be corrected. 1 for 0 on the right side of (1c), p. 234 and $\mathfrak{G}-\gamma_1$ for $\mathfrak{G}$ in (5), p. 235.}

T. Watanabe (Urbana, Ill.)

Hinderer, Karl 5482

Bemerkungen zu räumlich homogenen Gitterirrfahrten.

Math. Ann. **153** (1964), 14–20.

Formulas are derived for the n-step transition probabilities of an N-dimensional, spatially homogeneous, lattice-valued random walk—where by random walk is meant the sequence of partial sums of a sequence of independent random vectors. Special attention is given to the case when the walk is stationary, and when only unit steps in the direction of the coordinate axes are allowed. The method of derivation involves classical use of characteristic functions. Some remarks are added on the question of recurrence of possibly non-stationary walks.

P. E. Ney (Ithaca, N.Y.)

Moran, P. A. P. 5483

Some general results on random walks, with genetic applications.

J. Austral. Math. Soc. **3** (1963), 468–479.

The author starts by considering a finite-state continuous-parameter birth-and-death process with two reflecting barriers. He remarks that the transition probabilities for such processes are known in the form of Laplace-Stieltjes integrals [S. Karlin and J. L. McGregor, Trans. Amer. Math. Soc. **85** (1957), 489–546; MR **19**, 989; ibid. **86** (1957), 366–400; MR **20** #1363]. He shows here how by determinantal methods one can find the Laplace transforms of the $p_{ij}(t)$ with regard to t. Some examples are looked at in detail, and it is noted that similar techniques are available for random walks (discrete-parameter case). The author then turns to finite-state random walks with one absorbing and one reflecting state, and to chains with two absorbing states satisfying a symmetry condition such that by identifying pairs of states it can be reduced to the previous case. Mean passage times to absorption are calculated, and the results are then applied to a problem in genetics (mean time to homozygosity when the heterozygote is selectively advantageous over equally fit homozygotes).

D. G. Kendall (Cambridge, England)

Gupta, S. K.; Goyal, J. K. 5484

Queues with Poisson input and hyper-exponential output with finite waiting space.

Operations Res. **12** (1964), 75–81.

Customers arrive at a counter in accordance with a Poisson process of density λ, and are served by a single server. The service times are mutually independent, identically distributed random variables with distribution function $H(x)=\sum_{r=1}^n \sigma_r(1-e^{-\mu_r x})$ and independent of the arrival times. The maximum number of customers allowed in the system is N. If an arriving customer finds N customers in the system, then he immediately departs without being served. The authors reduce the problem of finding the generating function of the stationary distribution of the queue size to finding the zeros of a polynomial of degree n and to solving a system of $n+1$ linear equations.

L. Takács (New York)

Gupta, S. K.; Goyal, J. K. 5485

Queues with hyper-Poisson input and exponential output with finite waiting space.

Operations Res. **12** (1964), 82–88.

A single-server queue is considered. The inter-arrival times and the service times are independent sequences of mutually independent, identically distributed random variables with distribution functions $F(x)=\sum_{r=1}^n \sigma_r(1-e^{-\lambda_r x})$ and $H(x)=1-e^{-\mu x}$, respectively. The maximum number of customers allowed in the system is N. If an arriving customer finds N customers in the system, then he immediately departs without being served. The authors reduce the problem of finding the generating function of the stationary distribution of the queue size to finding the zeros of a polynomial of degree n and to solving a system of $n+1$ linear equations. {It seems that it escaped the authors' attention that the reviewer has already found an explicit expression for the stationary distribution of the queue size in the more general case of $m\geqq 1$ servers and general inter-arrival time distribution $F(x)$ [L. Takács, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **1** (1958), 73–82; MR **21** #1649].}

L. Takács (New York)

Chang, Wei 5486

Output distribution of a single-channel queue.

Operations Res. **11** (1963), 620–623.

Using an idea of E. Ventura [Management Sci. **6** (1959/60), 423–443; MR **22** #6032] the author calculates the equilibrium distribution of interdeparture times for $GI/G/1$ in terms of the equilibrium distribution of waiting time and the chance of not having to wait. It is verified that the interdeparture times have the same distribution as the interarrival times when the system has the special form $M/M/1$.

D. G. Kendall (Cambridge, England)

Craven, B. D. 5487

Asymptotic transient behaviour of the bulk service queue.

J. Austral. Math. Soc. **3** (1963), 503–512.

The author considers the queueing system $M/G/1$ (later specialised to $M/E_p/1$) with bulk service (during each service interval, a batch of N customers, or the whole queue if less than N, is served). For the stationary behaviour of this system he refers to N. T. J. Bailey

J. Roy. Statist. Soc. Ser. B **16** (1954), 80–87; MR **16**, 148]. He uses an imbedded chain associated with the epochs when service begins, using essentially double generating functions, Rouché's theorem, and Watson's lemma to obtain asymptotic formulae for the transition probabilities after a large number n of steps. Asymptotic formulae for the mean queue-length are also obtained.

D. G. Kendall (Cambridge, England)

Vere-Jones, D. 5488

A rate of convergence problem in the theory of queues. (Russian summary)

Teor. Verojatnost. i Primenen. **9** (1964), 104–112.

A Markov chain is called geometrically ergodic when the n-step transition probabilities converge (as $n\to\infty$) to their limits at geometric rates; the class-character of this phenomenon was noted by the reviewer [Bull. Amer. Math. Soc. **64** (1958), 358–362; MR **23** #A4174] and its uniformity within a class by the author [Quart. J. Math. Oxford Ser. (2) **13** (1962), 7–28; MR **25** #4571]. In this paper the author examines the Markov chains imbedded in the queueing systems $E_k/G/1$ and $GI/E_k/1$ from this point of view, and shows how the best rate of geometric convergence can be calculated from the elements of the system. The present paper should be read in parallel with another by the author [Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1963), 12–21; MR **28** #1666] in which he examines the spectral character of the linear operators associated with the same examples.

D. G. Kendall (Cambridge, England)

Heyde, C. C. 5489

On the stationary waiting time distribution in the queue $GI/G/1$.

J. Appl. Probability **1** (1964), 173–176.

A representation due to Smith [Proc. Cambridge Philos. Soc. **48** (1952), 698–717; MR **14**, 276] of the stationary waiting time distribution in the queue $GI/G/1$ is derived under the least restrictive conditions possible. This simple integral representation for the Laplace transform of the waiting time distribution is valid when the service time characteristic function $\varphi(\xi)$ is analytic in a half-plane $\operatorname{Im}\xi>-c$ $(c>0)$ and when the expected interarrival time exceeds the expected service time.

F. L. Spitzer (Ithaca, N.Y.)

Brockwell, P. J. 5490

The transient behaviour of the queueing system $GI/M/1$.

J. Austral. Math. Soc. **3** (1963), 249–256.

In the single server queueing system with general interarrival times, with distribution function $A(x)$ and exponentially distributed service times, let q_j^n denote the probability that the $(n+1)$st arrival finds more than j customers in the system. It is proved that

$$q_j^n = \sum_{\nu=j+1}^{} \frac{j+1}{\nu} \sum_{\Sigma i\alpha_i=\nu-j-1} \binom{\nu}{\sum\alpha_i} \frac{(\sum\alpha_i)!}{\prod(\alpha_i!)} \Psi_0^{\nu-\Sigma\alpha_i}\Psi_1^{\alpha_1}\cdots\Psi_{\nu-j-1}^{\alpha_{\nu-j-1}},$$

where the second summation is over all $(\nu-j-1)$-tuples $(\alpha_1,\alpha_2,\cdots,\alpha_{\nu-j-1})$ of non-negative integers such that $\sum i\alpha_i=\nu-j-1$, and Ψ_k is defined by

$$\Psi_k=\int_0^\infty \frac{x^k}{k!}e^{-x}\,dA(x).$$

It is observed that the limiting distribution of queue-size at an arrival as $n\to\infty$ is given by the infinite series of which the series for q_j^n is a partial sum. A similar technique is employed to obtain the distribution of the number of customers served in a busy period for the cases $M/M/1$ and $D/M/1$.

F. G. Foster (London)

Neuts, Marcel F. 5491

The distribution of the maximum length of a Poisson queue during a busy period.

Operations Res. **12** (1964), 281–285.

Customers arrive at a counter in accordance with a Poisson process of density λ and are served by a single server. The service times are mutually independent, identically distributed random variables with distribution function $H(x)=1-e^{-\mu x}$ $(x\geqq 0)$ and independent of the arrival times. Denote by Z_t the number of customers in the system at time t. The author proves that

$$\mathbf{P}\{0<Z_u<b \text{ for } 0\leqq u\leqq t \text{ and } Z_t=j\,|\,Z_0=i\} = (\lambda/\mu)^{(j-i)/2}(2/b)\sum_{r=1}^{b-1}\sin(ir\pi/b)\sin((b-j)r\pi/b) \times\exp[-(\lambda+\mu)t+2t\sqrt{(\lambda\mu)}\cos(r\pi/b)].$$

L. Takács (New York)

Avi-Itzhak, B.; Naor, P. 5492

Some queuing problems with the service station subject to breakdown.

Operations Res. **11** (1963), 303–320.

Customers arrive Poisson fashion at a single service station which is subject to breakdown from time to time, and then requires repair before further service can be offered. Service times and repair times have arbitrary distributions with finite second moments. An interrupted service is resumed after repair and no loss of service is involved.

Five different assumptions are made regarding the nature of the breakdowns corresponding to different industrial and other real situations: (A) breakdowns constitute a Poisson process homogeneous in time; (B) breakdowns can occur only during service; (C) breakdowns occur as in (A) but repair cannot start without customers present at the station; (D) breakdowns (interruptions of service) occur only at the initiative of a customer (who wants to improve the quality of service); (E) breakdowns occur only when no customer is present. The expected queue lengths and related operating characteristics of the five models are derived.

F. G. Foster (London)

Avi-Itzhak, B. 5493

Preemptive repeat priority queues as a special case of the multipurpose server problem. I, II.

Operations Res. **11** (1963), 597–609; *ibid.* **11** (1963), 610–619.

The author commences with the interesting remark that a queueing system with pre-emptive priorities is a special case of a multi-purpose service system, and that if we consider the service of one of the lower categories of customer we can utilise the theory (which he proceeds to construct) of a queueing system subject to random breakdowns.

The basic model is $M/G/1$ with first come, first served. Breakdowns occur Poisson-wise, and repair-time has a given arbitrary distribution. It is assumed that interrupted service is wasted; thus a deflected customer commences his service from the beginning as soon as the machine is running again. (Whence, in the application to pre-emptive priorities, it is the "repeat" model that is relevant, and not the "resume" one.) Two assumptions are possible about the service-time. (I) We may suppose that, on the several occasions when attempts are made to serve a particular customer, his successive demands for service are independently and identically distributed. The author justifies this assumption by referring to situations in which the service required is subject to variations depending rather on the fluctuating service-rate of the station than on the whim of the customer. Alternatively (II) we may suppose that each customer has a fixed demand for service (varying randomly from customer to customer, but not varying when repeated attempts are made to satisfy a particular customer). The two parts of this paper are concerned with the consequences of these two assumptions. Mean queue length and queueing time are derived, the consequences of (I) and (II) are contrasted, and some results for pre-emptive priority systems are obtained.

The author's analysis of the sources of service-time variability is very interesting and might be pressed further. Readers of these papers are advised to study the introductory sections to both, before embarking on a detailed study of either.

D. G. Kendall (Cambridge, England)

Takács, Lajos 5494

Priority queues.

Operations Res. **12** (1964), 63–74.

A single server has to serve customers of different priorities who arrive in accordance with a Poisson process. Two service disciplines are considered: (i) pre-emptive (service of a customer is interrupted at the arrival of a customer of higher priority); (ii) head of the line (no interruptions but after completion of a service period a customer of the highest priority class present is selected for the next service). In both cases customers of the same priority are served on a first come, first served basis. Let $\eta_n^*(p)$ be the time the nth arriving customer would wait if he belonged to priority class p. The author gives necessary and sufficient conditions for the existence of a stationary limiting distribution ($n\to\infty$) of $\eta_n^*(p)$ and finds the Laplace transform and the first 3 moments of the limiting distribution. These results are new for the "saturated case", i.e., when $\eta_n^*(q)\to\infty$ in probability for some $q>p$. (In formula (38) the author identifies without comment the limiting distribution of $\eta_n^*(p)$ and the limiting distribution of the waiting time of the nth departing customer. This is not obvious for a system with priorities. The undefined quantity $\psi_p(s)$ seems to be the Laplace transform of the service time distribution for priority class p.)

H. Kesten (Ithaca, N.Y.)

Takács, Lajos 5495

Combinatorial methods in the theory of dams.

J. Appl. Probability **1** (1964), 69–76.

Applications of so-called ballot theorems to theory of dams.

M. Dwass (Evanston, Ill.)

Gastwirth, Joseph L. 5496

On a telephone traffic system with several kinds of service distributions.

J. Appl. Probability **1** (1964), 77–84.

In a telephone exchange there are infinitely many available lines. Calls of s different types appear in accordance with a recurrent process. Independently of the others, p_i ($i=1, 2, \cdots, s$) is the probability that an arriving call is of type i. The holding times are mutually independent random variables and independent of the arrival times. The probability that the holding time of a call of type i is $\leqq x$ is $H_i(x)=1-e^{-\mu_i x}$ for $x\geqq 0$. For a stationary process the author determines the probability that an arriving call finds $k_1, k_2, \cdots, k_s$ calls of types $1, 2, \cdots, s$, respectively, in the system. A related particle counting problem is also discussed.

L. Takács (New York)

Beljaev, Ju. K.; Maksimov, V. M. 5497

Analytical properties of a generating function for the number of renewals. (Russian. English summary)

Teor. Verojatnost. i Primenen. **8** (1963), 108–112.

Let $\{t_i\}$ be a renewal process with positive, mutually independent, identically distributed increments with common distribution function F. Let $N_t=\max\{n|t_n<t\}$, $P_k=P\{N_t\geqq k\}$, $p_k=P\{N_t=k\}$ and $\pi_t(z)$ and $\tilde{\pi}_t(z)$ the corresponding generating functions. Theorem 1 asserts that π_t and $\tilde{\pi}_t$ are analytic in the circle of radius $1/r$ where $r=F(0+)$. Theorem 2 states that if $F(t)$ is continuous at zero, the order of the (entire) function $\tilde{\pi}_t$ does not depend on t.

H. P. Kramer (Santa Barbara, Calif.)

Finch, P. D. 5498

The limit theorem for aperiodic discrete renewal processes.

J. Austral. Math. Soc. **4** (1964), 122–128.

A new proof of the classical Erdős, Feller and Pollard theorem [Bull. Amer. Math. Soc. **55** (1949), 201–204; MR **10**, 367] is given. The idea of the proof is to regard the renewal process as a particular Markov chain, and to show that this chain is ergodic. It is claimed that the methods extend readily to continuous renewal processes.

P. E. Ney (Ithaca, N.Y.)

Širokov, F. V. 5499

On a paper of I. D. Čerkasov "On transforming the diffusion process to a Wiener process". (Russian)

Teor. Verojatnost. i Primenen. **9** (1964), 175–177.

The author remarks that a proof in a paper of I. D. Čerkasov [Teor. Verojatnost. i Primenen **2** (1957), 384–388; MR **20** #346] contains an inaccuracy and shows how this can be avoided.

Petr Mandl (Prague)

Bailey, Norman T. J. 5500

The simple stochastic epidemic: A complete solution in terms of known functions.

Biometrika **50** (1963), 235–240.

The so-called "simple stochastic epidemic" is a simplified epidemic model which involves infection but not recovery. The basic differential-difference equations for the process are

$$\frac{dp_r}{d\tau} = (r+1)(n-r)p_{r+1} - r(n-r+1)p_r \qquad (0 \leq r \leq n-1),$$

$$\frac{dp_n}{d\tau} = -np_n,$$

with initial condition $p_n(0) = 1$. The author finds an explicit solution for $p_n(\tau)$ and for its first moment.

J. Kiefer (Ithaca, N.Y.)

STATISTICS

See also 5452, 5453, 5780, 5845, 5846.

Brunk, H. D. 5501

A generalization of Spitzer's combinatorial lemma.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 395–405 (1964).

For many problems concerning identically distributed independent (or symmetrically dependent) random variables several authors have applied combinatorial identities. The present paper analyzes several such lemmas, dealing with certain one-to-one correspondences between ordered subsets of the first n integers and permutations decomposed into cycles. Statistical applications deal with generalized means, random permutations, records, and a test against trend for medians.

F. L. Spitzer (Ithaca, N.Y.)

Severo, Norman C.; Montzingo, Lloyd J., Jr. 5502

Convergence to normality of powers of a normal random variable. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 491–500.

Authors' summary: "Soit Y une variable aléatoire qui suit la distribution normale, et p un nombre réel tel que y^p est aussi réel, $-\infty < y < \infty$. Soit μ_y et σ_y la moyenne et l'écart-type de Y, respectivement. Nous avons établi que la distribution de $X = Y^p$ tend vers la distribution normale avec moyenne μ_x, et écart-type σ_x, si p est un entier positif, quand $\mu_y/\sigma_y \to \infty$. Autrefois, $X = Y^p$ tend vers la distribution normale avec moyenne $\mu_0 = \mu_y{}^p[1 + O(\eta^{-2})]$, et écart-type

$$\sigma_0 = (|p\mu_y{}^p|/\eta)[1 + O(\eta^{-2})]^{1/2}.$$

"Quand μ_x et σ_x existent, nous avons établi que pour que $\mu_x/\sigma_x \to \infty$ il faut et il suffit que $\mu_y/\sigma_y \to \infty$."

P. S. Dwyer (Ann Arbor, Mich.)

Csiszár, I.; Dobó, A. 5503

A method for the elimination of systematic errors. (Hungarian)

Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **12** (1962), 123–132.

An observation has a total error of magnitude $\zeta = \xi + \eta$, where ξ is the systematic error and η is the random error. The authors deal with the problem of how to correct the systematic error by making observations of the total error. If $\zeta_1, \zeta_2, \cdots, \zeta_n$ are the total errors in n observations, a correction of magnitude $\mathbf{E}\{\xi|\zeta_1, \cdots, \zeta_n\}$ is made. The following results are obtained: (i) If either a single correction is made after n observations or a correction is made after each of the n observations, then in both cases the corrected systematic error has the same distribution provided that in both cases the systematic error and the random errors have the same joint distribution. (ii) If the strong law of large numbers holds for the random errors, then

$$\lim_{n\to\infty} \mathbf{E}\{\xi|\zeta_1, \cdots, \zeta_n\} = \xi \text{ and } \lim_{n\to\infty} \mathbf{Var}\{\xi|\zeta_1, \cdots, \zeta_n\} = 0$$

with probability 1. Finally, some examples are also given.

L. Takács (New York)

Seber, G. A. F. 5504

The non-central chi-squared and beta distributions.

Biometrika **50** (1963), 542–544.

The author gives finite expansions (1) for the non-central chi-square distribution $G[y; n_1, \lambda]$ when n_1 is an odd integer, and (2) for the non-central beta distribution $B[x; a, b, \lambda]$ when b is an even integer. Here λ is the non-centrality parameter while u_1, a and b are the degrees of freedom.

E. Lukacs (Washington, D.C.)

Makabe, Hajime 5505

On approximations to some limiting distributions with applications to the theory of sampling inspections by attributes.

Kōdai Math. Sem. Rep. **16** (1964), 1–17.

Poisson approximation to the binomial, negative binomial, and Poisson binomial distribution are treated. In addition, the normal approximation to the Poisson binomial is also treated.

S. C. Port (Santa Monica, Calif.)

Walsh, John E. 5506

Approximate distribution of extremes for nonsample cases.

J. Amer. Statist. Assoc. **59** (1964), 429–436.

Author's summary: "The data are n univariate observations that are not necessarily from the same population, from continuous populations, or independent. The problem is to develop an approximate expression for a specified one of the upper (or lower) extremes of this set of observations when a type of m-dependence occurs. General expressions are developed that depend on n, the extreme considered, and the arithmetic average of the cumulative distribution functions (cdf's) for the individual observations. These results seem to be in a form that is satisfactory for practical applications, and a rule is given for deciding when n is large enough for their use. Also, a basis is furnished for deciding on the suitability of the model from the characteristics of the experimental situation, and consistent estimation of the average of the cdf's is considered. In practice, the asymptotic distributions that occur for the case of samples from continuous populations also seem to frequently occur for nonsample cases involving data that are not necessarily continuous. Some reasons are given which tend to explain the frequent occurrence of these types of asymptotic distributions."

E. J. Gumbel (Hamburg)

Sur, M. G. 5507

Kurtosis functions and additive functionals of Markov processes. (Russian)

Dokl. Akad. Nauk SSSR **143** (1962), 293–296.

Singh, S. N. 5508

A note on inflated Poisson distribution.

J. Indian Statist. Assoc. **1** (1963), 140–144.

The distribution function G of the inflated Poisson distribution is defined as $G(x)=(1-\alpha)F_1(x)+\alpha F_2(x;\lambda)$, where F_1 is the distribution function of a random variable whose distribution is degenerate at zero and F_2 is the distribution function of a random variable whose distribution is Poisson with parameter λ. The parameter space is $\Omega=\{(\alpha,\lambda): 0\leqq\alpha\leqq 1,\ 0<\lambda<\infty\}$; however, in finding the maximum likelihood estimates $\hat{\alpha}$ and $\hat{\lambda}$ of the parameters, the author finds the maximum of the likelihood function in the space $\Omega^*=\{(\alpha,\lambda): 0\leqq\alpha<\infty,\ 0<\lambda<\infty\}$; in Ω^*, $\hat{\alpha}$ and $\hat{\lambda}$ are solutions of $\hat{\lambda}=[n\bar{x}/(n-n_0)](1-\exp[-\hat{\lambda}])$ and $\hat{\alpha}\hat{\lambda}=\bar{x}$, where n is the sample size, n_0 is the number of zero observations in the sample and $\bar{x}$ is the sample mean. The reviewer wishes to point out that if the maximum of the likelihood function in Ω^* is not in Ω, then the maximum of the likelihood function in Ω occurs at $\hat{\alpha}=1$, $\hat{\lambda}=\bar{x}$. The asymptotic distributions of $n^{1/2}(\hat{\alpha}-\alpha)$ and of $n^{1/2}(\hat{\lambda}-\lambda)$ are obtained, and comparisons of the fit of the inflated Poisson distribution, the negative binomial distribution, the Neyman type A distribution and the Poisson binomial distribution are made for three sets of data for which $\lambda<1$; some table entries differ from their published source. *D. R. Barr* (Dayton, Ohio)

Blischke, W. R. 5509

Estimating the parameters of mixtures of binomial distributions.

J. Amer. Statist. Assoc. **59** (1964), 510–528.

Author's summary: "Let $Y_1,\cdots,Y_m$ be independent, each having distribution

$$P\{Y_i=y\}=\sum_{j=1}^{r}\alpha_j\binom{n}{y}(1-p_j)^{n-y}p_j^{\,y}\qquad (y=0,1,\cdots,n),$$

where $0<p_1<\cdots<p_r<1$, $0<\alpha_j<1$, $\sum\alpha_j=1$, and $n\geqq 2r-1$. This distribution is a mixture of r binomials. It is shown to be an identifiable mixture under the given restrictions on the parameters. Moment estimators for the $2r-1$ parameters $p_1,\cdots,p_r,\alpha_1,\cdots,\alpha_{r-1}$ are constructed and the covariance matrix of the joint asymptotic normal distribution of the estimators is obtained. It is found by comparison with the Cramér-Rao lower bound that the asymptotic efficiency of the moment estimators tends to unity as $n\to\infty$. Efficient estimators for any $n\geqq 2r-1$ are obtained as functions of the moment estimators by two procedures, one involving expansion of a χ^2 about the moment estimators to obtain BAN estimators, the other using Fisher's 'information' to compute a correction factor for the moment estimators. BAN estimators may also be computed using Neyman's linearization technique. A small empirical study of the behavior of several of these estimators for moderate sample sizes is included." *J. Neyman* (Berkeley, Calif.)

Aitchison, J. 5510

Inverse distributions and independent gamma-distributed products of random variables.

Biometrika **50** (1963), 505–508.

Let X be a random variable and $a(x)$ a measurable function, so that $Y=a(X)$ is a random variable. If the distribution of Y is known, then the distribution of X can be determined. The author uses this just to discuss properties of the Gamma, Beta and Dirichlet distributions. *E. Lukacs* (Washington, D.C.)

Parzen, Emanuel 5511

On statistical spectral analysis.

Proc. Sympos. Appl. Math., Vol. XVI, pp. 221–246. *Amer. Math. Soc., Providence, R.I.*, 1964.

The purpose of this paper is to give the theory of statistical spectral analysis by adopting what may be called a covariance approach rather than a spectral approach. There are four sections: definition of the spectrum, estimation of the spectrum, estimation of spectral density, estimation of spectral jumps. Some of the high points in this paper are: treatment of the role that weights play in estimating spectral averages from the point of view of approximation theory, presentation of different windows and how they may be compared by use of window generating functions, discussion of how, from the point of view of the practical use of the theory of spectral estimation, the fundamental questions to be settled concern the actual computing program to be used to estimate spectra. *E. A. Robinson* (Uppsala)

Aksomaitis, A. 5512

A statistical estimate of the amount of information in a discrete channel without memory. (Russian. Latvian and English summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 1, 5–8.

Let $I(X,Y)=\sum_i\sum_j p_{ij}\log(p_{ij}/p_ip_j)$ denote the amount of information in a discrete memoryless channel and let $\hat{I}_N=\sum_i\sum_j\hat{p}_{ij}\log(\hat{p}_{ij}/\hat{p}_i\hat{p}_j)$ denote its sample estimate, where the $\hat{p}$ are the relative frequencies in a sample of size N. The author shows that $\hat{I}_N$ is a biased asymptotically normal estimator of I. *H. P. Edmundson* (Pacific Palisades, Calif.)

Tiao, George C.; Guttman, Irwin 5513

Some useful matrix lemmas in statistical estimation theory.

Canad. Math. Bull. **7** (1964), 297–300.

John, Peter W. M. 5514

Pseudo-inverses in the analysis of variance.

Ann. Math. Statist. **35** (1964), 895–896.

Suppose that $\mathbf{EY}=\mathbf{X\beta}$, where the design matrix $\mathbf{X}$ has less than full rank. If $\hat{\boldsymbol\beta}=\mathbf{PX'Y}$ is any solution of the least-squares equations $\mathbf{X'X}\hat{\boldsymbol\beta}=\mathbf{X'Y}$, then $\mathbf{P}$ is described as a pseudo-inverse of $\mathbf{X'X}$. The author obtains a relationship between the two most common pseudo-inverses. *R. L. Plackett* (Newcastle upon Tyne)

Goodman, Leo A. 5515

Simultaneous confidence intervals for contrasts among multinomial populations.

Ann. Math. Statist. **35** (1964), 716–725.

Consider I multinomial populations, each of J classes, and let π_{ij} be the probability that an observation in the ith multinomial population falls in the jth class. Thus $\sum_{j=1}^{J} \pi_{ij} = 1$ for all i. A contrast θ is defined as $\sum_{i,j} c_{ij}\pi_{ij}$, where $\sum_{i=1}^{I} c_{ij} = 0$ for all j. Given a sample size n_i from the ith population, let the ML estimator of θ be $\hat\theta$ and its variance be estimated consistently by $S^2(\hat\theta)$. It is shown that as $\sum n_i \to \infty$ with $n_i/n \to p_i > 0$, the probability tends to $1-\alpha$ that, simultaneously for all θ,

$$\hat\theta - S(\hat\theta)L < \theta < \hat\theta + S(\hat\theta)L,$$

where L is the positive square root of the $100(1-\alpha)$th percentile of the χ^2 distribution with $(I-1)(J-1)$ degrees of freedom. The intervals thus obtained are shorter than those given by Gold [same Ann. **34** (1963), 56–74; MR **26** #5684]. For a specified set of G contrasts, it is sometimes possible to obtain shorter intervals than are given by L.

The usual test of homogeneity is modified by using the criterion

$$Y^2 = \sum_{i,j} (n_{ij} - n_i p_j)^2 / n_{ij},$$

where n_{ij} is the frequency in the ith class of the jth population and p_j is chosen to minimise Y^2. The author proves that a test of homogeneity using this criterion will reject the null hypothesis if and only if at least one estimated contrast is significantly different from zero. Applications are given to hypotheses about finite Markov chains. *R. L. Plackett* (Newcastle upon Tyne)

Goodman, Leo A. 5516

Interactions in multidimensional contingency tables.

Ann. Math. Statist. **35** (1964), 632–646.

Number the d_k classes in the kth dimension of an m-dimensional $d_1 \times d_2 \times \cdots \times d_m$ contingency table from $i_k = 0$ to $d_k - 1$ $(k = 1, 2, \cdots, m)$. Let p_i denote the multinomial probability associated with cell $i = \{i_1, i_2, \cdots, i_m\}$ of the table. Assume that $p_i > 0$ for all $\prod_{k=1}^{m} d_k$ possible values of i and write $\log p_i = b_i$. Let $j = \{j_1, j_2, \cdots, j_m\}$, where $j_k = 0, 1, \cdots, d_k - 1$ $(k = 1, 2, \cdots, m)$, and define $f(i, j) = 0$ unless, for each k, either $i_k = j_k$ or $i_k = 0$ or $j_k = 0$, and then $f(i, j) = (-1)^\nu$, where ν is the number of k for which $i_k j_k \neq 0$. Let $I^*(j) = \sum_i b_i f(i, j)$. The rth-order interactions are defined to be the $I^*(j)$ for all j containing $r+1$ non-zero elements.

When $d_k = 2$ $(k = 1, 2, \cdots, m)$ these interactions are the same as those defined by Good [same Ann. **34** (1963), 911–934; MR **27** #866], but when $d_k > 2$ for some k, Good's interactions are complex-valued, whereas the author's are real-valued. Methods are proposed for testing the hypothesis that any specified subset of interactions is equal to zero, and for obtaining simultaneous confidence intervals. The tests generalise methods given earlier by the reviewer [J. Roy. Statist. Soc. Ser. B **24** (1962), 162–166] and the author [ibid. **25** (1963), 179–188].

R. L. Plackett (Newcastle upon Tyne)

Kshirsagar, A. M. 5517

Wilks' Λ criterion.

J. Indian Statist. Assoc. **2** (1964), 1–20.

Author's summary: "So far, Wilks' Λ criterion has been given scanty treatment in existing textbooks. In this expository article, an attempt has been made to illustrate some of the numerous applications of the criterion and bring into focus some beautiful results hidden in it."

Philip, Johan 5518

Reconstruction from measurements of positive quantities by the maximum-likelihood method.

J. Math. Anal. Appl. **7** (1963), 327–347.

Consider a set of positive physical quantities $\{u_j \mid j = 1, \cdots, n\}$, which are being measured by a linear instrument. These measurements can be described by equations of the form $\int k_j(t)\, d\mu(t) = (k_j, \mu) = g_j$, $j = 1, \cdots, n$, where $k_j(t)$ describes the characteristics of the measuring instrument, and g_j are the actual measurements. The problem of reconstruction is then the problem of finding a measure μ satisfying these equations.

A more realistic model is one which takes account of the possibility of measurement error. This gives rise to the equations $(k_j, \mu) + x_j = d_j$, $j = 1, \cdots, n$. The author considers this model in detail and gives various hypotheses under which there exists a unique "most probable" solution μ to these equations. Here a "most probable" solution μ is one which minimizes a given positive definite form $\sum_{i,j=1}^{n} a_{ij}[d_i - (k_i, \mu)][d_j - (k_j, \mu)]$. Several examples are given. *J. R. Blum* (Albuquerque, N.M.)

Tranquilli, Giovanni Battista 5519

L'analisi della media basata sul campo di variazione. (French, English, Spanish, and German summaries)

Giorn. Ist. Ital. Attuari **26** (1963), 259–280.

Author's summary: "Given the problem of 'studentizing' a mean value $\bar x$ by a variation field w replacing the mean square deviation of samples, probability tables have been worked out for the test $v = n\bar x / w$, and applications have been shown particularly with regard to the case of block disposition."

Linnik, Ju. V. 5520

On A. Wald's test for comparing two normal samples. (Russian. English summary)

Teor. Verojatnost. i Primenen. **9** (1964), 16–30.

Consider two normal populations π_i, for $i = 1, 2$, with distributions $N(a_i, \sigma_i^2)$, where the parameters a_i and σ_i^2 are unknown. The problem is to test the hypothesis H that $a_1 = a_2$, using samples $X = (x_i)$ and $Y = (y_i)$ of equal size $n > 2$, randomly drawn from π_1 and π_2. Let $\bar X$, s_1^2, and $\bar Y$, s_2^2 be the means and variances of the two samples. A. Wald posed the problem of determining the critical region Z for testing H subject to the following four conditions: (i) Z is determined by the values of the sufficient statistics $\bar X$, $\bar Y$, s_1^2, s_2^2. (ii) If $(X, Y) \in Z$, then, for every c, $(X+c, Y+c) = (x_i + c, y_i + c) \in Z$. (iii) If $(X, Y) \in Z$, then, for every $k \neq 0$, $(kX, kY) = (kx_i, ky_i) \in Z$. (iv) If $(X, Y) \in Z$ and (X', Y') is another pair of samples with $|\bar X' - \bar Y'| \geq |\bar X - \bar Y|$ and $s_i' = s_i$, $i = 1, 2$, then $(X', Y') \in Z$. The four conditions imply that Z is determined by $|\bar X - \bar Y| \geq \Phi(s_1/s_2)(s_1^2 + s_2^2)^{1/2}$, where Φ is a measurable function of its argument. Wald's question is whether there exists an analytic function Φ with which the indicated test is exact, that is, such that $P\{(X, Y) \in Z \mid H\}$

has a fixed value independent of $\lambda = \sigma_1/\sigma_2$, at least for some set Λ of values of λ. In the present paper it is shown that if Λ is infinite, then no function $\Phi(\eta)$ exists which is continuously differentiable in $0<\eta<1$ and satisfies the Lipschitz condition in a sufficiently large closed interval $[0, \eta_1]$. *J. Neyman* (Berkeley, Calif.)

Linnik, Ju. V. 5521
Randomized homogeneous tests for the Behrens-Fisher problem. (Russian)
Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 249–260.
Randomized tests for the classical Behrens-Fisher problem are considered whose critical function $\Phi(\xi, \eta)$ depends only on $\xi = (\bar{x}-\bar{y})/s_2$ and $\eta = s_1/s_2$, in the usual notation for the sample means and variances. It is shown (assuming that one of the sample sizes is $\geqq 3$) that if the test is similar, we cannot have $\Phi(\xi, \eta)=0$ in a band containing the segment $\xi=0$, $0 \leqq \eta \leqq \eta_0$, where $\eta_0>1$, in its interior. *W. Hoeffding* (Chapel Hill, N.C.)

Cochran, William G. 5522
Approximate significance levels of the Behrens-Fisher test.
Biometrics **20** (1964), 191–195.
The accuracy of an approximation to the significance levels of the test statistic $d = x/\sqrt{(s_1^2+s_2^2)}$, where x is $N(0, \sigma_1^2+\sigma_2^2)$, and the s_i^2 are independent estimates of σ_i^2, $i=1, 2$, for the Behrens-Fisher problem previously proposed [W. G. Cochran and G. M. Cox, *Experimental designs*, 2nd ed., p. 101, Wiley, New York, 1957; MR **19**, 75] is studied by comparing actual significance levels with nominal significance levels.

From the author's summary: "It may be concluded that the approximation is adequate for routine tests of significance made at levels between 1% and 10%, although not for accurate mathematical calculations." *D. R. Barr* (Dayton, Ohio)

Patil, V. H. 5523
The Behrens-Fisher problem and its Bayesian solution.
J. Indian Statist. Assoc. **2** (1964), 21–31.
Author's introduction: "The Behrens-Fisher problem and the solution based on Behrens-Fisher distribution are reviewed in this paper. The criticisms and justifications of this solution by holders of different views of probability are explained briefly. The concepts of personal probability and Bayes' Theorem, Bayesian inference, stable estimation and Bayesian counterparts of hypothesis testing are discussed. Their application to the Behrens-Fisher problem is also considered." *J. Kiefer* (Ithaca, N.Y.)

Fabius, J. 5524
Asymptotic behavior of Bayes' estimates.
Ann. Math. Statist. **35** (1964), 846–856.
Freedman [same Ann. **34** (1963), 1386–1403; MR **28** #1706] obtained results about the asymptotic behavior of Bayes' estimates when the state space of the observations is discrete. The author gives two main extensions of Freedman's results.

For a rather restrictive class of prior distributions on the space of substochastic distributions on the positive integers, the author proves that along almost all sample sequences the posterior distributions of the expectations of bounded functions on the positive integers are, when appropriately normed, asymptotically normal.

To obtain consistency when the state space is denumerable, Freedman introduced the notion of tail-free prior distribution. The author extends the notion of tail-free prior distribution to obtain consistency when the state space is the closed unit interval. *D. L. Hanson* (Columbia, Mo.)

Castellano, Vittorio 5525
A deduction of significance tests from Bayes' formula. (French summary)
Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 449–454.
Author's summary: "L'auteur observe que l'ignorance des probabilités a priori P_H ne justifie pas la substitution de la probabilité ${}_AP_E$ à la formule de Bayes ${}_E\pi_A = P_{AA}P_E/\sum P_{HH}P_E$ et il montre à quelles conditions doivent satisfaire les P_H pour que l'on obtienne de la formule de Bayes les mêmes résultats que de la théorie des tests de significance, soit dans l'hypothèse où les P_H sont des véritables probabilités objectives (fréquences), soit dans l'hypothèse où les P_H sont seulement des probabilités subjectives." *H. D. Brunk* (Columbia, Mo.)

Störmer, Horand 5526
Ein Test zum Erkennen von Normalverteilungen.
Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 420–428 (1964).
Let the chance variables $x_1, \cdots, x_n$ be $N(\mu, \sigma^2)$, i.e., independently distributed with the common normal distribution with mean μ and variance σ^2. An obvious linear transformation takes the x's into $y_1, \cdots, y_n$, of which the first $(n-1)$ are $N(0, \sigma^2)$. The author gives a transformation (which does not depend on σ) which takes the latter y's into $z_1, \cdots, z_{n-1}$, of which the first $(n-2)$ are $N(0, 1)$. Suppose now that μ and σ^2 are unknown to the statistician, that $x_1, \cdots, x_n$ are observed, and that it is desired to test the null hypothesis that $x_1, \cdots, x_n$ are independently distributed with a common normal distribution against the alternative hypothesis that $x_1, \cdots, x_n$ are independently distributed with a common non-normal distribution. Then a test of exact size can be based on $z_1, \cdots, z_{n-2}$. The author proves that, when the alternative hypothesis is true, with probability one, the difference $D(x)$ between the empirical distribution function of $z_1, \cdots, z_{n-2}$ and the normal distribution function with mean zero and variance one converges to a non-zero constant for at least one point x on the real line (actually at least in a neighborhood of such a point). It follows that for most tests based on $z_1, \cdots, z_{n-2}$, including the one based on the supremum of $D(x)$, the corresponding test of normality based on $x_1, \cdots, x_n$ is consistent against any alternative. *J. Wolfowitz* (Ithaca, N.Y.)

Deshpande, J. V. 5527
Exact small sample significance points for the non-parametric V-test for problem of several samples.
J. Indian Statist. Assoc. **1** (1963), 167–171.
Author's summary: "In Ann. Math. Statist. **32** (1961),

1108–1117 [MR 27 #6348] Bhapkar has proposed a new non-parametric test (the V-test) for the problem of several samples. In this note the exact significance points of V for significance levels near about 0.05 and 0.01 are given for some small sample sizes for the case of three samples."

Bennett, B. M. 5528

A non-parametric test for randomness in a sequence of multinomial trials.

Biometrics **20** (1964), 182–190.

Author's summary: "A rank-order test is proposed for experimental situations where it is of concern to determine whether a sequence of multinomial probabilities may be varying significantly in n independent trials each with a fixed number of possible outcomes. An approximate χ^2-test for the hypothesis H_0 of constancy of probabilities is proposed, and the generating function is obtained for the distribution of the rank sums. An example using sibship data on congenital malformations (data of Milham [1962]) illustrates the use of the test for birth-order effect when several abnormalities are present."

H. B. Mann (Madison, Wis.)

Samuel, Ester 5529

Note on conditional tests and Dwass' modified procedure.

Skand. Aktuarietidskr. **1962**, 70–79 (1963).

In a paper by the reviewer [Ann. Math. Statist. **28** (1957), 181–187; MR **19**, 331], the suggestion was made that the practical difficulty in using permutation tests could be overcome by using a random sample of permutations, without harming efficiency too much. The author extends these ideas to a more general setting. She also obtains some finer information for the comparison of the powers of the original with the modified procedures.

M. Dwass (Evanston, Ill.)

Thompson, W. A., Jr. 5530

Negative estimates of variance components: An introduction. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 181–184.

This paper presents a brief examination of the problem of negative estimates of variance components. It describes a proposed resolution of the problem, using tree graphs, which is called the "pool the minimum violator" method. Supporting mathematical proofs and a more detailed account of the method are deferred till later papers [e.g., Ann. Math. Statist. **33** (1962), 273–289; MR **24** #A3736].

P. S. Dwyer (Ann Arbor, Mich.)

Kurkjian, B.; Zelen, M. 5531

Factorial designs and the direct product. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 509–519.

The authors present the analysis of variance of a general factorial experiment in a new notation using the direct or Kronecker product. The paper gives several lemmas concerning the combination of symbolic direct products and direct products (both defined in the paper) applied to special kinds of matrices that arise in the factorial e[xperi]ment. The general mathematical model of main [effects] and interactions are exhibited in the new notation [together] with certain contrasts. Also unbiased estimates of p[arameters,] interactions and variances are derived. The paper [gives] only an outline of the results; no proofs and no [numerical] examples illustrating the necessary computation[s are] given.

D. A. Sprott (Waterloo,

Zelen, M.; Federer, W. [Federer, Walter T.]

Applications of the calculus for factorial arrange[ments.] II. Two way elimination of heterogeneity.

Ann. Math. Statist. **35** (1964), 658–672.

The authors discuss the analysis and constructi[on of] designs for two-way elimination of heterogeneity[. Two] structural properties are defined, one defined a[s (A)] relating to the column incidence matrix, and the [other] defined as (B) relating to the row incidence matrix [of the] design. Based on properties (A) and (B), general expre[ssions] are obtained for treatment estimates and varia[nces of] elementary treatment contrasts, and the result[s are] applied to the analysis of balanced incomplete and [group] divisible designs. Some new classes of incomplete [block] designs which have properties (A) and (B) are ob[tained] for two-way elimination of heterogeneity, and [their] efficiency factors are evaluated.

M. Atiqullah (I

Federer, W. T. [Federer, Walter T.]; Atkinson, G. F.

Tied-double-change-over designs.

Biometrics **20** (1964), 168–181.

Authors' summary: "The construction and analysis [of a] class of experimental designs denoted as tied-d[ouble-] change-over designs are presented. These design[s are] useful for situations wherein the treatments are app[lied in] sequence to an experimental unit and where the eff[ect of] a treatment persists for one period after the per[iod in] which the treatment was applied; they allow estin[ation] of direct and residual treatment effects. Tied-d[ouble-] change-over designs are constructed utilizing one, ..., $t-1$ orthogonal latin squares for t treatm[ents.] Although the analysis is for r rows and for c colum[ns in] general, particular attention is given to the case [of] $r=tq+1$ rows and $c=ts$ columns for $sq=k(t-1)$, f[or k a] positive integer; explicit solutions are obtained f[or] situations where the first period results are omitted [from] the analysis and where the first period results are incl[uded.] A numerical example is used to illustrate the appli[cation] of the results to experimental data."

Singh, N. K.; Shukla, G. C.

The non-existence of some partially balance[d in]complete block designs with three associate classe[s.]

J. Indian Statist. Assoc. **1** (1963), 71–77.

Utilizing orthogonal arrays of strength two, the au[thors] develop an association scheme with three ass[ociate] classes. Necessary conditions for the existence of pa[rtially] balanced incomplete block designs having this assoc[iation] scheme are also presented.

S. Addelman (Durham,

Tyagi, B. N. 5535

A note on the construction of a class of second order rotatable designs.

J. Indian Statist. Assoc. **2** (1964), 52–54.

This paper deals with the construction of second order rotatable designs for three-level factors utilizing balanced incomplete block designs with $r > 3\lambda$, where r is the number of replicates and λ the number of blocks in which any pair of treatments occur. *S. Addelman* (Durham, N.C.)

Vartak, M. N. 5536

Disconnected balanced designs.

J. Indian Statist. Assoc. **1** (1963), 104–107.

A design is said to be "completely balanced" if every normalized estimable linear function of the treatment effects can be estimated with the same variance. After presenting this definition of balance, the author proves the following theorems. Theorem 1: Let N be a block design with v treatments and b blocks. Let r_i $(i = 1, 2, \cdots, v)$ be the number of times the ith treatment is replicated in the design and k_α $(\alpha = 1, 2, \cdots, b)$ be the size of the αth block. A necessary and sufficient condition that the design is balanced is that the non-zero characteristic roots of the matrix

$$C(N) = \operatorname{diag}(r_1, r_2, \cdots, r_v) - N \operatorname{diag}\left(\frac{1}{k_1}, \frac{1}{k_2}, \cdots, \frac{1}{k_b}\right) N'$$

are all equal.

Theorem 2: A connected design, N, is balanced with all the $v-1$ non-zero characteristic roots of $C(N)$ equal to r, the number of times each treatment is replicated, if and only if it is a randomized block design.

S. Addelman (Durham, N.C.)

Raghavarao, Damaraju 5537

Construction of second order rotatable designs through incomplete block designs.

J. Indian Statist. Assoc. **1** (1963), 221–225.

This paper is concerned with the construction of second order rotatable designs (SORD) for three level factors utilizing (1) balanced incomplete block (BIB) designs with $r \neq 3\lambda$, where r is the number of replicates and λ the number of blocks in which any pair of treatments occur, and (2) certain symmetrical unequal block designs with two unequal block sizes.

The BIB design with parameters $v = 6$, $b = 10$, $r = 5$, $k = 3$, $\lambda = 2$ is used to construct an SORD for a 3^6 experiment with 128 design points. The symmetrical unequal block arrangement with parameters $v = 15$, $b = 16$, $r = 6$, $k_1 = 5$, $k_2 = 6$, $n_1 = 6$, $n_2 = 10$, $\lambda = 2$ is used to construct an SORD for a 3^{15} experiment with 512 design points.

S. Addelman (Durham, N.C.)

Masuyama, Motosaburo 5538

La décomposition périodique dans le calcul des blocs. Le raffinement des décompositions. (English summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 3, 155–160.

Consider a ring of v elements (v prime) with a unit. To each element is associated a treatment or variety. Let H be a multiplicative subgroup of the ring. If a is an element of the ring, its products by the elements of H form a "periodic block". The elements of the ring are divided into disjoint classes consisting of such blocks and sums, or partial sums of these classes generate a partially balanced incomplete block design. Examples are given.

P. A. P. Moran (Canberra)

John, Peter W. M. 5539

Balanced designs with unequal numbers of replicates.

Ann. Math. Statist. **35** (1964), 897–899.

A simple proof is given of V. R. Rao's result [same Ann. **29** (1958), 290–294; MR **20** #401] for a design to be balanced, and examples are derived to show that some balanced designs exist even when they are not equireplicate. *M. Atiqullah* (Dacca)

Zacks, S. 5540

Generalized least squares estimators for randomized fractional replication designs.

Ann. Math. Statist. **35** (1964), 696–704.

In this paper the author considers the problem of estimating an entire vector of parameters in a factorial system of size $N = 2^m$, a slightly wider problem than discussed by him earlier [same Ann. **34** (1963), 769–779; MR **27** #3057], on the basis of randomised fractional replication of size $S = 2^s$ $(m > s)$. Some properties of generalised least squares estimators (g.l.s.e.) are discussed. It is shown that there is no randomised fractional replication procedure for which an unbiased g.l.s.e. of the entire vector of N parameters exists. When the parameters may assume arbitrary values, the randomisation procedure which assigns equal probabilities to various fractional replications is the only admissible procedure for the risk function, measured by the trace of the mean-square-error matrix, and the Bayes g.l.s.e. represents the minimum and admissible g.l.s.e. Besides, the author discusses the characterisation of estimable linear functions of the parameters having the property of uniformly minimum variance g.l.s.e. *M. Atiqullah* (Dacca)

Agrawal, H. 5541

On balanced block designs with two different number of replications.

J. Indian Statist. Assoc. **1** (1963), 145–151.

This paper presents the analysis of variance and some methods of construction of balanced incomplete block designs of b blocks of k treatments each such that the first v_1 treatments are replicated r_1 times and the remaining v_2 treatments are replicated r_2 times. A numerical example is given. *D. A. Sprott* (Waterloo, Ont.)

Ghosh, Sakti P. 5542

Post cluster sampling.

Ann. Math. Statist. **34** (1963), 587–597.

Each element of a finite population Π is marked with two characteristics C and X. A two-step sampling of Π is considered meant to estimate the arithmetic mean $\bar{X}$ of X. First an unrestrictedly random sample S_1 of n elements of Π is drawn. This sample is classified into "clusters" $c_1, c_2, \cdots$, each cluster being composed of those elements of S_1 that have the same value of C (presumably inexpensive to determine). Next a sample S_2 of m clusters

so formed is drawn, either with equal probabilities or with probabilities proportional to sizes of these clusters. The value of the "expensive" characteristic X is determined for each member of all the m clusters included in S_2. The paper studies the variances of several intuitively selected estimates of $\bar{X}$ and the problem of optimal selection of n and m subject to a preassigned cost of the survey.

J. Neyman (Berkeley, Calif.)

Dubey, Satya D. 5543

A simple test function for guarantee time associated with the exponential failure law.

Skand. Aktuarietidskr. **1962**. 25–38 (1963).

Whittle, P. 5544

Gaussian estimation in stationary time series. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 105–129.

This paper considers conditions for the consistency and efficiency of Gaussian estimates in time series analysis for schemes that contain a finite and fixed number of parameters. The following topics are discussed: estimates based on extremal criteria, special considerations for processes not representable as autoregressions, Gaussian estimates for stationary processes, distributions of test statistics, and efficiency of Gaussian estimates. The author states that this paper is only a first preliminary to a complete treatment.

E. A. Robinson (Uppsala)

Dobrušin, R. L.; Pinsker, M. S.; Širjaev, A. N. 5545

An application of the concept of entropy to signal-detection problems with background noise. (Russian. Latvian and English summaries)

Litovsk. Mat. Sb. **3** (1963), *no.* 1, 107–122.

The (large) number of observations needed to decide between two simple hypotheses when the errors of both types are required to be small can be given approximately in terms of what is now called the entropy of one hypothesis relative to another. (The first instance of such a computation is due to A. Wald [e.g., Ann. Math. Statist. **16** (1945), 287–293, equation (4.7); MR **7**, 209], who did not, however, use the term "entropy".) The authors compute these entropies for a number of interesting cases, including several in which a process is observed over continuous time.

J. Wolfowitz (Ithaca, N.Y.)

Yang, Chung-How 5546

Run tests.

Bull. Math. Soc. Nanyang Univ. **1963**, 51–58.

From the author's introduction: "In this paper, I shall give new derivations of some formulae in runs theory and shall discuss some applications of these formulae."

Gray, K. B. 5547

The application of stochastic approximation to the optimization of random circuits.

Proc. Sympos. Appl. Math., Vol. XVI, pp. 178–192. *Amer. Math. Soc., Providence, R.I.*, 1964.

The author proves still another version of the theorem on mean and almost everywhere convergence of the Kiefer-Wolfowitz procedure for finding the maximum of a function of several variables which is observable subject to random error. He then applies his results to a problem of optimum circuit design.

J. R. Blum (Albuquerque, N.M.)

NUMERICAL METHODS

See also 5231, 5232, 5266, 5307, 5584, 5813, 5860, 5869, 5871, 5880.

Jennings, Walter 5548

★First course in numerical methods.

The Macmillan Co., New York; Collier-Macmillan, Ltd., London, 1964. xiv + 233 *pp.* $7.50.

This book is designed for a one-term introductory course in numerical methods for advanced undergraduates, and as such it is very good. Without appearing to rush, it successfully covers the necessary material through the integration of ordinary differential equations. Essential ideas are often handled with the wise use of the phrase "It is known that . . ." so that the student is neither held back by having to prove everything he hears about, nor, alternatively, is he deprived of knowing about a wide range of material.

The book has a distinctly mathematical flavor as well as a good deal of practical content, so that it should appeal to both mathematicians and engineers.

R. W. Hamming (Murray Hill, N.J.)

Wynn, P. 5549

The numerical transformation of slowly convergent series by methods of comparison. I.

Chiffres **4** (1961), 177–210.

This paper is a discussion of an extension of the Euler transformation for the summation of slowly convergent series. It is preceded by an analysis of the computational misfortunes that are known as "slowly convergent".

The method is, briefly, as follows. If $\Phi(x) \sim \sum_s c_s x^s$, then

$$\Theta(x) \sim \sum_s c_s v_s x^s \sim \sum \frac{x^s}{s!} \frac{d^s\Phi(x)}{dx^s} \Delta^s v_0,$$

where $\sim$ means definition, equivalence, or equality, as appropriate.

The development is purely formal, and numerical examples in each of the categories of computational misfortune are exhibited.

A useful paper to computers.

J. C. P. Miller (Cambridge, England)

Handscomb, D. C. 5550

A rigorous lower bound for the efficiency of a Monte Carlo technique.

Proc. Cambridge Philos. Soc. **60** (1964), 357–358.

J. M. Hammersley and K. W. Morton, in the course of their important work [same Proc. **52** (1956), 449–475; MR **18**, 336] introducing a new Monte Carlo technique: antithetic variates, considered the estimation of $\theta = \int_0^1 f(x)\,dx$ by means of the functional

$$\phi_n f(\xi) = \frac{1}{n}\sum_{j=0}^{n-1} f\left(\frac{\xi+j}{n}\right),$$

where ξ is a random number uniformly distributed over the interval [0, 1].

The relative efficiency of $\phi_n f(\xi)$ as against the crude estimator $f(\xi)$ is given by $\mathrm{var}[f(\xi)]/n\ \mathrm{var}[\phi_n f(\xi)]$. Provided f has sufficiently many derivatives, these authors found, by means of the Euler-Maclaurin summation formula, an asymptotic expansion for the sampling variance of $\phi_n f(\xi)$.

In this short paper the author establishes the following inequality:

$$\mathrm{var}[\phi_n f(\xi)] \leqq n^{-2m} K_m{}^2 \int_0^1 [f^{(m)}(x)]^2\,dx,$$

where

$$K_m = \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu(m+1)}}{(2\nu+1)^{m+1}}$$

and m is chosen so that $\Delta_i = f^{(i)}(1) - f^{(i)}(0) = 0$ for $0 \leqq i \leqq m$.

This inequality has the merit of giving a definite bound to the variance of the estimator $\phi_n f(\xi)$.

D. D. Stancu (Cluj)

Ferguson, James 5551

Multivariable curve interpolation.

J. Assoc. Comput. Mach. **11** (1964), 221–228.

A method is described for defining a smooth surface through an array of points in 3-space. In the author's words, "the resulting solution is a smooth composite of parametric surface segments". The method is adapted to high-speed digital techniques. One numerical example is given with the resulting surface pictured from three different perspectives. Generalization to higher dimensions is briefly described. The content of this paper is related to that of Birkhoff and Garabedian [J. Math. and Phys. **39** (1960), 258–268; MR **22** #10151].

L. W. Ehrlich (Silver Spring, Md.)

Simauti, Takakazu 5552

Approximation formulas for some elementary functions.

Comment. Math. Univ. St. Paul. **12** (1964), 23–35.

These formulas were designed for certain Japanese computers but are of general interest. The functions dealt with are $\sin x$, $\cos x$, $\tan x$, $\arctan x$, $\exp x$, $\log x$; the representations are by straight power series which are generally obtained by splitting the range. As an example, $\tan x$ is to be obtained in $0 \leqq x \leqq \pi/4$ by the addition formula from $\tan u$, where $u = \pi z/32$ and $x = u + w$, where $32w/\pi = 1$, 3, 5 or 7; the first six terms of the power series in z gives a maximum error of 7×10^{-20}.

John Todd (Pasadena, Calif.)

Kublanovskaja, V. N. 5553

On a method of orthogonalizing a system of vectors. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 338–340.

For a given positive definite Hermitian matrix R, let it be required to find a system of vectors that are mutually orthogonal with respect to the metric determined by R. If the columns of S satisfy this condition approximately and if V is a matrix such that $V'S'RSV$ is diagonal, evidently the columns of SV satisfy it exactly. For the case of off-diagonal elements of $S'RS$ at most ε in magnitude, and the diagonal elements distinct, a known formula is applied to give a matrix V such that the off-diagonal elements of $V'S'RSV$ are of order ε^2. A similar method of successive approximation is shown to apply to the problem of finding biorthogonal systems, i.e., for a given matrix A, not necessarily Hermitian, to find matrices Q and $\hat{Q}$ such that $Q'A'\hat{Q}$ is diagonal.

A. S. Householder (Oak Ridge, Tenn.)

Massa, Emilio 5554

Sulla determinazione di autovalori con moduli poco diversi e delle relative autoennuple mediante procedimento di iterazione. (English summary)

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 394–416.

Author's summary: "The iteration procedure for computation of eigenvalues and eigenvectors, as for instance in problems of vibrations, critical speeds, etc., cannot be easily applied when there are closed eigenvalues. After a review of some basic characteristics of the iteration procedures, this paper examines in particular the case of two eigenvalues with closed moduli and equal or opposite sign. The two eigenvalues, together with the pertinent eigenvectors, are determined considering a linear combination with unknown coefficients of two vectors obtained one from the other through iteration (after conveniently applying the usual procedure of iteration and orthogonalisation) and using the stationary property of Rayleigh's quotient. Some practical examples are demonstrated and the approximation of the method is discussed."

L. Collatz (Hamburg)

Devjatko, V. I. 5555

On two-sided approximation in the numerical integration of ordinary differential equations. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 254–265.

Elaborating an idea of E. Ja. Remes, the author proposes to obtain upper and lower bounds for the exact solution $y = y(x)$ of the initial value problem $y(x_0) = y_0$, $y' = f(x, y)$ by simultaneously using two linear multistep formulas of identical order p whose error constants C have the sum zero. (For terminology see the reviewer's book [*Discrete variable methods in ordinary differential equations*, Wiley, New York, 1962; MR **24** #1772].) One such pair, given by Remes, is as follows:

$$y_n - y_{n-3} = \frac{3h}{4}(f_{n-3} + 3f_{n-1}),$$

$$y_n - y_{n-3} = \frac{3h}{4}(3f_{n-2} + f_n),$$

here $p = 3$ and $C = \pm\frac{3}{8}$. The method requires the total derivative $d^p f(x, y)/dx^p$ to have constant sign, which somewhat limits the practical applicability. An extension to systems is indicated, and there are several numerical examples.

P. Henrici (Zürich)

Fraboul, François 5556

Stabilisation d'une méthode d'intégration numérique des systèmes différentiels.

C. R. Acad. Sci. Paris **257** (1963), 4126–4128.

The author applies his stabilization method [same C. R. **256** (1963), 3242–3244; MR **26** #5732] to a difference scheme for $x' = Ax$, $x(t_0) = x_0$, where A is constant matrix.
M. Lees (Pasadena, Calif.)

Dąbrowski, Mirosław; Szoda, Zenon 5557
On the use of Runge-Kutta and Adams' methods in computer practice. (Polish. English summary)
Algorytmy Zeszyt Specjalny No. 1 (1963), 71–85.
Authors' summary: "In the present paper descriptions are given of Runge-Kutta, Runge-Kutta-Gill and Runge-Kutta-Merson's methods, as well as that of extrapolation and extrapolation-interpolation Adams' method of the 4th order. The suitability of these methods as standard processes is proved. A way is presented of choosing adequate methods and also a general arranging of computation with a constant and variable integration step. An example of the equation $y' = 5y/(x+1)$ with initial condition $y(0) = 1$ illustrates the above methods."

Knapp, H. 5558
Über eine Verallgemeinerung des Verfahrens der sukzessiven Approximation zur Lösung von Differentialgleichungssystemen.
Monatsh. Math. **68** (1964), 33–45.
This paper extends the formal method of Lie-Reihen for a regular system of differential equations to neighboring non-regular systems. The main result consists of a theorem which asserts that an iteration process based on successive approximations and using a truncated Lie-Reihen converges to the solution of the differential equation system

$$\dot{Z}_i(t) = \vartheta_i(Z_1(t), \cdots, Z_n(t), t) \qquad (i = 1, \cdots, n)$$

with the initial conditions $Z_i(t_0) = Z_i^{(0)}$. The functions ϑ_i and their derivatives up to the order of the truncation are required to be continuous and satisfy a special Lipschitz condition. It is noted that the suggested iteration processes may be useful for numerical calculations.
W. Sangren (San Diego, Calif.)

Pogorzelski, Ryszard 5559
On the stability of some difference methods. (Polish. English summary)
Algorytmy Zeszyt Specjalny No. 1 (1963), 9–20.
Author's summary: "The paper deals with a numerical solution of differential equations of the form

$$y' = f(x, y)$$
$$y(x_0) = y_0.$$

The stability of the formula

$$\alpha_k y_{n+k} + \alpha_{k-1} y_{n+k-1} + \cdots + \alpha_0 y_n = h(\beta_k y'_{n+k} + \cdots + \beta_0 y_n')$$

is investigated in the first part of the paper, and the formula is given that is stable for rather great stability regions $\bar{h} = f_y(x, y) \cdot h$. In the second part the stability of difference methods is considered; the predicator and modification influence upon the method stability being taken into account."

Pogorzelski, Ryszard 5560
On numerical integration of differential equations whose solutions have discontinuous higher derivatives. (Polish. English summary)
Algorytmy Zeszyt Specjalny No. 1 (1963), 109–113.
Author's summary: "The paper deals with initial value problems for ordinary differential equations. The existence of the first k continuous derivatives is assumed. The truncation error is evaluated when the difference method of order $p > k$ is applied to this problem. In addition, some results of experimental computations made on the ZAM-2 computer are given."

Jaroševskiĭ, V. A.; Voeĭkov, V. V. 5561
A method of accelerating the calculation of rapid quasi-periodic motions on a digital computer. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 168–171.
The authors consider the following system of ordinary differential equations

$$\ddot{y} + \varepsilon f(r_j, y)\dot{y} + F(r_j, y) = 0$$
$$\dot{r}_i - \varepsilon s_i(r_j, y) = 0 \qquad (i, j = 1, 2, \cdots, n),$$

where ε is the parameter of smallness, F and s_i are differentiable functions of their arguments.

This system describes a rapid quasi-periodic motion. The authors state that the above system requires a lot of machine time to solve on a digital computer, and they propose a special method to speed up the computations. This method is based on the fact that every r_i can be regarded as a sum of a slowly varying function and a rapidly oscillating supplement arising from the oscillations of the variable y. The function $F(r_j, y)$ can be regarded as the mechanical rigidity, and the authors call their method "the method of artificially decreasing rigidity". The gain in computation time does not seem to be worth the trouble.
T. Leser (Aberdeen, Md.)

Ząbek, Świątomir 5562
Some error estimates for approximate solutions of linear differential equations of even order. (Polish. English summary)
Algorytmy Zeszyt Specjalny No. 1 (1963), 37–43.
Author's summary: "The following boundary problem is considered

$$(1) \qquad L[y] \equiv \sum_{\mu=0}^{m} (-1)^\mu (p_\mu(x) y^{(\mu)})^{(\mu)} = r(x) \qquad (y^{(0)} \equiv y)$$

$$(2) \qquad y^{(\mu)}(a) = y^{(\mu)}(b) = 0 \qquad (\mu = 0, 1, \cdots, m-1),$$

where m is a natural number and $p_\mu(x)$ are real functions of class C^μ in $\langle a, b \rangle$, $p_\mu(x) \geqq 0$ for $\mu = 0, 1, \cdots, m$, $p_m(x) \geqq p > 0$ in $\langle a, b \rangle$. Let the function $\tilde{y}(x)$ of class C^{2m} in $\langle a, b \rangle$ satisfy conditions (2), and $Y(x)$ be the exact solution of the problem (1) and (2). Denote $\eta = \max |Y(x) - \tilde{y}(x)|$ and $\tilde{z}(x) = [Y(x) - \tilde{y}(x)]/\eta$. According to Bertram, for any natural m the [following] inequality holds:

$$(3) \qquad \eta \leqq \frac{\int_a^b |L[\tilde{y}] - r(x)|\, dx}{\int_a^b \sum_{\mu=0}^{m} p_\mu(x)[\tilde{z}^{(\mu)}(x)]^2\, dx}.$$

In order to obtain a concrete over-estimation for η the under-estimation of the integral in the denominator of

(3) should be found. The paper contains generalizations of results obtained recently by G. Bertram [Numer. Math. **1** (1959), 181–185; MR **21** #6096] and K. Tatarkiewicz [Ann. Polon. Math. **1** (1955), 346–359; MR **17**, 539]."

Lynn, M. Stuart 5563

On the round-off error in the method of successive over-relaxation.

Math. Comp. **18** (1964), 36–49.

The author examines, by means of a statistical model, the asymptotic behavior of the round-off error which accumulates when the iterative method of (point) successive over-relaxation is used to solve a large scale of linear equations $Cx=b$, where the $n\times n$ real matrix C is assumed to be symmetric positive-definite, to have property A and to be σ_1-ordered and to be written as $C=I-B$, $B=L+U$ (I being a unit matrix, L and U being lower and upper triangular matrices). Defining the iteration by

$$x^{(k+1)} = \omega(Lx^{(k+1)}+Ux^{(k)}+b)+(1-\omega)x^{(k)},$$
$$k = 0,1,2,\cdots,\ 0<\omega<2,$$

it is assumed that actually

$$\tilde{x}^{(k+1)} = \omega(L\tilde{x}^{(k+1)}+U\tilde{x}^{(k)}+b)+(1-\omega)\tilde{x}^{(k)}+e^{(k+1)}$$

are computed, $e^{(k)}$ being the local round-off error, treated as independent random variables such as $E[e^{(k)}]=\varepsilon_k=\varepsilon=\text{const}$, $De^{(k)}=\sigma^2R$, where R is a constant symmetric positive definite matrix that commutes with L and U. Letting $r^{(k)}=x^{(k)}-\tilde{x}^{(k)}$ (accumulated round-off error)$\to r^\infty$ as $k\to\infty$, the author shows the following results:

$$r_k \equiv E[r^{(k)}]\to r = \omega^{-1}C^{-1}\varepsilon,$$
$$V_k \equiv Dr^{(k)}\to V = \sigma^2/[\omega(2-\omega)]\cdot RC^{-1},$$

for all $0\leqq\beta\leqq 1$, $\Pr\{\|r^\infty-r\|^2\leqq[\sigma^2\operatorname{tr}(RC^{-1})]/[\omega(2-\omega)\beta]\}\geqq 1-\beta$. A numerical experiment was made on the one-dimensional Dirichlet problem $y''=f(x)$, $y(0)=\beta_0$, $y(1)=\beta_1$.

W. Sibagaki (Fukuoka)

Omarov, E. O. 5564

An approximate solution by the method of lines for the Dirichlet problem for a partial differential equation of elliptic type. (Russian. Uzbek summary)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk **1963**, *no.* 1, 21–25.

The author considers the equation

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}+cu = f,$$

with constant coefficients a, b, c, in a trapezoidal domain bounded above and below by two parallels to the x-axis. He discretizes in the y-direction only and thus reduces the problem to one of solving a system of ordinary differential equations with constant coefficients. By applying a clever transformation, he is able to write down the solution explicitly.

P. Henrici (Zürich)

Samarskiĭ, A. A. 5565

Homogeneous difference schemes on non-uniform grids for equations of parabolic type. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 266–298.

This is the eighth of a series of papers published in 1961–62 in the same journal by the author alone or in collaboration with A. N. Tihonov. It deals with numerous variants of the problem stated below. Though quasi-linear parabolic equations, systems of parabolic equations, and equations with discontinuous coefficients are discussed, the spirit and nature of the difference methods and convergence and stability theorems can be adequately described by sampling the treatment of the following linear problem: Approximate the solution of

$$(1)\quad c(x,t)\frac{\partial u}{\partial t} = L^{(k,q,f)}u = \frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right)-q(x,t)u+f(x,t),$$

$$(2)\quad u(0,t) = u_1(t),\qquad u(1,t) = u_2(t),\qquad 0\leqq t\leqq T,$$

$$(3)\qquad u(x,0) = u_0(x),\qquad 0\leqq x\leqq 1,$$

$$(4)\quad 0<c_1\leqq k(x,t),\qquad 0<c_2\leqq c(x,t),\qquad 0\leqq q(x,t),$$

where c_1 and c_2 are constants. On a grid x_i, t_j let $h_i=x_i-x_{i-1}>0$, $\tau_j=t_j-t_{j-1}>0$, $\sum_1^N h_i=1$, $\sum\tau_j=T$. Approximate $u(x_i,t_j)$ by y_i^j defined by the implicit difference equations

$$(5)\qquad \rho_i^{j+1(\alpha)}(y_i^{j+1}-y_i^j)/\tau_{j+1} = (L_h^{(k,q,f)}y_i^{j+1})^{(\alpha)},$$

where

$$(6)\quad L_h^{(k,q,f)}y^{j+1} = [a_{i+1}^{j+1}(y_{i+1}^{j+1}-y_i^{j+1})/h_{i+1}-a_i^{j+1}(y_i^{j+1}-y_{i-1}^{j+1})/h_i]/\hbar - d_i^{j+1}y_i^{j+1}+\varphi_i^{j+1}$$

with $\hbar=0.5(h_i+h_{i+1})$. The coefficients $a(x,t)$, d, φ, and ρ are defined by the following smoothing operations. Let A be a non-decreasing homogeneous functional (on the space of functions continuous on the stated interval) having a second differential, with $A[1]=1$, $A[s]=-0.5$, and let $a_i^{j+1}=A[f(x_i+sh_i,t_{j+1})]$, $-1\leqq s\leqq 0$. Let F be a non-negative linear functional such that $F[1]=1$, $F[s]=0$, and let

$$\hbar\varphi_i^{j+1} = h_iF[f(x_i+sh_i,t_{j+1})\eta_0^-(s)] + h_{i+1}F[f(x_i+sh_{i+1})\eta_0^+(s)]$$

for $|s|\leqq 0.5$, where $\eta_0^-(s)=1$ [0] for $s<[>]\,0$, and $\eta_0^+(s)=1-\eta_0^-(s)$. The definitions of d_i^{j+1} and ρ_i^{j+1} are identical with that of φ_i^{j+1}, except that f is replaced by q or c, and F by other functionals D or R. Finally, $\rho_i^{j+1(\alpha)}=\alpha\rho_i^{j+1}+(1-\alpha)\rho_i^j$, $0\leqq\alpha\leqq 1$. The difference equations (5) and (6) are to be supplemented by obvious initial and boundary conditions. The main novelty of this paper relative to its predecessors is a new definition of φ, d, and ρ.

For a solution u of (1)–(4) the truncation error is

$$\Psi = (L_h^{(k,q,f)}u_i^{j+1}-L^{(k,q,f)}u_i^{j+1})^{(\alpha)} - \rho_i^{j+1(\alpha)}(u_i^{j+1}-u_i^j)/\tau_{j+1}+(c\partial u/\partial t)_i^{j+1(\alpha)}.$$

Under conditions on c, k, q, and f shown in previous papers to assure $\Psi=O(\tau^{m_\alpha})+O(h^2)$, $m_\alpha=2$ [1] for $\alpha=[\neq]0.5$ for uniform grid $h_i=h$, $\tau_j=\tau$ the author now finds for non-uniform mesh $\Psi=O(\tau_{j+1}^{m_\alpha})+O(h_i^2)+O(h_{i+1}^2)$. The error $z_i^j=y_i^j-u_i^j$ satisfies the difference system

$$\rho_i^{j+1(\alpha)}(z_i^{j+1}-z_i^j)/\tau_{j+1} = (\Lambda z_i^{j+1})^{(\alpha)}+\Psi,$$
$$\Lambda z_i^j = L_h^{(k,q,f)}z_i^j-\varphi_i^j,$$

$$\Psi = (L_h^{(k,q,f)} u_i^{j+1})^{(\alpha)} - \rho_i^{j+1(\alpha)}(u_i^{j+1} - u_i^j)/\tau_{j+1},$$
$$z_0^j = z_N^j = z_i^0 = 0.$$

Then one can establish an error bound

$$\max_{x_i} |z(x_i, t_j)| \leq M\left(\sum_{j'=1}^{j} \tau_{j'} \|\Psi(x_i, t_{j'})\|_{2*}^2\right)^{0.5},$$

where the constant $M > 0$ depends only on the original and the smoothed coefficient functions; the x-mesh is arbitrary; the t-mesh satisfies $\tau_j - \tau_{j-1} = O(\tau_j \tau_{j-1})$ or $O(\tau_j^2)$; and $\|\Psi(x_i, t_j)\|_{2*}^2 = \sum_0^{N-1} (\Psi_i^j)^2 h_i$.

J. H. Giese (Aberdeen, Md.)

Bramble, J. H.; Hubbard, B. E. 5566

On a finite difference analogue of an elliptic boundary problem which is neither diagonally dominant nor of non-negative type.

J. Math. and Phys. **43** (1964), 117–132.

A finite-difference analogue to the differential equation

$$-y''(x) + q(x)y(x) = f(x), \qquad x \in [0, 1],$$
$$y(0) = y(1) = 0,$$

is considered. Let $h = 1/N$ and $Y_i = Y(ih)$. Then y_i'' is replaced by $h^{-2}\delta^2 Y_i$ for $i = 1$ and $N-1$, and by $h^{-2}(\delta^2 - (h^2/12)\delta^4)Y_i$ for $i = 2, 3, \cdots, N-2$. The resulting matrix of coefficients for $(Y_0, Y_1, \cdots, Y_N)$ is neither diagonally dominant nor of non-negative type. However, it is monotone and this property is used to show that the error in the solution is $O(h^4)$ even though the truncation error is $O(h^2)$ at $i = 1$ and $N-1$.

It is also shown that for h sufficiently small the Jacobi iteration diverges whereas the Gauss-Seidel iteration converges.

C. Froese (Cambridge, Mass.)

Hockney, R. W. 5567

A solution of Laplace's equation for a round hole in a square peg.

J. Soc. Indust. Appl. Math. **12** (1964), 1–14.

From the author's summary: "A series solution is obtained for Laplace's equation in the region between a circle and an enclosing square, which are at constant but differing pressures or potentials. The coefficients of the series are tabulated as functions of the relative size of the circle and square and enable the solution to be found with an error of a few parts in a thousand."

Thompson, Robert J. 5568

Difference approximations for inhomogeneous and quasi-linear equations.

J. Soc. Indust. Appl. Math. **12** (1964), 189–199.

Suppose the well-posed initial-value problem $u_t = Au$ has the convergent difference analogue $U(t+k) = C_k U(t)$, i.e., $U \to u$ as $k \to 0$ if $U(0) = u(0)$. Then the author shows that the same is true of the solutions of $u_t = Au + g(t, u)$ and $U(t+k) = C_k U(t) + g(t, U)$, assuming that g is continuous in t and Lipschitz in u.

W. G. Strang (Cambridge, Mass.)

Greenspan, Donald 5569

On the numerical solution of problems allowing mixed boundary conditions.

J. Franklin Inst. **277** (1964), 11–30.

Approximations of first- and second-order accuracy are made to the normal derivative and solutions by difference methods are considered to the Laplace equation with mixed Dirichlet and Neumann boundary conditions. It is shown that the first-order approximations lead to a system of algebraic equations which always has a solution, but little can be said theoretically about the system resulting from the second-order approximations. Indeed, an example is given using the latter procedure in which the determinant of the linear system is zero. Computing examples are given for the solution of the Laplace equation on a circle using 219 points on a rectangular grid. The problem is first solved with a Dirichlet condition on the lower half of the circumference and a Neumann condition on the upper half. Answers are compared using first- and second-order approximations to the normal derivative. The problem is then solved by extending the Dirichlet conditions over the entire circumference, and the exact solution to this problem is compared with the difference solution. Although it is not pointed out by the author, the mixed problem has its normal derivative chosen so that the exact solution of the mixed problem should be the same as for the Dirichlet. In spite of the author's claim that superior results were obtained by using the second-order approximations, the table of answers presented for the problem described would seem to indicate that far more accurate solutions can be obtained from the difference system for the Dirichlet problem than from either of the alternate approximations proposed for the mixed problem.

A. O. Garder, Jr. (Houston, Tex.)

Gastinel, N. 5570

Quelques procédés itératifs pour la résolution de systèmes linéaires associés, par la méthode des différences, à des équations aux dérivées partielles.

Deux. Congr. Assoc. Française Calcul et Traitement Information (Paris, 1961), pp. 39–46. *Gauthier-Villars, Paris,* 1962.

For the equation $Ax = b$, where A is a symmetric positive definite matrix, the author discusses iterative procedures of the form

$$x^{(n+1)} = x^{(n)} - (\alpha' I + \beta' K)(Ax^{(n)} - b), \qquad K = U_i \cdot U_i^T,$$

where U_i is an eigenvector of A.

J. R. Cannon (Upton, N.Y.)

Zajac, E. E. 5571

Note on overly-stable difference approximations.

J. Math. and Phys. **43** (1964), 51–54.

When a boundary-value problem for a partial differential equation of evolution (i.e., the heat or wave equation) is approximated by a stable and consistent finite-difference boundary-value problem, convergence takes place in any finite time interval. However, if the partial differential equation is without dissipation while the finite-difference approximation has dissipation, then the asymptotic behavior of the solution of the differential equation may differ radically from that of the finite-difference equation for a fixed mesh spacing. This phenomenon, which the author calls over-stability, is illustrated for the wave equation.

M. Lees (Pasadena, Calif.)

Petersen, I. 5572

Convergence of gradient methods for finding a local conditional minimum of a non-linear functional under linear conditions in a Hilbert space. (Russian)

Dokl. Akad. Nauk SSSR **151** (1963), 45–47.

The author considers the problem of minimizing a sufficiently differentiable (non-linear) functional $f(x)$ defined on some convex subset D of a Hilbert space H under side conditions of the form $(c^i, x) = b_i$, $i = 1, 2, \cdots, n$. He sets up a differential equation for the lines of steepest descent and gives conditions under which all its solutions will tend to the desired minimal point. He then shows how to approximate the minimal point by integrating the differential equation numerically by an arbitrary one-step method of order $\geqq 1$. Detailed error estimates are given for the cases where the numerical integration is carried out by the Euler method and by the modified Euler method. {An exponent 2 appears to be missing on the right of equation (10).} *P. Henrici* (Zürich)

Cherruault, Yves 5573

Sur l'approximation des formes linéaires.

C. R. Acad. Sci. Paris **258** (1964), 1992–1994.

The author proposes a method of approximating a linear operator T on a one-dimensional space by a sum of discrete weights at constant intervals $T = \sum_{p=-N}^{N} a_p \delta_{(ph)}$.

In the scheme suggested the weights a_p are determined by the condition that a set of scalar products in the space be approximated as closely as possible. For the linear operator $T = \int_A^B dx$, the method reproduces the usual one-dimensional integration rules, the rule reproduced depending on the scalar products considered.

L. M. Delves (Kensington)

Rjaben'kiĭ, V. S. 5574

Necessary and sufficient conditions for best conditionality of boundary-value problems for systems of ordinary difference equations. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 242–255.

The problem considered is that of solving

$$\text{(I)}\quad A[(n+1/2)h, h]U_n + B[(n+1/2)h, h]U_{n+1} = F_{n+1/2},$$

$$\text{(II)}\quad a(h)U_0 = \varphi, \qquad b(h)U_N = \psi,$$

$$n = 0, 1, \cdots, N-1, \text{ where } Nh = 1.$$

In this $A(x, h)$ and $B(x, h)$ are square matrices of order p, with elements that are continuous functions of x and h except for a finite number of discontinuities in x; and the matrices $a(h)$ and $b(h)$ have p columns each and a total of p rows, their elements being continuous in h. The system is said to be well-conditioned in case a unique solution exists for h arbitrarily small, satisfying

$$\max\|U_n\| \leqq M(\max\|F_{n+1/2}\| + \|\varphi\| + \|\psi\|),$$

where the norm is the maximum norm, and M is a constant independent of $F_{n+1/2}$, φ, ψ, and h.

Most of the paper is devoted to the statement and proof of one theorem giving a necessary and sufficient condition for the problem to be well-conditioned. This condition is that equation (I) be "regular", that the points of discontinuity be "regular", and that the boundary conditions be "regular". Regularity of (I) means essentially that the matrix $A+B$ be nonsingular for every x, but the other two notions of regularity cannot be stated in a few words. The final page indicates a generalization to a system involving matrices a_{ij}^k instead of the pair A and B.

A. S. Householder (Oak Ridge, Tenn.)

Položiĭ, G. N.; Čalenko, P. I. 5575

The strip method of solving integral equations. (Ukrainian. Russian and English summaries)

Dopovidi Akad. Nauk Ukraïn. RSR **1962**, 427–431.

From the authors' summary: "The method of strips combines the main ideas of the method of finite differences, the method of ordinary iteration and the method of approximation of kernels. In the integral equation

$$\varphi(x) = f(x) + \lambda \int_0^1 K(x, s)\varphi(s)\, ds$$

the domain of the kernel is divided into m strips, and in each of these strips the approximation of the kernel of the form

$$C_k(x, s) = C_k(x) + P_k(x)Q_k(s), \qquad k = 1, \cdots, m,$$

is carried out. The solution of the integral equation is obtained in the form

$$\varphi(x) = F(x) + \lambda \int_0^1 H(x, s; \lambda)F(s)\, ds."$$

Tihonov, A. N. 5576

On the solution of ill-posed problems and the method of regularization. (Russian)

Dokl. Akad. Nauk SSSR **151** (1963), 501–504.

The author considers the problem of solving the Fredholm equation of the first kind

$$(*)\quad A[x, z] \equiv \int_a^b K(x, s)z(s)\, ds = u(x), \qquad c \leqq x \leqq d,$$

which is not well-posed because (i) the solution may not exist and (ii) the solutions z belonging to two functions u which are close may not be close. Assuming the existence of a unique solution $\bar{z}$ of (*) and assuming that $\bar{z}$ is piecewise differentiable, the author shows how to approximate $\bar{z}$ by the solutions of a sequence of well-posed problems, defined as follows. Consider the "smoothing" functional M^α depending on the parameter $\alpha > 0$ and defined by

$$M^\alpha[z, u] = N[z, u] + \alpha\Omega[z],$$

where

$$N[z, u] = \int_c^d \{A[x, z] - u(x)\}^2\, dx,$$

$$\Omega[z] = \int_a^b [k(s)z'(s)^2 + p(s)z(s)^2]\, ds,$$

and where k and p are two fixed positive functions. Determine the function z^α minimizing $M^\alpha[z, u]$, which can be done by solving a Sturm-Liouville boundary-value problem. Then for $\alpha \to 0$, z^α converges to $\bar{z}$ uniformly. A finite-difference scheme for constructively minimizing M^α is also indicated. *P. Henrici* (Zürich)

Tihonov, A. N. 5577

On the regularization of ill-posed problems. (Russian)

Dokl. Akad. Nauk SSSR **153** (1963), 49–52.

The author considers ill-posed problems, notably Fredholm equations of the first kind,

$$A(x, z(s)) = \int_a^b K(x, s)z(s)\,ds = u(x), \qquad c \leq x \leq d.$$

He proves (roughly) that if the functional

$$\int_c^d [A(x, z(s)) - \bar{u}(x)]^2\,dx + \alpha \int_a^b \sum_0^{n+1} K_i(s)[z^{(i)}(s)]^2\,ds \qquad (K_i > 0)$$

is minimized by $z = z_n^\alpha$, and if $A(x, \bar{z}(s)) = \bar{u}(x)$, then z_n^α converges to $\bar{z}$ for $\alpha \searrow 0$, in the C^n-norm when $n \geq 0$, and weakly when $n = -1$. The method has with great success been tested on electronic computers.

J. Friberg (Göteborg)

COMPUTING MACHINES

See also 4988, 4998, 5557, 5561.

Lehmer, D. H.; Lehmer, E.; Mills, W. H.; Selfridge, J. L. 5578

Machine proof of a theorem on cubic residues.

Math. Comp. **16** (1962), 407–415.

Any set of three consecutive positive integers is called a triplet. It is known that for any prime $p \equiv 1 \pmod 6$, up to a finite number of exceptional primes, there exists a triplet whose members are $\leq p-1$ and cubic residues mod p. The authors proved, by using a computer method, that (a) the only exceptional primes are 2, 3, 7, 13, 19, 31, 37, 43, 61, 67, 79, 127, 283; (b) every non-exceptional prime has a triplet of cubic residues not exceeding (23532, 23533, 23534); (c) there are infinitely many primes whose smallest triplet of cubic residues is the above one, so that (b) is the best possible result.

The authors' method of proof is as follows. Since the $p-1$ reduced residue classes mod p form a multiplicative group of order 3 modulo the group of cubic residues, let $R(s)$ be an isomorphism of this group onto the additive group mod 3. Let $S = \{q_1, \cdots, q_t\}$ be a set of distinct primes $\equiv 1 \bmod 6$, and for any such prime $p \notin S$, $A = (R(q_1), \cdots, R(q_t))$ be called the S-vector belonging to p. Then, a number $n = q_1^{b_1} \cdots q_t^{b_t}$ is a cubic residue mod p if and only if $R(n) = b_1 a_1 + \cdots + b_t a_t \equiv 0 \pmod 3$. If this is the case, the authors say that n disposes of A. If a number N and a set S of primes are found such that for any of the 3^t possible S-vectors A there exists a triplet whose members do not exceed $N+1$ and do dispose of A, then all primes $p \notin S$ are non-exceptional.

By testing all primes less than 11243 in a preliminary machine run, the primes in (a) are shown to be exceptional. Then, the following ingenious selections of S and N lead the authors to success. For the first machine run the set of all primes ≤ 127 and 283 (see (a)) are used as S and 44224 as N, and (a) is already proved. N was chosen by a probabilistic discussion in order to supply enough triplets whose prime divisors are restricted to those in S. Then N is lowered successively to 2 to the power 17, 16, 15. In these three runs (a) is again proved either by the machine itself or by some simple additional computations by hand. Then in the fifth run the set of all primes less than the 51st was selected as S and $3 \cdot 2^{13}$ as N. These runs, especially the last, gave information as to what kind of S-vectors were hard to be disposed of. So in the sixth run (b) was proved with $N = 24389$ instead of 23533. Then with the intention of proving part (c), a case test program was written by using special S-vectors selected by the information of the previous machine runs. This run resulted in getting the number 23533. The final run was made with S of all primes less than the 56th and $N = 23533$. This run alone proves (a) and (b), and (c) was proved by the case test program.

Interesting techniques for reducing machine time by improving the computations so as to fit machine computation, and discussions for selecting suitable S and N are described. This paper represents a kind of machine proof of mathematical theorems which is absolutely impossible without using a computer, and the details of the proof lie in the programs the authors wrote and the output from the machine.

S. Kuroda (College Park, Md.)

Levine, Leon 5579

★Methods for solving engineering problems using analog computers.

Chapters 9 and 10 contributed by Arnold Levine.

McGraw-Hill Book Co., New York-Toronto-London, 1964. xiii+485 *pp.* $14.50.

The authors have produced a most welcome and long overdue addition to the literature on analog computing methods. The carefully described methods will become increasingly important as more groups begin to use hybrid computers. However, this should not be taken to mean that this text is limited to analog-digital combinations. On the contrary, the text is a most valuable guide to general analog methods.

It is important to note that both authors deal very well with both theory and practice. Analytic methods for treating differential equations receive an unusually good treatment considering this to be a "computer" text. This reviewer feels that we have passed well through the "enchantment" stage and it is important that our texts begin to reflect an awareness of the computer in perspective. As a second example of the important sophistication displayed in this text, both authors deal carefully with the problems of error detection and reduction. And, statistical procedures and applications of statistics in the area of analog computation receive the attention of no less than three of the eleven chapters in the text.

On the side of practice, it has been difficult for many authors to refrain from producing a kind of operating manual for a rather specific product line. In this text, however, very little of the material presented will become obsolete as new generations of machines are devised. By recognising those aspects of practice which are most basic and important and by avoiding the snares of particular types of hardware, the text should be valuable for a rather long time as these things go.

One of the more important criticisms of this text lies in its index. The wealth of material contained in this text deserves and almost requires careful cross-indexing. This text will be used often as a reference long after it has been used in the classroom. In this light, the brevity of the index is unfortunate.

Nonetheless, this text should receive careful attention from many quarters and should be required reading for budding authors in the field of analog computation.

F. H. Westervelt (Ann Arbor, Mich.)

★**Problems in the theory of mathematical machines. Collection II [Вопросы теории математических машин. Сборник II].** 5580
Edited by Ju. Ja. Bazilevskiĭ.
Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1962. 239 *pp.* 1.04 *r.*

Papers of mathematical interest will be reviewed individually.

Svoboda, A. 5581
The numerical system of residual classes in mathematical machines. (French, German, Russian, and Spanish summaries)
Information processing, pp. 419–422. *UNESCO, Paris; R. Oldenbourg, Munich; Butterworths, London,* 1960.

This is one of the early papers on residue class (or modular) arithmetic. The author describes an algorithm for the multiplication of two fractions, X/P and Y/P, where $P = p_0 \cdot p_1 \cdots p_n$ and the p_i are relatively prime in pairs. X and Y are integers in the closed interval $[0, P-1]$ and are not represented in the usual positional notation relative to a given base (or radix). Instead, X is represented by a sequence of digits $x_0, x_1, \cdots, x_{n+1}$, where $x_i \equiv X \pmod{p_i}$, $0 \leqq x_i \leqq p_i - 1$, $i = 0, 1, \cdots, n$. Y is similarly represented. The fact that every integer in $[0, P-1]$ can be uniquely represented in this manner is the starting point for modular arithmetic. (This application of the Chinese remainder theorem will both gladden and sadden the hearts of number theorists.) The present paper refers to previous ones for a development of the principles of modular arithmetic. It concentrates on the details of obtaining the product XY/P^2 with careful attention to proper rounding. *E. K. Blum* (Middletown, Conn.)

Fisher, F. Peter; Swindle, George F. 5582
★**Computer programming systems.**
Holt, Rinehart and Winston, New York, 1964. xi + 643 *pp.*

This book, written by two IBM account representatives, contains a good description of the IBM 1401 Programming System. It is not, however, a "comprehensive, detailed analysis of the field of programming systems", as the jacket claims. It is perhaps better described as a detailed defense of the programming systems IBM has chosen to support. There is exactly one reference to SOS (SHARE Operating System), and no other references to any other work in the field outside IBM. The short references to ALGOL are best illustrated by direct quotation, since those who understand ALGOL will then most easily see the extent to which the authors do not understand it. "Probably the most important single factor in the development of ALGOL is that it was developed concurrently with FORTRAN." "Certainly the fact that there are so many things common to the two languages is a reflection of the fact that they are functionally the same and both have approximately the same capabilities. One of the most interesting things in the development of ALGOL is the adoption of several of the more desirable features of the ALGOL language into the FORTRAN language itself." In the small chapter on ALGOL only the most insignificant features are mentioned, such as the use of :=, and the FORTRAN analogues are presented in every case. The following false statement appears: "There is no significant difference between the ALGOL array declaration and the FORTRAN dimension statement, since one-, two- and three-dimensional arrays have the same properties in any programming language". In the Bibliography, there are no references for the ALGOL chapter, and 39 of the 40 references are IBM form numbers. In the Glossary, "recursive" is defined as "the ability to be repeated". There is no attempt to provide historical information or to give credit to other workers; e.g., there is no mention of the Bell Laboratories macro facilities, the M.I.T. Linking Segment System, the Remington Rand GPX System, the General Motors 3-phase Input/Output System, the M.I.T. Summer Session System for Whirlwind, or any other milestones in the history of programming systems. This book is recommended to anyone who wishes to understand his IBM 1401 Programming System, but hardly for any other reason. *B. A. Galler* (Ann Arbor, Mich.)

Liminga, Rune; Olovsson, Ivar 5583
Some crystallographic programs for the computers BESK and FACIT EDB.
Acta Polytech. Scand. Math. and Comput. Mach. Ser. No. 10 (1964), 11 *pp.*

Authors' summary: "A series of programs for the calculation of structure factors and summation of Fourier series is described. These programs can handle practically all space groups without writing subroutines. The reflexions can occur in any order and no sorting is necessary between changes of the summation order. The intervals are completely general. A program for systematic evaluation of distances and angles is also described."

Marczyński, Romuald; Pulczyn, Wiesława 5584
Simulation of a digital analyzer for differential equations on a digital computer. (Polish. English summary)
Algorytmy Zeszyt Specjalny No. 1 (1963), 45–53.

Authors' summary: "The realization of a model of a digital analyzer of differential equations on a digital computer is described. The realization is based on separate unit realization such as for instance: integrator, function generator, summator, as well as on the method of an equivalent translation of the block scheme (applied in differential equation analyzer) to the digital computer language. The given autocode permits one to record the block scheme of a differential equation in a simple and fast way. The record of the scheme is transformed into an analyzer model by means of a translator. The conclusion gives an example of solving ordinary differential equations on the URAL-2 computer using the described method."

GENERAL APPLIED MATHEMATICS

See 5397.

MECHANICS OF PARTICLES AND SYSTEMS

See also 5213, 5214, 5793, 5865, 5868, 5874, 5876.

Beletskii, V. V. [Beleckiĭ, V. V.] 5585
A particular case of the motion of a rigid body about a fixed point in a Newtonian force field.
Prikl. Mat. Meh. **27** (1963), 175–178 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1963), 255–261.

The author considers an axi-symmetric body rotating about a fixed point, which coincides with the center of mass. The body is acted upon by gravitational attraction. Initially, the body rotates about its axis of symmetry. In this case the equations of motion can be integrated by quadratures, and by using the Eulerian angles the author investigates the motion of the point of intersection of the axis of symmetry with the unit sphere centered at the center of mass. This point traces on the unit sphere a curve which is bounded between two parallel circles. The shape of the curve depends on the values of certain parameters. This curve is sometimes tangent to one or to both of the circles, and sometimes forms cusps at them.

T. Leser (Aberdeen, Md.)

Diaz, J. B.; Metcalf, F. T. 5586

Upper and lower bounds for the apsidal angle in the theory of the heavy symmetrical top.

Arch. Rational Mech. Anal. **16** (1964), 214–229.

When a point fixed on the axis of a symmetrical top performs a periodic motion between two horizontal planes, the angle between the horizontal projection of the position vector of the point when the point is at the lowest level and the same projection for the point at the highest level is called the apsidal angle. This angle is expressed in the form of definite integrals, and upper and lower bounds are then found by replacing the integrands with larger and smaller integrands, respectively, and then carrying out the integrations.

T. R. Kane (Stanford, Calif.)

Rybarski, A. 5587

A minimum principle in the theory of conservative systems.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **12** (1964), 21–24.

Let $v(x)$ be positive and continuous on an interval J, where $v^{-1}(x)$ is integrable on J. The minimum principle

$$\left(\int_J v^{-1}(x)\,dx\right)^2 = \min\left(\int_J w(x)\,dx\right)^{-3}\int_J v(x)w^3(x)\,dx$$

is established, where $w(x)$ ranges over a class of non-negative, piecewise continuous functions on J. The principle is applied in studies of the oscillation frequencies of periodic solutions of differential equations of the type $y''+g(y)=0$, where $g(y)$ is assumed piecewise continuous and to have other properties which insure the existence of periodic solutions of the differential equation.

W. M. Whyburn (Chapel Hill, N.C.)

Borodina, R. M. 5588

Influence of an hysteresis motor on the stability of motion of a gyroscope in a Cardan suspension. (Russian)

Approximate methods of solving differential equations, pp. 11–18. *Izdat. Akad. Nauk Ukrain. SSR, Kiev*, 1963.

Das kinetische Auswandern kardanisch gelagerter Kreisel unter dem Einfluß von Nutationsschwingungen wurde bereits von zahlreichen Autoren untersucht. In der vorliegenden Arbeit werden die bisherigen Ergebnisse durch Untersuchungen des Einflusses eines Antriebsmotors ergänzt, dessen Motormoment $M=k(\omega t-\varphi)$ ist; ω ist die Winkelgeschwindigkeit des Rotors, φ ist der Eulerwinkel für die Relativbewegung zwischen Rotor und innerem Kardanring (Rotorgehäuse). Die nichtlinearen Bewegungsgleichungen werden nach der Methode der langsam veränderlichen Parameter behandelt, wobei die Lösungen nach einem Verfahren von Volossov als asymptotische Reihen nach einem kleinen Parameter erhalten werden. Das Ergebnis zeigt, daß das Verhalten des Systems durch das Vorhandensein eines Synchronmotors in erster Näherung qualitativ nicht geändert wird. Erst in zweiter Näherung sind Änderungen durch das Auftreten höherer Harmonischer zu erwarten.

K. Magnus (Stuttgart)

Schmidt, Günter; Weidenhammer, Fritz 5589

Gedämpfte erzwungene Schwingungen in rheolinearen Systemen.

Math. Nachr. **27** (1963/64), 215–228.

Die Verfasser untersuchen partikuläre Lösungen inhomogener linearer Gleichungen mit periodischen Koeffizienten, wie sie bei der Berechnung kleiner Schwingungen in gedämpften Systemen erhalten werden. Drei Aufgaben werden behandelt: (1) die Ermittlung der Zwangsschwingungen außerhalb der Resonanzstellen, (2) Ausrechnung von Resonanzstellen, an denen die Lösungen unbeschränkt anwachsen und (3) Untersuchung der Wachstumsgesetze für die Amplituden der Resonanzlösungen. Es wird gezeigt, daß trotz des Vorhandenseins von Dämpfungen unbeschränkte Resonanzlösungen dann auftreten können, wenn sich das System an der Stabilitätsgrenze der freien parametererregten Schwingungen befindet. Aber nicht alle diese Stabilitätsgrenzen sind resonanzgefährdet. In den Resonanzgebieten wachsen die Amplituden linear mit der Zeit an. Durch die Arbeit werden die bisherigen Erkenntnisse in zwei Richtungen erweitert: erstens werden Systeme endlich vieler linearer Gleichungen untersucht, zweitens sind die Zwangserregungen nicht als harmonisch vorausgesetzt, sondern als beliebige, in Fourier-Reihen entwickelte Funktionen angenommen worden.

K. Magnus (Stuttgart)

Schmidt, Günter 5590

Mehrfache Verzweigungen bei gelenkig gelagerten längs pulsierend belasteten Stäben.

Math. Nachr. **26** (1963), 25–43.

Im Anschluß an eine zwei Jahre zuvor veröffentlichte Arbeit untersucht der Verfasser das Stabilitätsverhalten von pulsierend auf Druck beanspruchten Stäben. Die periodischen Lösungen der Differentialgleichung werden dabei durch Iteration aus zwei unendlichen Systemen nichtlinearer Integrodifferentialgleichungen gewonnen. Die darin vorkommenden Verzweigungsparameter können aus entsprechenden Verzweigungsgleichungen ermittelt werden. Da die bis zum dritten Näherungsschritt ausgerechneten Gleichungen zu undurchsichtig bleiben, beschränkt sich der Verfasser weiterhin auf die Diskussion einiger Sonderfälle. Neben dem Vorhandensein von kritischen Bereichen erster und zweiter Art wird auf die Existenz von verschiedenartigen Kopplungen zwischen den Einzelamplituden hingewiesen, die in kritischen sogenannten Summen- und Differenz-Bereichen erster und zweiter Ordnung zum Ausdruck kommen.

K. Magnus (Stuttgart)

Roĭtenberg, Ja. N. 5591

On the motion of a non-linear gyroscopic system under the action of random forces. (Russian. English summary)

Applications of the methods of non-linear vibrations to the problems of physics and technology (Proc. Internat. Sympos. Non-linear Vibrations, Vol. III, 1961), *pp.* 441–447. *Izdat. Akad. Nauk Ukrain. SSR, Kiev,* 1963.

Bei einer sehr häufig verwendeten Konstruktion des Einkreiselkompasses wird zur Dämpfung von Einstellschwingungen ein Quecksilberdämpfer nach Art eines Schlingertanks verwendet. Bei Rollschwingungen des Schiffes können infolge der Einwirkung des Dämpfers und wegen der Schwerpunktstieferlage des Kreiselsystems Kursabweichungen hervorgerufen werden. Die Größe dieser Kursabweichung wird für den Fall berechnet, daß die Rollschwingung eine stationäre Zufallsbewegung mit gebrochen rationaler Spektraldichte ist. Aus einer Operatorform der Bewegungsgleichungen wird unter Berücksichtigung der gegebenen Spektraldichte ein Erwartungswert für den Azimutwinkel ausgerechnet, und diese Formel wird dann für ein konkretes Beispiel ausgewertet. Es zeigt sich, daß der Kursfehler für die Hauptkurse verschwindet und bei den Interkardinalkursen ein Maximum wird. Dieses Maximum kann bis zu 20° anwachsen, jedoch läßt es sich durch geeignete Abstimmung der Eigenschwingungszeit des Quecksilberdämpfers klein halten. *K. Magnus* (Stuttgart)

Arkhangel'skii, Iu. A. [Arhangel'skiĭ, Ju. A.] 5592

On the algebraic integrals in the problem of motion of a rigid body in a Newtonian field of force.

Prikl. Mat. Meh. **27** (1963), 171–175 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1963), 247–254.

The approximate equations of motion of a rigid body about a fixed point under the action of a central Newtonian force field in every case have three independent integrals: the kinetic energy integral, the integral of areas, and the trivial integral. A fourth integral exists only in two cases: when the fixed point coincides with the center of mass, and when the body possesses kinetic symmetry about one of its principal axes of inertia and the center of gravity is located on that axis.

The author had proved previously that the possession of kinetic symmetry is a necessary but not a sufficient condition. In this paper the author shows that the location of the center of gravity on the axis of kinetic symmetry is the sufficient condition for the existence of the fourth integral. *T. Leser* (Aberdeen, Md.)

Iarov-Iarovoi, M. S. [Jarov-Jarovoĭ, M. S.] 5593

Integration of the Hamilton-Jacobi equation by the method of separation of variables.

Prikl. Mat. Meh. **27** (1963), 973–987 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1499–1520.

A method is presented for the construction, by means of separation of variables, of the complete integral of the Hamilton-Jacobi equation, and necessary and sufficient conditions for integrability are established. *T. R. Kane* (Stanford, Calif.)

Krementulo, V. V. 5594

A problem on the stability of a spherical gyroscope.

Prikl. Mat. Meh. **27** (1963), 1005–1011 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1547–1556.

Sufficient conditions for asymptotic stability of particular solutions of the differential equations governing the motion of a spherical gyroscope are established. Motions of gyroscopes whose mass centers do not remain at rest, as well as motions with variable angular velocity, are considered. *T. R. Kane* (Stanford, Calif.)

Fujiwara, I.; Hemmer, P. Chr. 5595

New aspects in classical dynamics. I, II, III.

Norske Vid. Selsk. Forh. (*Trondheim*) **36** (1963), 46–56.

The authors derive several new results in the classical mechanics of one-dimensional systems characterized by Lagrangians which are quadratic in the velocity. Let $q(t; y, x)$ be the natural path given by Hamilton's stationary action principle and satisfying the boundary conditions $q(0; y, x) = x$, $q(\tau; y, x) = y$. They introduce a certain linear second-order ordinary differential equation, and show that $\beta(t) \equiv aq_x(t) + bq_y(t)$, $q_x(t) \equiv \partial q(t; y, x)/\partial x$, etc., a and b constants, is the general solution. From the constancy of the Wronskian they then derive the equation $\beta(0)\beta(\tau)\int_0^\tau dt\beta(t)^{-2} = (-m)[\partial^2 S(y, x, \tau)/\partial y\,\partial x]^{-1}$, where $S(y, x, \tau) \equiv \int_0^\tau dt L[\dot q(t), q(t)]$ is the classical action integral. Their second result is a reduction formula for $\int_0^\tau dt L[\dot z(t), z(t)]$, where $z(t)$ is an arbitrary path having the same end points as the natural path $q(t)$, which expresses it in terms of $S(y, x, \tau)$, the corresponding integral along the natural path. These formulas are generalized to the case of Lagrangians of degree higher than quadratic in the velocity. These results are useful as lemmas in the first author's investigation of functional integration in a later series of papers [#5597]. *R. Ingraham* (University Park, N.M.)

Fujiwara, I. 5596

New aspects in classical dynamics. IV, V.

Norske Vid. Selsk. Forh. (*Trondheim*) **36** (1963), 57–63.

The author generalizes two new results in classical mechanics proved in an earlier series of papers [#5595] to the case of a system of $f > 1$ degrees of freedom. The principal formal difference is that quantities which were formerly numbers become f-dimensional vectors or $f \times f$ matrices. *R. Ingraham* (University Park, N.M.)

Fujiwara, I. 5597

Correspondence principle and functional integration method in quantum mechanics. I–X.

Norske Vid. Selsk. Forh. (*Trondheim*) **36** (1963), 72–91; *ibid.* **36** (1963), 97–126.

In a one-dimensional non-relativistic quantum-mechanical system, all the dynamical information is contained in the transformation function $K(y, t; x, 0) \equiv \langle y, t | x, 0\rangle$, where $|q, t\rangle$ denotes the eigenfunctions of the position operator $q(t)$; K satisfies the Schrödinger equation and the boundary condition $\lim_{t\to 0} K(y, t; x, 0) = \delta(y - x)$. A formal solution for K is

$$(1)\quad K(y, t; x, 0) = \exp[-(it/\hbar)H(-i\hbar\partial/\partial y, y)]\delta(y-x).$$

Direct evaluation of the right member is very difficult

for a general potential $V(y)$ and has only been carried out for the case $V(y)=0$ (free particle) and $V(y)\propto y^2$ (harmonic oscillator). However, by using the composition rule for K, it is sufficient to evaluate the right member of (1) to first order in t only; one then obtains an expression for K for finite t in the form of a new mathematical construct, functional integration, or the Feynman "path integral" $K=\int d[z(s)]\Phi[z(s)|y, t; x, 0]$ with the functional integrand $\Phi=N(t, 0)\exp\{(ih)\int_0^t ds L[\dot{z}(s), z(s)]\}$, where L is the classical Lagrangian and $N(t, 0)$ a certain normalization facto·. Using new techniques and lemmas in classical mechanics developed in earlier papers [#5595; #5596] the author evaluates the functional integral and the path integral as power series in Planck's constant h (equations (15), (16) and (17), (18) in part IV, respectively). These solutions have the form $\Phi=K_c\mathscr{D}[z(s)-q(s)][1+O(h)]$ and $K=K_c[1+\sum_{n=1}^{\infty}(i/h)^n K_n(y, t; x, 0)]$, where $K_c\equiv K_c(y, t; x, 0)\equiv[(i/h)\partial^2 S/\partial y\,\partial x]^{1/2}\exp[(i/\hbar)S]$ is the "semi-classical kernel", $S\equiv S(y, t; x, 0)\equiv\int_0^t ds L[\dot{q}(s), q(s)]$ is the classical action integral taken for the natural path $q(t)$, and $\mathscr{D}[z(s)-q(s)]$ is the delta functional. Thus he proves the validity of the "Correspondence Principle" in the form $\lim_{h\to 0}\Phi=K_c\mathscr{D}$ or $\lim_{h\to 0}K=K_c$ for a general $V(y)$. These formulas are then extended to the case of any finite number of degrees of freedom and the case of a field (infinite number of degrees of freedom).

R. Ingraham (University Park, N.M.)

Kononenko, V. O. 5598

Some autonomous problems in the theory of non-linear oscillations. (Russian. English summary)

Applications of the methods of non-linear vibrations to the problems of physics and technology (Proc. Internat. Sympos. Non-linear Vibrations, Vol. III, 1961), pp. 151–179. Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963.

From the author's summary: "Dynamic systems are investigated, each of them consisting of an oscillating system and a source of energy. It is assumed that the oscillating system and the source of energy are connected, so that the systems are considered as autonomous ones. The basic problem is stated as follows: What is the influence of the properties of the source of energy upon the regime of the oscillating system in the resonance region? The results are obtained for some typical oscillating systems. The analysis shows that the steepness of the characteristic of the source of energy at the representing point is the important parameter for the stability of stationary oscillations. A method is proposed for solving similar problems in cases where the oscillating system has more than one degree of freedom."

J. K. Hale (Baltimore, Md.)

Frolov, K. V. 5599

The simulation of resonance properties of certain autonomous non-linear oscillatory systems. (Russian. English summary)

Applications of the methods of non-linear vibrations to the problems of physics and technology (Proc. Internat. Sympos. Non-linear Vibrations, Vol. III, 1961), pp. 498–513. Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963.

From the author's summary: "This paper deals with the influence of the properties of the energy source (motor) on the oscillations of nonlinear one-degree-of-freedom mechanical systems. The investigation was carried out by using electronic analog computers. Unstable regimes of motion were examined for both small and large non-linearities. It was established that the slope of the momentary characteristics of the source of energy is of essential importance to the stability of resonance regimes. The regions of stable and unstable slopes of characteristics were found."

J. K. Hale (Baltimore, Md.)

Parkyn, D. G. 5600

Spin integrals in dynamics.

Amer. Math. Monthly **71** (1964), 144–151.

The problem presented by the motion under gravity of a homogeneous sphere rolling on a rough surface (s) does not admit so-called spin integrals if (s) is arbitrary. The object of the paper is to ascertain for which surfaces (s) scalar integrals of the form $(\alpha\mathbf{n}+\beta\mathbf{k})\cdot\mathbf{\Omega}=$ const exist, where α and β are constants, $\mathbf{n}$ the unit vector directed from the point of contact towards the centre of the sphere, $\mathbf{k}$ the unit vertical vector and $\mathbf{\Omega}$ the angular velocity of the sphere. Results: (1) Calling $\mathbf{k}\cdot\mathbf{\Omega}=K=$ const a K-integral, the totality of surfaces (s) possessing K-integrals comprises planes, cylinders with vertical axes, and a group of surfaces including the right circular cone with vertical axes; (2) The integral $\mathbf{n}\cdot\mathbf{\Omega}=N=$ const is called an N-integral, and such integrals exist for motion on a plane or a sphere; (3) Except in the case of a plane, K- and N-integrals cannot exist simultaneously.

E. B. Schieldrop (Oslo)

Laval, Guy; Pellat, René 5601

Instabilités des systèmes stationnaires dues à un couplage entre deux modes normaux.

C. R. Acad. Sci. Paris **258** (1964), 1756–1759.

In another paper the authors [the authors, Cotsaftis, and Trocheris, Nuclear Fusion (to appear)] established a criterion for the marginal stability of the normal mode of a Lagrangian system. They now go on to extend their work to the case where the normal mode in question is degenerate.

F. D. Kahn (Manchester)

ELASTICITY, PLASTICITY

See also 5208, 5664, 5821.

Bramble, J. H.; Payne, L. E. 5602

Effect of error in measurement of elastic constants on the solutions of problems in classical elasticity.

J. Res. Nat. Bur. Standards Sect. B **67B** (1963), 157–167.

This paper is concerned with solutions to the usual boundary-value problems of linear elastostatics. In particular, the authors study the dependence of such solutions on the elastic constants. Let μ and σ, respectively, denote the shear modulus and Poisson ratio of the material, with $\mu\neq 0$, $-1<\sigma\leqq\frac{1}{2}$. The authors show that for given body forces and surface tractions (or surface displacements), the displacement vector and all of its derivatives are continuous functions of the elastic constants μ and σ. As a corollary they then have the fact that the solution of a given compressible boundary-value problem tends to the corresponding result in the classical

incompressible theory as $\sigma \to \frac{1}{2}$. Finally they derive inequalities which give upper and lower bounds for the pointwise error in the approximation of the exact solution by an arbitrary function. The error in measurement of the elastic constants is taken into account in these bounds.

M. E. Gurtin (Providence, R.I.)

Kuvshinskii, R. V. [Kuvšinskiĭ, E. V.]; Aéro, É. L. [Aèro, È. L.] 5603

Continuum theory of asymmetric elasticity. The problem of "internal" rotation.

Fiz. Tverd. Tela **5** (1963), 2591–2598 (*Russian*); *translated as Soviet Physics Solid State* **5** (1964), 1892–1897.

The authors continue their line of investigation [Fiz. Tverd. Tela **2** (1960), 1399–1409; MR **23** #B1717] aimed at bridging the gap between continuum and molecular theories of elastic solids from the continuum side. Apparently unacquainted with the work of E. and F. Cosserat [*Théorie des corps déformables*, Hermann, Paris, 1909] the authors rediscover here the equations of a three-dimensional, linear Cosserat continuum, but they omit the rotatory inertia of the Cosserat trièdre. As a consequence of the omission, there is no optical branch for transverse waves from their equations. The Cosserat theory, as usual, gives no optical branch for longitudinal waves because the deformation of the trièdre and the associated inertia are not taken into account. (An extension of the Cosserat theory to include the additional effects appears in a paper by the reviewer [Arch. Rational Mech. Anal. **16** (1964), 51–78; MR **28** #3569].)

R. D. Mindlin (New York)

Langenbah, A. [Langenbach, Arno] 5604

On some non-linear operators in the theory of elasticity in Hilbert space. (Russian. English summary)

Vestnik Leningrad. Univ. **16** (1961), *no.* 1, 38–50.

Author's summary: "Some non-linear functional equations whose operators have a positive definite linear Gâteaux differential in Hilbert space are investigated. Such equations are reduced to a certain variational problem. The problem has a solution in a suitable extension of the original domain of the operator. The results are applied to some problems of the theory of plasticity."

Cherepanov, G. P. [Čerepanov, G. P.] 5605

The hydrodynamic formulation of certain problems in the theory of cracks.

Prikl. Mat. Meh. **27** (1963), 1077–1082 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1649–1657.

The author notes conditions in which it is legitimate to use classical hydrodynamics to obtain approximate solutions in elasticity, and applies the method to a number of problems involving the generation or propagation of cracks.

J. W. Craggs (Melbourne)

Hayes, M. 5606

Uniqueness for the mixed boundary-value problem in the theory of small deformations superimposed on large.

Arch. Rational Mech. Anal. **16** (1964), 238–242.

The author investigates the question of uniqueness for the mixed boundary-value problem arising when a small deformation is superimposed on a large homogeneous deformation. The mathematical problem reduces to one of determining the solution of a certain second-order system of equations for the infinitesimal displacements u_i $(i=1, 2, 3)$ under the assumption that the u_i are prescribed on a portion S_1 of the boundary while the surface tractions T_i are prescribed on S_2, the remainder of the boundary. The author shows that if S_2 is a plane and S_1 is star-shaped with respect to a point on S_2, then strong ellipticity of the displacement equations is sufficient to insure uniqueness. He gives an example to show that the condition of strong ellipticity is not always necessary.

L. E. Payne (College Park, Md.)

Lu, Chian-ke [Lu, Chien-Ke] 5607

The periodic Riemann boundary value problem and its applications to the theory of elasticity.

Acta Math. Sinica **13** (1963), 343–388 (*Chinese*); *translated as Chinese Math.* **4** (1964), 372–422.

Dhaliwal, R. S. 5608

Stresses set up by rotation in a cardioid shaped plate.

Indian J. Math. **5** (1963), 67–70.

Author's summary: "The problem of steady rotation of discs about normal axes treated as generalised plane stress problem or of shafts considered as plane strain problem has been solved by many investigators. Mitra has considered the rotation of a plate in the shape of a cardioid about an axis passing through the origin and normal to the plate. The author has obtained the stresses in certain thin curvilinear plates rotating about an axis lying in their middle plane. In this paper steady rotation of a cardioid-shaped plate about the initial line has been considered. The problem has been considered as a statical one and inertia forces have been treated as body forces."

Benthem, J. P. 5609

A Laplace transform method for the solution of semi-infinite and finite strip problems in stress analysis.

Quart. J. Mech. Appl. Math. **16** (1963), 413–429.

The author considers the semi-infinite strip $0<x<\infty$, $-1<y<1$. The biharmonic equation is subjected to a Laplace-transform. The transformed equation may be solved if all boundary terms at $x=0$ are known. Since only two boundary conditions are specified at $x=0$, the two remaining boundary terms are unknown functions of y, which may be written as Fourier series with unknown coefficients. The solution must remain bounded for $x\to\infty$, and the Laplace transform therefore cannot have poles in the positive half-plane. This condition determines the unknown coefficients in the Fourier series. The procedure may be modified in case of singularities at the corners $x=0$, $y=\pm 1$ by introduction of additional boundary terms with an appropriate singularity at these corners, in addition to the Fourier series. In case of a finite strip the two-sided Laplace transform is used. The number of unknown Fourier coefficients is doubled, and the requirement is now that the Laplace transform cannot have poles in both the positive and negative half-planes.

W. T. Koiter (Delft)

Solomon, Liviu 5610

Une solution approchée du problème du poinçon rigide à base plane bornée convexe non elliptique.

C. R. Acad. Sci. Paris **258** (1964), 64–66.

Author's summary: "On propose une méthode de solution approchée du problème du poinçon rigide à base plane non elliptique. A ce but, on utilise la fonction de torsion de Prandtl—aisément déterminable exactement ou par voie expérimentale. Les exemples des poinçons à base plane triangulaire ou carrée donnent la possibilité de juger sur la précision de la méthode."

R. M. Morris (Cardiff)

Tonojan, V. S. 5611

On a plane contact problem for an elastic quadrant. (Russian. Armenian summary)

Akad. Nauk Armjan. SSR Dokl. **37** (1963), 121–130.

On the edges of a plane elastic quadrant the following boundary conditions are given: $\sigma_x(0, y)=0$ for $0<y<a$ and for $y>b$, $u(0, y)=f(y)$ for $a<y<b$, $\tau_{xy}(0, y)=0$ for $y>0$ and $\sigma_y(x, 0)=\tau_{xy}(x, 0)=0$ for $x>0$. The Airy function is expressed in the form of Fourier integrals. The problem is reduced to a system of "trial" integral equations and a Fredholm integral equation of the second kind. The solution of the trial integral equations is given by determining the coefficients in a pair of dual trigonometric series. The results of C. J. Tranter [Quart. J. Mech. Appl. Math. **14** (1961), 283–292; MR **23** #B2642] are used.

Z. Kączkowski (Warsaw)

Deb, Dipti 5612

Note on the rotation of an aeolotropic elliptic cylinder about its axis.

Indian J. Theoret. Phys. **10** (1962), 67–71.

Author's summary: "In this paper the problem of an aeolotropic cylinder of elliptic cross-section steadily rotating about its axis has been solved."

Collins, W. D. 5613

Some coplanar punch and crack problems in three-dimensional elastostatics.

Proc. Roy. Soc. Ser. A **274** (1963), 507–528.

The author considers three-dimensional problems of an elastic half-space indented by two circular punches and of an infinite solid with two or more coplanar penny-shaped cracks opened by internal pressure. The problems are reduced to boundary-value problems in potential theory. Exact solutions in closed form do not seem possible, but the problem may be reduced to an infinite set of simultaneous linear Fredholm integral equations of the second kind. In case of two identical punches on a half-space, and in case of two equal penny-shaped cracks in an infinite solid, an expansion of the solution is obtained in powers of a/f, where a is the punch or crack radius and f is the half-distance between centre lines. A similar expansion is obtained in the case of an infinite row of coplanar penny-shaped cracks. The results are compared with those of similar two-dimensional problems.

W. T. Koiter (Delft)

Pal'mov, V. A. [Pal'mov, V. A.] 5614

State of stress in the neighborhood of a rough surface of elastic bodies.

Prikl. Mat. Meh. **27** (1963), 963–969 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1479–1489.

The roughness of the surface of machine parts produced by shaping, milling and grinding is known to produce a state of stress in the neighbourhood of the surface which differs from that corresponding to an ideal smooth surface. This paper is concerned with clarification of the question of the magnitude of this change in the state of stress.

The author considers an isotropic elastic semi-infinite space $z \geqq H(x, y)$ subjected at infinity $(z \to \infty)$ to the action of normal stresses σ_1 and σ_2 in the x- and y-directions, and assumes that the stress-free boundary $z=H(x, y)$ of the semi-space differs little from the plane $z=0$. The rough surface itself is considered "as the realisation of a homogeneous statistically anisotropic random field with a normal law of distribution". The results obtained are in a convenient form for practical computation, since they contain easily measurable statistical characteristics of the profile of the rough surface.

J. Hubert Wilkinson (Battersea)

Wesołowski, Zbigniew 5615

The axially symmetric problem of stability loss of an elastic bar subject to tension. (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 383–395.

The author considers the stability problem of a solid circular cylinder subject to finite elongation. The material is assumed to be isotropic and incompressible. The theory of small additional deformation superimposed on finite axisymmetric deformation is employed. The results show that no instability in tension occurs for neo-Hookean materials. Instability at finite extension does occur, however, for a particular form of elastic potential developed for rubber-like materials.

W. T. Koiter (Delft)

Sewell, M. J. 5616

A general theory of elastic and inelastic plate failure. I.

J. Mech. Phys. Solids **11** (1963), 377–393.

The problem of the bifurcation instability of elastic and inelastic plates is treated. The general approach follows that of Hill's bifurcation theory [same J. **7** (1959), 209–225; MR **21** #3978]. A differential equation for the bifurcation condition of thin plates of arbitrary contour and fairly general material properties subjected to in plane loading is derived. Among other items of interest, the equivalent of the tangent modulus buckling load for plates can be obtained from this basic equation.

S. R. Bodner (Providence, R.I.)

Sapondžjan, O. M. 5617

A case of the bending of a thin rectangular plate. (Russian. Armenian summary)

Akad. Nauk Armjan. SSR Dokl. **37** (1963), 137–141.

The problem under consideration is that of a rectangular plate simply supported on two opposite edges and satisfying arbitrary boundary conditions on the remaining edges. On a part of the straight line dividing the plate

perpendicular to the simply supported edges certain discontinuities are assumed. As an example the author considers a plate simply supported on the periphery with a partial cut on the axis of symmetry. The deflection of the plate is expressed by simple Fourier series. The boundary and continuity conditions lead to a regular infinite system of linear algebraic equations.

Z. Kączkowski (Warsaw)

Bhargava, R. D.; Pande, D. 5618

Misfitting shells under uniform pressures.

J. Sci. Engrg. Res. **7** (1963), 361–367.

The authors solve the rather elementary problem of shrink fits of two spherical or cylindrical shells with internal and external pressures. The mating cylinders are assumed of equal lengths, and longitudinal stresses and strains are ignored. *L. H. Donnell* (Chesterton, Ind.)

Naghdi, P. M. 5619

A new derivation of the general equations of elastic shells. (French, German, Italian, and Russian summaries)

Internat. J. Engrg. Sci. **1** (1963), 509–522.

From the author's summary: "This paper is concerned with a new derivation of the general equations of the linear theory of elastic shells under the Kirchhoff-Love hypothesis. The entire boundary-value problem of shell theory is recast in terms of new variables for the strain measures as well as the stress and couple resultants. Particular attention is paid to an exact derivation of the constitutive equations and their first approximations which meet all invariance requirements. The natural boundary conditions for stress and couple resultants and all field equations consisting of compatibility, equilibrium and the constitutive equations (or their first approximations) involve only symmetric tensors and are, moreover, remarkably free of the anti-symmetric parts of both the middle surface strains and the couple resultants."

W. T. Koiter (Delft)

Ahund-Zade, M. Ju.; Tagiev, I. G. 5620

A new approximation method of solving the equilibrium equations for arbitrary shallow shells. (Russian. Azerbaijani summary)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **1963**, *no.* 4, 127–134.

The system of two equations of V. Z. Vlasov in the technical theory of shallow shells is reduced, by the use of the perturbation method, to an infinite system of recurrence differential equations with constant coefficients. For a shell of arbitrary shape having positive Gaussian curvature as the small parameter it is assumed that the larger of the two quantities: $\nu_i = 1 - r_{i\,\min}/r_{0\,\max}$ $(i = 1, 2)$, where $r_i(\alpha, \beta)$ are the principal radii of curvature and $r_{0\,\max}$ is the largest radius of curvature in the whole shell. Each of the recurrence equations deduced has the same form as in the case of a spherical shell. For the shell with vanishing Gaussian curvature each of the equations has the same form as in the case of a circular cylindrical shell. The convergence of the solution is rather good.

Z. Kączkowski (Warsaw)

Prasad, C. 5621

On vibrations of spherical shells.

J. Acoust. Soc. Amer. **36** (1964), 489–494.

The paper extends analysis of vibrations of shallow spherical shells as developed by E. Reissner, Kalnins, and Naghdi to non-shallow spherical shells. The basic differential equations, taken in a form which includes the effects of transverse shear and rotatory inertia, are reduced to a system of three equations: an uncoupled sixth-order differential equation for the displacement w, and two second-order partly coupled equations for two auxiliary variables ψ, Λ. The introduction of auxiliary variables follows closely the method of Kalnins [same J. **33** (1961), 1102–1107; MR **24** #B1359] for the shallow shell. The equations are solved by means of associated Legendre functions, and the nature of the solutions is discussed briefly for a number of special cases, e.g., complete shell, axisymmetric vibrations, neglect of transverse shear and rotatory inertia. The free vibrations of a clamped (non-shallow) cap are considered in somewhat more detail. *H. J. Weinitschke* (Hamburg)

Šamiev, F. G. 5622

On the design of shells of minimal weight. (Russian. Azerbaijani summary)

Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **1963**, *no.* 5, 37–41.

A rigid-plastic circular cylindrical sandwich shell under axisymmetrical radial load is considered. A procedure for minimum volume design of a short shell (1) with simply supported edges, and (2) with clamped edges, is established under the assumption of Tresca's yield condition. The author refers to the paper of R. T. Shield [J. Appl. Mech. **27** (1960), 316–322; MR **22** #3261].

Z. Kączkowski (Warsaw)

Kaliski, Sylwester; Michalec, Jerzy 5623

Magnetoelastic resonance vibration of a perfectly conducting cylinder in a magnetic field. (Polish and Russian summaries)

Arch. Mech. Stos. **15** (1963), 359–369.

The problem of magnetoelastic resonance vibration of a circular cylinder placed in an axisymmetrical magnetic field is considered, supposing that around the cylinder is a vacuum. First the internal friction of the material of the cylinder is taken into account by supposing that it is a Voigt body. Then the solution for the perfectly elastic cylinder is obtained. *N. Cristescu* (Bucharest)

Flemming, Manfred 5624

Die matrizielle Berechnung von Querschnittswerten und Einheitsspannungen bei beliebiger Querschnittsform. (English and French summaries)

Z. Flugwiss. **12** (1964), 171–178.

Author's summary: "The paper shows how to calculate, by matrix methods, the cross-section properties and unit stresses, including diffusion loads for arbitrary multi-cell closed sections with attached open sections. The characteristics of the section are expressed in matrix form and these matrices are then identified in the form of index pairs."

Ufliand, Ia. S. [Ufljand, Ja. S.] 5625

Oscillations of elastic bodies with finite conductivity in a transverse magnetic field.

Prikl. Mat. Meh. **27** (1963), 740–744 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1135–1142.

The author derives an exact formal solution for the two-dimensional (plane strain) waves in an infinite elastic solid, assumed to have finite conductivity, and to be under the influences of a transverse magnetic field and arbitrary body forces. Use is made of a multi-integral transform (Laplace and two Fourier) to derive the formal solutions for the dilatational and equivoluminal potentials from the governing coupled elasto- and electro-dynamic equations. Interest focuses predominantly on dilatational waves since the governing equations show that only these are influenced by electro-magnetic effects in the present case.

The author assumes the body force is an impulse concentrated at a point. Then with the aid of known potential-body force relations from elasticity theory, and certain integral forms for Bessel functions, he inverts the Fourier transforms and reduces the formal solution for the dilatational potential to the complex inversion integral for the Laplace transform (Bromwich integral). Use of a well-known Tauberian theorem then yields wave-front information on the dilatational waves. The results are interesting, showing that the solution for the dilatational potential has two components, one wave, and one non-wave, in nature. The wave component propagates with the usual elastic dilatational body-wave velocity. The non-wave component is diffusive, exhibiting the influence of the finite conductivity. The author also gets information from his general solution on the perfect conductor and the case of a line distribution of impulsive body forces.

J. Miklowitz (Pasadena, Calif.)

Kvinikadze, G. P. 5626

The third and fourth boundary-value problems of planar elasticity theory for steady-state vibrations of isotropic bodies. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **32** (1963), 535–542.

The fundamental vibration equation considered is

$$(\lambda+\mu)\ \mathrm{grad\ div}\ \mathbf{u}+\mu\Delta\mathbf{u}+\omega^2\sigma\mathbf{u}=\mathbf{0},$$

σ being the density and ω the circular frequency. Two boundary-value problems are discussed by setting up and then solving certain singular integral equations. The paper is highly compressed and contains misprints.

J. Hubert Wilkinson (Battersea)

Massa, Emilio 5627

Effetto di stati piani di coazione sulle vibrazioni libere con piccola inflessione di piastre circolari. (English summary)

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 319–345.

Author's summary: "Free vibrations of thin circular plates with internal plane stresses are examined. After obtaining the pertinent differential equations and the expressions for the kinetic and elastic energy, the small vibrations about the pre-buckling equilibrium position (when non-linear terms may be neglected in the differential equations) are studied in the presence of internal plane stresses distributed with circular symmetry. The way of calculating the normal mode of vibration is established with the Rayleigh-Ritz method. The analysis is applied to the case of a free plate with parabolic distribution of temperature, maximum at the center and zero at the boundary. The variation of the natural frequencies with increasing temperature is determined, for some of the principal modes, up to the onset of buckling; the main characteristics of the vibrations under examination are indicated."

Massa, Emilio 5628

Sulle vibrazioni libere con inflessione non piccola di piastre circolari sottoposte a stati piani di coazione. (English summary)

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 346–368.

Author's summary: "Following a previous investigation [#5627] this paper deals with the free vibrations of thin circular plates with internal plane stresses in the case of large deflection, where the non-linear terms can no longer be neglected in the pertinent differential equations. The modes of vibration that with decreasing deflection tend to the linear normal modes of the plates are determined with an approximate method based upon an application of Hamilton's principle. Furthermore, the vibrations in the postbuckling regime are determined. The analysis is applied to the case of a free plate with a parabolic distribution of temperature, maximum at the center and zero at the boundary."

Feodos'ev, V. I. [Feodos'ev, V. I.] 5629

Application of a step-by-step method to the analysis of stability of a compressed bar.

Prikl. Mat. Meh. **27** (1963), 833–841 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1272–1283.

The author discusses the computer analysis of deformable systems in successive steps of time. He then gives extensive results of such analyses of initially curved hinged-end struts whose ends are brought together at a constant rate, inertia forces being considered. The cross-section is rectangular and the strain-hardening material follows a two-straight-line stress-strain law. At the higher of the two rates of shortening studied the inertia forces are very important, and linear damping to reduce the effects of vibration is considered. The most striking finding is the great dependence of the peak load upon the amplitude of the one-half sine component of the initial curvature.

L. H. Donnell (Chesterton, Ind.)

Movčan, A. A. 5630

On the stability of deformation processes of continuous media. (Russian. Polish and English summaries)

Arch. Mech. Stos. **15** (1963), 659–682.

Author's summary: "A generalisation of Lyapunov's stability definition is discussed in the present paper, which makes it possible to prove the fundamental theorems of Lyapunov's direct method applicable to solid bodies (systems of infinite degrees of freedom). To make the exposition of the said theorems free of secondary details the axiomatic method of description is used. The principal

properties of the processes taking place in material systems, relating to stability problems, are postulated by some axioms. In determining the stability problems as well as in proving the theorems only these axioms are utilised. All the results obtained in this way of abstraction will be true for the material processes for which these axioms are satisfied. The application of the developed method to solid bodies is illustrated by the solution of a linear problem of stability of a compressed plate as also in proving the analogous of the theorems of Lagrange (Lejeune-Dirichlet) and Kelvin for the case of solid body finite displacement and deformation."

Kaul, R. K.; McCoy, J. J. 5631

Propagation of axisymmetric waves in a circular semi-infinite elastic rod.

J. Acoust. Soc. Amer. **36** (1964), 653–660.

Authors' summary: "The Mindlin-McNiven equations for axially symmetric waves are used to determine the radial strains in a circular, semi-infinite, isotropic, elastic rod for both pure and mixed end conditions when a constant pressure is suddenly applied to its end. Using double-integral transforms, the solution is obtained in terms of three Fourier integrals, each one representing one mode of propagation. These integrals are too complex to be evaluated exactly and, therefore, an asymptotic solution, valid for large distance of travel, is obtained for the head of the pulse by using the method of steepest descent. The results predict the existence of edge resonance and further demonstrate its influence on the strain field."

Flavin, J. N. 5632

Surface waves in pre-stressed Mooney material.

Quart. J. Mech. Appl. Math. **16** (1963), 441–449.

Propagation of waves over the surface of an initially stressed isotropic half-space, consisting of either Mooney or neo-Hookean material, is considered. In particular, the author treats the propagation of a wave along a direction other than that of a principal axis of extension.

M. Hayes (Newcastle upon Tyne)

Sen, Bibhutibhusan 5633

Note on the transient response of a linear visco-elastic plate in the form of an equilateral triangle.

Indian J. Theoret. Phys. **10** (1962), 77–81.

Author's summary: "Using trilinear co-ordinates the problem of the dynamic response of a linear visco-elastic plate in the form of an equilateral triangle subjected to uniform load has been solved in this note."

Lee, T. M. 5634

Dilatation constants and complex ratio from forced vibration of a free viscoelastic sphere.

J. Acoust. Soc. Amer. **36** (1964), 458–462.

The author solves the problem of a hollow viscoelastic sphere with a traction-free outer boundary acted upon by a harmonic oscillating uniform internal pressure. He then uses this solution to obtain expressions which relate the properties of the test material to quantities measurable in the laboratory. *W. S. Edelstein* (Providence, R.I.)

Bhargava, R. D.; Sharma, C. B. 5635

Elasto-plastic medium containing a cylindrical cavity under uniform internal pressure.

J. Franklin Inst. **277** (1964), 422–427.

Authors' summary: "The problem of energy distribution in an infinite elasto-plastic medium containing a cylindrical cavity under uniform pressure has been considered in this paper. This gives a quantitative idea about energy dissipated in such a medium. The result has important applications in ballistics. Further, some calculations have been made for the first time for cylindrical inclusions. The equilibrium boundary and the stress-strain fields reveal many interesting features."

Mandel, Jean 5636

Sur une généralisation de la théorie du potentiel plastique de Koïter.

C. R. Acad. Sci. Paris **258** (1964), 2007–2009.

Author's summary: "En considérant des glissements dont les résistances sont 'couplées', on obtient, entre les vitesses de contrainte et de déformation, des relations qui généralisent celles qu'a proposées W. T. Koïter. Lorsque les glissements obéissent à la loi de Schmid, on montre que, moyennant deux hypothèses complémentaires, tous les théorèmes de la théorie du potentiel plastique s'étendent aux relations obtenues."

W. T. Koiter (Delft)

Singh, Manohar 5637

A linearized theory of tube drawing. (German summary)

Z. Angew. Math. Phys. **15** (1964), 1–12.

A thin-walled tube is drawn through a conical die. Variation of stress across the tube thickness is neglected so that the only non-vanishing stresses, meridional (σ_1) and circumferential (σ_2), are functions of a single variable, taken to be the radius r of the die. Choice of $-r$ as the quasi-static "time" variable allows the entire steady-state plastic flow problem to be formulated in terms of ordinary differential equations for σ_1, σ_2, thickness h, and radial velocity v.

A closed form solution for the Mises yield condition is given, provided the exit radius is at least 0.40377 times the entrance radius. Various piecewise linear approximations to the Mises condition are also used and found to give good agreement for σ_1, fair agreement for σ_2 and v, and poor agreement for h. This suggests a modified procedure in which σ_1 is first found from a linearized theory and this value then substituted into the nonlinear equations of the Mises theory to determine σ_2, v, and h. Figures of the dependent variables show that this latter procedure gives good agreement in all variables.

P. G. Hodge, Jr. (Stanford, Calif.)

FLUID MECHANICS, ACOUSTICS

See also 5247, 5766, 5773.

Surovihin, K. P. 5638

A group-theoretic classification of equations describing one-dimensional non-stationary gas flows. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 533–536.

The paper presents an extension of the group-theoretical classification of Ovsiannikov to more general adiabatic flows in which the effective cross-section of the stream-tube is an arbitrary function of the coordinate. Such flows nclude cases of cylindrical and spherical symmetry.
W. Eckhaus (Paris)

Soo, Shao L.; Sarafa, Zuhair N. 5639
Flow of rarefied gas over an enclosed rotating disk. (German summary)
Z. Angew. Math. Phys. **15** (1964), 21–39.
A theoretical solution for the slip flow of an incompressible fluid between a rotating disk and a stationary plate is presented and compared with experiment.
K. R. Enkenhus (Silver Spring, Md.)

Borisenko, A. I.; Myškis, A. D. 5640
Plane flow of an ideal incompressible fluid around thin profiles with large camber. (Russian. English summary)
Ukrain. Mat. Ž. **15** (1963), 119–134.
This paper considers the plane flow of an ideal fluid past a thin airfoil section with finite camber, and the corresponding problem for a lattice. The perturbation flow is generated by an unknown vortex distribution on the mean camber line, and integral equations are presented for the determination of the unknown vortex strength. Numerical methods of solution are discussed and a detailed solution is presented for the case of a single parabolic arc. A brief discussion is included of the electrolytic analogy to this problem. *J. N. Newman* (Washington, D.C.)

Tuck, E. O. 5641
Some methods for flows past blunt slender bodies.
J. Fluid Mech. **18** (1964), 619–635.
The author considers the problems of axially symmetric potential flow and of axially symmetric Stokes flow past a slender body. He formulates both problems in prolate spheroidal coordinates as well as dipolar coordinates, and compares the results with those obtained by standard methods using cylindrical coordinates. He finds that use of spheroidal coordinates requires less complicated analysis than the other two systems and is more likely to lead to results which are uniformly valid for blunt bodies. As a byproduct of the comparison of results using different coordinate systems, the author obtains some apparently new relations involving Legendre functions.
L. E. Payne (College Park, Md.)

Newman, J. N. 5642
A slender-body theory for ship oscillations in waves.
J. Fluid Mech. **18** (1964), 602–618.
Author's summary: "A linearized theory is developed for the oscillations of a slender body which is floating on the free surface of an ideal fluid, in the presence of incident plane progressive waves. Green's theorem is used to represent the velocity potential and the first-order slender-body potential is developed from asymptotic approximation. The general theory is valid for arbitrary slender bodies in oblique waves, and detailed results are presented for a body of revolution."

To this may be added that the asymptotic expression developed for the velocity potential is intended only for the region near the body. The equations for swaying and yawing (of the second order) involve hydrodynamic forces but no free-surface effects. The hydrodynamic forces for heaving and pitching do involve free-surface effects but are dominated by lower-order hydrostatic and wave-exciting forces. *J. V. Wehausen* (Berkeley, Calif.)

Timman, R. 5643
The wave pattern of a moving ship.
Simon Stevin **35** (1961/62), 53–67.
An exposition of the classical problem of the waves caused by a moving concentrated pressure point over the free surface of a liquid subject to gravity. The linearized boundary conditions are used. The asymptotic behavior at large distances, but away from the track of the point and from the lines making an angle arc cot $2^{3/2}$ with it, is investigated and the method of stationary phase, used for this purpose, is also developed in detail.
J. V. Wehausen (Berkeley, Calif.)

Wehausen, John V. 5644
Effect of the initial acceleration upon the wave resistance of ship models.
J. Ship Res. **7** (1963/64), *no.* 3, 38–50.
The well-known integral representation for the wave resistance of a thin ship with unsteady forward velocity is examined for its asymptotic behavior at large time, and with the velocity of the ship a constant after an initial acceleration from rest. This problem is of importance in estimating the transient effects of initial acceleration in ship model resistance experiments. Asymptotic expressions are obtained for the wave resistance of an arbitrary thin ship with arbitrary acceleration history, and detailed computations are given for a simple ship with constant acceleration. These results show that for large time the wave resistance oscillates about its steady-state limit with amplitude inversely proportional to time.
J. N. Newman (Washington, D.C.)

Erošin, V. A.; Poručikov, V. B. 5645
The motion of a cone in a fluid of finite depth. (Russian. English summary)
Vestnik Moskov. Univ. Ser. I Mat. Meh. **1964**, *no.* 2, 41–48.
This paper derives expressions for the velocity potential and hydrodynamic force acting on a cone with vertical axis, moving in a fluid of finite depth. The fluid is bounded below by a rigid plane surface and above by a free surface. The free surface boundary condition is taken to be that of vanishing velocity potential. The method of images is employed and the potential is represented by an axial source distribution, the strength of which is determined by assuming that the cone is slender. Details are given for a cone moving upward from the bottom, downward from the free surface, and upward from the bottom in the presence of plane progressive gravity waves. The approximations both of the free surface condition and the slender-body condition appear to be inconsistent.
J. N. Newman (Washington, D.C.)

Bhattacharyya, Panchanan 5646

Note on the flow of a viscous fluid standing on a rigid base due to a single periodic pulse of tangential force on the surface.

Indian J. Theoret. Phys. **10** (1962), 73–76.

Author's summary: "In this note, the motion of a viscous fluid has been investigated when a tangential pulse of a periodic force is applied on the surface. The motion has been assumed laminar."

Hays, D. F. 5647

A variational formulation applied to Couette and Poiseuille flow.

Acad. Roy. Belg. Bull. Cl. Sci. (5) **49** (1963), 576–602.

Glansdorff, Prigogine, and the author [Phys. Fluids 5 (1962), 144–149] have shown that in a slowly moving viscous incompressible fluid the rate of entropy production is a minimum, i.e., any other fluid motion consistent with the boundary conditions but not satisfying the equations of motion generates entropy at a greater rate. This result is used by the present author as a basis for approximating Couette and Poiseuille flows. Using simple expressions for the velocity and temperature and determining the arbitrary constants by means of the above theorem, he obtains approximations to the known exact solutions in error by amounts varying up to about 10%.

K. Stewartson (Durham)

Lal, Krishna 5648

On the steady laminar flow of a viscous, incompressible fluid through a tube whose cross-section is the intersection of two circles.

Gaṇita **13** (1962), 9–15.

Author's summary: "The maximum velocity, mean velocity and the discharge of flow of a viscous, incompressible fluid through a cylindrical tube are obtained, the cross-section of the tube being formed by the intersection of two circles. Expressions for k and k', as introduced by Boussinesq, are also obtained. These are compared with similar quantities for tubes of circular and elliptic sections. The present tube is shown to be more efficient."

Kapur, J. N.; Srivastava, P. N. 5649

On the unsteady flow of viscous incompressible fluid in an annulus under toroidal pressure gradient.

Gaṇita **13** (1962), 17–24.

Authors' summary: "An exact solution of the equations of motion has been found for the unsteady tangential flow of an incompressible viscous fluid in an annulus under a variable toroidal pressure gradient, which is an arbitrary function of time. The cases when this pressure gradient decreases exponentially and when it is periodic have been discussed as particular examples. The steady state solution has been obtained as a limiting case."

Kapur, J. N. 5650

On incompressible viscous flows having constant velocity magnitude along each stream line.

Bull. Calcutta Math. Soc. **54** (1962), 67–73.

Author's summary: "In the present paper the most general viscous incompressible flows in which the velocity magnitude is constant along streamlines are obtained for (a) the two-dimensional case, (b) the axially-symmetric case with no toroidal velocity component, and (c) the axially-symmetric case with no radial velocity component. An equation characterising the general three-dimensional flow of the given type is also obtained."

Kapur, J. N.; Bhatia, B. L. 5651

Flow of a viscous incompressible fluid in an annulus with variable suction and injection along the walls.

J. Phys. Soc. Japan **19** (1964), 125–129.

Authors' summary: "Another exact solution of the Navier-Stokes equations of motion is discussed corresponding to the flow of a viscous incompressible fluid in the annulus between two coaxial cylinders when the walls are porous and allow suction and injection at a rate varying linearly with distance from some fixed point on each of the walls. Some interesting new phenomena arise in this case, for example, the fluid can flow by this suction and injection alone in the absence of the motion of the walls and/or a pressure gradient. Another interesting result is that if the walls move in one direction, portions of the fluid can move in the opposite direction. The modifications in the flow when the fluid belongs to a class of non-Newtonian fluids have also been discussed."

R. R. Burnside (Toronto, Ont.)

Kapur, J. N.; Gupta, R. C. [Gupta, R. K.] 5652

Flow of Reiner-Rivlin fluids in the inlet region of a channel.

J. Phys. Soc. Japan **19** (1964), 386–392.

Under the assumptions that all the material coefficients of the Reiner-Rivlin fluid are of much smaller order than the boundary-layer thickness, the boundary-layer equations are derived for a two-dimensional flow of an incompressible Reiner-Rivlin fluid. The Kármán-Pohlhausen method is used to obtain a solution for the flow conditions in the inlet region of a channel. The procedure of approximating the equations of motion is essentially an asymptotic approximation valid in the limit of small material coefficients. In contrast to the above there is the approximation valid for the case of large inlet velocity, which the authors find cannot be consistently applied for a Reiner-Rivlin fluid. This is not surprising, for a Reiner-Rivlin constitutive equation can represent reality only at infinitesimally small fluid velocity.

P. P. Niiler (Cambridge, Mass.)

Pavlovskiĭ, Ju. N. 5653

Numerical calculation of a plane laminar boundary layer for a blunt plate in a compressible gas. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 178–183.

This is a continuation of a paper by the author [same Ž. **2** (1962), 884–901; MR **27** #1051]. The numerical example of the earlier paper is reworked with additional terms retained in the approximation formulas, and with radiation taken into account. The effects of different order approximations on the computed results are shown for both the radiating and non-radiating cases.

H. M. Sternberg (Silver Spring, Md.)

Libby, Paul A.; Fox, Herbert 5654
Some perturbation solutions in laminar boundary-layer theory. I. The momentum equation.
J. Fluid Mech. **17** (1963), 433–449.
The solution given is a small perturbation on the Blasius solution. If, as usual, exponential decay at infinity is required, a discrete eigenvalue problem arises. The first ten eigenvalues and related eigenfunctions for the first-order solution are obtained numerically.

Applications are to (1) an initial profile slightly differing from that of Blasius, (2) flows in which the product of density and viscosity varies slightly, and (3) flows with mass transfer at the wall. Satisfactory agreement with some exact solutions is found. *J. C. Cooke* (Farnborough)

Nickel, Karl 5655
Ein Eindeutigkeitssatz für instationäre Grenzschichten. II.
Math. Z. **83** (1964), 1–7.
An error is corrected in the previous paper with the same title [same Z. **74** (1960), 209–220; MR **22** #9734]. In its new form uniqueness of solution of the unsteady boundary-layer equations is established for conventional boundary conditions, but with the additional restriction that $\partial u/\partial y > 0$. The method of proof is to apply Crocco's transformation which converts the equation into a convenient form for the Nagumo-Westphal lemma (in the earlier paper the von Mises form was used). A suitable trial function is constructed to which the lemma can be applied, and a contradiction is immediately obtained if the equations have two solutions.

K. Stewartson (Durham)

Gershuni, G. Z. [Geršuni, G. Z.]; Zhukhovitskii, E. M. [Žuhovickiĭ, E. M.] 5656
On parametric excitation of convective instability.
Prikl. Mat. Meh. **27** (1963), 779–783 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1197–1204.
A linear stability analysis is made on the Boussinesq equations with two free horizontal boundaries when the undisturbed temperature gradient β can be given at time t by $\beta = -A + a \operatorname{sgn} t$, $|t| < \frac{1}{2}T$; $\beta(t+T) = \beta(t)$, $|t| > \frac{1}{2}T$; where A, a, and T are constants. The problem reduces to a Hill equation with non-small damping. Results for stable and unstable mean gradients A are similar to the parametric effects on a regular and inverted pendulum with oscillating supports. *L. A. Segel* (Cambridge, Mass.)

Weinbaum, Sheldon 5657
Natural convection in a horizontal circular cylinder.
J. Fluid Mech. **18** (1964), 409–437.
In an xy-coordinate system with gravity pointing in the direction of the negative y-axis, the author studies convection inside the circle $x^2 + y^2 = r_0^2$ on whose boundary the temperature is given by (i) $T_0 + y\Delta T$ (bottom heating), (ii) $T_0 + x\Delta T$ (side heating), (iii) $T_0 + (x+y)\Delta T$ (oblique heating). T_0 and ΔT are constants. The critical value of the Rayleigh number λ is determined for (i). The first two terms of the λ power series are found for the other cases. When λ is large, inviscid core motions are matched to peripheral viscous boundary layers by means of Carrier-Oseen linearizations of the boundary-layer equations. [The core rotates like a rigid body for (i) and (iii) but is stationary for (ii).] Solutions are given in detail and are compared with experiment.

L. A. Segel (Cambridge, Mass.)

Skinner, L. A.; Bankoff, S. G. 5658
Dynamics of vapor bubbles in spherically symmetric temperature fields of general variation.
Phys. Fluids **7** (1964), 1–6.
The authors consider the spherically symmetric growth of a vapor bubble within a volume of superheated inviscid liquid. Thermal diffusion in the liquid causes evaporation at the liquid-gas interface, thus feeding the growing bubble. In the full generality of the problem, the hydrodynamic equations are coupled to the equation governing diffusion with convection. Under properly dilineated conditions, however, the bubble growth soon passes over into an asymptotic stage, in which hydrodynamical resistance is negligible, so that the growth rate is completely diffusion-controlled.

Two difficulties persist even in the asymptotic stage: the necessity of applying boundary conditions at the bubble wall, a surface whose motion is not known a priori, and the presence of the convection term in the diffusion equation. The first difficulty is removed and the second mitigated by passing to a Lagrangian formulation. The explicit convection term disappears, but convection makes its presence felt implicitly as a nonlinear coefficient in the conduction term.

For large values of the Jakob number, the Lagrangian formulation of the problem can be treated as a perturbation of a linear conduction problem. The growing bubble is influenced primarily by the temperature field in its immediate neighborhood, so that the nonlinear coefficient can be approximated by a tractable substitute. This approach, the thin-shell approximation, had previously been used to study various aspects of the bubble-growth problem under conditions of uniform initial superheat. The authors now generalize to the case where the initial distribution of superheat is spherically symmetric, but otherwise arbitrary (except for certain tacit assumptions of smoothness inherent in the very use of a thin-shell approximation). Further, they account for the fact that the asymptotic stage does not extend down to the time of zero bubble radius. However, by carrying out the detailed analysis for the case of uniform initial superheat, they show that the growth enters the asymptotic stage while the bubble radius is still quite small compared with the values of interest, and that the error involved in setting the initial radius equal to zero drops rapidly to 2% as growth proceeds.

The results are applied to the problem of a bubble growing on a heated surface. The authors point out that such a model of surface boiling has certain obvious limitations; essentially, these limitations can be summarized by stating that conditions must be such that the temperature field and its mirror image across the heated surface approximate the spherically symmetric picture required by the authors' technique.

W. Langlois (San Jose, Calif.)

Wang, K. C. 5659
Unsteady linearized relaxing flow past slender bodies.
Phys. Fluids **7** (1964), 25–32.

Integral transform techniques are used to solve the three-dimensional, unsteady, linearised equation for the potential of a reacting gas flow. To the order of accuracy of the slender body theory, the differences between this solution and the one pertaining to the non-reacting case arise solely as a result of differences in the propagation of one-dimensional disturbances. It follows that only streamwise forces are affected by the reactions. The theory necessary for the evaluation of these forces in accelerating flight is exemplified by a consideration of uniform accelerations.
J. F. Clarke (Cranfield)

Luke, Yudell L. 5660
Approximate inversion of a class of Laplace transforms applicable to supersonic flow problems.
Quart. J. Mech. Appl. Math. **17** (1964), 91–103.
Solutions of numerous problems require the inversion of a Laplace transform involving functions of hypergeometric type. A procedure for inverting rational function approximations of these transcendents is presented. The technique is illustrated by applying it to the solution of the wave equation with particular boundary conditions. The functions which arise have aerodynamic applications in problems such as the pressure distribution of quasi-cylinder bodies of revolution, and others.
K. R. Enkenhus (Silver Spring, Md.)

Mihaĭlos, A. N. 5661
On the calculation of a supersonic gas flow about a blunt body of revolution at an attack angle. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 171–177.
Il s'agit d'une extension de la méthode de Belocerkovskiĭ [Dokl. Akad. Nauk SSSR **113** (1957), 509–512; MR **19**, 1121]. On utilise ici une double interpolation : polynomiale (linéaire dans l'application) entre l'obstacle et le choc détaché, par polynôme trigonométrique (linéaire dans l'application) le long des parallèles. L'exemple trivial de la sphère est traité pour éprouver la méthode. On donne aussi la répartition de pression sur un disque circulaire à $M=5, 8$, placé sous une incidence de 8°.
J. P. Guiraud (Paris)

Bochenek, Krystyn 5662
On shock wave profiles and equations describing them. (Polish. English summary)
Algorytmy Zeszyt Specjalny No. 1 (1963), 103–108.
Author's summary: "The paper deals with equations of the form $x=f(x)$ with boundary conditions $x(-\infty)=x_1$ and $x(+\infty)=x_2$, where $f(x_{1,2})=0$; the above equations have been considered by Richtmyer, P. Lax and C. Morawetz. Boundary conditions are singular. The system of equations may be reduced by means of an effective integration to functional dependences of the character of the laws of conservation. One equation remains, in which the dissipation mechanism appears. In principle, the problem cannot be reduced to a typical initial problem. Two ways of approach are suggested. The change of variables and the development into a series in the neighbourhood of the initial point, or the solution by using classical numerical methods. In that case, small perturbations of initial conditions might cause a transfer from the solution $x=0$ to the trajectory of the appropriate solution. Numerical solutions are examined in the neighbourhood of typical singular points."

Powell, Alan 5663
On flow fields driving a contiguous acoustic field.
J. Acoust. Soc. Amer. **36** (1964), 830–832.
The radiation of sound from a region (free of solid boundaries) containing a rotational flow, such as a ball of turbulent fluid or interacting vortex rings, is considered. The method is first to use pure incompressible theory to compute the pressure fluctuations on a surface surrounding the rotational flow. These fluctuations are then considered as an equivalent source causing acoustic radiation into the (now compressible) fluid outside the rotational region. The flow in the source region is specified in terms of vorticity. It turns out that the final result for the far-field sound pressure is the same as that given earlier by another method [the author, same J. **35** (1963), 1133–1143; MR **27** #4470]. Some attention is also given to the case in which small solid surfaces are present.
D. T. Blackstock (Rochester, N.Y.)

Slepian, L. I. [Slepjan, L. I.] 5664
On the displacements of a deformable body in an acoustic medium.
Prikl. Mat. Meh. **27** (1963), 918–923 (*Russian*): *translated as J. Appl. Math. Mech.* **27** (1964), 1402–1411.
By resolving the forces acting on the body in a certain way, restricting the acoustic medium to be linear and making certain assumptions regarding the nature of the external forces, several general properties of the system are demonstrated. Unfortunately, the derivation is clouded by the introduction of a "fictitious body" which is never defined so that the reader must fill in many gaps. The results of this analysis are of some interest as they indicate the relationship between the limits which the displacements approach at infinite time in terms of the properties of the external forces over all time.
P. R. Paslay (Houston, Tex.)

Mishra, S. K. 5665
Diffraction of sound pulses by a fluid cylinder.
Proc. Cambridge Philos. Soc. **60** (1964), 295–312.
The following two-dimensional problem is posed: an acoustic pulse from a line source is incident on a circular domain in which the velocity of sound is inferior to that of the surrounding medium. The solution is obtained through an extension of Friedlander's treatment of the corresponding problem for a rigid cylinder [Comm. Pure Appl. Math. **7** (1954), 705–732; MR **16**, 538]. The behaviour of the solution for small time is inferred from the asymptotic behaviour of its Laplace transform, the determination of which requires some rather involved analysis. The final results are shown to be in agreement with Keller's heuristic theory of geometric diffraction [New York Univ. Inst. Math. Sci. Div. Electromagnetic Res., Res. Rep. No. EM-115 (1958)] and also with a conjecture made by Friedlander as to the form of the diffracted pulse. The author suggests that there may be discrepancies between his results and those obtained by Knopoff and

Gilbert [Bull. Seismol. Soc. America **51** (1961), 35–49; MR **22** #6591] for a fluid cylinder imbedded in an elastic solid, but his remarks on this point are rather vague.
J. W. Miles (Canberra)

Pipkin, Allen C.; Rivlin, Ronald S. 5666
Normal stresses in flow through tubes of non-circular cross-section. (German summary)
Z. Angew. Math. Phys. **14** (1963), 738–742.
From the authors' summary: "The slow flow of an incompressible viscoelastic fluid through a tube of non-circular cross-section is considered. From the first-order perturbation of the solution of the corresponding problem for a Newtonian fluid a general expression is obtained for the normal surface traction on the walls of a tube of arbitrary cross-section. The result is specialized to the case of tubes with elliptical, equilateral-triangular, and rectangular cross-sections." *B. Gross* (Vienna)

Thomas, R. H.; Walters, K. 5667
The motion of an elastico-viscous liquid due to a sphere rotating about its diameter.
Quart. J. Mech. Appl. Math. **17** (1964), 39–53.
From the authors' summary: "Consideration is given to the flow of an idealized elastico-viscous liquid due to a sphere immersed in it rotating about a vertical diameter. It is shown that the streamline projections on any plane containing the axis of rotation are strongly dependent on the liquid parameters. An expression is obtained for the couple on the sphere due to the liquid, which may be useful in determining an estimate of the liquid parameters when experimental results are available."
R. C. DiPrima (Troy, N.Y.)

Mishra, Shankar Prasad 5668
Shear flow of a finitely electrically conducting elastico-viscous fluid past a porous flat plate in the presence of a transverse magnetic field.
J. Sci. Engrg. Res. **7** (1963), 388–397.
The steady uniform shear flow of an incompressible, elastico-viscous, electrically conducting fluid past a porous flat plate is considered. A transverse magnetic field and fluid suction or injection at the plate are included. Asymptotic profiles are obtained and the nature of the boundary-layer is discussed. It is interesting to note that the skin friction at the wall is affected both by the magnetic field and elastic elements in the present solution. *R. R. Gold* (Los Angeles, Calif.)

Iordanskiĭ, S. V.; Kulikovskiĭ, A. G. 5669
Stability of the higher correlation functions in a plasma. (Russian)
Dokl. Akad. Nauk SSSR **152** (1963), 849–852.

Rosciszewski, Jan 5670
Magnetohydrodynamic Rayleigh problem for magnetic Prandtl number close to one.
Arch. Rational Mech. Anal. **16** (1964), 230–237.
The problem of indicial motion of an infinite flat plate in its own plane subject to a perpendicular magnetic field, i.e., the magnetohydrodynamic Rayleigh problem, is considered once again. Additional information is provided, however, from the extension of a previous solution for small magnetic Prandtl numbers to the case of Prandtl number close to one. The effects of both small and large conductivity of the wall are included in the solutions for the distributions of velocity, magnetic field, and vorticity. A lucid, informative discussion of the physical meaning of the solutions is presented.
R. R. Gold (Los Angeles, Calif.)

Gupta, R. K. 5671
Magneto hydrodynamic flow of a viscous fluid past axisymmetric bodies.
Indian J. Math. **5** (1963), 1–11.
Using the Oseen approximation the author discusses the steady flow of a viscous, incompressible, electrically conducting fluid past an axially symmetric body, the flow being such that the ambient flow field is collinear with the uniform magnetic field. He obtains in the case of flow about a general spheroid an approximate expression for the drag coefficient in terms of the Reynolds number, the magnetic Reynolds number and the Hartman number.
L. E. Payne (College Park, Md.)

Kirstein, P. T. 5672
Thermal velocity effects in axially symmetric solid beams.
J. Appl. Phys. **34** (1963), 3479–3490.
The author investigates the effect of a Maxwellian distribution of emission velocities on the longitudinal current density in axially symmetric solid beams. On the basis of Liouville's Theorem, general integral expressions are derived for the axial current density in the paraxial approximation, taking into account both transverse and longitudinal emission velocities, and allowing for the presence of a non-uniform magnetic field along the beam. Explicit expressions are found under certain conditions for the on- and off-axial beam current densities in the special case of uniform emitter current density. The method is illustrated in detail by a simple example in which a parallel beam from a magnetically shielded cathode emerges through an aperture in the anode into a region of constant potential and uniform magnetic field.
L. B. Wetzel (Providence, R.I.)

Gundersen, Roy M. 5673
Entropy perturbations in one-dimensional magneto-hydrodynamic flow.
AIAA J. **1** (1963), 969–971.
L'auteur considère un écoulement unidimensionnel non stationnaire d'un fluide parfait et parfaitement conducteur soumis à l'action d'un champ magnétique transversal, et il étudie sous quelle condition une perturbation de l'état thermodynamique est compatible avec le champ des vitesses (c'est-à-dire conduit à une perturbation de vitesse nulle). Le résultat est obtenu lorsque la masse volumique et l'induction magnétique sont proportionnelles; ceci

comprend en particulier le cas où l'écoulement non perturbé est un écoulement par ondes simples.
P. Germain (Paris)

Singh, Dilip 5674
Hydromagnetic flow illustrating the response of a laminar boundary layer to a given change in the free stream velocity.
J. Phys. Soc. Japan **18** (1963), 1676–1685.
The author examines the unsteady flow of a viscous, incompressible, electrically conducting fluid past an infinite porous flat plate. A uniform transverse magnetic field along with time-dependent free stream velocities and pressure gradients (in direction along plate) are prescribed. By neglecting the effect of the induced magnetic field, closed-form analytical solutions to linear equations are obtained for the velocity fields. From these the laminar skin friction and displacement thickness are calculated for each case. When properly interpreted, such solutions can give valuable insight into corresponding non-linear cases.
R. M. Mark (Palo Alto, Calif.)

Chu, C. K. 5675
Linearized hyperbolic steady magnetohydrodynamic flow past nonconducting walls.
Phys. Fluids **7** (1964), 707–714.
The author considers linearized two-dimensional steady hyperbolic flow of a perfectly conducting fluid past a slightly perturbed infinite insulating wall. The magnetic field satisfies an elliptic equation in the wall and must be continuous across the wall boundary; therefore, local disturbances are not merely propagated into the fluid along the positive fast and slow characteristics but are generally felt everywhere in the fluid, as in a shock-boundary-layer interaction.

The author solves this problem by first assuming that the magnetic field is specified on the boundary; using Riemann invariants he finds there will be incoming waves unless the two components of the magnetic field satisfy a certain linear relation at the wall. This relation provides a boundary condition for the determination of both components of the magnetic field in the insulator and therefore determines both components at the wall. The flows past a wavy wall and past a wall with an inclined step are solved explicitly. It is shown that the flow past a wall with a single corner does not have a meaningful solution. *T. S. Lundgren* (Minneapolis, Minn.)

Regirer, S. A. 5676
Flow in a conducting fluid over a porous plate in the presence of a uniform magnetic field.
Prikl. Mat. Meh. **27** (1963), 1095–1099 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1677–1683.
Previously, the author studied the general problem of steady, longitudinal flow of a conducting fluid over a cylindrical surface under the influence of a magnetic field and suction or blowing. A class of exact solutions was found for the case where the longitudinal components of the velocity and magnetic field depend only on the coordinate normal to the surface. In this paper an example of such a solution is constructed for the problem of flow over a porous plate. *S. I. Pai* (College Park, Md.)

Thibault, Roger 5677
Sur les chocs stationnaires au voisinage de la vitesse de Alfvén.
C. R. Acad. Sci. Paris **258** (1964), 1736–1739.
This note deals with the orders of magnitude of jumps in various flow and field quantities across shock waves and shock layers in magnetohydrodynamics. For example, it is shown that an infinitesimally small jump in density across a shock wave can correspond to finite jumps in pressure and the tangential components of velocity. The discussion is based on the assumption that the normal velocity of the fluid across these discontinuities is in the neighborhood of the Alvén speed. Furthermore, the treatment is limited to the study of slow and fast waves; the intermediate and rotational shocks are excluded.
R. P. Kanwal (University Park, Pa.)

Liu, Ching Shi; Cambel, Ali Bulent 5678
One-dimensional magnetogasdynamic flow with heat transfer.
Phys. Fluids **7** (1964), 564–567.
Authors' summary: "The case of one-dimensional magnetogasdynamic flow with heat transfer is analyzed. It is shown that the flow variables can be expressed as functions of the dimensionless K number, which is defined as the ratio of the local flow speed to the Alvén wave propagation speed. The important flow regimes are delineated and it is observed that although the flow is reminiscent of classical flow with heat transfer, there do appear significant differences." *M. E. Gurtin* (Providence, R.I.)

Moreau, René 5679
Jet libre, plan, laminaire, d'un fluide incompressible dont la viscosité est égale à la diffusivité magnétique.
C. R. Acad. Sci. Paris **258** (1964), 440–443.
In his previous notes [same C. R. **256** (1963), 2294–2296; MR **27** #1078; ibid. **256** (1963), 4849–4852] the author has discussed the free, plane, laminar jet of an incompressible and magnetized fluid of very high magnetic diffusivity. The present paper is a continuation of that work. It deals with the special situation when the coefficients of viscosity and magnetic diffusivity are equal. Two alignments of the magnetic field are considered. In one case the field is aligned longitudinally along the axis of the jet, while in the second case it is aligned transverse to it.
R. P. Kanwal (University Park, Pa.)

Ludford, G. S. S.; Singh, M. P. 5680
The hydromagnetics of an ellipsoid moving in a cross-field.
Proc. Cambridge Philos. Soc. **60** (1964), 341–351.
The authors' previous work on the flow of an electrically conducting fluid past a sphere in the presence of a strong transverse magnetic field [same Proc. **59** (1963), 615–624; MR **27** #3239a; ibid. **59** (1963), 625–635; MR **27** #3239b] is extended to include ellipsoids. The ultimate flow is again cylindrical, the magnetic fluid undisturbed, and the general features of the flow are the same as for a sphere. One especially interesting result is that the flow past a perfectly conducting ellipsoid never becomes even approximately planar, no matter how wide the ellipsoid may be.
K. Stewartson (Durham)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

Hu, Ming-kuei [Hu, Ming-Kuei] 5681
Modified Zernike polynomials and their application to the analysis of Fresnel region fields of circular apertures with nonuniform and nonsymmetric illumination.
J. Opt. Soc. Amer. **53** (1963), 261–266.
Author's summary: "Modified Zernike polynomials and also a set of generalized functions $W_m^{\nu}(\gamma, u)$ are introduced. These polynomials and functions are then applied to the analysis of the Fresnel region field of circular apertures with nonuniform and nonsymmetric illumination. With $m=0$, the general result obtained in this paper reduces to the case of nonuniform but circular symmetric illumination obtained by the author in a previous paper. A general expression for the far field (Fraunhofer field) of circular apertures with general illumination is also derived. Finally, a simple example is given as an illustration."

Poincelot, Paul 5682
★Précis d'électromagnétisme théorique.
Préface de Louis de Broglie.
Dunod, Paris, 1963. xxi + 456 *pp.* 76 *F.*
As the title implies, this book consists of a systematic development in concise summary form of the fundamentals of electromagnetic theory together with illustrative applications. The contents are outlined in the preface by Louis de Broglie:

"L'ouvrage débute par des préliminaires mathématiques étendus et par l'exposé critique des équations de base du champ électromagnétique. Un chapitre spécial est consacré à l'introduction, sous sa forme rationalisée, du système MKSA qui est aujourd'hui légal et dont l'emploi comporte, à côté de quelques inconvénients, d'importants avantages bien connus. Puis vient l'étude de l'influence sur la forme du champ électromagnétique des conditions aux limites et des surfaces de discontinuité des milieux qui en sont le siège. Le critérium d'unicité des équations de Maxwell est soigneusement analysé.

"Toute une partie du livre est ensuite dévolue à l'Électrostatique dont les problèmes parfois difficiles sont étudiés avec soin en utilisant tous les résultats obtenus par divers auteurs et en particulier par M. Émile Durand dans son important Traité. Puis vient une analyse détaillée des lois de l'Électromagnétisme et de leurs expressions tensorielles avec des chapitres où sont discutées en détail la forme que prend dans divers cas la relation de Lorentz entre les potentiels, les propriétés fondamentales du champ électromagnétique et l'analyse du théorème de Poynting.

"Les chapitres 17 et suivants traitent de l'émission des rayonnements par les doublets, de leur représentation par la méthode de Bromwich et les potentiels de Hertz et enfin de l'effet Kelvin avec des applications aux cavités électromagnétiques, aux guides d'ondes et aux lignes coaxiales. L'ouvrage se termine par l'étude de divers problèmes annexes."

Since nearly all the material in this book is available in numerous textbooks on classical electromagnetic theory, a question arises as to the particular function of this handbook-like presentation. From a student's viewpoint the mathematical level is about that of a first-year graduate course. However, the lack of interpretation or explanation limits its usefulness since a high order of physical understanding on the part of the reader is implied, which a student presumably does not yet possess. For the teacher of such a course the book might have some value as a source of topics from which material set in a standard mathematical framework could be selected. A research worker, on the other hand, will already have his favorite ready-reference book, and for him this volume has the serious disadvantage of not including any contemporary problems or techniques.

About all that can be said, therefore, is that the author has skillfully organized and lucidly presented what are today the standard topics of classical electromagnetic theory. *R. D. Kodis* (Lexington, Mass.)

Lipkin, Daniel M. 5683
Existence of a new conservation law in electromagnetic theory.
J. Mathematical Phys. **5** (1964), 696–700.
This paper deals with the following relation, which is a consequence of the Maxwell-Lorentz equations of the electromagnetic field (Gaussian units)

$$(1)\quad \frac{\partial}{\partial T}(\mathbf{E}\cdot\operatorname{curl}\mathbf{E}+\mathbf{H}\cdot\operatorname{curl}\mathbf{H}) + \operatorname{div}\left(\mathbf{E}\times\frac{\partial\mathbf{E}}{\partial T}+\mathbf{H}\times\frac{\partial\mathbf{H}}{\partial T}\right) = -8\pi\mathbf{j}\times\operatorname{curl}\mathbf{E}$$

and certain generalizations suggested by it. These relations are considered only for the case of free space for which the right-hand side of (1) vanishes identically ($\mathbf{j}=0$), when they take the form of conservation equations.

The bracketed quantities appearing on the left side of (1) are first expressed in terms of a tensor of rank 3, Z^{ijk}, from which a total of ten such differential conservation laws are obtained. This tensor is bilinear in the electric and magnetic field quantities and their first partial derivatives with respect to the space-time coordinates, as is evident in (1). The resulting conservation laws hence involve differential invariants of higher order than those which occur in the stress-energy tensor of the field.

The author raises the question of the physical interpretation of these quantities, which he labels with the designation of "zilch" of the field. A brief examination is made of their nature for such solutions as plane waves but no general answer is found. {Since the natural state of the electromagnetic field is the vacuum state, it would seem clear that a physical interpretation could be arrived at only by use of the set of inhomogeneous equations of the form (1) which describe the generation of the field from the vacuum state. One then must make use of physical interpretations for the first partial derivatives of the field quantities, which takes one beyond the usual arguments based on the Lorentz force equation. It seems likely that there exist similar differential invariants involving still higher-order derivatives of the field vectors.}
E. L. Hill (Minneapolis, Minn.)

Goldberg, J. N.; Kerr, R. P. 5684
Asymptotic properties of the electromagnetic field.
J. Mathematical Phys. **5** (1964), 172–176.
In this paper the authors derive the so-called peeling theorems of outgoing spherical electromagnetic waves, i.e., they show that each algebraically special component of the field decreases along an outgoing null cone with a different, and typical, negative power of r. These peeling

theorems were recently derived by several different authors for gravitational outgoing fields. Since the above paper appeared, corresponding results have been obtained and published for different fields as well, and for coupled fields. In a rapidly developing area this is one of the "classical" papers. *P. G. Bergmann* (New York)

Shail, R. 5685
On the multipole expansion in electrodynamics.
Canad. J. Phys. **42** (1964), 1011–1015.

In a recent discussion by J. Fiutak [same J. **41** (1963), 12–20; MR **26** #4649] concerning the multipole expansion of the interaction between a system of charged particles and an external field, the derivation of the long-wavelength dipole approximation interaction Hamiltonian due to M. Göppert-Mayer [Ann. Physik (5) **9** (1931), 273–294] is reproduced. In the present note it is shown that the Göppert-Mayer method, which involves the subtraction of a total time derivative from the Lagrangian, may be used to obtain higher multipole couplings. A localized hydrogen atom is considered. The system consisting of the atom together with the radiation field is treated as a closed dynamical system, and accordingly the derivation of the higher multipole interactions is performed under more general conditions than those considered by Fiutak.
H. Rund (Pretoria)

Hoffman, W. C. 5686
Wave propagation in a general random continuous medium.
Proc. Sympos. Appl. Math., Vol. XVI, pp. 117–144. *Amer. Math. Soc., Providence, R.I.*, 1964.

This is an extensive survey of work dealing with the solution of the (scalar) wave equation in a medium with a randomly fluctuating index of refraction. Methods of treatment depend, of course, on the character of the random distribution assumed. The initial sections of this article deal, accordingly, with the random function and its spectral representation. The wave equation may then be solved by perturbation methods (Born approximation), through the reformulation as an integral equation, as a Riccati equation, or the wave equation may be replaced by the eikonal equation, assuming the applicability of a WBK approach. The bibliography contains 52 items.
P. G. Bergmann (New York)

Barlow, H. M.; Brown, J. 5687
★Radio surface waves.
Clarendon Press, Oxford, 1962. xi+200 *pp.* $6.75.

The authors have made many important contributions to the subject of electromagnetic surface waves in recent years. Therefore, it is quite fitting that they should co-author a monograph on surface waves. Essentially, it is a detailed exposition of their own views on this rather controversial subject. The electromagnetic waves discussed in the monograph are of a non-radiating type. Thus, the subject matter really deals with evanescent waves and this might have been a better title for the book. It is just this evanescent character which is the basis of a working definition of a "surface wave", although, as the authors point out, not all evanescent fields are associated with waves which propagate along an interface. Explicitly, their definition reads: "A surface wave is one that propagates along an interface between two different media without radiation, such radiation being construed to mean energy converted from the surface wave field to some other form". As the authors admit, the definition is highly idealistic and realizable only with a smooth and plane interface between homogeneous media. They propose that waves on actual structures such as when a wave propagates around a smooth cylindrical supporting surface be regarded as a "perturbed surface wave". However, no suggestions are put forth as to how one would classify the various field components produced by a dipole in the presence of a plane interface between homogeneous media.

The book discusses in detail the conditions for the support of surface waves on various planar and cylindrical structures. Many interesting physical descriptions are put forth, such as the significance of a complex Brewster angle and its relation to surface waves. By first leaving aside the question of excitation, the early chapters use only quite elementary methods. Later on in the book the question of launching of surface waves is considered in an elegant fashion. Unfortunately, however, the reader is never told if the Zenneck surface wave can be excited over a homogeneous conducting half-space.

Most noteworthy is the material on approximate methods of analysis and design procedures for surface wave antennas. Such problems as discontinuities in surface properties and the resultant radiation are dealt with by several approaches which have various degrees of rigor. The section on Wiener-Hopf techniques is particularly valuable.

On the whole, the book is clearly written and is a worthwhile purchase. The quality of the printing and the illustrations is excellent. *J. R. Wait* (Boulder, Colo.)

Ballieu, Robert J. 5688
Application de la méthode du moment angulaire complexe à un problème de diffusion classique.
Ann. Soc. Sci. Bruxelles Sér. I **78** (1964), 69–89.

This paper describes in detail the application of the Watson transformation to the scattering of a plane wave by a perfectly conducting cylinder in the small wavelength limit. According to the author, Watson's treatment of the problem contains errors while that of Franz is too succinct. *I. Kay* (Ann Arbor, Mich.)

Kido, Masao 5689
Foundations for the boundary problems of the polyphase transmission lines considering the initial conditions. I.
Bull. Univ. Osaka Prefecture Ser. A **12** (1963/64), *no.* 2, 21–30.

Taking the Laplace transform of the differential equations of the n-phase system, there are obtained the matrix equations:

$$(2)\qquad \begin{aligned} -\frac{d[e]}{dx} &= \{[R]+s[L]\}[i]-s[L][I_0],\\ -\frac{d[i]}{dx} &= \{[G]+s[C]\}[e]-s[C][E_0], \end{aligned}$$

where s denotes the complex variable, $[R]$, $[L]$, $[G]$, and $[C]$ denote the nth-degree square matrices whose elements

are resistances, inductances, leakances and capacitances of the transmission lines per unit length, x denotes the distance, $[e]$ and $[i]$ the transforms of the voltages and the currents, and $[I_0]$ and $[E_0]$ the values of currents and voltages at $t=0$. At the sending end, let the series impedances be $[Z_0]$ and the shunt admittances be $[Y_0]$. It can be shown that, for $Z_0=R_0+sL_0$ and $Y_0=G_0+sC_0$,

$$(4)\quad [i_{x=0}] = [Z_0]^{-1}\{[e_{10}]+s[L_0][I'_{0\,t=0}]\} + s[C'_0][E_{x=0,\,t=0}]-\{[Z_0]^{-1}+[Y_0]\}[e_{x=0}].$$

Substituting the first equality of (2) in (4) and using the transformation of

$$(5)\quad \begin{aligned} [e'] &= [e]-[e_t], \\ [e_t] &= \left(1-\frac{x}{l}\right)\{[U]+[Z_0][Y_0]\}^{-1}[e_{10}] + \frac{x}{l}\{[U]+[Z_1][Y_1]\}^{-1}[e_{l1}], \end{aligned}$$

where Z_1 and Y_1 denote the series impedance and the shunt admittance at the receiving end, respectively, the boundary conditions at $x=0$ reduce to

$$(6)\quad \left(\frac{d[e']}{dx}\right)_{x=0}+[a]([e'])_{x=0} = [e_i']+[e_f] = [e_f'],$$

where $[a]$, $[e_f]$ and $[e_i']$ are defined in equations (7), (8) and (9) of the paper. Similarly, for the receiving end, there is

$$(10)\quad \left(\frac{d[e']}{dx}\right)_{x=l}+[b]([e_t'])_{x=l} = [e_t']+[e_g] = [e_g'].$$

Differentiating the first equality of (2) with respect to x and making use of the transformation of (5) gives

$$(13)\quad \frac{d^2[e']}{dx^2}-[k]^2[e'] = [k]^2[e_t]+[Q].$$

By introducing the diagonal matrix $[q]^2$, which is related by

$$(15)\quad [u]^{-1}[k]^2[u] = [q]^2,$$

there are obtained the new equations:

$$(17)\quad \frac{d^2[e'']}{dx^2}-[q]^2[e''] = [u]^{-1}\{[k]^2[e_t]+[Q]\};$$

$$(18)\quad \begin{aligned} \left(\frac{d^2[e'']}{dx^2}\right)_{x=0}+[a']([e''])_{x=0} &= [u]^{-1}[e_f'], \\ \left(\frac{d^2[e'']}{dx^2}\right)_{x=l}+[b']([e''])_{x=l} &= [u]^{-1}[e_g'], \end{aligned}$$

where $[a']=[u]^{-1}[a][u]$ and $[b']=[u]^{-1}[b][u]$.

Y. H. Ku (Philadelphia, Pa.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 5268.

Sastry, U. A. 5690

Heat transfer of laminar forced convection in doubly connected regions.

Boll. Un. Mat. Ital. (3) **18** (1963), 351–357.

Author's summary: "In this paper the heat transfer of laminar forced-convection in a pipe whose outer cross-section is a Booth's lemniscate and inner cross-section a circle has been investigated by using Schwarz's alternating method."

Biot, M. A. 5691

Lagrangian thermodynamics of heat transfer in systems including fluid motion.

J. Aerospace Sci. **29** (1962), 568–577.

Author's summary: "The Lagrangian thermodynamic equations of irreversible processes are extended to convective heat transfer. This generalization provides equations for the unified analysis of transient heat flow in complex systems comprising solid structures and moving fluids in either laminar or turbulent flow. The concept of a surface-heat-transfer coefficient is eliminated from the formulation. The theory is developed along two different lines. In one approach a new concept referred to as the 'trailing function' is introduced. It represents the surface-heat-transfer properties and may be evaluated by quite simple but remarkably accurate variational procedures. The method of 'associated fields' is also generalized to convective phenomena. The second line of approach extends to convective heat transfer the thermodynamic concept of entropy production for both laminar and turbulent flow. The theory amounts to an extension of the thermodynamics of irreversible processes to systems for which Onsager's relations are not valid."

QUANTUM MECHANICS

See also 5597, 5758, 5759, 5760, 5787.

Spitzer, Richard 5692

The breakdown of reflection symmetry and the principle of restricted coherence.

Nuclear Phys. **51** (1964), 553–560.

Author's summary: "It is argued that existing information on parity violation suggests that this phenomenon is, in all cases of its occurrence, an expression of the restrictions on the set of observables imposed by the intrinsic space-inversion properties of the states connected by parity-violating processes. Parity conservation in strong phenomena and the absence of purely strong transitions between subspaces connected by parity-violating processes are then interpreted as different manifestations of a single principle that restricts the measurement of purely strong phenomena to be compatible with invariance under space inversion."

S. Bludman (Philadelphia, Pa.)

Fonda, Luciano; Ghirardi, Gian Carlo 5693

Properties of the bound states embedded in the continuum.

Ann. Physics **26** (1964), 240–246.

Authors' summary: "The energy dependence of the matrix element connecting the state vector describing the generic bound state embedded in the continuum with the unperturbed states is investigated in detail. The energy behavior is such that the scattering matrix turns out to be

not infinite at the energy of the bound state embedded in the continuum. The asymptotic spatial behavior of the state vectors relative to these kinds of eigenvalues is analyzed. It results that the two-particle projected part of these vectors determines in most cases the asymptotic behavior of such states." *I. Kay* (Ann Arbor, Mich.)

Falk, Harold 5694
Erratum: "Variational property of free-energy perturbation theory".
Phys. Rev. Lett. **12** (1964), 211.
An erratum to the paper published in same Lett. **12** (1964), 93–94 [MR **28** #2862], indicating that the theorem stated there is invalid.

Bronzan, John B. 5695
Overlapping resonances in dispersion theory.
Phys. Rev. (2) **134** (1964), B687–B697.
Author's summary: "The Khuri-Treiman dispersion representation is applied to the discussion of overlapping resonances among particles in production and decay final states. The kernel of the dynamical equation following from the Khuri-Treiman representation has branch points overlapping the integration contour, but recently reported work permits us to select the correct branch of the kernel. We thus eliminate all restrictions on the masses of the final-state particles or strengths of the resonances. An iteration procedure is developed for the solution of the dynamical equation when three spinless particles are present in the final state. There is no restriction on the angular momentum of the resonances, but for simplicity only s-wave resonances are considered here. Plausibility arguments are given which indicate that for narrow resonances the once-iterated approximation to the solution is a good approximation. A detailed study of all higher approximations supports this assertion. In the once-iterated approximation, one finds a branch point on the second sheet of the transition amplitude which may cause a characteristic variation of the amplitude near the low-energy boundaries of the physical region. This variation is studied quantitatively for the kinematically favorable reaction $N+N \to N+N+\pi$, and is found to be of negligible importance. The suppression of the variation is related to the threshold behavior of two-particle scattering amplitudes." *A. C. Hurley* (Melbourne)

Agranovich, Z. S. [Agranovič, Z. S.]; Marchenko, V. A. [Marčenko, V. A.] 5696
★The inverse problem of scattering theory.
Translated from the Russian by B. D. Seckler.
Gordon and Breach Science Publishers, New York-London, 1963. xiii+291 *pp.* $14.50.
Problems where it is required to ascertain the spectral data which determine a differential operator uniquely, and where a method is sought for constructing this operator from the data, are usually called "the inverse problem" of spectral theory. Probably the most important class of such problems is the inverse problem of scattering theory.

The present monograph discusses in detail the particular methods developed previously by the same authors to deal with the inverse scattering problem. The book does not purport to be a comprehensive review of all methods dealing with the inverse problem. The authors' method consists essentially of establishing an inhomogeneous Volterra type integral equation of the second kind, from the solution of which the scattering potential is obtained by simple differentiation. The kernel and the inhomogeneity term of the integral equation are constructed from the scattering data. By scattering data we mean the S-matrix, the bound state energies, and the asymptotic normalization of the bound state wave functions.

In the first part of the book the problem is treated for the case when the potential satisfies the usual condition

$$\int_0^\infty x|V(x)|\,dx < \infty.$$

In the second part of the book the more complicated case is treated when the potential is permitted to have singularities of the order of x^{-2} near zero and at infinity. The treatment for this case is made possible by applying transformations of a special type. In the subsequent chapters, special consideration is given to the problems arising in the theory of the deuteron.

This book is one on pure mathematics and not on physics as such. However, in view of the lucid presentation and excellent pedagogical approach, this volume may be used with great advantage by any theoretical physicist. Incidentally, quite apart from the principal topic, various methods of functional analysis and algebra which are used should also be of great interest to both pure and applied mathematicians. *P. Roman* (Boston, Mass.)

Bell, W. W. 5697
The analytic properties of the scattering amplitude in nonrelativistic quantum mechanics. (Italian summary)
Nuovo Cimento (10) **29** (1963), 644–654.
Author's summary: "The analytic properties of the scattering amplitude in potential theory are investigated using an iterative momentum-space method involving both the Lippmann-Schwinger and the Low equations. The usual cuts in the energy and momentum transfer are obtained together with the bound-state poles."
B. W. Lee (Philadelphia, Pa.)

Schenter, R. E.; Downs, B. W. 5698
Target-momentum and nonlocal effects in the high-energy optical potential.
Phys. Rev. (2) **133** (1964), B522–B528.
The optical potential obtained by means of the multiple-scattering and impulse approximations is evaluated without the usual additional approximations which result in a local potential. The dependence of the effective potential on the target nucleon momenta, on the center of mass of the two interacting momenta, and finally the form factor dependence on momentum transfer are evaluated in the high-energy approximation [see also B. Mulligan, Ann. Physics **26** (1964), 159–180].
H. Feshbach (Cambridge, Mass.)

Hahn, Yukap; O'Malley, Thomas F.; Spruch, Larry 5699
Bounds on multichannel scattering parameters.
Phys. Rev. (2) **134** (1964), B397–B404.

Authors' summary: "Using the projection techniques recently developed in the formal theory of reactions, it is shown that a bound on the exact reactance matrix K is provided by the close-coupling reactance-matrix approximation K^P; that is, K-K^P is, in a sense that can be made precise, a positive definite operator. This is of more than formal interest since the numerical solution of the finite number of coupled equations which arise when we allow the target system to be excited to only a restricted number of virtual states, and the determination of K^P is feasible for a variety of three-body problems which includes, of course, three-body model problems. Furthermore, K^P improves monotonically as one includes more and more virtual states. The recognition of this monotonicity property is useful in self-consistency analyses during the course of numerical calculations, and provides a more precise meaning for the numerical results obtained. Choosing a particular representation, the bound on K generates bounds on the appropriately defined eigenphase shifts. The question of the absolute definition of phase shifts and of eigenphase shifts is discussed in some detail and it is shown that the presently used definition has serious deficiencies." *M. Blažek* (Bratislava)

Hahn, Yukap; O'Malley, Thomas F.; Spruch, Larry 5700

Improved minimum principle for multichannel scattering.

Phys. Rev. (2) **134** (1964), B911–B919.

Authors' summary: "If the open-channel target states are known, the minimum principle formulation of scattering theory provides a systematic approach whereby one can, to arbitrary precision, monotonically approach the reactance matrix **K**. The scattering wave function and the Green's function for the open-channel approximation, that in which the closed channels are not taken into account at all, must be solved numerically. An explicit method for constructing the Green's function is given. The minimum principle approach is probably limited at present, in practice though not in principle, to the three-body problem with just a few open channels. A very useful simplification is possible at the threshold for a new channel; one need not there include the new channel in the equations that must be solved exactly."

Nguyen-Van-Hieu [Nguyen Van Hieu] 5701a

Asymptotic relations between photoproduction amplitudes.

Phys. Lett. **9** (1964), 81–82.

Nguyen-Van-Hieu [Nguyen Van Hieu] 5701b

Invariance properties and asymptotic relations between scattering amplitudes.

Phys. Lett. **9** (1964), 83–84.

These two brief notes announce results concerning relationships between the high-energy behaviour of a scattering process and its "crossed" process. The first relates the polarization of final baryons in photoproduction to the asymmetry parameter of the crossed process, and also the asymptotic equality of the differential sections of the processes $\gamma+p\to\pi^++n$ and $\gamma+n\to\pi^-+p$. The second letter derives similar results without using invariance under parity. *R. F. Streater* (London)

Teplickii, È. Š. 5702

On the meromorphy of the partial scattering amplitude in the complex angular momentum plane. (Russian. Georgian summary)

Soobšč. Akad. Nauk Gruzin. SSR **32** (1963), 543–547.

Meromorphy of the partial wave scattering amplitudes in the complex l-plane can be studied either by the N/D method or by analytic continuation through the "elastic" branch cut in the physical Riemann surface [R. Oehme, Phys. Rev. (2) **121** (1961), 1840–1848; MR **22** #11868]. The author derives the first method from the second by neglecting the contribution of inelastic processes.

J. M. Cook (Argonne, Ill.)

Fong, R.; Sucher, J. 5703

Relativistic particle dynamics and the S matrix.

J. Mathematical Phys. **5** (1964), 456–470.

From the authors' summary: "Direct-interaction theories are examined from the viewpoint of relativistic scattering theory and the associated concept of 'asymptotic covariance'. It is shown that the requirement of asymptotic covariance ensures both the covariance of the S-matrix and the existence of a unique representation of the inhomogeneous Lorentz group to be associated with the relativistic two-particle system. The form of H given by Bakamjian and Thomas is shown to satisfy asymptotic covariance and, moreover, to be the most general form of interest from the viewpoint of relativistic scattering theory, thereby including as a special case a form of H suggested by Sudarshan. It is also proved that relativistic Hamiltonians of this type do not admit the usual notion of a coupling constant." *I. Białynicki-Birula* (Warsaw)

Gol'fand, Yu. A. [Gol'fand, Ju. A.] 5704

The space-time structure of the relativistic scattering matrix.

Ž. Èksper. Teoret. Fiz. **45** (1963), 1067–1080 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 738–746.

Author's summary: "A general, relativistically invariant method is proposed for the construction of the S-matrix in field theory in which T-products are not employed. Based on this method two different representations of the S-matrix are constructed. It is shown that the problem of construction of the S-matrix may be reduced to the solution of a certain set of integral equations. The equivalence of the method of construction here proposed with the conventional field theory formulation is proven. The unitarity of the S-matrix is demonstrated explicitly."

G. Källén (College Park, Md.)

Weinberg, Steven 5705

Perturbation theory for strong repulsive potentials.

J. Mathematical Phys. **5** (1964), 743–747.

Author's summary: "A conformal mapping of the coupling-constant plane is used to rearrange the Born series. The new series is guaranteed to converge for any decent repulsive potential. The first terms do well in actual calculations of the scattering length."

Arbuzov, B. A.; Filippov, A. T.; Khrustalev, O. A. [Hrustalev, O. A.] 5706

On the correlation between the exact solution and the perturbation theory series in the case of Schroedinger equation with singular potential.

Phys. Lett. **8** (1964), 205–207.

The authors consider the scattering problem for potentials singular at the origin like $-g(\ln r)/r^2$. Such potentials are supposed to be the nonrelativistic counterpart of a $g\varphi^4$ theory. It is shown that the so-called "principal" logarithms approximation in this case does not work. The reason for this is that the "principal" logarithms form a convergent series while the exact solution has only an asymptotic expansion. Further, due to the asymptotic character of the perturbation series, it is impossible to construct unambiguously the function which generates the series, starting from the series itself. *E. Predazzi* (Turin)

Lee, C. H. 5707

Nuclear potential in axiomatic quantum field theory.

Nuclear Phys. **51** (1964), 369–382.

An approach is given to define a nuclear potential from axiomatic quantum field theory. Extrapolation of the scattering matrix off the energy shell is performed in a certain way. To express the potential in closed form in terms of field theory matrix elements, the appropriate equation is the "Dirac equation" from which the Schrödinger equation is obtained as an approximation.

S. Azuma (Fukusima)

Yanase, M. M. 5708

Remarks on the theory of measurement.

Amer. J. Phys. **32** (1964), 208–211.

E. P. Wigner has pointed out [see H. Araki and the author, Phys. Rev. (2) **120** (1960), 622–626; MR **22** #13103] that an observable cannot be measured precisely if it does not commute with all conserved quantities which, like linear or angular momentum, are additive (i.e., are represented by operators of the form $L_1 \otimes I + I \otimes L_2$ on the tensor product of the state spaces of the measured object and the measuring apparatus).

The author classifies quantum mechanical quantities into three categories: (1) Those which commute with all additive conserved quantities and are therefore exactly measurable as discussed in Chapter 6 of von Neumann's *Mathematical foundations of quantum mechanics* [Princeton Univ. Press, Princeton, N.J., 1955; MR **16**, 654]. (2) Those which do not, but are approximately measurable by using very large measuring apparatus according to Wigner's modification of von Neumann's theory. (3) Those which depend on phase differences in orthogonal super-selecting subspaces [G. C. Wick, A. S. Wightman, and E. P. Wigner, Phys. Rev. (2) **88** (1952), 101–105; MR **14**, 827] and therefore cannot be measured at all.

He then discusses these categories from more general points of view and concludes that the problem of measurement is far from a satisfactory solution.

J. M. Cook (Argonne, Ill.)

Sato, Shigeo 5709

An approach to the formulation of quantum electrodynamics without use of indefinite metric.

Progr. Theoret. Phys. **31** (1964), 256–268.

An attempt is made to derive the Hamiltonian in quantum electrodynamics from the invariance under the inhomogeneous Lorentz group or more specifically from commutation relations among the generators of this group. Only transverse photons are introduced and the usual form for the free Hamiltonian and the first-order interaction Hamiltonian are assumed. It is then argued that the second-order terms (Coulomb interaction) can be deduced from the relativistic invariance. The final part of the paper is devoted to the study of divergencies appearing in this formulation. Since the whole procedure is not manifestly Lorentz-covariant, the reviewer is very skeptical concerning the success of such an approach.

Unfortunately, no comparison is made with the more standard methods of formulation of quantum electrodynamics (especially with the radiation gauge formalism of Schwinger). The problem of gauge invariance is not even mentioned, and, as a result, such gauge-violating terms as the photon mass appear in the discussion.

I. Białynicki-Birula (Warsaw)

Ladányi, K. 5710

On the nonlinear spinor theory. (Italian summary)

Nuovo Cimento (10) **31** (1964), 809–826.

Für die Heisenberg-Dürr'sche Urmaterie-Gleichung wird in gewisser Näherung die Zweipunktfunktion (propagator) ausgerechnet. Der Verfasser geht von den Fockschen Gleichungen für die Matrixelemente von Feldoperatoren, führt aber (um diese Gleichungen lösen zu können) eine Reihe von Vereinfachungen ein. Der Referent kann schwer übersehen, ob alle diese Vereinfachungen gerechtfertigt sind. Verfasser schlägt ein in gewissem Sinne selfconsistentes Verfahren zur Bestimmung der Massen und Kopplungskonstanten vor. *G. Heber* (Leipzig)

Just, K. 5711

Equality of renormalizations. (Italian summary)

Nuovo Cimento (10) **29** (1963), 997–1002.

The author considers the Lorentz covariant field equations of a Dirac spinor with electric and gravitational interaction. The Einstein metric is written as a Minkowski metric plus a first-order deviation. Following the Gupta approach, the latter field is assumed to satisfy a Lorentz-covariant wave equation. Upon assuming certain commutation rules between the spinor field and the unrenormalized form of the total charge and momentum operators, the author derives the result that the ratio of renormalized to unrenormalized coupling constants is the same for both gravitational and electric coupling constants. It should be observed, however, that gravitation theory is unrenormalizable in perturbation theory. Nevertheless, one can assume the existence of a mass spectrum integral, and it is the latter which is equal to the above ratio. In some sense, the unrenormalized coupling constants referred to by the author should be considered as being defined by the ratio of the renormalized coupling constant and the mass spectrum integral.

M. Schwartz (Garden City, N.Y.)

Gačok, V. P. 5712

Models of relativistic field theory and the Mandelstam representation. (Russian)

Ukrain. Mat. Ž. **16** (1964), 225–232.

The model of relativistic field theory discussed in this paper has been originally proposed by O. W. Greenberg [Ann. Physics **16** (1961), 158–176; MR **24** #B281]. Two years later it was rediscovered independently by J. T. Łopuszański [Phys. Lett. **8** (1964), 85–87; MR **28** #2812] and by the author of the paper under review. Neither Greenberg nor Łopuszański and the author have noticed, however, that the use of the Jacobi identity for commutators of field operators reduces this model to the generalized free field model of Greenberg. This fact has been pointed out by D. W. Robinson [ibid. **9** (1964), 189–190]. Since the S-matrix in the generalized free field case is equal to unity the question of the existence of a model of the local relativistic field theory with interaction remains still open. *I. Bialynicki-Birula* (Warsaw)

McKean, H. P., Jr. 5713

Kramers-Wannier duality for the 2-dimensional Ising model as an instance of Poisson's summation formula.

J. Mathematical Phys. **5** (1964), 775–776.

Author's summary: "The well-known Kramers-Wannier high-low-temperature duality for the 2-dimensional Ising model is derived by means of the Poisson summation formula for a commutative group."

Koštál, K. 5714

On the transformation of the Schrödinger wave equation into a system of equations of stochastic processes which have first and second mean square derivatives.

Phys. Lett. **9** (1964), 26–29.

Let $X(t)$, $-\infty < t < \infty$, be a square-integrable stochastic process with values in R^n, with first and second mean square derivatives $\dot{X}(t)$, $\ddot{X}(t)$. Set

$$v_j(x,t) = E\{\dot{X}_j(t) \mid X(t) = x\},$$

$$\overline{v_j v_k}(x,t) = E\{\dot{X}_j(t)\dot{X}_k(t) \mid X(t) = x\},$$

$$a_j(x,t) = E\{\ddot{X}_j(t) \mid X(t) = x\}.$$

Let $X(t)$ have a smooth density $f(x,t)$. Then

$$(*)\qquad \frac{\partial f}{\partial t} + \sum_j \frac{\partial}{\partial x_j}(f v_j) = 0,$$

$$\frac{\partial}{\partial t}(f v_j) + \sum_k \frac{\partial}{\partial x_k}(f \overline{v_j v_k}) = f a_j.$$

Now, let $\psi = Ae^{i\varphi}$ be a function on $(-\infty, \infty)$ with values in $L_2(R^n)$, and set $f(x,t) = A^2(t,x) = |\psi(t,x)|^2$. Then the Schrödinger equation for ψ is equivalent to a pair of equations involving f and φ, in which f plays a role analogous to that in (*), with $(\hbar/m)\partial\varphi/\partial x_j$ as v_j. $\overline{v_j v_k}$ and $\bar{a}_j$, however, are not uniquely determined, and three alternative possibilities for these are mentioned, having various physical interpretations, some of which had been previously considered by others.

J. Feldman (Berkeley, Calif.)

Hellman, O. 5715

A possible generalization of quantum electrodynamics.

Nuclear Phys. **52** (1964), 609–629.

Author's summary: "The formalism of quantum electrodynamics is generalized to a formalism in which the differential equations are replaced by covariant functional equations, a kind of difference equations. The expressions contain the dimensionless constant $\beta = mcl_0/\hbar$, where l_0 is a length which is introduced into the equations by the process of generalization. For $\beta > 1$ all free field solutions of the generalized Dirac equation, a denumerably infinite family of Dirac fields with complex masses, have a finite lifetime. For $\beta < 1$ the free field solutions will include, in addition to the above decaying Dirac fields, also two stationary waves, those of a Dirac equation with mass $m \ln(1-\beta)^{-1/\beta}$. The ordinary perturbation method converges for $\beta > 1$, and also for $\beta < 1$ provided that the stationary waves are first excluded. The problem of finding, say, the total energy of the system in the case of a value of $\beta = \beta_1 < 1$ may be solved by first going over to a value $\beta_2 > 1$, by applying the now convergent perturbation method for the equations with $\beta = \beta_2$, and then by letting $\beta_2 \to \beta_1$ in the expression for the energy."

J. M. Jauch (Geneva)

Henneberger, Walter C. 5716

Hermiticity of Hamiltonians and existence of eigenstates in soluble field theories.

Nuclear Phys. **49** (1963), 321–327.

This is another investigation of the model consisting of a charged particle, elastically bound to a point and interacting with the dipole part of the radiation field. The purely formal treatment of this model leads to inconsistencies which have been noted many times before. For instance, in the limit of a point particle there seem to exist eigenstates of the Hamiltonian with imaginary eigenvalues, which is impossible if the Hamiltonian is self-adjoint. The Hamiltonian is formally hermitian and so the problem arises where this inconsistency comes from.

The author tries to answer this question by showing that under the assumption that the Hamiltonian has eigenstates, one obtains imaginary eigenvalues. Since the Hamiltonian must be hermitian and therefore cannot have imaginary eigenvalues, the author concludes that this assumption is false and no eigenstates exist in the limit of a point particle.

{In the reviewer's opinion no such conclusion can be drawn from the considerations of the author. The real problem is whether the Hamiltonian is essentially self-adjoint, or has equal defect indices which would permit at least a self-adjoint extension of the operator. The eigenvalues are then all real and for a discrete spectrum a complete set of eigenfunctions always exists.}

J. M. Jauch (Geneva)

Gotō, T. 5717

Bohr-Sommerfeld's quantum conditions and rigid body rotation. (Italian summary)

Nuovo Cimento (10) **31** (1964), 397–401.

Bopp and Haag [Z. Naturforsch. **5a** (1950), 644–653; MR **13**, 239] have shown that both integral and half-integral spin states, as given by the representation theory of the three-dimensional rotation groups, are obtained in the quantum theory of a spherically symmetric rigid rotator. The author supports this conclusion by reconsidering the problem in a more intuitive way by the application of the Bohr-Sommerfeld quantum condition of the old quantum theory to the rotational motion of a rigid body.

using the Cayley-Klein parameters, and showing that half-integral states come into play in this semi-classical step. *B. S. Madhavarao* (Poona)

Tiktopoulos, George 5718

Analytic continuation in complex angular momentum and integral equations.

Phys. Rev. (2) **133** (1964), B1231–B1238.

Author's summary: "An attack is made on the problem of the analytic continuation in the angular momentum variable l of amplitudes defined by integral equations beyond the value of Re l at which the kernel ceases to be of the Schmidt type and the Fredholm theory cannot be applied. A general technique is developed and applied to the Yukawa potential case and to the ladder graph series in the φ^3 theory. In both cases meromorphy is established for Re $l > -\frac{3}{2}$ and a procedure is indicated for a stepwise continuation to the entire l plane."

The continuation is effected by decomposing the kernel into a part of finite rank and the rest which is of the Schmidt type, so that the part of finite rank can be continued explicitly. *B. W. Lee* (Philadelphia, Pa.)

Bose, A. K. 5719

A class of solvable potentials. (Italian summary)

Nuovo Cimento (10) **32** (1964), 679–688.

Author's summary: "The problem of the construction of solvable one-variable Schrödinger potentials is formulated. A class of simple potentials for which the Schrödinger equation can be solved in terms of special functions of physics is constructed."

Bradley, Joe C.; Kammel, Frank J. 5720

The determination of irreducible representations of symmorphic space groups using semidirect product groups.

J. Math. Anal. Appl. **8** (1964), 474–502.

In their introduction the authors state that the irreducible representations of space groups are needed for the investigation of the solutions of the time-independent Schrödinger wave equation. They therefore proceed to give in Part 1 of this paper a clear account of certain standard material on the irreducible representations of direct and semi-direct products of groups. This account suffers from a number of inaccuracies; for example, Theorem II.2 is clearly false, since the elements of the claimed basis of $E_\gamma{}^G$ do not even belong to this space. Furthermore, certain theorems of G. W. Mackey are used in places where their applicability is in doubt. Thus the proof of Theorem II.6 refers to a theorem of Mackey on finite groups (the corresponding results for infinite groups are more complicated). The final part of the paper deals at length with the group of symmetries of the CsCl lattice. This group is a semidirect product of the translation group T (free abelian on 3 generators) and of the group S of symmetries of a cube. There are seven pages of tables of irreducible representations of S. The reviewer feels that the authors have laboured unnecessarily to obtain these results. The group S is the direct product of a group of order 2, and of a subgroup of order 24, namely the subgroup of proper rotations. The determination of the representations of this subgroup is a standard illustrative example to be found in a number of textbooks; see, for example, M. Hall, Jr. [*The theory of groups*, pp. 304–310, Macmillan, New York, 1959; MR **21** #1996]. Finally, the reviewer is grateful to the authors for a very readable clear account. The above remarks, made in an entirely constructive spirit, would have been impossible with many a paper in the same field. *H. K. Farahat* (Sheffield)

Shima, Kazuhisa 5721

Local isotopic gauge transformation.

Progr. Theoret. Phys. **31** (1964), 139–150.

In this paper the theory of non-Abelian gauge fields (see, for example, the paper by V. B. Adamskiĭ [Uspehi Fiz. Nauk **74** (1961), 609–626]) is formulated in a space endowed with a coordinate-dependent metric tensor and an arbitrary affine connection. It is also shown that the mathematical properties of a coordinate-dependent gauge transformation permit it to be identified as a homomorphism between sheaves. *G. R. Allcock* (Liverpool)

Reynolds, J. T.; Onley, D. S.; Biedenharn, L. C. 5722

Some exact radial integrals for Dirac-Coulomb functions.

J. Mathematical Phys. **5** (1964), 411–419.

Authors' summary: "The zero energy loss Dirac-Coulomb integrals are evaluated using the technique of contour integration. The expressions obtained have a closed analytic form, showing that these integrals are formally similar to the corresponding classical and nonrelativistic quantum mechanical, zero energy loss integrals which also have exact elementary solutions. Application of the zero energy loss Dirac-Coulomb integrals occurs in inelastic electron scattering and similar problems. The investigation of the finite energy loss Dirac-Coulomb integrals requires a study of the zero energy loss integrals as a preliminary."

Salem, L. 5723

Quantum-mechanical sum-rule for infinite sums involving the operator $\partial H/\partial\lambda$.

Phys. Rev. (2) **125** (1962), 1788–1791.

Author's summary: "The sum rule

$$\sum_n{}' \frac{\langle m|A|n\rangle\langle n|\partial\Omega/\partial\lambda|m\rangle + \langle m|\partial\Omega/\partial x|n\rangle\langle n|A|m\rangle}{\varepsilon_n-\varepsilon_m} = \left\langle m\left|\frac{\partial A}{\partial\lambda}\right|m\right\rangle - \frac{\partial}{\partial\lambda}(\langle m|A|m\rangle)$$

is derived. In this relation $|m\rangle, \cdots, |n\rangle, \cdots$ form a complete set of orthonormal vectors, which are the eigenvectors of the Hermitian linear operator Ω, with eigenvalues $\varepsilon_m, \cdots, \varepsilon_n, \cdots$; λ is a parameter which occurs in Ω, and A is an arbitrary linear operator. In many sums of this type, Ω is the Hamiltonian operator H. Particular examples are considered, and a differential equation relating the mass dependence and coordinate dependence of the wave function ψ is derived." *S. Olszewski* (Warsaw)

Barut, A. O. 5724

A symmetry group containing both the Lorentz group and SU_3.

Nuovo Cimento (10) **32** (1964), 234–236.

The author points out that both the Lorentz group

(space-time symmetry group) and SU_3 (the internal symmetry group of the strongly interacting elementary particles) are contained as subgroups of the complex extension G of the homogeneous Lorentz group. The structure of G and its Lie algebra are discussed briefly.

W. T. Sharp (Toronto, Ont.)

Garczynski, W. 5725

The necessary and sufficient condition in terms of Wightman functions for a field to be a generalized free field.

J. Mathematical Phys. **5** (1964), 714–719.

The author proves that the necessary and sufficient condition for a field to be a generalized free field is the vanishing of all the truncated Wightman functions of this field except for the two-point function. The main point of the proof seems to be that of sufficiency where the Wightman axioms and the c-number commutation relation are honestly checked. In item A of that proof, it is not necessary to quote results of Bogoliubov because the relevant product can be considered as the product of distributions of independent variables. *H. Araki* (Kyoto)

Schwinger, Julian 5726

Non-Abelian vector gauge fields and the electromagnetic field.

Rev. Modern Phys. **36** (1964), 609–613.

Gauge vector fields, whose prototypes are the Maxwell and Yang-Mills fields, cannot be coupled to each other, or to sources, in an arbitrary way, owing to their gauge-invariance properties. The consistency problem for charged massless vector fields or a charged Yang-Mills triplet interacting with the Maxwell field has been explicitly discussed, for example, by R. Arnowitt and the reviewer [Nuclear Phys. **49** (1963), 133–143; MR **28** #1883]. The author discusses here the compatibility problem of fitting in the electromagnetic field as one member of an $(n+1)$-dimensional vector multiplet, containing some charged components and interacting with a spin $\frac{1}{2}$ field. It then turns out that, irrespective of the value of n, only one pair of charged components is allowed in the multiplet, the remainder being necessarily neutral. Further, this charged pair combines with another component to make an isotopic spin (Yang-Mills like) triplet. The problem now arises in identifying the Maxwell field as that component coupled purely to the (conserved) electric current. For, as shown also in the paper, such a coupling which picks out the photon's "direction" simultaneously destroys the theory's consistency (gauge invariance). The author concludes that this failure may actually be exploited as follows. To restore consistency, the original gauge invariance must be broken by another term. One way is to endow the other vector components with a mass, which then lifts the contradiction. Now experimentally, it is indeed the case that the vector mesons coupled to isospin and hypercharge are massive. Whether this approach would yield the quantitative T_3 dependence of mass multiplets is left open at this stage.

S. Deser (Waltham, Mass.)

Ogievetskiĭ, V. I. [Ogieveckiĭ, V. I.]; Polubarinov, I. V. 5727

On interacting fields with definite spin.

Ž. Èksper. Teoret. Fiz. **45** (1963), 237–245 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 166–171.

A quantum theory of free fields describes quanta which contain intrinsic angular momenta if the field has several components which transform according to a representation of the Lorentz group. In general, there may occur several different quanta corresponding to several irreducible representations of the Lorentz group. A field which carries only one value of spin is obtained by subjecting the field components to a suitable subsidiary condition.

It is shown in this paper that the same condition is needed for interacting fields. It is further shown that there are types of interactions which are inconsistent with such a subsidiary condition (for instance, a vector field with axial vector coupling to a spinor field). In such a case the interacting field carries several spin values.

J. M. Jauch (Geneva)

Ogievetskiĭ, V. I. [Ogieveckiĭ, V. I.]; Polubarinov, I. V. 5728

Interacting spin 1 fields and symmetry properties.

Ž. Èksper. Teoret. Fiz. **45** (1963), 966–977 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 668–675.

This is an "inverted" Yang-Mills derivation for interaction symmetries [C. N. Yang and R. L. Mills, Phys. Rev. (2) **96** (1954), 191–195; MR **16**, 432]. The usual derivation postulates a local gauge symmetry $\exp(i\alpha^n\omega^n)$, where ω^n is a set of generators of an algebra; the existence of vector fields transforming like the ω^n and coupling through them is then a plausible, though not unique, conclusion [see the authors' article in Nuovo Cimento (10) **23** (1962), 173–180; MR **26** #5900]. In this new approach, the symmetry is an automatic result from the postulation (or experimental discovery) of vector meson fields. It is interesting that the same result has been derived by R. Cutkosky using dispersion relations without a field theory [Phys. Rev. (2) **131** (1963), 1888–1890].

Y. Ne'eman (Pasadena, Calif.)

Cutkosky, R. E. 5729

Self-consistency of superglobal multiplet assignments.

Phys. Rev. Lett. **12** (1964), 530–531; *erratum, ibid.* 12 (1964), 572.

In this short note, the possibility is discussed that the multiplets of strongly interacting particles have the symmetry properties of the group SU(3) ⊗ SU(3), as proposed by Schwinger. On the basis of self-consistency or "bootstrap" arguments, it is concluded that certain irreducible representations of this group are incompatible with experiment, but the model itself is not ruled out. Through no fault of the author, a table referred to in the paper does not appear with it, but appears separately as the erratum listed in the heading above.

D. B. Lichtenberg (Bloomington, Ind.)

Jones, C. E.; Teplitz, V. L. 5730

Singularities of partial-wave amplitudes arising from the third double-spectral function. (Italian summary)

Nuovo Cimento (10) **31** (1964), 1079–1085.

Authors' summary: "The partial-wave amplitudes are shown to have singularities along the left-hand cut, for both integral and nonintegral angular momenta, at positions where there are peaks in the third double-spectral function. The nature of these singularities and their connection with the Gribov-Pomeranchuk essential singularities in the angular-momentum plane are discussed." *J. W. Moffat* (Baltimore, Md.)

Fischer, J. 5731
Scattering of pions on pions in threshold region.
Czechoslovak J. Phys. **14** (1964), 77–88.
Author's summary: "It is shown that the integral equations for isoscalar $\pi\pi$-scattering, obtained from the Mandelstam representation by the method of so-called differential approximation, admit the threshold behaviour of partial amplitudes which is in agreement with the quantum theory of scattering and, moreover, they also admit solutions with other possible threshold behaviours. This result does not depend on whether the conformal mapping technique is used or not."
J. Franklin (Livermore, Calif.)

Bjorken, James D. 5732
Regge behavior of forward elastic scattering amplitudes.
J. Mathematical Phys. **5** (1964), 192–198.
Author's summary: "A theorem concerning the asymptotic behavior of forward elastic scattering amplitudes in relativistic theories is stated and proved. The assumptions made are (1) identical spinless particles interact via $G\phi^3$ and $\lambda\phi^4$ couplings; (2) a cutoff of the propagators is introduced; (3) the forward scattering amplitude satisfies a Bethe-Salpeter equation in the crossed channel; (4) the kernel of the equation is an arbitrary finite subset of the Feynman graphs which compose the exact kernel. The theorem states that under these assumptions, the forward scattering amplitude exhibits Regge behavior, i.e., $A(s,0)\to s^\alpha+O(1)$ as $s\to\infty$."
B. W. Lee (Philadelphia, Pa.)

Diu, B.; Rubinstein, H. R. 5733
Incompatibility of the exact and approximate bootstrap conditions in a soluble model.
Phys. Lett. **8** (1964), 203–205.
This paper considers a very simple one-channel dispersion relation for which the exact solution is known. The authors show that the bootstrap procedure including the approximation of including only one-particle exchange to calculate the left-hand discontinuities leads to a set of equations having no solutions. They conclude from this that the procedure generally used in bootstrap calculations might lead to misleading results. *J. Franklin* (Livermore, Calif.)

Kaeser, C. 5734
Analytic structure of partial-wave amplitudes for production and decay processes.
Phys. Rev. (2) **132** (1963), 2712–2721.
A model of "overlapping" interactions in a three-body final state is studied. The model is based on a dispersion relation of the type proposed by Khuri and Treiman [same Rev. (2) **119** (1960), 1115–1121] and Sawyer and Wali [ibid. (2) **119** (1960), 1429–1435] for the τ decay $K\to3\pi$. Let q_1, q_2, and q_3 be the four-momenta of the three particles with masses m_1, m_2, m_3, and let $s_i=(q_j+q_k)^2$, i, j, k all different. Then the dispersion relation takes the form

$$M(s_1, s_2, s_3) = \mathscr{M}_0+\mathscr{M}_1(s_1)+\mathscr{M}_2(s_2)+\mathscr{M}_3(s_3),$$

where $\mathscr{M}_0$ is a constant, and $\mathscr{M}_i(s_i)$ is analytic in the s_i-plane with a cut on the real axis between $(m_j+m_k)^2$ and infinity. $\mathscr{M}_i$ has the representation

$$\mathscr{M}_i(s_i) = \frac{s_i-s_0}{\pi}\int_{(m_j+m_k)^2}^{\infty}\frac{\mu_i(s)\,ds}{(s-s_i-i\varepsilon)(s-s_0-i\varepsilon)}.$$

The density function μ_i factors: $\mu_i(s)=f_i^*(s)M_i(s)$. Here f_i is the s-wave amplitude for scattering of particles j and k, while M_i is a particular projection of M. Specifically, in the frame where the three-momentum q_2+q_3 is zero, define z_1 as the cosine of the angle between q_2-q_3 and q_1. Then at least for s_1 not too far above its threshold $(m_2+m_3)^2$, M_1 is given by

$$M_1(s_1) = \tfrac{1}{2}\int_{-1}^{1} M(s_1, s_2(s_1, z_1), s_3(s_1, z_1))\,dz_1,$$

and similarly for M_2 and M_3. The author's goal is to perform similar projections on the dispersion relation, in order to obtain a closed system of equations for the M_i. The projection $M_i(s_i)$ is taken for arbitrary, complex s_i, so that standard methods for the factorization of right-hand cut contributions may be applied in the solution of the equations. The bulk of the paper is concerned with the slightly involved technical problems that arise in defining the projections for general s_i. One question that gives some trouble is how to define an integral obtained by a formal interchange of the projection integration and the integration over s in the dispersion relation. The author answers the question by taking a hint from perturbation theory, but remarks in passing that another prescription is possible. He does not investigate possible inequivalence of equations obtained by the two different routes. A system of equations for the M_i is obtained, but their solutions for given f_i are not discussed.
R. L. Warnock (Chicago, Ill.)

Atkinson, D. 5735
Removal of the divergence in zero momentum-transfer dispersion relations. (Italian summary)
Nuovo Cimento (10) **30** (1963), 551–564.
Partial wave dispersion relations can be derived from zero momentum-transfer dispersion relations by taking derivatives with respect to the momentum-transfer variable. This method has the advantage that the input information required to use the dispersion relations is needed only in physical regions, but the difficulty is that the expansion for a partial wave amplitude in terms of the derivative amplitudes does not converge above a certain (usually low) energy. In this paper, a conformal transformation is used to give convergent series for all energies, thus extending the validity of the method. In the paper under review, the technique is applied to a phenomenological analysis of π-N scattering, but it can also be applied to dynamical calculations.
J. Franklin (Livermore, Calif.)

Volkov, D. V. 5736

Factorization at the Regge pole of scattering amplitudes of particles with spin.

Ž. Èksper. Teoret. Fiz. **45** (1963), 742–745 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 509–511.

Author's summary: "It is shown that by making use of derivatives with respect to particle momenta the contribution of a boson Regge pole to the amplitude for scattering of particles with spin may be represented in an explicitly factorized form." *C. G. Bollini* (Buenos Aires)

Vaks, V. G.; Larkin, A. I. 5737

Amplitude singularities at $l = -1$ in the Bethe-Salpeter equations.

Ž. Èksper. Teoret. Fiz. **45** (1963), 1087–1101 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 751–760.

Singularities of the scattering amplitude for $l = -1$ are studied with the aid of the equations of field theory. In this approach, the mass ratios are not essential so that the anomalous case can be considered together with the normal one. These singularities are related to the singularity of the irreducible four-pole network in the Bethe-Salpeter equation. Under some reasonable assumptions concerning the properties of the network, it is shown that the contribution to the asymptotic amplitude due to accumulation of the poles is small, provided the poles are close to $l = -1$. *J. N. Chahoud* (Bologna)

Frautschi, S. C.; Kaus, P. E.; Zachariasen, F. 5738

Method for the self-consistent determination of Regge pole parameters.

Phys. Rev. (2) **133** (1964), B1607–B1615.

Authors' summary: "A method is suggested for approximately bootstrapping Regge trajectories, thereby avoiding the cutoff problems of the usual bootstrap calculation. The method is based on dispersion relations for Regge trajectories and on unitarity applied at $l = \alpha$. Successively more realistic approximations are described which bring in more information on the potential, and more trajectories. The approximate Regge parameters are guaranteed to have the desired threshold and asymptotic properties." *J. N. Chahoud* (Bologna)

Omnes, R. 5739

Asymptotic behavior of partial-wave amplitudes.

Phys. Rev. (2) **133** (1964), B1543–B1548.

Calculations of the partial-wave amplitudes in pion-pion scattering originally met with difficulties due to the increase of the partial-wave left-hand cuts (or equivalently, forces) at a rate in conflict with unitarity when the energy became negative and infinite and the spin of the resonance was greater than or equal to one [G. F. Chew and S. Mandelstam, same Rev. (2) **119** (1960), 467–477; MR **22** #11880]. In spite of this, it was found that by assuming that the left-hand discontinuities were rapidly damped in the asymptotic region, results were obtained for the first few partial waves in essential agreement with experiment [cf. B. H. Bransden and the reviewer, Nuovo Cimento (10) **21** (1961), 505–518; MR **23** #B2846]. This circumstance corresponds to the nearest singularity hypothesis. The present work is an interesting attack on the basic problem of determining high-energy divergences and obtaining the exact behavior of partial-wave discontinuities. The following three assumptions are adopted: (a) The scattering amplitude satisfies the Mandelstam representation. (b) The asymptotic behavior of the amplitude is of the Regge type. (c) It is supposed that the Regge trajectory $\alpha(s)$ ($s = (\text{total energy})^2$) is an analytic function of s in the complex s-plane cut from $s = 4$ to $+\infty$.

With these assumptions it is shown that the discontinuity at infinity is determined by the position of the leading Regge pole at zero total energy $\alpha(0)$. In view of the unitarity limitation $\alpha(0) \leq 1$, it follows that the left-hand discontinuity is rapidly damped without oscillations as the energy tends to large negative values. The paper ends with a discussion of the number of subtractions required in partial-wave dispersion relations and the problem of cuts in the angular momentum plane. *J. W. Moffat* (Baltimore, Md.)

Jouvet, B. 5740

Anomalous properties of the vector fields in interaction with Dirac's current. (Italian summary)

Nuovo Cimento (10) **26** (1962), 283–297.

Author's summary: "We have studied the properties of the solutions of the equations of a vector field (of arbitrary mass) in interaction with Dirac's current, fulfilling successively the postulates of covariance, causality, and canonical quantization. We have found that such solutions may have various forms, all of which violate, as stressed by Schwinger [Phys. Rev. Lett. **3** (1959), 296–297], the postulates either of unitarity, or of vacuum as a minimal energy state, but not necessarily for observable states. Indeed the solutions contain contributions, unexpected from the spectral postulate, associated to states $|h_0(\mathbf{p})\rangle$ of zero mass and zero spin and unobservable. If the observable states have a positive norm: (a) the norms of the states $|h_0(\mathbf{p})\rangle$ are negative; (b) the commutation relations are only compatible if the renormalization constant Z_1 is zero; (c) the unrenormalized polarization tensor should have the gauge-invariant form

$$\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2 - i\varepsilon}\right) F(p^2), \qquad F(0) \neq 0,$$

which is quite different from the usual perturbation expression. If one omits the contributions of the states $|h_0(\mathbf{p})\rangle$, the equation which determines their norm becomes the new definition, recently proposed [Johnson, Nuclear Phys. **25** (1961), 431–434, p. 433; ibid. **31** (1962), 464–470; MR **25** #2821] for the bare mass of the vector boson, and ghosts should appear in observable states."

Matthews, P. T.; Salam, A. 5741

A remarkable representation of the weak and electromagnetic Lagrangian.

Phys. Lett. **8** (1964), 357–358.

Cabibbo has suggested that all weak interactions of strongly interacting particles are mediated by a linear coupling of the weak vector boson W to the strong vector bosons ρ and K^*, the relative strengths of the two interactions being specified through a (ρ, K^*) mixing angle. The authors of the present note point out that the

phenomenological magnitude of this angle is very close to arc sin$\{(\sqrt{3}-1)/(2\sqrt{2})\}$, and most ingeniously relate this curious value to certain properties which apply to the electromagnetic current in a unitary-symmetric gauge theory of strong interactions. In brief, they are led to consider a novel but simple recipe, motivated at least in part by unitary symmetry, by which the charged ρ and K^* mesons are assigned respectively to two charge triplets of unfamiliar appearance. The neutral member of either triplet is in each case a well-defined linear combination of ρ^0 and ϕ-mesons. When the two triplets are taken in linear combination with the above mixing angle, the charged components represent weak interaction currents, while the neutral component corresponds precisely to the electromagnetic current. *G. R. Allcock* (Liverpool)

Stewart, A. L. 5742
On the $Z^{-1/2}$ expansion of unrestricted Hartree-Fock wave functions.
Proc. Phys. Soc. **83** (1964), 1033–1037.
Author's summary: "It is pointed out that the perturbation expansions of unrestricted Hartree-Fock orbitals, based on zero-order uncorrelated orbitals, contain terms of order $Z^{-1/2}$ which do not appear in the similar expansions of Hartree-Fock orbitals. It is shown that these terms give about 90% of the total radial correlation in the second-order energy of the ^1S ground state of the two-electron systems."

Schild, A. 5743
Electromagnetic two-body problem.
Phys. Rev. (2) **131** (1963), 2762–2766.
An unpublished observation by Corben, using Bohr quantisation for a point charge moving relativistically in the field of a magnetic dipole, produces orbital radii of sub-nuclear dimensions. Could, therefore, some elementary particles be composite with an interaction between constituents depending essentially on magnetic moments? A fully relativistic simple case is explored.
C. Strachan (Aberdeen)

Perelomov, A. M. 5744
Possible determination of additional characteristics of an unstable particle.
Dokl. Akad. Nauk SSSR **146** (1962), 75–78 (*Russian*); *translated as Soviet Physics Dokl.* **7** (1963), 809–812.
Neutral stable particles scattered in an electric field through an intermediate state which interacts with virtual charged particles have a rotation of spin which is zero on a time-average but the first moment with respect to time of the rotation is not zero. *C. Strachan* (Aberdeen)

de Swart, J. J. 5745
The crossing matrix in the octet model. (Italian summary)
Nuovo Cimento (10) **31** (1964), 420–426.
Author's summary: "A general expression is given for the elements of the crossing matrix in the octet model in terms of the Clebsch-Gordan coefficients for SU(3). Several useful relations about the crossing matrices are derived. For two specific cases the crossing matrix is given."
J. Franklin (Livermore, Calif.)

Trueman, T. L.; Wick, G. C. 5746
Crossing relations for helicity amplitudes.
Ann. Physics **26** (1964), 322–335.
Authors' summary: "Crossing relations for helicity amplitudes for particles of arbitrary spin are formulated without recourse to the introduction of scalar amplitudes. The basic assumption is that the amplitudes are simply related by analytic continuation; the path of continuation is carefully specified. The relations are given a simple geometrical interpretation. The relation between $\pi N\to\pi N$ and $\pi\pi\to N\bar{N}$ obtained in this way agrees with that obtained by direct elimination of scalar amplitudes."
J. Franklin (Livermore, Calif.)

Secrest, Don; Holm, Lloyd Martin 5747
Eigenfunctions of the electron spin operators.
J. Mathematical Phys. **5** (1964), 738–742.
Authors' summary: "The problem of finding linear combinations of Slater determinants which are eigenfunctions of both the S^2 and S_z operators is considered. The method of projection operators is used in forming eigenfunctions of S^2. A recursion relation is given for the coefficients in the linear combination. Rules are given for selecting a complete linearly independent set of eigenfunctions and an explicit formula is given for the members of the set." *H. L. Frisch* (Murray Hill, N.J.)

Królikowski, W. 5748
Baryons and leptons and group SU_4.
Nuclear Phys. **52** (1964), 342–344.
The author suggests using SU(4) to describe the baryon-lepton system. The lepton number is identified with the third diagonal operator; eight operators correspond to the usual SU(3) in the octet model. Leptons (e^-, ν_{eL}, ν_{eR}) and (μ^+, $\nu_{\mu L}$, $\nu_{\mu R}$) are assigned as an SU(3) triplet-antitriplet set to the adjoint **15**, with the baryon spin ½ octet and Y_0 (1405). It seems to this reviewer that the inclusion of leptons in a symmetry scheme should describe properly their interactions with the baryons, i.e., the weak interaction. The present scheme does not lend itself to a natural description of weak lepton currents since the μ-ν current would get a wrong chirality operator. It would be worth checking whether this can be corrected in this scheme. It is also not clear whether the author has studied the implications of the Y_0 (1405) odd parity if the present experimental situation is validated.
Y. Ne'eman (Pasadena, Calif.)

Musha, Toshimitsu 5749
Amplification of waves due to quanta with negative energy.
J. Appl. Phys. **35** (1964), 137–141.
It has been pointed out that the phenomenological treatment of the quantum theory of radiation in a dispersive medium may lead to states of photons with negative energy [J. M. Jauch and K. M. Watson, Phys. Rev. (2) **74** (1948), 950–957; MR **10**, 346].

In this paper it is shown that there exist several other systems which exhibit similar properties, for instance, travelling-wave-type amplifiers, resistive-wall amplifiers, amplifiers of ultrasound in semiconductors and masers. In the classical description such systems are characterized by

instabilities which lead to amplification of waves through mode coupling. In the quantum description these systems are characterized by spontaneous pair creation processes where one pair has a positive and the other a negative energy. *J. M. Jauch* (Geneva)

Lieb, Elliott H.; Liniger, Werner 5750
Simplified approach to the ground-state energy of an imperfect Bose gas. III. Application to the one-dimensional model.
Phys. Rev. (2) **134** (1964), A312-A315.
The study is continued of an integro-differential equation proposed by Lieb [same Rev. (2) **130** (1963), 2518-2528] which permits one to evaluate the ground-state energy of an imperfect Bose gas. The method is applied to the one-dimensional delta-function gas, where the exact result is known for all values of the coupling constant γ. For small γ the equation gives the correct first two terms in an asymptotic series.

The equation is also solved numerically. This solution shows that the maximum relative error occurs for $\gamma=\infty$ (19%). For very strong coupling ($\gamma=\infty$) the exact solution may be compared with the numerical solution given by the integral equation. The agreement is satisfactory.
E. J. Verboven (Nijmegen)

Ruamps, Jean 5751
Calcul des intégrales de recouvrement des fonctions d'onde de deux oscillateurs harmoniques différents. Application aux intensités dans les spectres des molécules diatomiques. I. (English summary)
J. Phys. Radium **22** (1961), 759-763.
The overlapping integrals between the wave functions of two harmonic oscillators, out of phase, and for the cases of equal and different frequencies, have been calculated in terms of the Laguerre polynomials.
S. Olszewski (Warsaw)

Bolotin, A. B.; Šugurov, V. K. [Šugurovas, V.] 5752
Transformation of many-center integrals to one center. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 560–564.
The paper deals with a very important problem of theoretical molecular physics: how to calculate the many-center integrals which appear in the LCAO method when the interaction between the electrons of the molecule is included. The problem seems to be not solved satisfactorily up till now, although several serious attempts have been made [see, in particular, M. P. Barnett and C. A. Coulson, Philos. Trans. Roy. Soc. London Ser. A **243** (1951), 221–249; MR **12**, 702].

The authors present the method of carrying the many-center integral to that of one center by applying the Fourier transformation of atomic orbitals. Then the dependence of atomic orbital on the position vector $\mathbf{r}_u$ centered at the atom u is enclosed in the term $e^{i\mathbf{k}\mathbf{r}_u}$, and this term can be easily transformed into the dependence on the position vectors centered in a point which is chosen as the center of the whole molecule. The procedure can be repeated with each atomic orbital under the integral.

As a result, one obtains the many-center integral as the one-center integral over the positions of both electrons, where the expression under the integral is equal to the series of products composed of two types of factors: one type consists of the products of spherical harmonics relating to the directions by which the atom centers are seen from the molecular center. The second type of factors are polynomials expressed in terms of distances from the molecular center multiplied by infinite integrals over products of hypergeometrical functions.

Each term of the series under the integral can be easily computed, but the convergence of the result is not investigated in the paper. The numerical applications of the method are promised by the authors.
S. Olszewski (Warsaw)

Didry, Jean-René; Cabaret, Françoise; Guy, Jean 5753
Considérations topologiques sur les contributions des orbitales moléculaires localisées aux constantes d'écran magnétique. (English summary)
J. Phys. Radium **23** (1962), 65–72.
The paper presents the calculus of the magnetic screening constant for the hydrogen molecule as the function of the location in space. It is pointed out generally that if the eigenfunction in the ground state of the molecule equals a simple product of molecular orbitals, then the magnetic screening constant is a sum of contributions coming separately from each orbital. These contributions can be calculated following the lines established in the previous papers of the authors and their associates. In particular, the authors compare the results coming from two approximate equations for orbital components of the magnetic screening constant. One of the equations represents the average magnetic screening constant of the orbital as equal to one third of the trace of the corresponding magnetic screening tensor; the parameters entering in tensor components are determined in terms of those which appear in the molecular wave function and are given by a variational procedure. The other equation—valid only at large distances from the molecular center—determines the orbital magnetic screening constant in terms of anisotropy of the magnetic susceptibility tensor.
S. Olszewski (Warsaw)

Capel, H. W. 5754
The hole-equivalence principle.
Nederl. Akad. Wetensch. Proc. Ser. B **67** (1964), 80–97.
A theory is established which gives a relation between an electron configuration with n electrons in a shell and the corresponding one with n electrons missing from a closed shell. It is independent of the particular transformation properties of the one-electron states used in the basis. Correspondences are also derived for the matrices of one- and two-electron operators in the configurations.
D. F. Mayers (Oxford)

Capel, H. W. 5755
The Van Vleck relation.
Nederl. Akad. Wetensch. Proc. Ser. B **67** (1964), 98–107.
Author's summary: "A general equivalence is established between the high-spin states (maximal S) of the configurations with n and $n+p$ electrons of an electron shell

in which $2p$ electrons can be placed. The proof is independent of the transformation properties of the one-electron wave functions which determine the electron shell. Together with the hole-equivalence principle this provides a simple calculation scheme for high-spin states."
D. F. Mayers (Oxford)

Wong, K. W. 5756
Application of nonlocal field operators to a system of hard-sphere Bose gas.
J. Mathematical Phys. **5** (1964), 637–642.
Um ein Bose-Gas, welches aus harten Kugeln mit endlichem Radius besteht, mit dem Formalismus der zweiten Quantelung behandeln zu können, muss man nichtlokale Feldoperatoren einführen. Verfasser diskutiert den Zusammenhang dieser nichtlokalen mit normalen, lokalen Feldoperatoren. Die Wechselwirkung kann auch durch gewisse Pseudopotentiale ausgedrückt werden, wobei aber nicht nur zwei, sondern auch Mehrkörper-Pseudo-Potentiale auftreten. Verfasser untersucht schliesslich auch die Gestalt der dynamischen Gleichungen des superfluiden Zustands eines solchen Systems für sehr tiefe Temperaturen. Es ergibt sich u.a. eine gewisse Modifikation der Londonschen Gleichungen für die Suprafluidität.
G. Heber (Leipzig)

Maki, Kazumi 5757
Effect of magnetic fields on heat transport in superconductors.
Progr. Theoret. Phys. **31** (1964), 378–387.
Author's summary: "The thermal conductivity of a superconducting thin film in magnetic fields is calculated by the use of the thermal Green's functions. It is shown that the anisotropic terms due to the induced flow vanish of the order of l/ξ_0 when l is short, where l is the electronic mean free path and ξ_0 the coherence length of the electron pair."

STATISTICAL PHYSICS, STRUCTURE OF MATTER

Grant, W. J. C. 5758
General theory of cross relaxation. I. Fundamental considerations.
Phys. Rev. (2) **134** (1964), A1554–A1564.
Author's summary: "A statistical analysis is made of spin transitions induced by dipole interactions which change the total magnetization while exactly conserving energy. The first-order effect of the dipole operator can be described by a function $\Phi(\omega)$, which is related to the level broadening observed in resonance lines. The second-order effect leads to a function $\chi(\omega)$ which represents the power spectrum of the dipole operator. The cross-relaxation probability $W_{CR}(\omega)$ is given by the convolution of these two functions. W_{CR} is calculated explicitly in various approximations, without appeal to moments. For single-spin flips in magnetically dilute systems, the magnitude of W_{CR} depends linearly on the concentration n. There is a very sharp peak at $\omega=0$ with a width proportional to the geometric mean of the resonance width and of the nearest-neighbor dipole energy."

Grant, W. J. C. 5759
General theory of cross relaxation. II. Higher order processes.
Phys. Rev. (2) **134** (1964), A1565–A1573.
Author's summary: "The theory of Part I [#5758] is extended to multiple spin processes. The lowest order two-spin process yields a cross-relaxation probability W_{CR} whose principal characteristics are the same as those of W_{CR} for single spin flips. Higher order two-spin processes occur with much lower probability, and this probability is not sharply peaked at harmonic coincidence. Three-spin and higher multiple K-spin processes yield functions $W_{CR}(\omega)$ whose width becomes progressively larger and less dependent on the concentration, and whose magnitude depends on the concentration n as n^K. The effects of short-range forces (exchange) are easily embodied in the theory, but in the absence of such forces special attention must be paid to the dipole fields of near neighbors, which are then likely to dominate the cross-relaxation process."

Grant, W. J. C. 5760
General theory of cross relaxation. III. Application to experiment.
Phys. Rev. (2) **134** (1964), A1574–A1581.
Author's summary: "The theory developed in Parts I [#5758] and II [#5759] is applied to the classical experiments of Mims and McGee, and Pershan. The predictions of the theory regarding the dependence of W_{CR} on the energy imbalance $\hbar\omega$ on concentration and on the exchange radius, as well as the prediction of the actual magnitude of W_{CR}, are confirmed in the Mims and McGee experiment on ruby. The crucial significance of near-neighbor dipole interactions, in particular, the effect of the associated power spectrum, which is quite broad and quite sensitive to crystal direction, is illustrated by application of the theory to Pershan's experiments on LiF."

Andersen, H. C.; Oppenheim, I.; Shuler, Kurt E.; Weiss, George H. 5761
Exact conditions for the preservation of a canonical distribution in Markovian relaxation processes.
J. Mathematical Phys. **5** (1964), 522–536.
Authors' summary: "Necessary and sufficient conditions have been determined for the exact preservation of a canonical distribution characterized by a time-dependent temperature (canonical invariance) in Markovian relaxation processes governed by a master equation. These conditions, while physically realizable, are quite restrictive so that canonical invariance is the exception rather than the rule. For processes with a continuous energy variable, canonical invariance requires that the integral master equation be exactly equivalent to a Fokker-Planck equation with linear transition moments of a special form. For processes with a discrete energy variable, canonical invariance requires, in addition to a special form of the level degeneracy, equal spacing of the energy levels and transitions between nearest-neighbor levels only. Physically, these conditions imply that canonical invariance is maintained only for weak interactions of a special type between the relaxing subsystem and the reservoir. It is also shown that canonical invariance is a sufficient condition for the exponential relaxation of the mean energy. A number of systems (hard-sphere Rayleigh gas,

Brownian motion, harmonic oscillators, nuclear spins) are discussed in the framework of the above theory. Conditions for approximate canonical invariance valid up to a certain order in the energy are also developed and then applied to nuclear spins in a magnetic field."

E. J. Verboven (Nijmegen)

Aleksandrov, I. B.; Kuharenko, Ju. A.; Niukkanen, A. V. 5762

Two-time one-particle Green's functions for a non-ideal Fermi system. (Russian)

Vestnik Moskov. Univ. Ser. III Fiz. Astronom. **1964**, *no.* 2, 43–51.

Owing to the damping of the elementary excitation it is difficult to obtain a kinetic equation on the basis of an approximate shredding of the set of equations for the one-time correlation functions. In this paper, the authors investigate the derivation of the equations for the two-time correlation functions and the one-particle Green's functions for a weak-nonideal Fermi system; they suppose that the binary interaction potential energy is small in comparison with the average kinetic energy. By means of the introduced mass operator, a Dyson-type equation in the quadratic approximation is obtained by using the small parameter series of the perturbation theory. Further, the authors use the condition of correlation attenuation for the space-distant parts of the system. The expression for the mass operator in the second approximation is found. By means of this expression the energy and the damping of the elementary excitation are calculated.

M. Blažek (Bratislava)

Iwata, Giiti 5763

A method for evaluating a class of partition functions. I.

Progr. Theoret. Phys. **31** (1964), 67–82.

The classical partition function of a system of N particles interacting through a pair-potential is approximately evaluated by a method which is a modification of the "spherical model", introduced by Berlin and Kac. The method consists of (1) expressing the partition function as an integral of the product of a Boltzmann factor and a weighting factor over the space of all variables, chosen in such a way that the energy becomes a quadratic form; (2) dividing the whole space of variables into cells by quadratic and linear surfaces so selected that the Boltzmann factor slowly varies in each cell; (3) replacing the weighting factor by its mean value in each cell. General expressions are derived for the equation of state, points of phase transition and the rule of equal areas. The result is compared to that of the virial expansion. As an example, the equation of state of a system of particles interacting through a repulsive Yukawa potential is calculated. Further applications are announced for succeeding papers.

K. Schram (Utrecht)

Northcote, Robert S.; Potts, Renfrey B. 5764

Energy sharing and equilibrium for nonlinear systems.

J. Mathematical Phys. **5** (1964), 383–398.

The authors attack the controversial item of the equipartition of energy among the linear normal modes in a one-dimensional, nonlinear system. In particular, considerable interest has been shown in that respect after the results obtained by Fermi, Pasta and Ulam [Los Alamos Sci. Lab. Rep. LA-1940 (1955)] who carried out numerical computations and found very little tendency towards equipartition. They considered an assembly of point particles coupled by forces which were linear except for small nonlinear terms. The authors of the present paper consider a one-dimensional assembly of particles in which nonlinearity is introduced by assuming that collisions occur between adjacent particles. The advantages of the model are that it has physical realism, nonlinear effects can be made small or large merely by altering the mean energy per particle of the system and the assembly behaves linearly between collisions. In the first part of the work the authors define the mathematical model they deal with, and next propose a mathematical solution of it. The main equation is in the matrix form. The numerical method of computation is of an iteration character, with the coefficients derived from the Chebyshev series expansions for the trigonometric functions. Numerical computations have been carried out on the IBM 1620 and 7090. In the second part the authors develop the statistical mechanics of the model with the resulting equation similar to that derived previously by D. Koppel [Phys. Fluids **6** (1963), 609–616]. In the next section there are computed the thermodynamic variables, i.e., the mean temperature and pressure. In the conclusion the authors state that, in contrast to the systems studied by Fermi, Pasta and Ulam, their nonlinear systems show a behavior which can reasonably be described as ergodic. This is evidenced both by the equipartition of the energy of each system among all the modes, in the time average, and by the rapid approach of the temperature and pressure to their equilibrium values. But, of course, a question may be left in the mind of a reader (remark of the reviewer) how much of the controversy is due to formal differences in the numerical schemes (approximations of differential equations by difference ones) used by various authors.

M. Z. v. Krzywoblocki (E. Lansing, Mich.)

Watabe, Mitsuo 5765

A note on the thermodynamic potential of an interacting electron-phonon system.

Progr. Theoret. Phys. **31** (1964), 326–328.

In this note the author gives, by using the "diagram technique", an expression for the thermodynamic potential of a coupled electron-phonon system in terms of the electron and phonon Green's functions and the polarisation part of the phonon Green's function and self-energy part of the electron Green's function. This enables one to discuss the low-temperature properties of a system of electrons interacting with phonons. A similar expression for a many-fermion system is also derived.

F. C. Auluck (Delhi)

Saltanov, N. V.; Tkalič, V. S. 5766

On a non-stationary magneto-gasdynamical problem. The analogue of the Riemann wave. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 529–532.

From the relativistic magneto-hydrodynamic equations, under the assumption that $\partial/\partial x_2 = \partial/\partial x_3 = 0$, two scalar equations are deduced for determining the total pressure

and the first velocity component. After several transformations, the Riemann type waves are set into evidence.

The gas is considered perfectly conducting and the constitutive equations of the medium are not taken under the covariant Einstein-Minkowski form.

L. Dragos (Bucharest)

Dreicer, H. 5767

Kinetic theory of an electron-photon gas.

Phys. Fluids **7** (1964), 735–753.

In this paper direct electron-photon collisions by spontaneous and induced emission, induced absorption and Compton scattering are accounted for by additional collision terms in the electron and photon kinetic equations. The collisions are characterized by three transition probabilities and a cross-section and are assumed to be two-body processes. The electrons are not allowed to change position during the collision but may experience forces which alter the various collision probabilities. Collective effects are not treated. The low frequency (Fokker-Planck) limit of the electron collision terms is given for the cases of cyclotron radiation and free Compton scattering. Finally, the results are applied to three problems—relaxation of electron in a strong magnetic field and a radiant heat bath, the saturation of a cyclotron maser and the enhanced Compton scattering of electrons in a radiation field excited in two modes.

In addition to collision terms of the familiar quadratic type, this treatment also yields linear and cubic terms (from spontaneous emission and induced Compton scattering, respectively) which have no analogue in particle-particle scattering theory. All collision terms vanish when the electrons and photons are in thermodynamic equilibrium.

This theory is necessarily incomplete, since collective effects are not dealt with, and a precise delimitation of its range of validity must await a more exact theory. But the theory is also incomplete within itself, since it does not specify which forces are assumed to be acting on the electron when collision probabilities are being computed. Presumably forces due to the averaged fields should be used—they are the simplest to treat and are often dominant—but other choices might be more suitable. Fortunately, this ambiguity is not important for the examples treated in the paper.

R. C. Mjolsness (Los Alamos, N.M.)

Bassichis, W. H. 5768

Generalization of the Boguliubov method applied to mixtures of Bose-Einstein particles.

Phys. Rev. (2) **134** (1964), A543–A549.

The ground state energy and quasi-particle excitation spectrum of a mixture of several different kinds of bosons are discussed using the method developed by Bogoliubov for a one-component system, i.e., the zero momentum state creation and annihilation operators for each type of boson are replaced in the Hamiltonian by c-numbers, and then only terms quadratic in the field operators are retained. The resulting Hamiltonian is explicitly diagonalized in the case of a mixture of two types of oppositely charged spin-zero bosons, interacting only through Coulomb forces. In this case the excitation spectrum has one branch which is quadratic for small momentum, and a second branch which approaches the plasma frequency (calculated from the density of particles in the zero momentum states) as $k \to 0$.

N. D. Mermin (La Jolla, Calif.)

Chernikov, N. A. [Černikov, N. A.] 5769

The relativistic gas in the gravitational field.

Acta Phys. Polon. **23** (1963), 629–645.

This is a comprehensive and detailed review of work of the author, previously published in a number of shorter papers [Dokl. Akad. Nauk SSSR **144** (1962), 89–92; MR **26** #4744; ibid. **144** (1962), 314–317; MR **26** #4745; ibid. **144** (1962), 544–547; MR **26** #4746]. The general and covariant theory of the Liouville equation in an arbitrary gravitational field is discussed by introducing the invariant seven-dimensional volume element in ordinary space-time and on the mass shell. The Boltzmann collision integral is derived in its relativistic form and its main properties deduced; in particular, the Maxwell-Enskog equations for the momenta and the H-theorem are proved. The principle of detailed balancing for a relativistic gas is also considered. The range of the intermolecular forces is assumed to be small with respect to the radius of curvature, so that collisions are not affected by gravitation and are dealt with according to special relativity.

B. Bertotti (Frascati)

Infeld, E. 5770

On the solutions of linearized equations of magneto-hydrodynamics in non homogeneous magnetic fields.

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **11** (1963), 707–713.

A set of sufficient conditions is given for the solutions to the linearized magneto-hydrodynamic equations to be exact for the nonlinear equations. Some specific examples are given. *D. C. Montgomery* (College Park, Md.)

Silin, V. P. 5771

Contribution to the theory of oscillations of a weakly inhomogeneous plasma.

Ž. Èksper. Teoret. Fiz. **45** (1963), 1060–1066 (*Russian. English summary*); *translated as Soviet Physics JETP* **18** (1964), 733–737.

Author's summary: "The characteristic oscillations of a plasma which is weakly inhomogeneous in three dimensions are studied in the geometric optics approximation. The corresponding dispersion relations are derived from multi-dimensional quantization rules. High-frequency longitudinal and transverse oscillations of an electron plasma, low-frequency ion-acoustic oscillations of a quasi-isotropic electron-ion plasma, and low-frequency oscillations of a cold magneto-active plasma are considered by applying the dispersion relations."

N. D. Mermin (La Jolla, Calif.)

Timofeev, A. V. 5772

Convection of weakly ionized plasma in a nonuniform external magnetic field.

Ž. Tehn. Fiz. **33** (1963), 776–781 (*Russian*); *translated as Soviet Physics Tech. Phys.* **8** (1964), 586–589.

The known results for the convection instability of a fully ionized plasma in a non-uniform magnetic field are

extended to the case of a weakly ionized plasma in a strong external magnetic field. The plasma is assumed to occupy the space between two coaxial cylinders with a gap width d which is small compared to R, the radius of the inner cylinder. Thermal ionization is assumed to occur on the inner cylinder and ions and electrons diffuse ambipolarly through the neutral gas to the outer dielectric cylinder. The system is axially symmetric with an azimuthal magnetic field decreasing as $1/r$. The density of charged particles is small. The neutral component is assumed stationary and the densities of ions and electrons are taken equal. The equations of continuity and equations of motion are established in the limit of large magnetic field and/or large charge particle neutral collision time. The effects of curvature are neglected except for the terms which lead to the axial density gradient as a consequence of the non-uniform azimuthal magnetic field. The density of charged particles in the inner boundary is assumed known and the potential is assumed to be zero there. The charged particle density is taken as zero on the outer boundary and charged particle flux is equal on the outer boundary. A steady state solution is established and the stability of the system is investigated for density and potential perturbations of the form $f(x)\exp(-iwt+ikz)$.

Stability is investigated for the first harmonic. An approximation technique is used in which $f(x)$ is replaced by $p \sin kx$. It is shown that the critical magnetic field for instability of the first harmonic is proportional to $(R/d)^{1/2}$. The critical field for the second harmonic is twice as large.

A quasi-linear analysis which takes into account quadratic terms for the small amplitudes of oscillation is used to analyze the laminar convection which occurs when the magnetic field is greater than the critical field strength. The total change in ambipolar mass flux is estimated. It is shown that the additional flux resulting from convection may become equal to the diffusional flux.

R. L. Peskin (New Brunswick, N.J.)

Jackson, J. L.; Klein, L. S. 5773

Continuum theory of a plasma.

Phys. Fluids 7 (1964), 232–241.

In a companion paper [Phys. Fluids 7 (1964), 228–231] the authors show how the chemical potential and the two-particle distribution function of the classical Debye-Hückel theory are related to the probability distribution function of the micropotential at a typical point in a fluid at equilibrium. In the present paper this distribution function is calculated for a classical plasma by considering it to be a functional of the net charge density in the plasma. This is done both for a typical point and for the joint distribution of the potential at two points. A first-order correction to the potential distribution of the continuum theory is obtained by treating the nearest-neighbor charge as a discrete charge and the remainder of the plasma as a continuum. The resulting correction to the free energy resembles the results of diagrammatic calculations, the existing discrepancy between the two results being ascribed to the inadequacy of the nearest-neighbor approximation. *L. B. Wetzel* (Providence, R.I.)

Nekrasov, F. M. 5774

A class of nonlinear solutions of the collisionless kinetic equation.

Ž. Tehn. Fiz. **33** (1963), 769–775 (*Russian*); *translated as Soviet Physics Tech. Phys.* 8 (1964), 581–585.

A class of solutions to the collisionless kinetic equation (Boltzmann-Vlasov equation) is considered. In particular, a one-dimensional, steady-state, non-linear motion of the plasma in a self-consistent field is studied. The space-time dependence of the variables is written $F(r,t)=F$ $(\zeta\equiv z-v_\phi t)$, where v_ϕ is the assumed constant phase velocity of the propagating wave. The distribution function is assumed to take the form of the product of a function of the peculiar velocity in the direction of propagation times, a function of the remaining velocities normal to the direction of propagation, and the space-time variable. This implies that only those distribution functions which are Maxwellian in the direction of propagation are considered. The temperature and the directed velocity along the axis of propagation are constant. Magnetic field strength in the direction of propagation is also taken constant. Several cases are considered. In the first case both the magnetic field and the directed velocity along the propagation axis are assumed different from zero. In this case the electric field along the axis of propagation vanishes and the solutions are transverse waves with constant phase velocity propagating with respect to the plasma along the constant magnetic field. These solutions represent a particular case of Alfvén waves. For the case of a plasma whose mean velocity along the axis of propagation vanishes and which exhibits approximately equal temperatures parallel and normal to the propagation direction, the relation between the phase velocity, the fixed field and the plasma parameters reduces to the familiar Alfvén phase velocity equation.

In the case that the directed velocity and the magnetic field in the direction of propagation both vanish, the electro-magnetic field moves with the plasma and a solution which is stationary in the coordinate system moving with the plasma must be obtained. Three non-linear differential equations of second order for the electro-magnetic potential are integrated using certain simplifying assumptions. The number density of ions and electrons is assumed equal. The temperatures in the y and z directions are assumed equal for ions and electrons, respectively, and the x and y components of the mean velocity at the point $z=0$ are assumed to vanish (z is the propagation axis). When the electron temperature in the x direction is less than the electron temperature in the z direction, the results show that the plasma is concentrated in a layer of finite thickness. The magnetic field far from the layer is essentially constant.

The case when the magnetic field in the direction of propagation vanishes and with the velocity of propagation different from zero is trivial. With the magnetic field not zero and the directed velocity zero one finds the magnetic field in the x and y direction vanishes and the electric field to be the Debye shielding field for the one-dimensional geometry.

The assumption of Maxwellian distribution along the z-axis leads to two types of solutions in this paper. The first of these corresponds to transverse waves with constant phase velocity. The second type of solution represents single pulses propagating with a fixed phase velocity equal to the velocity of the plasma. For this type of solution the magnetic field along the direction of the propagation vanishes. The second type of solution may or

may not have an electric field in the direction of propagation and this paper presents two particular solutions without the longitudinal electric field.

R. L. Peskin (New Brunswick, N.J.)

Butcher, P. N.; McLean, T. P. 5775

The non-linear constitutive relation in solids at optical frequencies.

Proc. Phys. Soc. **81** (1963), 219–232.

Authors' summary: "The constitutive relation between the current density and the electric field in a solid is extended to include terms in all powers of the field. By calculating the density matrix of the charged particles in the solid, a general expression is obtained for the nth-order conductivity tensor which connects the current density with the nth power of the field. The second- and third-order tensors produce the most significant non-linear effects for the radiation fields currently available. They are discussed in more detail and the restrictions placed on them by the symmetry requirements of the thirty-two crystal classes are evaluated. The electronic contribution to the conductivity tensors is treated in the one-electron approximation; for low frequencies and odd values of n, the nth-order conductivity tensor may be expressed in terms of the $(n+1)$st derivative of the one-electron energy with respect to the wave vector."

Callaway, Joseph 5776

★Energy band theory.

Pure and Applied Physics, Vol. 16.

Academic Press, New York-London, 1964. x + 357 *pp.* $10.00.

The author's book is a monograph on the band theory of solids. It is written from an advanced point of view; no subject is avoided or diluted because of mathematical difficulties. The subject is developed logically and systematically from first principles.

Remarkable in this book is the elegance and clarity of the discussion. A reader previously unacquainted with many topics discussed in the book can read it without difficulty, provided he acquires the necessary mathematical background, particularly a thorough knowledge of group theory. Although many standard topics are covered, a number of subjects are discussed that have not previously appeared in book form. They are nevertheless related to, and incorporated in, the general theory.

The book contains five well-chosen and instructive appendices on "Symmetrized linear combinations of plane waves", "Summation relations" (closure relations involving summations over the direct and the reciprocal lattices), "The effective mass equation in the many-body problem", "Evaluation of tunneling integral", and "Spin density waves". Band structures of actual materials are presented in one chapter: the alkalis, the noble metals, the valence crystals, bismuth, graphite, and the transition metals. This is a subject to which the author himself has made extensive original contributions.

The bibliography is unusually extensive and carefully selected. Altogether, the book can be regarded as a future classic in its field, combining as it does clear and logical organization, and an elegant and readable style with a high degree of technical competence.

W. Franzen (Boston, Mass.)

Ribarič, M. 5777

The relation between the reflection properties of the body and the reflection properties of its parts. II.

Arch. Rational Mech. Anal. **16** (1964), 196–213.

The content of this paper is supplementary and related to that of two previous papers [same Arch. **8** (1961), 381–407; MR **24** #B1967; ibid. **15** (1964), 54–68; MR **28** #961]. There, an attempt is made to present a general function-analytical treatment for a global reactor theory. Here, in several small chapters, it is shown that a number of assumptions made in the previous papers are not necessary.

E. H. Bareiss (Argonne, Ill.)

Mullikin, T. W. 5778

Nonlinear integral equations of radiative transfer.

Nonlinear Integral Equations (Proc. Advanced Seminar Conducted by Math. Research Center, U.S. Army, Univ. Wisconsin, Madison, Wis., 1963), *pp.* 345–374. *Univ. Wisconsin Press, Madison, Wis.,* 1964.

This is an excellent review of results concerning the Ambarzumian-Chandrasekhar approach to radiative transfer problems. The author starts with the standard time-independent transport equation, assuming slab geometry and isotropic scattering, and obtains a variety of integral equations and integro-differential equations satisfied by the functions now classically denoted by S, T, X, and Y. There follows an interesting discussion of the lack of uniqueness of the solutions to these non-linear equations, a matter frequently overlooked when calculations are being made. Side conditions are derived which specify the physically pertinent solutions. It is shown that even the numerical integration of the integro-differential equation for S and T may produce unexpected difficulties. Though the solutions are unique, a peculiar type of instability exists. The author also gives a complete solution of the X, Y problem in terms of the solutions of relatively "trouble-free" linear Fredholm equations.

Proofs are only sketched, and the reader is referred to a rather extensive bibliography, including many of the author's own papers on this important class of problem.

G. M. Wing (Albuquerque, N.M.)

Bellman, R. E.; Kagiwada, H. H.; Kalaba, R. E.; Prestrud, M. C. 5779

★Invariant imbedding and time-dependent transport processes.

Modern Analytic and Computational Methods in Science and Mathematics, Vol. 2.

American Elsevier Publishing Co., Inc., New York, 1964. x + 256 *pp.* (4 *inserts*) $7.50.

In the first book of this series the problem of a time-independent plane parallel flux of particles impinging at a specified angle on a finite slab, assuming absorption and isotropic scattering, was investigated [R. E. Kalaba, R. E. Bellman, M. C. Prestrud, *Invariant imbedding and radiative transfer in slabs of finite thickness*, RAND Corp., Santa Monica, Calif., 1962; MR **26** #1171]. In the present volume the steady beam is assumed to impinge at $t=0$ and time-dependence of the reflected flux is accounted for.

The authors reduce the invariant imbedding equations for the reflected flux to the time-independent case by means of the Laplace transform, then apply a scheme for the numerical inversion of this transform to obtain the

desired information. The first part of the book contains a description of the imbedding method and develops the inversion scheme in some detail. Tables of the reflection function for various times, angles, etc., occupy the next hundred or so pages. Results agree for large t with those obtained in the first volume. Details of the FORTRAN program, etc., are also given.

G. M. Wing (Albuquerque, N.M.)

Valat, Jean 5780

Sur différentes méthodes de corrélation. Application à la mesure des fortes antiréactivités d'un réacteur nucléaire sous-critique.

C. R. Acad. Sci. Paris **258** (1964), 1704–1707.

Author's summary: "Pour une excitation donnée, l'intercorrélation entre l'entrée excitatrice et la sortie résultante s'avère meilleure que l'autocorrélation de la sortie dans le cas d'un bruit de fond interne important. Par contre, l'autocorrélation de l'intercorrélation permet d'améliorer encore la précision statistique, mais aux dépens de la simplicité de l'extraction de l'information désirée."

Jannussis, A. 5781

Doppler-Verbreiterung. (English summary)

Bull. Soc. Math. Grèce (N.S.) **4** (1963), *no.* 1, 127–131.

Author's summary: "Es werden explizit die Funktionen $\psi(x, \Theta)$ und $\chi(x, \Theta)$, die bei der Berechnung der Doppler-Verbreiterung auftreten, nach steigenden Potenzen von Θ und nach fallenden Potenzen von $1/(1+x^2)$ entwickelt. Hieraus lassen sich leicht die üblichen asymptotischen Lösungen herleiten."

Mori, H. 5782

On the correlation-function theory of transport in non-uniform systems.

Phys. Lett. **9** (1964), 136–137.

If a set of collective variables are properly chosen for a macroscopic system, it is possible to project the dynamical motion of the system onto the motion of such variables. The author derives here such a formal equation of motion and obtains the linear relation between the fluxes and the forces. He hopes in this way to establish the correlation function formulae for non-uniform thermal disturbances.

R. Kubo (Tokyo)

Pomraning, G. C. 5783

Variational boundary conditions for the spherical harmonics approximation to the neutron transport equation.

Ann. Physics **27** (1964), 193–215.

The mono-energetic neutron transport equation has been shown to be characterized by the Lagrangian and the associated spherical harmonics or P-N approximation can in turn be derived as Euler-Lagrange equations by a suitable choice of the trial function. This paper shows how the variational method can be used to derive both interface and outer surface boundary conditions. Although the derivations are given for a slab geometry, the formalism appears applicable to other geometries. While the resulting interface conditions are those currently in use, the conditions derived at outer surfaces are more accurate than those now commonly used.

W. Sangren (San Diego, Calif.)

RELATIVITY

See also 5769, 5818, 5819.

Fock, V. [Fok, V. A.] 5784

★The theory of space, time and gravitation.

Second revised edition. Translated from the Russian by N. Kemmer. A Pergamon Press Book.

The Macmillan Co., New York, 1964. xii+448 pp. $15.00.

The Russian second edition was published in 1961 by Fizmatgiz, Moscow.

From the Preface to Second Edition: "The second edition differs from the first [Pergamon, New York, 1959; MR 21 #7042] by some additions and reformulations. The question of the uniqueness of the mass tensor is treated in more detail (Section 31*: A system of equations for the components of the mass tensor as functions of the field) and is illustrated by two examples (Appendices B and C). The notion of conformal space is introduced and used as a basis for the treatment of Einsteinian statics (Sections 56 and 57). Greatest care has been applied to the formulation of the basic ideas of the theory and to the elucidation of those points on which the author's views differ from the traditional (Einsteinian) ones. Thus, in order to discuss the general aspects of the relativity principle, Section 49* (Remark on the relativity principle and the covariance of equations) has been added."

Zeeman, E. C. 5785

Causality implies the Lorentz group.

J. Mathematical Phys. **5** (1964), 490–493.

In a Minkowski space M two events x and y can be ordered if the vector $x-y$ is time-like; in this case x is said to follow y if the time component of $x-y$ is positive. Consider a one-to-one mapping f of M onto itself, such that it preserves the time ordering; it is shown that f coincides with the group generated by the orthochronous Lorentz group, the translations and the multiplication by a scalar. It is interesting to note that this conclusion does not hold if M is two-dimensional.

B. Bertotti (Frascati)

Gotusso, Guido 5786

Sulle simmetrie nello spazio tempo.

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 194–215.

The title concept is studied from various points of view, and a programme of classification by symmetry properties is indicated for space-times V_4. However, isometry groups are not discussed in any detail, and generally there are few equations, except for those arising in the construction, by way of example, of a cylindrically symmetric static vacuum space-time.

F. A. E. Pirani (Waltham, Mass.)

Lovelock, D. 5787

Classical relativistic dynamics of "spin" particles. (Italian summary)

Nuovo Cimento (10) **29** (1963), 1126–1142.

A general development of the classical relativistic mechanics of spin particles is given. The Lagrangian is a function of the position vector and its first two derivatives: the spin angular momentum tensor (i.e., total minus

orbital angular momentum) is initially arbitrary. The most general form of a conserved angular momentum is given for a Lagrangian involving the position vector and its first derivatives. Discussion is given of the above more general case and previous theories are classified into two incompatible groups. *C. Strachan* (Aberdeen)

Lubkin, E. 5788
A critique of the space-time variables in physical theory. (Italian summary)
Nuovo Cimento (10) **32** (1964), 171–179.
The author thinks aloud on reasons for removing the space-time variables from the axioms of physical theory. The treatment is entirely qualitative. *A. Peres* (Haifa)

Cornish, F. H. J. 5789
Møller's energy momentum pseudo-tensor and its application to the Born-Infeld model of a charged particle.
Proc. Phys. Soc. **82** (1963), 807–815.
The Einstein and Møller canonical momentum-energy pseudo-tensors are simply derived from Komar's covariant density [Phys. Rev. (2) **113** (1959), 934–936; MR **21** #1196]. Their differences and their limitations are briefly discussed. The conservation law based on Møller's pseudo-tensor is then applied to find the energy distribution in the field associated with the Born-Infeld model of a charged particle incorporated in general relativity by Hoffman [ibid. (2) **47** (1935), 877–880]. In particular, the total energy of the field is formally equal, apart from a constant of order unity, to the total energy of the Nordstrom-Reissner field [Florides, Proc. Cambridge Philos. Soc. **58** (1962), 102–109; MR **27** #6588; ibid. **58** (1962), 110–118; MR **27** #6589].

The paper ends by showing that, by introducing an infinite negative mass of non-electromagnetic origin, the Nordstrom-Reissner solution may be regarded as a limiting case of the Born-Infeld model. *P. S. Florides* (Dublin)

Robaschik, Dieter 5790
Bestimmung einiger geometrischer Grössen für das Schwarzschildfeld.
Acta Phys. Polon. **24** (1963), 313–316.

Robaschik, Dieter 5791
Zur Störungsrechnung in der Gravitationstheorie.
Acta Phys. Polon. **24** (1963), 299–311.
Der Verfasser behandelt die Einsteinschen Gravitationsgleichungen erster Näherung für ein Störungsfeld $\overset{1}{g}_{\mu\nu}$ in einem vorgegebenen Riemannschen Raum mit der Metrik $\underset{0}{g}_{\mu\nu}$. Dieser Metrik wird ein Materie-Tensor nullter Näherung $\underset{0}{T}_{\mu\nu}$ zugeordnet. {Der Referent möchte auf die dem Verfasser anscheinend unbekannte Arbeit von T. Regge und J. A. Wheeler [Phys. Rev. (2) **108** (1957), 1063–1069; MR **19**, 1021] hinweisen.} *H. Treder* (Berlin)

Petrow, A. S. [Petrov, A. Z.] 5792
★Einstein-Räume.
Autorisierte bearbeitete Ausgabe. In deutscher Sprache herausgegeben von Hans-Jürgen Treder.
Akademie-Verlag, Berlin, 1964. x+394 *pp.* *DM* 58.00.

The original version of this book [*Einstein spaces* (Russian), Fizmatgiz, Moscow, 1961; MR **25** #4897] has been occasionally augmented in the present translation by both the author and H.-J. Treder. The latter has contributed an appendix on the most recent developments concerning the invariant representations of free gravitational fields and has also added to the bibliography which now lists almost 500 items. Thus the book is completely up-to-date. The excellent translation is due to Helmut Koch.

The treatise of the author is not only the first monograph devoted to the study of Einstein spaces as such but is also unusually comprehensive: it has, in fact, an almost encyclopaedic character. Chapter I provides a concentrated survey of Riemannian geometry, including the theory of groups of motions. Special attention is paid to the canonical representation of quadratic forms of arbitrary signature. After a very brief summary of the foundations of the theory of relativity, Einstein spaces are introduced in Chapter II, and a list of known solutions of Einstein's field equations (Schwarzschild, Kottler, Weyl, etc.) is given. These provide the motivation for Chapter III, which is devoted to the classification of Einstein spaces in terms of the algebraic structure of the curvature and energy-momentum tensors. This very important chapter is based to a very considerable extent on original work of the author, most of which is already widely known. By means of the set of all alternating tensors at a point P of the Riemannian space V_n, with metric tensor $g_{\alpha\beta}$, a centro-affine space E_N [with $N=\frac{1}{2}n(n-1)$] is defined, which is associated with P. This is the so-called Klein space. The mapping of the curvature tensor $R_{\alpha\beta\gamma\delta}$ of V_n as well as of $g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}$ onto E_N yields the symmetric tensors R_{ab}, g_{ab} $(a, b=1, \cdots, N)$, which give rise to the λ-matrix $(R_{ab}-\lambda g_{ab})$. The classification of the spaces V_n for a given n results from the transition to the canonical form of this matrix, whose characteristic determines the type of V_n. The development now proceeds on the basis of the resulting theory of elementary divisors. After the presentation of a few general theorems, such as the result that V_n with $n>2$ is a space of constant curvature if and only if the characteristic of the λ-matrix is of the form $[(1, \cdots, 1)]$, the author restricts himself to Einstein spaces $(R_{\alpha\beta}=\kappa g_{\alpha\beta})$ with $n=4$. According to the fundamental theorem there are precisely three distinct types of such spaces with the signature $(---+)$. These are denoted by T_i or $\dot{T}_i$ $(i=1, 2, 3)$ according as $R_{\alpha\beta}=0$ or $R_{\alpha\beta}=\kappa g_{\alpha\beta}$. The canonical forms of (R_{ab}) in T_i and $\dot{T}_i$ are discussed in detail. This is followed by a classification of general gravitational fields (i.e., fields for which the energy-momentum tensor $T_{\alpha\beta}\neq 0$). In Chapter IV a classification of empty gravitational fields by means of the theory of r-parameter groups of motions G_r is given. The following questions are posed: (I) Suppose that T_i admits G_r. What values can r assume for a given value of i? (II) What is the structure of G_r if i and r are given? Theorem: If the curvature of T_i $(i=1, 2, 3)$ is not constant, there does not exist an admissible G_r with $r\geqq 7$. Theorem: If T_1 admits a G_r with $r>4$, it is a Minkowskian (flat) space. For T_2 the maximal value of r is 6, while for T_3 we have $r\leqq 4$. The spaces T_2, T_3 cannot be spaces of constant curvature. A large number of special results concerning the structure of the metric of particular spaces T_i admitting certain G_r are derived. This leads to an

almost complete sub-classification of the types T_i, which is applied to the above-mentioned solutions of the field equations. In Chapter V a similar programme is carried out for general fields ($T_{\alpha\beta}\neq 0$). Conformal spaces ($\overset{*}{g}_{\alpha\beta}=e^{2\sigma}g_{\alpha\beta}$) form the subject matter of Chapter VI, with particular emphasis on the conformal mapping of Riemannian spaces onto Einstein spaces. These mappings depend on the compatibility conditions of certain systems of differential equations. In the theory of conformal mapping of Einstein spaces onto Einstein spaces a careful distinction has to be made between isotropic and non-isotropic mappings, of which the former are defined by the condition $g^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta}=0$. Chapter VII is devoted to a detailed discussion of the outer and inner Cauchy problems associated with the field equations. In Chapter VIII special types of gravitational fields are considered. These correspond to certain conditions imposed for physical or geometrical reasons. Besides decomposable and conformal-decomposable Einstein spaces the so-called symmetric spaces T_i (in which the covariant derivatives $R_{\alpha\beta\gamma\delta,\lambda}=0$) are treated. It is found that a symmetric T_1 is either a space of constant curvature or possesses a well-determined metric. If $\kappa=0$ every symmetric T_1 is flat. For a symmetric T_2 we always have $\kappa=0$, while the spaces T_3 cannot be symmetric. The following are also discussed in detail: static fields, fields with central or axial symmetry, harmonic fields, cylindrical gravitational waves and gravitational fields with boundary conditions.

No doubt this remarkable treatise will serve as an indispensable aid to workers in this field for many years to come. *H. Rund* (Pretoria)

Peretti, Giuseppe 5793

Sulla velocità di fuga di un grave.

Ist. Lombardo Accad. Sci. Lett. Rend. A **97** (1963), 57–63.

The author calculates the escape velocity from the Schwarzschild field and for various other similar situations. *F. A. E. Pirani* (Waltham, Mass.)

Uhlmann, Armin 5794

The closure of Minkowski space.

Acta Phys. Polon. **24** (1963), 295–296.

By adding a closed null-cone at infinity it is possible to complete Minkowski space to a compact conformal manifold (a projective quadric 4-fold of signature $++----$) [cf. H. Rudberg, Ph.D. Diss., Univ. of Uppsala, Uppsala, 1957, and the reviewer's report to the 1962 Warsaw conference on gravitation]. The author suggests an ingenious new way of obtaining this structure. Any point of Minkowski space can be represented by a hermitian 2×2 matrix H and this, in turn, by the unitary 2×2 matrix $(H-iE)(H+iE)^{-1}=U$. Those unitary matrices not corresponding in this way represent the points of the null cone at infinity. The conformal transformations of Minkowski space are given by $U\rightarrow(AU+B)(CU+D)^{-1}$ with $AA^*-CC^*=DD^*-BB^*=E$ and $AB^*=CD^*$. *R. Penrose* (Austin, Tex.)

Sachs, M. 5795

A spinor formulation of electromagnetic theory in general relativity. (Italian summary)

Nuovo Cimento (10) **31** (1964), 98–112; *erratum, ibid.* (10) **31** (1964), 1383.

This paper is a continuation of an earlier paper by the author and Schwebel [Nuovo Cimento (10) **21** (1961), suppl., 197–229; MR **24** #B2374] (cf. also, the author [ibid. (10) **27** (1963), 1138–1150; MR **27** #2274]). The electromagnetic field is cast into spinor form in the presence of a non-flat underlying manifold, whose metric also is represented in terms of two-component spinors. The author asserts that the invariance group of the theory must be a topological group that is locally compact (this appears questionable to the reviewer), connected, and that it must satisfy the second axiom of countability (this also appears questionable), hence isomorphic with the field of quaternions. Invariants are derived, then conservation laws, a new Lagrangian formulation given, and the new theory compared to conventional theory, to which it is equivalent at this stage of development, though there appears a fundamental length which might play a role when the Dirac field is added. *P. C. Bergmann* (New York)

Perrin, Henri 5796

Sur l'interprétation des relations de choc en magnétohydrodynamique relativiste.

C. R. Acad. Sci. Paris **258** (1964), 3203–3206.

Author's summary: "L'interprétation des relations de choc dans des repères convenablement choisis, montre que la vitesse du fluide et le champ magnétique peuvent être rendus colinéaires si le vecteur $i^\beta=n_\alpha\overset{*}{H}^{\alpha\beta}$ est orienté dans le temps; le champ magnétique peut être rendu tangentiel à l'onde de choc si i^β est orienté dans l'espace; le cas où i^β est vecteur nul, est irréductible à l'un des cas précédents." *Y. Bruhat* (Paris)

Cavaillès, Paul 5797

Théorie intrinsèque des spineurs. Adjonctions hermitienne et de Dirac. Conjugaison de charge.

C. R. Acad. Sci. Paris **258** (1964), 1166–1169.

A Hermitian form, invariant over the spinor group, is obtained intrinsically over the space of spinors of R^4. An analogous invariant Hermitian form is defined over the space of spinors of Minkowski space M^4. Conjugation of charge and the adjunction of Dirac are defined intrinsically, and it is shown that the product of these two operators gives rise to an interesting invariant 2-spinor metric. The treatment follows the book by C. Chevalley [*The algebraic theory of spinors*, Columbia Univ. Press, New York, 1954; MR **15**, 678] and reference is made to mimeographed notes of lectures by A. Lichnerowicz ["Champs spinoriels et propagateurs en Relativité générale", Cours du Collège de France, 1962/63]. *T. J. Willmore* (Liverpool)

Cavaillès, Paul 5798

Théorie intrinsèque des spineurs. Décomposition de la représentation de Dirac; interprétation géométrique du spineur.

C. R. Acad. Sci. Paris **258** (1964), 2759–2762.

This is a continuation of the preceding paper of the author [#5797 above]. It is shown that certain elements of the Clifford algebra admit a unique factorisation, and this gives rise to an intrinsic description of the decomposition of the Dirac representation without using matrix representation. This factorisation also leads to a geometrical

interpretation of the spinor relative to R^4 or to Minkowski M^4. The two semi-spinors into which a spinor relative to R^4 or M^4 is decomposed can be identified with two spinors relative to R^3. A physical interpretation of this algebraic decomposition is given. *T. J. Willmore* (Liverpool)

Levashev, A. E. [Levašev, A. E.]; Ivanitskaya, O. S. [Ivanickaja, O. S.] 5799
Generalized tetrad formulation of general relativity theory (G.R.T.).
Acta Phys. Polon. **23** (1963), 647–653.
The world coordinates and metric are not given in advance, as is the case in the usual tetrad formulation of general relativity theory [F. J. Belinfante, Physica **7** (1940), 305–324; MR **1**, 273; F. A. E. Pirani, Acta Phys. Polon. **15** (1956), 389–405; MR **19**, 509]. Instead, Galilean coordinates referred to a local frame of reference are used, and a general asymmetric connection is introduced in the transformation of the tetrad components. The gravitational potentials are determined from quantities which describe the deformation of the tetrad.
C. Gilbert (Newcastle upon Tyne)

Levashev, A. E. [Levašev, A. E.]; Vorontsov, V. I. [Voroncov, V. I.] 5800
The orientability of the physical tetrad in relativistic electrodynamics.
Acta Phys. Polon. **23** (1963), 655–656.
It is maintained that if the orientation of the physical tetrad is dependent on the charge and current 4-vector j^m, then the dual tensor $\check{j}_{[lmn]}$ should also be significant in physics. *C. Gilbert* (Newcastle upon Tyne)

Kreisel, E.; Liebscher, E.; Treder, H. [Treder, Hans-Jürgen] 5801
Gravitationsfelder mit Nullstellen der Determinante der $g_{\mu\nu}$. II.
Ann. Physik (7) **12** (1963), 195–208.
Part I (by Treder) appeared in same Ann. (7) **9** (1961/62), 283–294 [MR **26** #7398]. This paper deals with Riemannian manifolds that are static (i.e., they possess a hypersurface-orthogonal Killing field which is time-like in a four-dimensional domain) and which contain hypersurfaces on which—in a static coordinate system—the metric determinant g vanishes. The prototype situation for this occurrence is, of course, the Schwarzschild radius. The authors show generally, and even in the presence of other fields, that such manifolds are incomplete, in that there are geodesics reaching the $g=0$ hypersurface along an arc of finite proper length. *P. G. Bergmann* (New York)

Chandrasekhar, S. 5802
Dynamical instability of gaseous masses approaching the Schwarzschild limit in general relativity.
Phys. Rev. Lett. **12** (1964), 114–116; *errata, ibid.* **12** (1964), 437–438.
By a method extensively used in his book [*Hydrodynamic and hydromagnetic stability*, Clarendon, Oxford, 1961; MR **23** #A1270], the author finds, within the framework of general relativity, a sufficient condition for spherically symmetric gaseous masses to become dynamically unstable with respect to adiabatic radial oscillations. In particular, it is shown that in the case of a homogeneous sphere of uniform energy density, the system becomes dynamically unstable long before the radius of the system reaches its Schwarzschild limit. This result, however, is obtained by choosing the "Lagrangian displacement", ξ, with respect to the time coordinate, equal to the radial coordinate; to what extent the above result depends on this choice of ξ is not made clear.
Some mistakes, which do not alter the above result, are amended in the "errata" listed in the heading above.
P. S. Florides (Dublin)

Halpern, Leopold 5803
On alternative approaches to gravitation.
Ann. Physics **25** (1963), 387–399.
This paper is a fairly detailed and critical review of so-called flat-space theories of gravitation and their relationship to general relativity. *P. G. Bergmann* (New York)

Jordan, Pascual; Ehlers, Jürgen; Kundt, Wolfgang; Sachs, Rainer K.; Trümper, Manfred 5804
Beiträge zur Theorie der Gravitations-Strahlungsfelder. Strenge Lösungen der Feldgleichungen der allgemeinen Relativitätstheorie. V.
Akad. Wiss. Lit. Mainz Abh. Math.-Natur. Kl. **1962**, 965–1000.
Part IV (by Jordan, Ehlers, Kundt, and Sachs) appeared in same Abh. **1961**, 791–837 [MR **27** #6602]. A discussion of interior Einstein fields and Einstein-Maxwell-Jordan fields with algebraically special Weyl tensors.
The first part of the paper analyzes fields of Petrov type N generated by a perfect fluid, making use of Ehlers' covariant decomposition of the velocity gradient. A number of interesting properties of the streaming are derived and used to illustrate how various special assumptions about the kinematics restrict the equation of state. The main results are as follows. Consider fluid fields of type N in which the flow is irrotational and the energy density depends only on pressure and an invariantly defined time. Then the condition that the flow be shear-free uniquely characterizes Friedmann's conformally flat space-time, and there are no physically realistic models with non-vanishing rate of shear.
"Purely radiative fields" form the subject of the second part of the paper. An Einstein-Maxwell matter-free field is called purely radiative if it admits a shear-free geodesic null congruence which is also an eigencongruence of the electromagnetic field tensor. Special attention is given to the construction of canonical coordinates and classification of solutions in the case where the congruence is twist-free. This goes some way towards extending the results for twist-free purely radiative fields in vacuo obtained by Robinson and Trautman [Proc. Roy. Soc. Ser. A **265** (1961/62), 463–473; MR **24** #B1970].
W. Israel (Edmonton, Alta.)

Mariot, Louis; Pigeaud, Pierre 5805
Commutateurs en théorie n-dimensionnelle.
C. R. Acad. Sci. Paris **258** (1964), 2763–2765.
Through analogy with physical theories in the four-dimensional space-time manifold the authors postulate

commutation relations for the metric, and for other fields in manifolds possessing higher dimensionality. Their emphasis on five- and six-dimensional situations indicates that they are motivated by the possible applications to unified field theories having five or six dimensions.

P. G. Bergmann (New York)

Souriau, J.-M. 5806

Five-dimensional relativity. (Italian summary)

Nuovo Cimento (10) **30** (1963), 565–578.

The author studies the hypothesis that the universe is a 5-dimensional manifold U, homeomorphic to $R^4 \times S_1$, periodic in the distinguished dimension with period $4\pi\hbar G^{1/2}/ec \sim 2\cdot 10^{-31}$ cm (where G is the Newtonian gravitational constant). From this version of the Kaluza-Klein unified field theory he infers, among other things, gauge invariance, and parity violation in β-decay [see also the author, C. R. Acad. Sci. Paris **247** (1958), 1559–1562; MR **20** #6312; *Géométrie et relativité*, Hermann, Paris, 1964]. For the usual Kaluza-Klein theory see W. Pauli [*Theory of relativity*, Pergamon, New York, 1958; MR **22** #3534].

F. A. E. Pirani (Waltham, Mass.)

Gilbert, C. 5807

The generalization of the Born-Infeld electrodynamics.

Proc. Phys. Soc. **83** (1964), 181–188; *corrigendum, ibid.* **83** (1964), 682.

Author's summary: "A unified theory of gravitation and electromagnetism is derived from a variational principle. The characteristic action function is an invariant density formed from the Ricci tensor of a semi-symmetric connection. The field equations have the same general character as those of the Born-Infeld electrodynamics [Proc. Roy. Soc. London Ser A **144** (1934), 425–451] and its generalization by Hoffmann and Infeld [Phys. Rev. (2) **51** (1937), 765–773]. The theory presupposes the existence of a fundamental length of the order of magnitude of the electron radius. The Einstein-Maxwell equations are obtained from a special choice of the action function."

W. Israel (Edmonton, Alta.)

Mavridès, Stamatia 5808

Trajectoires des satellites dans le champ de gravitation d'une sphère en rotation, en théorie minkowskienne linéaire.

C. R. Acad. Sci. Paris **258** (1964), 4655–4658.

The author calculates, according to the linear Lorentz-invariant theory of gravitation described by her previously [same C. R. **257** (1963), 4139–4142; MR **28** #2890], the perihelion motion of a planetary orbit and the additional motion resulting from the rotation of the central body. The former agrees with the usual general relativity value while the latter, which for planets and for their natural satellites is unobservably small, is $\frac{3}{4}$ of the value found by Lense and Thirring for general relativity [Phys. Z. **19** (1918), 156–163].

F. A. E. Pirani (Waltham, Mass.)

Lanczos, C. 5809

Signal propagation in a positive definite Riemannian space.

Phys. Rev. (2) **134** (1964), B476–B480.

In a recent paper the author [J. Mathematical Phys. **4** (1963), 951–959; MR **27** #2330] endeavoured to show that a "highly curved" Riemannian space-time with a positive definite metric can nevertheless simulate in macroscopic relations the behaviour of a Minkowskian line element. This conclusion supposedly depends on the concept of a 4-fold periodicity in the world with a very small lattice constant. In the present paper this theory is amplified (along lines of reasoning not always clear to the reviewer), and some of the final conclusions are reflected in the following excerpts from the author's summary: "The proof is given that a genuinely Riemannian (positive definite) space of fourfold lattice structure is well suited to the propagation of signals, if the g_{ik} assume very large values along some narrow ridge surfaces. The resulting signal propagation is strictly translational and has the nature of a particle which moves with light velocity (photon). According to this theory the discrepancy between classical and quantum phenomena is caused by the misinterpretation of a Riemannian metric in Minkowskian terms. The Minkowskian metric comes about (in high approximation) macroscopically, in dimensions which are large in comparison to the fundamental lattice constant. Since this constant is of the order 10^{-32} cm, this condition is physically always fulfilled."

H. Rund (Pretoria)

ASTRONOMY

See also 5808, 5875.

Kuzmak, G. E.; Kopnin, Ju. M. 5810

A new form of the equations of motion of a satellite and its application to the study of motions which are nearly Keplerian. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 730–741.

The differential equations of satellite motion introduced by the authors consist of two groups. The first group determines the motion in the instantaneous plane of the orbit, and the second group controls the rotation of this plane caused by the j_z component of the perturbing acceleration (perpendicular to the plane of the orbit).

The authors investigate the applicability of the method of averaging to the numerical solution of the equations of motion when the perturbing acceleration is small compared with gravitational acceleration. The authors claim that for orbits sufficiently distant from the center of attraction and for components of the perturbing acceleration in the plane of the instantaneous orbit decreasing sufficiently rapidly, the method of averaging computes accurately the almost elliptic and almost hyperbolic trajectories. The authors demonstrate the saving of computation time (machine time) in one special case.

T. Leser (Aberdeen, Md.)

Volkov, M. S. 5811

Rotational motion of artificial satellites in an elliptic orbit. (Russian. English summary)

Bjull. Inst. Teoret. Astronom. **9** (1963), 274–282.

Author's summary: "The present paper deals with the integration of differential equations of rotational motion of an artificial satellite in elliptical orbit. The solution has been

constructed in the form of a series in $\sigma = \frac{3}{2}n(A-C)/L_0A$, where n is the mean motion of the satellite in the orbit, L_0 is the precessional rotation velocity of the satellite, and A, C are moments of inertia of the satellite with dynamical symmetry."

Volkov, M. S. 5812

A periodic solution of the second kind which represents the rotation of a satellite in a circular orbit. (Russian. English summary)

Bjull. Inst. Teoret. Astronom. **9** (1963), 283–291.

Author's summary: "The present article deals with the rotational motion of a satellite moving along a circle. The periodical solution of the second kind resembling the solution of the second kind in the three body-problem has been constructed by applying Poincaré's method."

Egorova, A. V. 5813

Results of numerical integration of the equations of motion of an artificial satellite of the Earth. (Russian. English summary)

Bjull. Inst. Teoret. Astronom. **9** (1963), 323–329.

Author's summary: "The figures 1–8 show the perturbations in the elements of the artificial Earth satellite from the Moon, the Sun and the first two powers of the Earth's flattening. These values have been obtained by means of numerical integration of the equations of motion of the satellite during 100 revolutions."

Petrovskaja, M. S. 5814

Estimates of the remainder terms in Hill's series. (Russian. English summary)

Bjull. Inst. Teoret. Astronom. **9** (1963), 257–273.

Author's summary: "The power series in $m = n_0/(n_1 - n_0)$ in Hill's lunar theory are considered, n_0, n_1 being the mean angular velocities of the Sun and the Moon. Some estimates of errors are given for the cases when the series are replaced by polynomials of 2, 3, 4, 5, 6 powers of m. Analogous estimates are also obtained for Hill's series expanded in m^2 which were examined in Ljapunov's paper [*Collected works* (Russian), Vol. I, Izdat. Akad. Nauk SSSR, Moscow, 1954; MR **19**, 109]. All the estimates concern the case $|m| \leq \sigma$ (≈ 0.080849), σ being the value of m for the Moon."

Roberts, P. H. 5815

On highly rotating polytropes. I.

Astrophys. J. **137** (1963), 1129–1141.

Author's summary: "This paper is concerned with the equilibrium of a highly rotating mass of fluid which obeys a polytropic equation of state. Since exact solutions are possible only when the angular velocity Ω of the configuration is small compared with $(G\bar{\rho})^{1/2}$, where $\bar{\rho}$ is the mean density and G is the gravitational constant, the methods expounded in this paper are necessarily approximate. It is supposed that the equidensity surfaces are similar and similarly situated oblate spheroids, and the problem is reduced to finding the appropriate eccentricity of these spheroids and the variation of density with equatorial radius. Three methods of making these determinations are proposed and are examined in detail for the polytrope of index unity. These methods, respectively, depend on (i) a variational principle, (ii) satisfying hydrostatic equilibrium exactly in the equatorial plane, and (iii) satisfying hydrostatic equilibrium exactly on the polar axis. Method (i) is shown to lead to results which are a compromise between those of methods (ii) and (iii). Some attention is given to the question of whether or not, as Ω is increased, the polytrope will bifurcate to a Jacobi form before equilibrium is broken by centrifugal forces at the equator. It is concluded that if the polytropic index is less than unity (approximately), bifurcation will indeed occur."

Roberts, P. H. 5816

On highly rotating polytropes. II.

Astrophys. J. **138** (1963), 809–819.

Author's summary: "In the first paper with this title [#5815] a variational method was given for determining approximately the structure of a rotating polytrope. On specifying an n-parameter family of equidensity surfaces, the variational method gave the 'best' values for these n-parameters and the 'best' density to assign to each surface. In this paper we assume a two-parameter family of surfaces, namely, oblate spheroids of variable eccentricity. The variational method is used to derive differential equations for the eccentricity $e(m)$ and the density $\rho(m)$, as a function of the equatorial radius m. It is proved that, for small e, the theory agrees with that of Clairaut and also with that of Chandrasekhar for a slightly distorted polytrope."

Chandrasekhar, S. 5817

A general variational principle governing the radial and the non-radial oscillations of gaseous masses.

Astrophys. J. **139** (1964), 664–674.

In this paper a general variational principle, applicable to radial as well as non-radial oscillations of gaseous masses, is formulated and proved. It is further shown that when the normal modes are analyzed in vector spherical harmonics, the variational principle requires that the square of the characteristic frequency of oscillation, σ^2, belonging to a particular spherical harmonic, is stationary with respect to simultaneous variations of two independent radial functions. A consequence of this result is that σ^2 (belonging to a particular harmonic) emerges as a characteristic root of a 2×2 matrix.

Two simple illustrations of the variational principle are given. *P. L. Bhatnagar* (Bangalore)

Fock, V. A. [Fok, V. A.] 5818

The researches of A. A. Fridman on the Einstein theory of gravitation.

Uspehi Fiz. Nauk **80** (1963), 353–356 (*Russian*); *translated as Soviet Physics Uspekhi* **6** (1964), 473–474.

A. A. Friedman [Z. Physik **10** (1922), 377–386; ibid. **21** (1924), 326–332] discovered non-static cosmological solutions of Einstein's field equations some time before the general expansion of the universe had been established empirically. The paper under review, celebrating the 75th anniversary of his birth, summarizes Friedman's results,

without details, but with several anecdotes and a polemical paragraph about the cosmological constant.
F. A. E. Pirani (Waltham, Mass.)

Zel'dovich, Ya. B. [Zel'dovič, Ja. B.] 5819
The theory of the expanding universe as originated by A. A. Fridman.
Uspehi Fiz. Nauk **80** (1963), 357–390 (*Russian*); *translated as Soviet Physics Uspekhi* **6** (1964), 475–494.

An introductory article for non-specialists, roughly on the level of the *Scientific American*, but with equations; these are intelligible to anyone with a grounding in undergraduate calculus. Besides giving an exposition of standard results in relativistic cosmology, the author also surveys recent studies by himself and others of the early state of a 'big bang' universe.
F. A. E. Pirani (Waltham, Mass.)

Raychaudhuri, A. K.; Banerji, S. 5820
Cosmological evolution with creation of matter.
Z. Astrophys. **58** (1963/64), 187–191.

The authors consider cosmological evolution in the steady-state theory of Hoyle [Monthly Notices Roy. Astronom. Soc. **120** (1960), 256–262; MR **22** #6581] (hereafter I), and of Hoyle and Narlikar [Proc. Roy. Soc. Ser. A **270** (1962), 334–341; MR **26** #3508] (II).

They are interested in the question whether the steady-state model will evolve from an arbitrary distribution of pressure-free perfect fluid. The scalar creation field of (I) is assumed to be a function of time only. Then the authors generalise Raychaudhuri's equation [Phys. Rev. (2) **98** (1955), 1123–1126; MR **16**, 1059] to cover (I). They conclude that the evolution remains indeterminate for a completely arbitrary system. {Reviewer's remark: However, they consider only four of eleven field equations.}

A similar investigation for (II) also yields inconclusive results, but it is claimed that in this theory rotational motion of pressure-free material is precluded by the field equations. However, this conclusion rests on the assumption that dust in (II) always follows geodesics: to the reviewer this is not obvious and indeed seems likely to be false.
W. B. Bonnor (London)

GEOPHYSICS

Knopoff, L.; Aki, K.; Archambeau, C. B.; Ben-Menahem, A.; Hudson, J. A. 5821
Attenuation of dispersed waves.
J. Geophys. Res. **69** (1964), 1655–1657.

The authors show an alternative way of establishing the known relation $UQ_T=cQ_X$ between the dimensionless attenuation factors $1/Q_T$ and $1/Q_X$ for standing and progressive wave experiments, respectively. U is the group velocity. The proof follows from a Taylor expansion to first-order terms of the eigenvalue equation which is considered as dependent on the geometry of the medium, elastic parameters, wave number and frequency. A slight variation from the loss-less case brings the approximate result.
S. C. Das Gupta (Howrah)

ECONOMICS, OPERATIONS RESEARCH, GAMES

See also 4985, 4992, 5248, 5471, 5496.

Bahtin, I. A. 5822
On an extremal problem. (Russian)
Ž. Vyčisl. Mat. i Mat. Fiz. **4** (1964), 120–135.

The problem studied in this paper is that of minimizing the function $\phi=\sum_{i=1}^{L}\prod_{j=1}^{N}\alpha_{ij}^{x_{ij}}$, $0<\alpha_{ij}\leq 1$, where $\sum_{i=1}^{L}x_{ij}=m_j$, $m_j>0$, $j=1,2,\cdots,N$; $x_{ij}\geq 0$.

For $N=1,2$ the problem was studied by the author, M. A. Krasnosel'skiĭ and A. Ju. Levin [same Ž. **3** (1963), 400–409; MR **27** #6193]. In the present article a way is given for reducing the case $N>1$ to the case $N-1$.
P. L. Ivănescu (Bucharest)

Kincaid, W. M.; Darling, D. A. 5823
An inventory pricing problem.
J. Math. Anal. Appl. **7** (1963), 183–208.

A seller has a stock of one unit which is valueless at the end of a given time period. He receives offers from potential buyers that appear in Poisson fashion at an expected rate of 1 per unit time. The probability that a potential buyer at time t prior to the conclusion of the sale is willing to pay at least an amount z for the item is a specified function $\varphi(z,t)$. A strategy s is any rule giving the probability, $p(s,z,t)$, with which the seller should accept an offer of amount z at time t. An optimal strategy, σ, is one that maximizes the seller's expectation. Denote by $E(s,t)$ the seller's expectation during a sale of length t using strategy s. It is proved that σ may be defined by the rule:

$$p(\sigma,z,t) = 1 \quad \text{if } z \geqq \sup_s E(s,t),$$
$$= 0 \quad \text{if } z < \sup_s E(s,t).$$

More generally, let the seller have an initial stock of n items for which he receives offers for single units at a time. Let $E_k(s,t)$ be the expectation, starting with k items at time t, from following a strategy s. It is proved that an optimal strategy is defined by the rule that an offer z at time t will be accepted if and only if z satisfies the inequality

$$z \geqq \sup_s E_k(s,t)-\sup_s E_{k-1}(s,t).$$

The alternative model is studied in which the seller is required to announce a price, and a strategy s is defined by the functions $Z_k(s,t)$ specifying the price when k items are in stock at time t. Let the expectation from following strategy s be $F_k(s,t)$, and let $\zeta(x,t)$ denote, for given x and t, the least value of z for which the maximum of the function $(z-x)\varphi(z,t)$ is attained. The analogous theorem is proved that an optimal strategy is defined by the price functions

$$Z_k(t) = \zeta[\sup_s F_k(s,t)-\sup_s F_{k-1}(s,t), t].$$

It is noted that there is a relationship between the method of approach used in this paper and the principle of optimality in dynamic programming, but that the problem has been developed independently here.
F. G. Foster (London)

Fürst, Dario 5824

Su un problema di ottimo in un caso di prenotazione di posti. (French, English, Spanish, and German summaries)

Giorn. Ist. Ital. Attuari **26** (1963), 64–78.

Author's summary: "If on a train, or an aircraft, etc., there are booked seats only, but seats can be booked in two or more different offices (either for subsequent stations or for the same station) and such offices are not in communication with one another, it is necessary for the total number of seats to be allotted in advance among the various offices. The best form of allotment is defined and determined from various viewpoints."

Heineken, W. 5825

★Exakte Methoden der Unternehmensführung.

Verlag G. Braun, Karlsruhe, 1963. vii + 134 *pp.* *DM* 18.60.

The author gives a survey on Operations Research for non-mathematicians, whereby practical examples are shown in a very adapted way. The two parts, namely, (1) Die Planung in den Funktionsbereichen, and (2) Hilfsmittel für Planungsaufgaben, are very well chosen. This book is an excellent guide for managing directors showing Operations Research. *H. Künzi* (Zürich)

Verhovskiĭ, B. S. 5826

On multi-index transportation problem with axial sums. (Russian)

Dokl. Akad. Nauk SSSR **156** (1964), 282–285.

The paper deals with the problem of minimizing the function $\sum_{i_1,\cdots,i_s} P_{i_1,\cdots,i_s} x_{i_1,\cdots,i_s}$ where $x_{i_1,\cdots,i_s} \geqq 0$ and

$$\sum_{i_2,\cdots,i_s} x_{i_1,\cdots,i_s} = a_{i_1}{}^{(1)}, \cdots, \qquad \sum_{i_1,\cdots,i_{s-2},i_s} x_{i_1,\cdots,i_s} = a_{i_{s-1}}^{(s-1)},$$

$$\sum_{i_1,\cdots,i_{s-1}} x_{i_1,\cdots,i_s} = a_{i_s}{}^{(s)}$$

and where all the sums are taken for i_k from 1 to n_k.

The basic theorems of the common transportation problem are extended to this problem. A method of reducing the original problem to a common transportation problem is given. *P. L. Ivănescu* (Bucharest)

Afriat, S. N. 5827

Gradient configurations and quadratic functions.

Proc. Cambridge Philos. Soc. **59** (1963), 287–305.

The utility functions of an individual, on which equilibrium theory in economics is based, are generally not available from empirical data. All that may be available are a certain finite set of commodity bundles $x_1, \cdots, x_k$ that the individual has purchased and the price vectors $p_1, \cdots, p_k$ at which he has purchased them. This gives partial information about the utility functions, because the indifference curve passing through a point x_r must be supported by the budget plane through x_r corresponding to prices p_r (the plane $x \cdot p_r = x_r \cdot p_r$). The author discusses certain problems connected with fitting a quadratic utility function with convex indifference curves to such a set of empirical data. *R. J. Aumann* (Jerusalem)

Kwan, Mei-ko 5828

Graphic programming using odd or even points.

Acta Math. Sinica **10** (1960), 263–266 (*Chinese*); *translated as Chinese Math.* **1** (1962), 273–277.

Suppose given a finite, connected graph in which each edge has associated with it a non-negative number, its "length". This paper considers the interesting problem of adding duplicate edges so that (1) the resulting graph is unicursal, and (2) the total length of duplicated edges is minimal. It is shown that a necessary and sufficient condition for minimality is that the sum of the lengths of duplicated edges in each simple cycle be at most half the length of the cycle. The author describes how this minimality test can be used to construct a solution to the problem. Unfortunately, the construction involves examining all simple cycles to see whether the minimality test is met or not, and this is easier said than done.

D. R. Fulkerson (Santa Monica, Calif.)

Guan, Mèĭ-gu [Kwan, Mei-ko] 5829

Graphical method of the odd-and-even point. (Russian)

Sci. Sinica **12** (1963), 281–287.

This paper was originally published in Acta Math. Sinica **10** (1960), 263–266; the English translation is reviewed above [#5828].

Gomory, R. E. 5830

Large and nonconvex problems in linear programming.

Proc. Sympos. Appl. Math., Vol. XV, pp. 125–139. *Amer. Math. Soc., Providence, R.I.,* 1963.

This paper provides the reader with a comprehensive review of methods proposed for solving large-scale systems. According to the author, work of Gilmore and Gomory on the cutting stock problem, Gomory and Hu on network flows, Dzielinsky and Gomory on lot-size programming, Dantzig and Wolfe on the decomposition principle, Ford and Fulkerson on multi-commodity flows can all be viewed "as special cases of an extremely simple extension to the basic calculations that must be gone through in performing . . . simplex method". In the problems cited there can be an enormous number of variables or inequalities. The objective is one of finding a way of avoiding working explicitly with the full system. In the decomposition approach one begins with the system: minimize cx subject to $x \geqq 0$, $A_1x = b_1$, $A_2x = b_2$. This is replaced by: minimize $\sum_i (cx^i)w_i$ subject to $w_i \geqq 0$, $\sum w_i = 1$, $\sum (A_1x^i)w_i = b$, where $x^1, x^2, \cdots$ are extreme points of the convex $C = \{A_2x = b_2, x \geqq 0\}$. Thus if the original system has $m_1 + m_2$ equations in non-negative variables, the new system has $m_1 + 1$ equations. This reduction in itself is not interesting because it could result in an enormous increase in the number of variables w_i (which correspond to the number of extreme points of C). However, instead of doing column selection (in the simplex method) explicitly, it is done implicitly at each iteration by solving a sub-linear program over C with a modified objective form. The author next discusses how non-convex problems can be brought into the framework "by convexifying them". This is illustrated first for integer programs. The convex of feasible solutions is reduced as required by the addition of inequalities by a procedure due to Gomory until an integer optimal solution is obtained. The decomposition of Benders is next discussed and shown to be the dual of the

decomposition principle. It has the valuable feature that it permits reduction of a problem involving some integer and some real variables to an integer program. Finally a combination of integer programming methods and the decomposition principle is used to solve the lot-size model due to A. S. Manne. *G. B. Dantzig* (Berkeley, Calif.)

Kunzi, Hans P. [**Künzi, Hans Paul**]; **Oettli, Werner** 5831
Integer quadratic programming.
Recent advances in mathematical programming, pp. 303–308. *McGraw-Hill, New York*, 1963.
The authors consider the problem of minimizing a strictly convex quadratic function of variables that must be integers and must satisfy linear inequality constraints. They show that it can be treated as a finite sequence of integer linear programming problems with several integer variables and one continuous variable. The approach is analogous to that of Benders [Numer. Math. **4** (1962), 238–252; MR **26** #4820]. *E. M. L. Beale* (Brentford)

Bui, Trong Lieu [**Bui Trong Lieu**] 5832
On a problem of convexity and its applications to nonlinear stochastic programming.
J. Math. Anal. Appl. 8 (1964), 177–187.
Some results of O. L. Mangasarian and J. B. Rosen ["Inequalities for stochastic nonlinear programming problems", Rep. P.1127, Shell Development Co.] concerning the problem of minimizing $\varphi(x)+\psi(y)$ with the constraints (*) $g(x)+h(y)\gtrless b$ are generalized. Let B_b be the set of the elements (x, y) satisfying (*) and let $B_b{}^x$ be the set of the elements y satisfying (*) for a given x. Under certain assumptions, $\alpha(b)=\inf_{(x,y)\in B_b}[\varphi(x)+\psi(y)]$ is a real-valued convex and continuous function on a certain W, and $\gamma(b, x)=\varphi(x)+\inf_{y\in B_b{}^x}\psi(y)$ is a real-valued convex function of x. The second part is concerned with the random case when b is a random variable defined on a basic probability space with range in an open parallelepiped of R^n. Let $f(x, y)$ be a real-valued convex function; the "two-stage solution" consists in fixing x, then computing the mathematical expectation of $\inf_{y\in B_b{}^x}f(x, y)$ and finally in taking the infimum with respect to x of $E\inf_{y\in B_b{}^x}f(x, y)$. The author studies this solution. *D. Vaida* (Bucharest)

Rosen, J. B.; Ornea, J. C. 5833
Solution of nonlinear programming problems by partitioning.
Management Sci. **10** (1963/64), 160–173.
This paper discusses some applications of the nonlinear partition programming program described by Rosen [*Recent advances in mathematical programming*, McGraw-Hill, New York, pp. 159–176, 1963; MR **29** #1063]. The computer program and some computational experiences are outlined. It is pointed out that it can be used to find a (possibly local) minimum for non-convex problems. *E. M. L. Beale* (Brentford)

Warga, J. 5834
Minimizing certain convex functions.
J. Soc. Indust. Appl. Math. **11** (1963), 588–593.
The author points out that the standard one-variable-at-a-time method of minimizing a function may fail if the function is not differentiable, even if it is convex. But he establishes conditions in which this method must converge, and suggests that it could be used as a modified Gauss-Seidel procedure for minimizing a sum of squares of linear functions of variables constrained to lie in preassigned intervals. *E. M. L. Beale* (Brentford)

Warga, J. 5835
A convergent procedure for convex programming.
J. Soc. Indust. Appl. Math. **11** (1963), 579–587.
This is a very abstract paper dealing with a problem that is related to, but not necessarily identical with, convex programming as it is normally understood. *E. M. L. Beale* (Brentford)

Rvačev, V. L. 5836
An analytic description of certain geometric objects. (Russian)
Dokl. Akad. Nauk SSSR **153** (1963), 765–767.
The author treats some properties of the functions
$$(a+b\pm\sqrt{(a^2+b^2)})/2, \qquad a\pm\sqrt{(a^2+b^2)}$$
in order to decide the non-negativity of certain functions. Without referring to any bibliography it is mentioned that B. L. Juščenko has used this operation in mathematical programming. *M. Hosszú* (Miskolc)

Owen, Guillermo 5837
Tensor composition of nonnegative games.
Advances in game theory, pp. 307–326. *Princeton Univ. Press, Princeton, N.J.*, 1964.
Let y be an n-vector and x^j, $j=1, \cdots, n$, be m-vectors. The composition $(x^j)\otimes y$ is the matrix with entries $x_i{}^jy_j$. Consideration of games which are composed of several games (e.g., the electoral college whose members are chosen by the voters of the several states) led the author to consider the following notion. Let v and w_j, $j=1, \cdots, n$, be non-negative games in characteristic function form on, respectively, $N=\{1, \cdots, n\}$ and $M=\{1, \cdots, m\}$, and suppose that $w_j(M)=1$. Let $u=(w_j)\otimes v$ be the characteristic function of a game on $M\times N$. It is shown that u is uniquely characterized by a plausible functional equation; that it is a constant-sum game if v and w_j are; that if V and W_j are solutions of v and w_j, then $(W_j)\otimes V$ is a solution of u; that if x and x^j are in the cores of v and w_j, then $(x^j)\otimes x$ is in the core of u; and if
$$b_v(j) = \max_{T\subset N}[v(T\cup\{j\})-v(T)],$$
etc., then $b_u(i, j)=b_{w_j}(i)b_v(j)$. *R. D. Luce* (Philadelphia, Pa.)

Bondareva, O. N. 5838
Some theorems in the theory of Ψ-stability in cooperative games. (Russian)
Dokl. Akad. Nauk SSSR **153** (1963), 61–63.
The concept of Ψ-stability was introduced by R. D. Luce [Ann. of Math. (2) **59** (1954), 357–366; MR **15**, 975] as a solution concept in the theory of n-person games.

Given a 0-1 normalized, n-person cooperative game with characteristic function $v(S)$, let α be an imputation, $\alpha_i \geq 0$, $\sum \alpha_i = 1$. Let $\mathcal{T}$ be a partition of the n players into coalitions, and let $\Psi(\mathcal{T})$ be a collection of coalitions which includes the coalitions of $\mathcal{T}$. Then $[\alpha, \mathcal{T}]$ is called Ψ-stable if (1) $\sum_{i \in S} \alpha_i \geq v(S)$, for every $S \in \Psi(\mathcal{T})$; (2) $\alpha_i = 0$ implies $\{i\}$ is a member of $\mathcal{T}$.

In the present paper, the above two conditions on the existence of a Ψ-stable pair are replaced by equivalent conditions through the use of a transposition theorem for finite systems of linear inequalities [Ky Fan, *Linear inequalities and related systems*, pp. 99–156, Princeton Univ. Press, Princeton, N.J., 1956; MR **19**, 432].

J. H. Griesmer (Yorktown Heights, N.Y.)

Shapley, L. S. 5839

Solutions of compound simple games.

Advances in game theory, pp. 267–305. *Princeton Univ. Press, Princeton, N.J.*, 1964.

The simple games in this paper are allowed to be "pseudo-games", with non-superadditive (but monotonic) characteristic function. The compound structure considered here is the compounding of players P_{ij} according to games G_i and Q; $P_{i1}, P_{i2}, \cdots$, play G_i for the ith seat in Q. The basic theorem is that any solutions X_i of G_i and Z of Q may be composed in the obvious way to form a solution of the compound game. The author goes on to investigate other solutions, which might be (1) composed similarly from non-solutions or (2) not composite. Case (2) always occurs except in some trivial examples; this follows already from Gillies' solution for any semisimple game [*Contributions to the theory of games*, Vol. IV, pp. 47–85, Princeton Univ. Press, Princeton, N.J., 1959; MR **21** #4850]. The author finds more solutions than Gillies; in fact, his well-known arbitrarily bad solutions [ibid., pp. 87–93; MR **22** #635] are non-composite solutions of compound games. He conjectures that (1) never occurs, and proves that if (1) occurs, then several curious (perhaps inconsistent) conditions must hold. {There is a typographical error in condition (e), where x_i should be X_i.}

J. R. Isbell (Seattle, Wash.)

Blackwell, David 5840

Memoryless strategies in finite-stage dynamic programming.

Ann. Math. Statist. **35** (1964), 863–865.

From the author's introduction: "Given three sets X, Y, A and a bounded function u on $Y \times A$, suppose that we are to observe a point $(x, y) \in X \times Y$ and then select any point a we please from A, after which we receive an income $u(y, a)$. In trying to maximize our income, is there any point to letting our choice of a depend on x as well as on y? We shall give a formalization to this question in which sometimes there is a point. If (x, y) is selected according to a known distribution Q, however, we show that dependence on x is pointless, and apply the result to obtain memoryless strategies in finite-stage dynamic programming problems."

W. P. Ziemer (Bloomington, Ind.)

Gross, O. 5841

The rendezvous value of a metric space.

Advances in game theory, pp. 49–53. *Princeton Univ. Press, Princeton, N.J.*, 1964.

For a compact connected metric space of diameter $2r$, there is a unique constant K such that for every finite set of points P there is a point whose average distance from the points in P is K. The possible values for K are exactly the numbers in $[r, 2r)$.

J. R. Isbell (Seattle, Wash.)

BIOLOGY AND BEHAVIORAL SCIENCES

See also 5483.

Rosen, Robert 5842

Abstract biological systems as sequential machines.

Bull. Math. Biophys. **26** (1964), 103–111.

Author's summary: "It is shown that a rather close relationship exists between the (M, R)-systems, defined previously as prototypes of abstract biological systems, and the sequential machines which have been studied by various authors. The theory of sequential machines is reformulated in a way suitable for its application to the study of the intertransformability of (M, R)-systems as a result of environmental alteration. The important concept of strong connectedness is most useful in this direction, and is used to derive a number of results on intertransformability. Some suggestions are made for further studies along these lines."

S. Ginsburg (Van Nuys, Calif.)

Bellman, R. E.; Jacquez, J. A.; Kalaba, R. 5843

Mathematical models of chemotherapy.

Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. IV, pp. 57–66. *Univ. California Press, Berkeley, Calif.*, 1961.

This paper overlaps to some extent two earlier papers by the same authors [Bull. Math. Biophys. **22** (1960), 181–198; MR **22** #9317; ibid. **22** (1960), 309–322; MR **22** #9318]. Physiological considerations relating to the course of an injected drug through the body, and the problems involved in constructing appropriate mathematical models are debated now a little more fully. A set of differential-difference and integro-differential-difference equations is written down, representing the case of a simplified animal with two organs connected in parallel to the circulation, which is maintained by a simplified heart. In view of the analytical difficulties which these equations present, two methods of applying simplifying approximations are put forward, and the resulting systems have been programmed for the IBM 704 computer. Some reasonable results have been obtained (to be published by B. Kotkin). The possibilities of a thoroughgoing use of computers in this way are discussed.

I. M. H. Etherington (Edinburgh)

Perret, C. J.; Levey, H. C. 5844

The theory of uncatalysed linear expanding systems.

J. Theoret. Biol. **1** (1961), 542–550.

One of the authors [Perret, Australian Sci. Teachers' J. **5** (1959), 9; J. Gen. Microbiol. **22** (1960), 589–617] has suggested that the phenomena of balanced growth which occur in a growing cell population, such as a bacterial culture, can be interpreted in terms of the simple chemical reaction kinetics of what he calls expanding systems. As compared with customary models, such a system in a suitable environment approaches asymptotically a state

in which it is expanding exponentially, instead of a steady state of constant volume. The theory is now put on a firmer basis by proving, for a highly simplified mathematical model, some of the contentions previously supported only by non-mathematical arguments. The crucial assumption is that the volume V is proportional to the amount of one (or more, but for simplicity of one, the last) of the substances into which absorbed nutrient is transformed in a sequence of metabolic steps. The behavior of the system is then represented by a set of linear differential equations, in matrix notation $d\xi/dt = A(\alpha)\xi$. The variables $\xi_1, \cdots, \xi_n, \xi_{n+1}$ ($\propto V$) are (in standardized units) the amounts of metabolites present in the system, and α measures the concentration of surrounding nutrient. The matrix $A(\alpha)$ resembles a matrix considered by Ledermann and Reuter [Philos. Trans. Roy. Soc. London Ser. A **246** (1954), 321-369; MR **15**, 625], and their technique is applied to obtain information about its latent roots and hence about the solution of the equations. The discussion dwells on various aspects of the solution, in particular on the way in which the asymptotically approached exponential state varies with α, and good resemblance to observed phenomena is claimed. The concluding remark is that "the expanding system concept . . . indicates how some possible aspects of the origin of life might be experimentally investigated in artificial systems".

I. M. H. Etherington (Edinburgh)

INFORMATION, COMMUNICATION, CONTROL

See also 5220, 5234, 5361, 5462, 5512, 5545, 5547, 5842.

Rényi, A. 5845

Statistical laws of accumulation of information. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 2, 311-316.

Author's summary: "Soit S_n un ensemble fini ayant n éléments. Supposons que nous recevons des informations sur un élément inconnu x de S_n dans la forme suivante: la $k$ième information répond à la question si x appartient ou non au sous-ensemble A_k de S_n ($k = 1, 2, \cdots, N$).

"Il est un fait bien connu que si $N \geqq \log_2 n$ on peut choisir les ensembles A_k d'une telle façon, que l'information mentionnée détermine sans équivoque l'élément inconnu x. La question se pose alors: si les sous-ensembles A_k de S_n sont choisis au hasard (d'une telle façon que A_k peut être n'importe quel sous-ensemble de S_n avec la même probabilité $1/2^n$, indépendamment du choix des A_j pour $j < k$), pour quelles valeurs de N serait déterminé sans équivoque l'élément inconnu x avec une probabilité au moins égale à $p > 0$. On peut montrer facilement que dans ce cas N doit être plus grand que $\log_2(n/\log p^{-1})$ si n est suffisamment grand (voir Théorème 1).

"Nous disons que les sous-ensembles A_k ($k = 1, 2, \cdots, N$) de S_n forment un système séparateur pour S_n si pour chaque couple x, y des éléments différents de S_n il y a entre les A_k au moins un qui contient l'un des éléments x, y et ne contient pas l'autre. Il existe des systèmes séparateurs pour S_k ayant N éléments si $N \geqq \log_2 n$. On peut montrer facilement que si les sous-ensembles A_k sont choisis au hasard (dans le sens mentionné plus haut) alors le système $A_1, \cdots, A_N$ serait un système séparateur pour S_n avec une probabilité $\geqq p$, lorsque N est plus grand que $2\log_2(n/\log p^{-1})$ et si n est suffisamment grand (voir Théorème 2).

"Ces résultats sont des cas particuliers d'un théorème plus général (Théorème 4)."

L. L. Campbell (Kingston, Ont.)

Bonnet, Georges 5846

Sur certaines propriétés statistiques de fonctions aléatoires issues de transformations non linéaires.

C. R. Acad. Sci. Paris **258** (1964), 4917–4920.

Let B be a normal random variable with mean m and standard deviation σ and let S be another random variable which is independent of B, has mean zero, and possesses moments of all orders. Let $X = B + S$. It is shown that

$$\left(\frac{\partial}{\partial\sigma^2}\right)^k E[h(X)] = 2^{-k}E[h^{(2k)}(X)],$$

$$\left(\frac{\partial}{\partial m}\right)^k E[h(X)] = E[h^{(k)}(X)].$$

The proof uses a Laplace integral representation of $h(X)$. As an example, this result is used to find the mean value of the output of a particular type of detector. The theorem supplements one due to Price [IRE Trans. Information Theory **IT-4** (1958), 69–72; MR **22** #13341] on covariance functions. *L. L. Campbell* (Kingston, Ont.)

Sawaragi, Yoshikazu; Sunahara, Yoshifumi; Nakamizo, Takayoshi 5847

On the response of non-linear time-variant control systems subjected to a pre-assigned signal in the presence of gaussian random noise.

Tech. Rep. Engrg. Res. Inst. Kyoto Univ. **13** (1963), 13–26.

Authors' summary: "This paper presents a graphical method of evaluating the response of non-linear time-variant control systems excited by a step input in the presence of a stationary random noise. Descriptions in this paper contain the theoretical treatment on various problems as follows. First, in order to motivate the general procedure an extension of the phase plane method is carried out and the indicial response of non-linear time-variant control systems under the condition of disturbance free is graphically calculated. Second, the non-stationary response of non-linear time-variant control systems with a random input is evaluated by a similar manner to phase plane analysis. The remainder of this paper is devoted to the evaluation of the indicial response of non-linear time-variant control systems in the presence of gaussian random disturbance. The advantage of the procedure is to be applicable to the calculation of the response of non-linear time-variant control systems with various types of pre-assigned inputs, even in the presence of random disturbance." *A. V. Balakrishnan* (Los Angeles, Calif.)

Sawaragi, Yoshikazu; Sunahara, Yoshifumi; Hara, Kei 5848

An analysis on non-stationary response of non-linear control systems by the method of Taylor-Cauchy transform.

Tech. Rep. Engrg. Res. Inst. Kyoto Univ. **13** (1963), 51–64.

Authors' summary: "In this paper, a new analytical technique for evaluating the non-stationary response of non-linear control systems subjected to a suddenly applied gaussian random input is developed by introducing the Taylor-Cauchy transform. The time-dependent mean squared value of the response is easily calculated by solving an algebraic recursion formula derived from the calculation of Taylor-Cauchy transform operated on a non-linear equation of a control system. Special emphasis is placed on not only the possibility of evaluation of the response of higher order non-linear control systems but also the removal of complexity due to the statistical equivalent linearization technique for several special types of non-linear transfer characteristics. Numerical results are compared with those obtained by another method established previously by the authors. Detailed discussion concerning the accuracy is carried out."

A. V. Balakrishnan (Los Angeles, Calif.)

Birch, John J. 5849

On information rates for finite-state channels.

Information and Control **6** (1963), 372–380.

Author's summary: "This paper presents sufficient conditions for the direct computation of the entropy for functional (non-Markovian) processes and thus also of the rate of information for finite-state channels. The condition for exponential convergence of the upper and lower bounds on the entropy [the author, Ann. Math. Statist. **33** (1962), 930–938; MR **25** #4573], viz., that the transition probability matrix be strictly positive, is here relaxed. Examples are presented involving simple finite-state channels."

Nedoma, Jiří 5850

Die Kapazität der periodischen Kanäle.

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2**, 98–110 (1963).

The author investigates further properties of the capacity of a periodic channel with finite past previously considered by Jacobs [Math. Ann. **137** (1959), 125–135; MR **23** #B1048; Trans. 2nd Prague Conf. Information Theory, pp. 231–249; Publ. House Czechoslovak Acad. Sci., Prague, 1960; MR **24** #B2502] and Winkelbauer [ibid., pp. 685–831; MR **23** #B2093]. The ergodic capacity is defined as the supremum of the transmission rate for a certain class of sources, and the capacity is defined with the help of the length of the ε-code. The class of stationary sources and the class of periodic sources with period of length r are two natural classes of sources for which the capacity may be defined. It is shown that for the class of periodic sources of period length r the larger ergodic capacity is obtained. An example is given of a periodic channel of period 2 for which the ergodic capacity based on a periodic source with period 2 is larger than that based on a stationary ergodic source. It is also shown that in the case of an ergodic periodic channel the ergodic capacity based on the transmission rate for the class of periodic sources with the period of the channel agrees with the capacity. Further definitions of channels related to periodic channels and their properties are given. The paper includes a resumé of the various relations between capacity and ergodic capacity. The various technical definitions and details cannot be summarized here.

S. Kullback (Washington, D.C.)

Liu, Bede 5851

Matching of initial behavior in time domain approximation.

J. Franklin Inst. **277** (1964), 107–118.

Author's summary: "The purpose of this paper is to present an extension of the method of least-squares as applied to time domain approximation. A technique is developed by which one is able to match the initial time domain behavior of the desired impulse response while performing the usual least-squares approximation. The method begins with the construction of a set of functions, all with proper initial behavior, through linear combination of exponentials. Then the desired impulse response is expanded in terms of this set of functions. The closure property of sets of exponential functions is studied to show that, if the error of approximation without the matching of initial behavior can be made arbitrarily small by using more and more terms, then the same is true when the initial behavior is matched. An example is included to illustrate the method."

Urkowitz, Harry 5852

Pre-envelopes of nonstationary bandpass processes.

J. Franklin Inst. **277** (1964), 31–36.

Author's summary: "Many of the useful properties of pre-envelopes of real waveforms are shown to hold when the notion of pre-envelope is applied to random processes which are not wide-sense stationary. In particular, if $x(t)$ represents a real random process which is not wide-sense and $\hat{x}(t)$ is its Hilbert transform, then $x(t)$ and $\hat{x}(t)$ have the same autocovariance function and have zero cross-covariance at the same instant. The autocovariance function of the pre-envelope $z(t)$, given by $z(t)=x(t)+j\hat{x}(t)$, is twice the pre-envelope of the autocovariance function of $x(t)$. The notion of a time-dependent power density spectrum allows a simple interpretation of a bandpass random process which is not wide-sense stationary. The well-known form of the autocovariance function of a wide-sense stationary bandpass process carries over with simple changes to processes which are not wide-sense stationary."

Bose, R. C.; Chakravarti, I. M. 5853

A coding problem arising in the transmission of numerical data. (French summary)

Bull. Inst. Internat. Statist. **39** (1962), *livraison* 4, 345–355.

The integers $i=1, 2, \cdots, 2^n$ are to be encoded (mapped 1-1) as n-bit vectors α of 0's and 1's. If i is encoded as α, if α is sent, and if α' corresponding to i' is received (over some channel), then the error is defined as $|i-i'|$.

Under the natural encoding $i\rightarrow\alpha=(a_1, \cdots, a_n)$, where $i=\sum a_r 2^{n-r}$, $a_r=0$ or 1, the expected error is minimum (assuming only single errors occur, and making some equal-likelihood assumptions). The authors find other encodings which also produce this minimum expected error. They do this by rigidly moving the n-cube to produce several encodings with the same expected error for a given n; then they put two of these encodings together to get one for $n+1$.

Among such encodings f the authors show (by similar methods) how to find f^* such that the minimum expected error among the integers i has the value $p(2^{n+1}+3n-\delta)/6n$,

which is known to be best possible for $n=2$ and 3. ($\delta=1$ [2], n odd [even]; p is the probability of an error in a binary digit.) *H. F. Mattson* (Waltham, Mass.)

Bartee, Thomas C.; Schneider, David I. 5854
Computation with finite fields.
Information and Control **6** (1963), 79–98.
Authors' summary: "A technique for systematically generating representations of finite fields is presented. Relations which must be physically realized in order to implement a parallel arithmetic unit to add, multiply, and divide elements of finite fields of 2^n elements are obtained. Finally, techniques for using a maximal length linear recurring sequence to modulate a radar transmitter and the means of extracting range information from the returned sequence are derived."
W. W. Peterson (Gainesville, Fla.)

Gorog, F. 5855
Les codes cycliques détecteurs & correcteurs.
Deux. Congr. Assoc. Française Calcul et Traitement Information (Paris, 1961), pp. 195–206. Gauthier-Villars, Paris, 1962.
By the author's definition, an error-correcting code corrects a "type of error" if it corrects an error pattern and all cyclic shifts of it. He proves several theorems, which in some cases determine and in some cases bound the number of distinct types of errors that can be corrected by a cyclic code or by certain shortened cyclic codes.
W. W. Peterson (Gainesville, Fla.)

Takahasi, H.; Goto, E. 5856
Application of error-correcting codes to multi-way switching. (French, German, Russian, and Spanish summaries)
Information processing, pp. 396–400. *UNESCO, Paris; R. Oldenbourg, Munich; Butterworths, London,* 1960.
The multi-way switches that the authors consider are such logical circuits as address selectors for computer core memories and decoding circuits for computer control units. They illustrate their ideas with the design of an address selector for a 4096-word core memory. Their solution is a circuit consisting of 24 drivers coupled to the 4096-address lines through a linear network. An address is specified by a 24-bit error-correcting code input to the drivers. The linear network consists of transformer coils suitably wound and connected. The maximum output of 24 is applied to the selected line, while an output of at most 8 is applied to the other lines. This discrimination ratio of 24/8 allows address selection to function correctly even if one driver fails, since the resulting ratio of 23/9 is within the operating range of the cores. A Golay code is used to obtain the minimum distance of 8 for the 4096 24-bit codes. The authors compare this with the ordinary core-matrix arrangement in a 64×64 plane, which requires 128 drivers and has a 2/1 discrimination ratio.
E. K. Blum (Middletown, Conn.)

Bandyopadhyay, G. 5857
A simple proof of the decipherability criterion of Sardinas and Patterson.
Information and Control **6** (1963), 331–336.
Author's summary: "A simple alternative proof is given for a necessary and sufficient condition for the decodability of a sequence of codes. The proof involves no application of linear syntactical bases or other sophisticated algebraic criteria." *S. Kotz* (Toronto, Ont.)

Schwartz, Eugene S. 5858
An optimum encoding with minimum longest code and total number of digits.
Information and Control **7** (1964), 37–44.
Author's summary: "It is shown that the maximum code length and the sum of all code lengths are dependent upon the method of merging combined frequencies in a Huffman minimum redundancy encoding. By bottom merging a combined frequency below a set of frequencies with equal weight an optimum encoding is obtained that has minimum $\sum p_i L_i$, minimum $L_{\max}$, and minimum $\sum L_i$."

Kiyasu, Zenichi 5859
Error detecting and correcting codes. (Japanese)
Sûgaku **15** (1963), 6–12.
The paper is tutorial in nature and discusses such things as mathematical definition of error detecting and correcting codes, cyclic group codes, Bose-Chaudhuri codes and methods for discovering errors in codes.
M. Aoki (Los Angeles, Calif.)

Mozgovaya, É. A. [Mozgovaja, È. A.] 5860
A method of search for the minimum of a function in the presence of constraints.
Avtomat. i Telemeh. **23** (1962), 1654–1661 (*Russian. English summary*); *translated as Automat. Remote Control* **23** (1963), 1552–1553.
From the author's summary: "We have obtained an algorithm for the search for the minimum of a function of several variables in the presence of constraints, when the initial point is chosen in the neighborhood of the minimum. The operations necessary for carrying out this search can be performed on standard blocks of scalar products. This algorithm is based on the gradient method. The derivation of the algorithm has been broken up into two problems: (1) the choice of an optimal direction; (2) the choice of an optimal step in a given direction."
This paper contains a careful and systematic application of classical gradient projection techniques to the search for a local minimum of a function of several variables in the presence of constraints.
H. Halkin (Whippany, N.J.)

Hanson, M. A. 5861
Upper and lower bounds for a certain class of constrained variational problems.
J. Austral. Math. Soc. **3** (1963), 449–453.
This paper is an earlier and slightly weaker version of another paper by the same author [J. Math. Anal. Appl. 8 (1964), 84–89; MR **28** #2020]. A convexity assumption in the present paper is replaced in the later paper by a slightly weaker property, called functional convexity by the author. *L. D. Berkovitz* (Lafayette, Ind.)

Jakovlev, B. S. 5862

A method of choice of parameters of automatic control systems according to given regions of distribution of poles and zeros. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 177–187.

This paper indicates the choice of parameters of automatic control systems so as to satisfy the location of poles and zeros in the predetermined regions; it utilizes some papers cited in references in which details of the method are more fully explained. The present paper is concerned mostly with examples of application of these methods; more specifically, it indicates with some details the construction of nomogrammes by which the above mentioned specification concerning the regions is carried out. Conversely, if one prescribes the quality of transient processes, the method can be used for synthesis; in such cases the use of majorant (minorant) functions in conjunction with these methods permits ascertaining conditions to be imposed on the parameters so as to maintain the operating point within the prescribed regions; the latter are determined mostly by stability considerations.

N. Minorsky (Aix en Provence)

Lur'e, K. A. [Lur'e, K. A.] 5863

The Mayer-Bolza problem for multiple integrals and the optimization of the performance of systems with distributed parameters.

Prikl. Mat. Meh. **27** (1963), 842–853 (*Russian*); *translated as J. Appl. Math. Mech.* **27** (1964), 1284–1299.

A system whose state is described by the vectors $z=(z^1, \cdots, z^n)$, $\zeta=(\zeta^1, \cdots, \zeta^n)$ is governed by a system of first-order partial differential equations $\partial z^i/\partial x = X_i(z,\zeta,u;x,y)$; $\partial z^i/\partial y = Y_i(z,\zeta,u;x,y)$; $\partial X_i/\partial y - \partial Y_i/\partial x = 0$, where (x, y) is in the plane and $u=u(x, y)=(u^1, \cdots, u^r)$ is a control vector. A simple closed curve Σ_1 in the plane is given, and the values of $n_1 \leqq n$ components of z are given on Σ_1. On a second simple closed curve Σ_2 containing Σ_1 in its interior $n_2 \leqq n$ components of z are required to satisfy a system of ordinary differential equations $dz^{i_k}/dt = T_{i_k}(z, v, t)$, where t is the parameter along the curve Σ_2 and $v=v(t)=(v^1, \cdots, v^s)$ is a control function. The controls u and v are required to satisfy both equality and inequality constraints. A function $F(z, \zeta, u; x, y)$, a function $f_1(z, t)$, and a function $f_2(z, v, t)$ are given. The problem is to choose the controls u, v and the curve Σ_2 so that the functional $J(u, v, \Sigma_2)$, defined as the sum of the integral of F over the region between Σ_1 and Σ_2, the integral of f_1 over Σ_1, and the integral of f_2 over Σ_2, is minimized.

The author gives necessary conditions, analogous to the Euler equations, that must hold at a stationary value of $J(u, v, \Sigma_2)$, and gives the forms that the necessary conditions of Weierstrass and Clebsch assume for this problem. These necessary conditions are then restated in terms of a Hamiltonian function, and an analogue of the Pontryagin maximum principle for the present problem is thereby obtained.

L. D. Berkovitz (Lafayette, Ind.)

Edelbaum, Theodore N. 5864

Theory of maxima and minima.

Optimization techniques, *pp.* 1–32. *Academic Press*, *New York*, 1962.

This paper summarizes certain well-known and elementary facts concerning the following problems: (i) Extremizing functions of several variables, both with and without constraints. (ii) Extremizing integrals of the form

$$\int_0^{t_1} f(t, x_1, \cdots, x_n)\, dt,$$

i.e., f independent of x'. As an example of the latter type of problem the paper treats the problem of determining the thrust program that minimizes the fuel consumption required to transfer a low thrust and propulsion system satellite from one circular orbit to another. The problem of optimum proportioning of mass between propellant and power supply is also studied.

L. D. Berkovitz (Lafayette, Ind.)

Faulkner, Frank D. 5865

Direct methods.

Optimization techniques, *pp.* 33–67. *Academic Press*, *New York*, 1962.

The author considers a class of optimization problems associated with planar rocket motion without drag in a constant or at most linearly varying gravitational field. A sufficiency theorem applicable to these problems is first proved. Numerical methods for their solution, using the system of differential equations adjoint to the equation of motion, are then presented. A graphical solution technique that is based on the known form of the optimal thrust program is also presented. Finally, the problem of programming thrust along a given curve in space is treated; here drag is included.

L. D. Berkovitz (Lafayette, Ind.)

Miele, Angelo 5866

Extremization of linear integrals by Green's theorem.

Optimization techniques, *pp.* 69–98. *Academic Press*, *New York*, 1962.

Let φ and ψ be given functions of (x, y) and let R be a sufficiently well-behaved region in the plane bounded by a sufficiently well-behaved curve C. It is required to find a curve Γ in the class of curves defined by equations $y=y(x)$, having initial and terminal points on C, and never leaving the closure of R, such that the integral $\int \varphi\, dx + \psi\, dy$ evaluated along Γ is a minimum (or maximum). It is first shown, using Green's theorem, how one can often determine the extremizing curve by analyzing the behavior of the function $W \equiv \varphi_y - \psi_x$. This problem with a linear isoperimetric constraint $\int(\varphi_1\, dx + \psi_1\, dy) = c$ adjoined is also analyzed. Several problems in flight mechanics are then reduced to these formats, from which the forms of the solutions are deduced.

L. D. Berkovitz (Lafayette, Ind.)

Miele, Angelo 5867

The calculus of variations in applied aerodynamics and flight mechanics.

Optimization techniques, *pp.* 99–170. *Academic Press*, *New York*, 1962.

The author reviews all of the necessary conditions, except the Jacobi condition, that must hold along a curve that furnishes a minimum in the Bolza problem of the calculus of variations. A large number of problems in aerodynamics and flight mechanics are then cast in the format of a Mayer problem and are analyzed using the above necessary conditions.

L. D. Berkovitz (Lafayette, Ind.)

Leitmann, G. 5868

Variational problems with bounded control variables.

Optimization techniques, pp. 171–204. *Academic Press, New York*, 1962.

The author uses the necessary conditions of the calculus of variations to study a variety of optimization problems in the area of flight mechanics. Mass flow rate limited systems, propulsive power limited systems, and thrust acceleration limited systems are considered, and a numerical example is treated.

L. D. Berkovitz (Lafayette, Ind.)

Kelley, Henry J. 5869

Method of gradients.

Optimization techniques, pp. 205–254. *Academic Press, New York*, 1962.

The author first discusses the following techniques for minimizing a differentiable function of n variables $(x_1, \cdots, x_n)$, both with and without side conditions. The gradient, or a steepest descent, technique for problems without side conditions; a gradient projection method for problems with $m<n$ side conditions of the form

$$\varphi_i(x_1, \cdots, x_n) = 0;$$

and the penalty function technique for problems with side conditions of the form $\varphi_i(x_1, \cdots, x_n)=0$ and also for problems with side conditions of the form $\varphi_i(x_1, \cdots, x_n) \geqq 0$.

Control problems in which the control function is real-valued and whose performance index is a function of terminal position are then discussed. The author proposes indirect computational methods for attacking such problems. He gives formulas for determining how one should change a given control $y(t)$ to achieve an improvement in the performance of the system. The formulas are obtained by heuristic arguments based on formal analogy with the gradient projection technique and the penalty function technique discussed in the preceding paragraph.

The numerical solution of two problems by the methods proposed in the paper are summarized.

The author states that, "... the present chapter amounts to a status report on research still underway rather than a definitive exposition of a standard technique". *L. D. Berkovitz* (Lafayette, Ind.)

Kopp, Richard E. 5870

Pontryagin maximum principle.

Optimization techniques, pp. 255–279. *Academic Press, New York*, 1962.

The author derives the maximum principle and discusses its relationship with the calculus of variations and with dynamic programming. The treatment is formal and heuristic. *L. D. Berkovitz* (Lafayette, Ind.)

Bellman, Richard 5871

On the determination of optimal trajectories via dynamic programming.

Optimization techniques, pp. 281–290. *Academic Press, New York*, 1962.

In applying the dynamic programming formalism to variational problems one is led to a certain equation whose discrete version can be written as follows:

$$(*) \quad f(c, k\Delta) = \min_v [g(c, v)\Delta + f(c+v\Delta, (k-1)\Delta)],$$

$f(c, 0)=\Gamma(c)$. Here, Δ is fixed, $k=1, \cdots, K$, the functions g and Γ are given, the function f is to be determined and v ranges over some set that is defined by the particular problem at hand. The author points out that if the state variable c is a vector of dimension two or more, then the memory requirements imposed by the recursive scheme (*) for determining f are prohibitive. The following alternate scheme is then proposed. For any k, $f(c, k\Delta)$, considered as a function of c, is to be approximated by the Nth partial sum $S_N(c, k\Delta)$ of its Fourier expansion in Legendre polynomials, where N is suitably chosen. The coefficients α_{jk}, $j=1, \cdots, N$, of the expansion are to be determined by a quadrature method for which one need only know the zeros t_j, $j=1, \cdots, M$, of the Legendre polynomial of order M, for suitable M, and M constants W_j, $j=1, \cdots, M$, known as Christoffel numbers. Instead of determining the functions $f(c, k\Delta)$ recursively by means of (*), one determines the approximating partial sums $S_N(c, k\Delta)$ recursively by means of (*). In this method, only the coefficients α_{jk} are to be stored; the values of the Legendre polynomials are to be computed using the well-known recursion formula connecting any three consecutive polynomials. Moreover, in the recursion procedure one need only know $S_N(c, k\Delta)$ at the points $c=t_j$.

L. D. Berkovitz (Lafayette, Ind.)

Kalaba, Robert 5872

Computational considerations for some deterministic and adaptive control processes.

Optimization techniques, pp. 291–309. *Academic Press, New York*, 1962.

The application of the dynamic programming formalism to variational problems is reviewed and is then used to treat a simple example, which is also treated classically. A proposed method for solving two-point boundary-value problems associated with second-order non-linear equations is illustrated by applying the method to the equation $u''-e^u=0$, $u(0)=c$, $u'(1)=0$. A problem of adaptive control is then discussed using dynamic programming.

L. D. Berkovitz (Lafayette, Ind.)

Kashmar, C. M.; Peterson, E. L. 5873

General imbedding theory.

Optimization techniques, pp. 311–321. *Academic Press, New York*, 1962.

The authors present certain transformations for reducing a control problem with specified terminal conditions to a Mayer problem involving the maximization of a single coordinate. Some of the transformations are well known, and others are justified only on formal grounds. A brief discussion of the maximum principle and of dynamic programming is also given.

L. D. Berkovitz (Lafayette, Ind.)

Lawden, Derek F. 5874

Impulsive transfer between elliptical orbits.

Optimization techniques, pp. 323–351. *Academic Press, New York*, 1962.

A rocket is moving in unpowered flight in a central force field with an inverse square law of attraction. The rocket is to be transferred by the application of a series of n impulsive thrusts from a given initial orbit to a given

terminal orbit that is coplanar with the initial orbit. The number n is specified. The thrusts, or equivalently, the $(n-1)$-intermediate orbits are to be chosen in such a way as to minimize the sum of the velocity increments resulting from the thrusts. The author develops a set of equations that in principle determine the intermediate orbits for an optimal transfer in the sense defined. He then remarks that in general the solution is not even known for $n=2$. Two special cases for $n=2$ are then treated. In the first, it is desired to rotate the major axis of an elliptical orbit through an angle 2α in the orbital plane, the shape and size of the orbit being unchanged. The solution is effected by a combination of analytical and graphical techniques. In the second problem the initial and terminal orbits have their axes aligned, and the author assumes that the optimal transfer is effected along one of the ellipses by the application of two impulses in a direction perpendicular to the radius from the center of attraction. On the basis of this assumption the author determines the transfer orbit. *L. D. Berkovitz* (Lafayette, Ind.)

Breakwell, John 5875

The optimum spacing of corrective thrusts in interplanetary navigation.

Optimization techniques, pp. 353–375. *Academic Press, New York*, 1962.

Two problems are considered. In the first a vehicle is in unpowered flight from Earth to Mars. The orbits of Earth, Mars, and the transfer ellipse are assumed to be coplanar. Corrective velocity thrusts can be applied at a finite number of points. The velocity correction V_n applied at point P_n is determined by the estimate $\hat{D}_{n-1}$ of the miss distance D_{n-1} that would occur if the vehicle continued in unpowered flight from P_n, i.e., without any corrections. The author takes $\hat{D}_{n-1}-D_{n-1}$ to be a normally distributed random variable with zero mean and with a variance that is a consequence of his assumption as to how one makes the measurements on which $\hat{D}_{n-1}$ is based. These measurements are made close to P_n with an uncertainty in velocity having a greater effect than an uncertainty in position. The problem is to choose the number N of correction points and the location of the correction points $P_1, \cdots, P_N$ so as to achieve in an average sense a required terminal accuracy and to minimize the sum $\sum_{n=1}^{N} E\{|V_n|\}$. The author solves the problem for certain terminal accuracy requirements and under certain other assumptions.

In the second example a vehicle has a nominal straight-line motion from A to B with constant speed. Its actual position, however, has a small lateral component perpendicular to AB but in a fixed plane. Corrective thrusts are to be applied perpendicular to AB. At time t the miss distance is to be estimated from closely spaced measurements of the lateral deviations at times less than t. The problem is again to find the optimal spacing of corrective thrusts. *L. D. Berkovitz* (Lafayette, Ind.)

Leitmann, G. 5876

Propulsive efficiency of rockets.

Optimization techniques, pp. 377–387. *Academic Press, New York*, 1962.

The author points out that according to two commonly given definitions, propulsive efficiency is dependent on the motion of the observer. Difficulties also arise in determining a variable exhaust speed that maximizes propulsive efficiency. The author then proposes an invariant definition of propulsive efficiency and determines the exhaust speed that maximizes propulsive efficiency for both constant and variable exhaust speeds. For the latter problem a direct and a variational solution are given. *L. D. Berkovitz* (Lafayette, Ind.)

Aleškov, Ju. Z. 5877

Optimal location of a point on a trajectory corresponding to a required navigation method. (Russian. English summary)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. **18** (1963), no. 4, 85–91.

The Pontrjagin maximum principle is applied in a straightforward manner to analyze two minimum-time guidance problems. *L. W. Neustadt* (Ann Arbor, Mich.)

Kalman, R. E. 5878

The theory of optimal control and the calculus of variations.

Mathematical optimization techniques, pp. 309–331. *Univ. California Press, Berkeley, Calif.*, 1963.

In the present informal exposition the author discusses a point of the theory of optimal control. The actual minimum $V(x,t)$ of the cost functional—thought of as a function of the starting point (x,t) for a given target T—satisfies, if sufficiently smooth, the Hamilton-Jacobi partial differential equation. Conversely, the knowledge of a solution of this equation—satisfying suitable initial data—allows one to determine the optimal strategy and the control law, or feedback. Some notations are confusing (as the free use of the words function and functional). *L. Cesari* (Ann Arbor, Mich.)

Paraev, Ju. I. 5879

A problem in the analytic design of controllers. (Russian. English summary)

Avtomat. i Telemeh. **25** (1964), 167–176.

The author considers the linear control problem of minimizing a quadratic functional of the state vector and the control vector, subject to the condition that the terminating state lie only in a given subspace. A method of constructing the optimal control is given for the case in which the control values are restricted. *J. K. Hale* (Baltimore, Md.)

Stahovskiĭ, R. I. 5880

An algorithm for solving boundary-value problems. (Russian. English summary)

Avtomat. i Telemeh. **24** (1963), 962–974.

In this paper the author proposes an algorithm for computer solution of the boundary-value problems encountered in the optimal control of dynamical processes. The author does not prove the convergence of the proposed algorithm and does not give any significant example of successful applications to particular problems. *H. Halkin* (Whippany, N.J.)

Pittel', B. G. 5881

Some optimal control problems. I. (Russian. English summary)

Avtomat. i Telemeh. **24** (1963), 1187–1201.

The author considers a plant described by the linear differential equations $dx/dt=(A+\sum_{i=1}^{r} u_i(t)B_i)x$ and satisfying the boundary condition $x(0)=x_0$. The problem is to find a control function $u(t)$ taking its values in a given bounded, closed convex polyhedron U and such that the improper integral $\int_0^{+\infty} \varphi(x(t))\,dt$ is a minimum. The given function $\varphi(x)$ satisfies the conditions: (a) $\varphi(x)$ is continuously differentiable with respect to x; (b) $\varphi(x)>0$ for $x\neq 0$ and $\varphi(x)=0$ for $x=0$; (c) $\varphi(\lambda x)=|\lambda|^k\varphi(x)$, $k>0$.

This problem is a particular case of the problem considered in Chapter IV, § 24, of *The mathematical theory of optimal processes* by L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko [Interscience, New York, 1962; MR **29** #3316b]. The author gives some theorems on qualitative behavior of the optimal control for the cases of second-order equations and for a particular system of automatic control.

H. Halkin (Whippany, N.J.)

Derman, Cyrus 5882

On sequential control processes.

Ann. Math. Statist. **35** (1964), 341–349.

Let $\{Y_t, \Delta_t, t=0,1,\cdots\}$ be a vector process where $Y_t=0,1,\cdots,L$ is the state of a dynamic system at time t, and $\Delta_t=d_1,d_2,\cdots,d_K$ is the decision made at time t. It is supposed that $\mathbf{P}\{Y_{t+1}=j\mid Y_0,\Delta_0,\cdots,Y_t=i,\Delta_t=d_k\}=q_{ij}(k)$, where $q_{ij}(k)\geq 0$ and $\sum_{j=0}^{L} q_{ij}(k)=1$, and

$$\mathbf{P}\{\Delta_t=d_k\mid Y_0,\Delta_0,\cdots,\Delta_{t-1},Y_t\}=D_k(Y_0,\Delta_0,\cdots,\Delta_{t-1},Y_t),$$

where $\sum_{k=1}^{K} D_k=1$. Let C denote the class of all possible decision rules $\{D_k\}$, C' the class of decision rules for which $D_k(Y_0,\Delta_0,\cdots,\Delta_{t-1},Y_t=i)=D_{ik}$, and C'' the finite subclass of C' for which the D_{ik}'s are either 0 or 1. In the case C', $\{Y_t\}$ is a finite-state Markov chain with stationary transition probabilities.

For every $R\in C$ and any initial state $Y_0=i$ let $\Phi_T^R(i)=\{x_{Tjk}, j=0,\cdots,L;\ k=1,\cdots,K\}$, where

$$x_{Tjk}=(T+1)^{-1}\sum_{t=0}^{T}\mathbf{P}\{Y_t=j,\Delta_t=d_k\mid Y_0=i\}.$$

Denote by $H_R(i)$ the set of limit points of $\{\Phi_T^R(i)\}$ as $T\to\infty$. Let $H(i)=\bigcup_{R\in C} H_R(i)$, $H'(i)=\bigcup_{R\in C'} H_R(i)$, and $H''(i)=\bigcup_{R\in C''} H_R(i)$; and let $\bar{H}'(i)$ denote the closure of the convex hull of $H'(i)$, and $\bar{H}''(i)$ the convex hull of $H''(i)$. The author proves the following theorems: (i) $\bar{H}'(i)=\bar{H}''(i)\supset H(i)$. (ii) If the Markov chain corresponding to R is irreducible for every $R\in C''$, then $\bar{H}''(i)=H'(i)=H(i)=\bigcup_{i=0}^{L} H(i)$. These theorems are generalizations of the author's earlier results [C. Derman, J. Math. Anal. Appl. **6** (1963), 257–265; MR **27** #2000]. (iii) Let $\mathbf{P}\{Y_0=1\}=1$. Define $Z_{tjk}=1$ if $Y_t=j$, $\Delta_t=d_k$, and $=0$ otherwise. Let $\tilde{\Phi}_T^R(i)=\{\tilde{x}_{Tjk}, j=0,\cdots,L;\ k=1,\cdots,K\}$, where $\tilde{x}_{Tjk}=(T+1)^{-1}\sum_{t=0}^{T} Z_{tjk}$. For a fixed $R\in C$ denote by ω a sample sequence of the vector process $\{Y_t,\Delta_t, t=0,1,\cdots\}$. Let $U^R(i,\omega)$ be the set of limit points of $\tilde{\Phi}_T^R(i)$ as $T\to\infty$. Then for each $R\in C$, $\mathbf{P}\{U^R(i,\omega)\subset\bar{H}''\}=1$, where $\bar{H}''$ is the convex hull of $\bigcup_{j=0}^{L} H''(j)$. The author also deals with the problem of finding optimal decision rules.

L. Takács (New York)

Butkovskii, A. G. [Butkovskiĭ, A. G.] 5883

The broadened principle of the maximum for optimal control problems.

Avtomat. i Telemeh. **24** (1963), 314–327 (*Russian. English summary*); *translated as Automat. Remote Control* **24** (1963), 292–304.

This paper is concerned with a general variational problem, formulated in terms of functional analysis in such a way as to include a large class of optimal control systems, including some which have been studied by the author [Avtomat. i Telemeh. **22** (1961), 17–26; MR **26** #7471; ibid. **22** (1961), 1288–1301; MR **25** #4967]. Then an extended maximum principle is derived.

The following problem of conditional minimum is studied: to determine a function $w=W(P)$ defined in a region $D\subset E_s$ with values in a Banach space, such that the functional $\Phi^0(\int_D K(P_1,S,W(S))\,dS)$ reaches a minimum and the condition (*) $\Phi(W(P),\int_D K(P,S,W(S))\,dS)=\theta$ is satisfied, where P_1 is a fixed point in D, Φ an operator defined in the Banach space B acting in the Banach space B_1, and θ the zero element of B_1. Supposing that B is the direct sum of $\{h\in B: \Phi'(W_0)\cdot h=\theta\}$ and a certain subspace, that the set M of allowed functions W is a convex set with internal points in B, and calling W_0 an ordinary point of the manifold defined by (*) if $\Phi'(W_0)$ maps B in the entire B_1, the author establishes the following extended maximum principle, which gives a necessary condition to be satisfied by a solution of the above problem: If W_0 is internal to the set M and an ordinary point of the manifold defined by (*), or if W_0 is not an internal point of M and is an ordinary point of the manifold defined by (*) and $\lambda(W)=0$ ($\lambda(W)=0$ is the equation of the supporting hyperplane of M at W_0), then we can construct a function $\pi(R,w)$ that reaches a maximum at $w=W_0(R)$ for almost all $R\in D$.

The author proves that this principle is also a sufficient condition, under the hypothesis that the left side of (*) can be written $L_1(W(P))+L_2(\int_D K(P,S,W(S))\,dS)$, where L_1 and L_2 are linear operators, and Φ^0 is a linear functional.

In case the function W can be represented as $W(P)=(Q(P),U(P))$, the function U has the meaning of a control and the function Q describes the state of the object under control, and a weaker form of the above principle is given.

U. D'Ambrosio (Providence, R.I.)

Bryson, A. E., Jr.; Denham, W. F.; Dreyfus, S. E. 5884

Optimal programming problems with inequality constraints. I. Necessary conditions for extremal solutions.

AIAA J. **1** (1963), 2544–2550.

The authors consider the following variational problem. Maximize the functional $J=\phi(\mathbf{x}(t_f),t_f)$, where ϕ is a given function and $\mathbf{x}(t)$ is a vector-valued function subject to the constraints: (1) $\dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x}(t),\alpha(t),t)$ for $t_0\leq t\leq t_f$ with $\mathbf{x}(t_0)=\mathbf{x}_0$, (2) $m_i(\mathbf{x}(t_f),t_f)=0$ for $i=1,\cdots,p$, and (3) $C(\mathbf{x}(t),\alpha(t),t)\leq 0$ for $t_0\leq t\leq t_f$, where $\mathbf{f}$, the m_i and C are sufficiently smooth given functions, t_0 and $\mathbf{x}_0$ are given, and $p\leq n$, the dimension of $\mathbf{x}$. The functional is to be maximized by appropriately choosing the scalar-valued "control" function $\alpha(t)$ and the time $t_f\geq t_0$. Let $C^0=C$ and let $C^k=(\partial C^{k-1}/\partial t)+(\partial C^{k-1}/\partial\mathbf{x})\mathbf{f}$ for $k\geq 1$. If C^k, for $k=0,1,\cdots,q-1$, is "not an explicit function of $\alpha(t)$", but C^q is, then the inequality constraint (3) is said to be of order q. The case of α a vector, and of more than one

inequality constraint—but each of order zero or one—has been previously treated by Gamkrelidze [Izv. Akad. Nauk SSSR Ser. Mat. **24** (1960), 315–356; MR **22** #11192] and Berkovitz [J. Math. Anal. Appl. **5** (1962), 488–498; MR **25** #4961]. The new result in this article is the extension to higher-order constraints. Although the paper is confined to scalar controls and a single inequality constraint, the more general case is claimed to have been treated by the second author in his doctoral dissertation.

The Euler-Lagrange equations for this problem are obtained together with the corresponding transversality conditions. The adjoint differential equations contain an additional unconventional term when the trajectory lies on the boundary $C=0$. If the constraint (3) is of order zero, the adjoint variable is continuous. For higher-order constraints, the adjoint variable may be discontinuous (but in a prescribed manner) at the instants of time when $\mathbf{x}$ enters or leaves the boundary $C=0$. Two examples are presented.

The mathematical derivations are formal, intuitive, and lacking in rigor, but should provide an excellent point of departure for a proper and more precise treatment of this important problem.

L. W. Neustadt (Ann Arbor, Mich.)

Denham, Walter F.; Bryson, Arthur E., Jr. 5885

Optimal programming problems with inequality constraints. II. Solution by steepest-ascent.

AIAA J. **2** (1964), 25–34.

In this paper the authors define a computational procedure ("steepest ascent") to obtain, by successive approximations, an approximate solution to the variational problem described in the preceding review [#5884]. A steepest-ascent method of computation for problems without inequality constraints has previously been described by the same authors [J. Appl. Mech. **29** (1962), 247–257; MR **26** #4822]. Unfortunately, the presented derivation of the method consists of purely formal manipulations without any trace of justification, together with heuristic verbal arguments. In addition, the text is marred by a sloppy terminology and an excessively informal and offhand style. There is a brief, somewhat naive, description of the "bang-bang" control problem which contains erroneous statements on the number of switchings of an optimal control. No convergence proofs are presented.

Two atmospheric re-entry problems, to which both the steepest-ascent and the older "penalty-function" technique were applied, are described, and the steepest-ascent method in each case proved to be superior.

L. W. Neustadt (Ann Arbor, Mich.)

Murphy, G. J.; Meksawan, T. 5886

Optimal synthesis of linear pulse-width-modulated computer-controlled systems.

J. Franklin Inst. **277** (1964), 128–139.

Authors' summary: "The purpose of this paper is to present a technique for optimal synthesis of linear control systems with pulse-width modulation. The criterion of system performance is a quadratic index, which is taken in an illustrative example to be the sum of the squares of the error at sampling instants for the case in which the input to the system is a step function. The technique, which is based on the state-variable concept and on dynamic programming, leads to the determination of a computer program for calculating the optimum sequence of signals to be applied to the input of the pulse-width modulator."

Fleishman, B. A. 5887

The approximate determination of periodic regimes in systems containing relay elements with dead zones.

IEEE Trans. Automatic Control **AC-8** (1963), 292–296.

Author's summary: "The problem of finding periodic solutions of relay (or 'flip-flop') systems with dead zones is considered. A method for approximating such solutions is offered here, in which the original problem is reduced to a similar problem for an ideal relay system. The reduction is effected by 'sloppy superposition', i.e., by modification of the principle of convex superposition in piecewise-linear systems. A successful application and a possible extension of the method are mentioned briefly."

Nečiporuk, È. I. 5888

On the design of circuits from threshold elements. (Russian)

Dokl. Akad. Nauk SSSR **154** (1964), 763–766.

Bestehe die Basis eines Schemas aus den T-Elementen [$T=$"threshold", siehe V. I. Varšavskiĭ, dieselben Dokl. **139** (1961), 1325–1328; MR **25** #1973] mit den Gewicht Eins. Es bezeichne $L(n)$ die minimale Anzahl der T-Elemente, die die folgende Eigenschaft besitzen: Jede Boolesche Funktion mit n-Veränderlichen kann man mit Hilfe dieser Elemente realisieren. Es gilt der folgende Satz:

$$2(2^n/n)^{1/2} \lesssim L(n) \lesssim 2\sqrt{2}\cdot(2^n/n)^{1/2}.$$

Z. Daróczy (Debrecen)

Waligórski, Stanisław 5889

On superpositions of zero-one functions.

Algorytmy **1**, *no.* 2, 91–98 (1963).

Author's summary: "The paper deals with the problem what are the zero-one functions f and g, respectively, determined on partially ordered sets P and Q for which exists an isotone mapping $h: P\to Q$ such that for $x\in P$ we obtain $f(x)=g(h(x))$. It is shown that the existence of the function h depends on the variations of the functions f and g defined in this paper. The solution of this problem is useful for the synthesis of multilevel switching circuits."

Zakrevskii, A. D. [Zakrevskiĭ, A. D.] 5890

On the theory of linear converters for binary series.

Avtomat. i Telemeh. **23** (1962), 1492–1496 (*Russian. English summary*); *translated as Automat. Remote Control* **23** (1963), 1399–1403.

The author considers certain operations on binary sequences and states a number of elementary results concerning the relationship between these operations. The operations in question are: (1) addition modulo 2 of corresponding terms in two sequences; (2) right shifting of a sequence; (3) integration of a sequence, i.e., forming a new sequence whose terms are partial sums of the given sequence; and (4) differentiation, the inverse of (3). The notions of canonical forms for networks consisting of these operations are introduced and transformations are given

for passing from one form to another. The results on canonical forms are related to those obtained by D. E. Muller [I.R.E. Trans. Electronic Computers **EC-3** (1954), no. 3, 6–12] and P. Calingaert [Trans. Amer. Inst. Elect. Engrs. Part I **79** (1960), 808–814].

J. H. Griesmer (Yorktown Heights, N.Y.)

Jeffrey, Richard C. 5891

Finite state transformations.

Information and Control 7 (1964), 45–54.

Author's summary: "A notion of index is defined for transformations which map n-tuples of words into m-tuples of words, and it is proved that the transformations of finite index are precisely those which are effected by finite-state sequential machines, provided these are allowed to have certain unusual features. Further conditions are formulated which, together with the finite index requirement, characterize the transformations which can be effected by finite-state sequential machines of the usual sort."

Lazarev, V. G.; Pill', E. I. 5892

Some classes of finite automata. (Russian)

Ž. Vyčisl. Mat. i Mat. Fiz. **2** (1962), 695–702.

On concrete examples of a general class of finite automata (whose transition-function and output depend not only on the inner state at the momentum t but on the change of this state on the passage of t to $t+1$) the authors illustrate a method to eliminate superfluous inner states.

V. Vučković (Notre Dame, Ind.)

Turski, Wladyslaw 5893

On a possible realization of an automaton determining the domain of practical stability. (Polish. English summary)

Algorytmy Zeszyt Specjalny No. 1 (1963), 21–28.

Author's summary: "The paper contains a description of an automaton that determines the shape of the domain of practical stability Q_0. A digital model of such an automaton is proposed, based on the principle of the Braverman preceptron. The differential equation considered is the following:

$$\text{(R)} \qquad \dot{x} = X(x, t),$$

where x is the n-dimensional vector, $\dot{x}$ its derivative and X a vector function. If the function X satisfies the condition $X(0, t) = 0$ for $t \geqq 0$, the point 0 is the equilibrium solution of the equation (R). The equilibrium solution is said to be practically stable if for any solution of the equation

$$\dot{x} = X(x, t) + p(x, t)$$

with initial conditions

$$x(0) = x^0 \in Q_0,$$

p being an arbitrary function such that $|p(x, t)| < \delta$, there is $x(t) \in Q$, $0 \leqq t \leqq T$, with closed domains Q and Q_0, $0 \in Q_0 \subset Q$, δ and T being non-negative constants."

Liu, C. L. 5894

The memory orders of states of an automaton.

J. Franklin Inst. **277** (1964), 37–49.

The author considers a finite, deterministic automaton with a finite set of input symbols, output symbols, and internal states. Complete knowledge of the initial state and input sequence then determines uniquely the output sequence. As the author points out, an automaton will usually have a finite, fixed memory capability. It behooves one, then, to investigate the possibility of determining the final output under certain memory constraints. This is the problem to which he addresses himself.

D. S. Adorno (Wilton, Conn.)

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Mould, Richard A. **4908**: Internal absorption properties of accelerating detectors.

Mousinho Leite Lopes, Maria Laura. **2470**: ★Fundamental concepts of geometry.

Movčan, A. A. **5630**: On the stability of deformation processes of continuous media.

Moyal, J. E. **1664**: (*with* Adke, S. R.) A birth, death, and diffusion process.

Mozart-Boisvert, Louis. = **Boisvert, Louis-Mozart.**

Mozgovaja, È. A. **5860**: A method of search for the minimum of a function in the presence of constraints.

Mozgovaya, É. A. = **Mozgovaja, È. A.**

Mozrzymas, J. **3717**: The solution of the Feynman-Gell-Mann equation in the spinor space. The finite dimensional representations of the Lorentz group. **3718a**: Rzewuski's iso-space as a curved complex space. **3718b**: The inversion of the angle of helicity as a conformal transformation in Minkowski's space-time.

Mrówka, S. **4516**: An elementary proof of Katětov's theorem concerning Q-spaces. **4515**: (*with* Pervin, W. J.) On uniform connectedness.

Mróz, Zenon. **773**: (*with* Xu, Bing-ye) The load carrying capacities of symmetrically loaded spherical shells.

KEY INDEX

SUBJECT CLASSIFICATION

This is a list of headings, with code numbers, under which articles were classified during 1964. The headings in capital letters are essentially identical with those appearing in the 1964 monthly issues.

00. GENERAL
- 00 General mathematics
- 04 Collections of papers
- 20 Bibliographies
- 30 Dictionaries and other reference works
- 40 Reports of meetings
- 50 Methodology and philosophy of mathematics
  Philosophy of science, general

01. HISTORY AND BIOGRAPHY
- 05 Ancient
- 10 India, Far East, Maya, etc.
- 15 Medieval and Renaissance
- 20 17th century
- 25 18th century
- 30 19th century
- 35 20th century
- 40 Contemporary
- 50 Biographies, obituaries, personalia
- 60 Collected or selected works; reprintings or translations of classics
- 70 Source books

02. LOGIC AND FOUNDATIONS
- 15 Formalizing verbal reasoning
- 20 Syntax, semantics, deducibility
- 30 Propositional calculus: two-valued
- 32 Logical calculus: first-order, two-valued (functional calculus)
- 40 Logistic, axiomatic set theory, paradoxes
- 50 Many-valued logic, modal logic
- 60 Recursive function theory, Turing computability
- 70 Intuitionism
- 75 Axioms for mathematical systems
- 80 Metatheorems for mathematical systems
- 90 Undecidability, unsolvability in mathematical systems
- 95 Abstract automata theory (see also 94.90)

04. SET THEORY
- 15 Point sets
- 20 Relations
- 30 Transfinite numbers
- 35 Problem of the continuum
- 60 Combinatorial (partitions, etc.)

05. COMBINATORIAL ANALYSIS
- 10 Partitioning, subsets, etc.
- 20 Matrices, magic squares, block designs, configurations
- 30 Factorials, binomial coefficients, polynomials, power series
- 40 Graphs
- 50 Enumeration of graphs

06. ORDER, LATTICES
- 10 Total order
- 20 Partial order
- 30 Lattices
- 40 Modular lattices, continuous geometries
- 50 Distributive lattices
- 60 Boolean algebras and rings
- 70 Boolean algebras with operators

08. GENERAL MATHEMATICAL SYSTEMS

10. THEORY OF NUMBERS
- 005 Tables
- 01 Elementary
- 05 Power residues and reciprocity laws, primitive roots, indices; general binomial congruences
- 10 Diophantine equations
- 20 Diophantine approximation
- 25 Continued fractions and other expansions
- 28 Irrationality, transcendence
- 30 Forms
- 40 Geometry of numbers
- 50 Exponential and power sums
- 54 Turán's method
- 55 Modular and automorphic functions, forms, and groups
- 60 Multiplicative asymptotic theory
- 62 Zeta-function and other Dirichlet series
- 64 Character sums, asymptotic theory
- 65 Distribution of primes
- 70 Additive asymptotic theory
- 75 Sequences of integers (additive bases, density theorems, etc.)
- 80 Probabilistic number theory
- 82 Distribution mod 1
- 85 Arithmetic of hypercomplex numbers
- 90 Algebraic number theory
- 95 Class field theory
- 96 Algebraic function fields
- 97 *p*-adic fields
- 98 Finite fields

12. FIELDS AND POLYNOMIALS
- 20 Finite fields
- 30 Polynomials
- 40 Galois theory
- 50 Algebraic number fields
- 60 Algebraic function fields
- 70 Valuations, topological fields
- 80 Differential and difference algebra

14. ABSTRACT ALGEBRAIC GEOMETRY (for classical algebraic geometry, see 50.00)
- 10 Theories of equivalence
- 15 Transformations and correspondences
- 20 Curves
- 35 Algebraic properties of function fields
- 40 Arithmetical properties of varieties
- 45 Algebraic geometry over special fields or over rings; rationality questions
- 50 Group varieties, abstract analytic groups
- 52 Abstract derivations and differentials
- 55 Sheaf-theoretic and homological methods
- 60 Birational invariants, genera, etc.

15. LINEAR ALGEBRA
- 05 Vector spaces
- 15 Linear inequalities
- 20 Inequalities involving eigenvalues and eigenvectors
- 25 Miscellaneous inequalities involving matrices
- 30 Eigenvalues and eigenvectors
- 33 Canonical forms, reductions
- 36 Linear sets of linear transformations
- 38 Matrix equations and identities
- 39 Invariants

40 Multilinear algebra
48 Determinants
49 Permanents
50 Linear equations, matrix inversion

16. ASSOCIATIVE RINGS AND ALGEBRAS
10 Ideal theory
15 Commutative ideal theory
20 General commutative theory
24 Polynomials
26 Formal power series
28 Local rings
30 Division rings, simple rings and algebras
35 Structure of rings
40 Radical theory
45 Rings with chain condition
50 Linear representations of rings and algebras (including orders)
60 Modules
70 Galois theory
80 Ordered rings
90 Topological rings
95 Semi-rings, etc.

17. NON-ASSOCIATIVE RINGS AND ALGEBRAS

18. HOMOLOGICAL ALGEBRA

20. GROUP THEORY AND GENERALIZATIONS
10 Free groups, relations
14 General products of groups
16 Group extensions
20 Permutation groups
25 Finite groups (including finite abelian)
28 Burnside problem, conditions for finiteness
30 Abelian groups, general theory
40 Nilpotent, solvable, p-groups
45 Groups with local properties
50 Lattices of subgroups
55 Special subgroups
60 Crystallographic and discrete geometric groups
65 Modular groups, discrete subgroups of classical groups
70 Classical groups
75 Linear groups, general
80 Representations, characters, group algebras
85 Representations of symmetric groups
88 Automorphisms and endomorphisms, groups with operators
90 Ordered groups, ordered semigroups
92 Semigroups
94 Groupoids
96 Quasigroups
98 Loops

22. TOPOLOGICAL GROUPS AND LIE THEORY
10 Topological groups
12 Haar measures
15 Topological algebras
40 Transformation groups
50 Lie groups
55 Classical groups
60 Representations and analysis
70 Homogeneous spaces (topology, differential geometry)
80 Lie algebras
90 Abstract Lie theory
95 Infinite Lie groups
99 Lie theory of differential equations

26. FUNCTIONS OF REAL VARIABLES
10 Foundations: limits and generalizations [elementary topology of the line (or plane) if used for real variables]
20 Calculus: elementary, advanced, etc.
21 Elementary functions
30 Continuity and related questions (modulus of continuity [if not used in boundary behaviour of analytic functions], discontinuities, etc.)
40 Differentiation: general theory
41 Non-differentiability (non-differentiable functions, points of non-differentiability), discontinuous derivatives
43 Generalized derivatives, derivatives of fractional order
47 Mean-value theorems
50 Monotonic functions, general
51 Functions of bounded variation; absolute continuity
52 Convexity; generalizations
53 Generalized monotonic functions
55 Several variables, implicit function theorems, Jacobians. Transformations with several variables
56 Length, area, volume
60 Polynomials
70 Inequalities (of a general and miscellaneous character)
80 Real-analytic functions
81 Quasi-analytic classes of real functions, functions of class C^∞
85 Superposition of functions

28. MEASURE AND INTEGRATION
10 Measurable sets, general: Borel fields, non-measurable sets
12 Analytic and Suslin sets
13 Finitely additive set functions, measures; generalizations
15 Special and generalized measures, Carathéodory theory of measure
17 Transfinite diameter and capacity. Hausdorff measures and capacities
20 Measurable and non-measurable functions
25 Integration, general
26 Riemann integral, Lebesgue integral, Denjoy integral, etc.
30 Measure-preserving transformations
31 Invariant measures
32 Ergodic theorems
33 Dynamical systems
40 Length, area, volume
41 Other geometric measure theory

30. FUNCTIONS OF A COMPLEX VARIABLE
01 General questions, foundations
02 Monogenic properties of complex functions (also polygenic and areolar monogenic functions)
09 Inequalities in the complex domain
10 Polynomials
11 Zeros of polynomials
20 Power series
21 Behavior on the boundary of convergence. Overconvergence
24 Dirichlet series
25 Continued fractions
28 Analytic continuation
30 Sequences and series of analytic functions; uniform convergence, normal families, normal functions
35 Integration; integral representations; Cauchy integral, Poisson and Poisson-Stieltjes integrals
36 Moment problems; interpolation problem
40 Conformal mapping, general theory
41 Capacity, harmonic measure
42 Univalent and p-valent functions
43 Coefficient problems
45 Riemann surfaces, uniformization
46 Algebraic functions, Abelian integrals
47 Quasiconformal mapping
49 Automorphic functions and modular functions
50 Maximum principle. Schwarz's lemma, Phragmén-Lindelöf theorems
52 Extremal problems in complex domain
55 Entire functions
60 Meromorphic functions, general theory

61 Distribution of values, Picard-type theorems, Nevanlinna theory, defect (deficiency) theory
62 Theory of cluster sets, prime ends, boundary behavior
65 Bounded functions, positive real part, Blaschke products
66 Bounded mean modulus, bounded characteristic
67 H_p classes and other restrictions
70 Approximation by polynomials and rational functions, Faber polynomials
75 Boundary-value problems, Dirichlet problem
77 Special boundary-value problems; Riemann-Hilbert problem. Inverse boundary problems
79 Algebroidal functions
80 Topological function theory, functions topologically equivalent to analytic mappings
81 Generalized analytic mappings. Pseudo-analytic functions of Bers, Vekua, etc.
83 Functions of hypercomplex variables and generalized variables
85 Spaces of analytic functions
86 Kernel function and applications
87 Algebras of analytic functions

31. POTENTIAL THEORY
10 Harmonic functions in the plane
11 Harmonic functions, general
15 Subharmonic, superharmonic functions
20 Boundary-value problems, Dirichlet problem
21 Capacity and harmonic measure
22 Generalized capacity and potential
30 Biharmonic and polyharmonic functions and equations
35 Poisson's equation

32. SEVERAL COMPLEX VARIABLES
02 Foundations, general questions
10 Multiple power series, formal theory
15 Analytic and meromorphic curves
20 Holomorphic mappings, domains of holomorphy, analytic spaces (see also 57.00)
25 Automorphic functions
30 Approximation theorems

33. SPECIAL FUNCTIONS
04 Number-theoretic functions, special sequences
05 Tables of special functions
10 Exponential and trigonometric functions
15 Gamma and Beta functions
17 Error function, probability integral
19 Elliptic functions and integrals
20 Hypergeometric functions
21 E-functions
25 Cylindrical functions; Bessel functions
27 Spherical functions; Legendre polynomials and functions, spherical harmonics, ultraspherical polynomials
28 Lamé, Mathieu, spheroidal wave functions
30 Wave functions
32 Gegenbauer functions
40 Orthogonal polynomials, general (Chebyshev, Hermite, Jacobi, Laguerre polynomials and functions)

34. ORDINARY DIFFERENTIAL EQUATIONS
02 Solutions in closed form; integration by quadratures; reduction of differential equations
03 Operational calculus
04 General existence and uniqueness theorems, Picard approximations and generalizations
07 Continuous dependence of solution on parameter
10 Analytic theory
20 Linear equations and systems, general coefficients
21 Linear equations with constant coefficients
22 Linear equations with periodic and almost periodic coefficients
30 Boundary-value problems, linear equations
31 Sturm-Liouville theory
32 Eigenvalues, eigenfunctions
33 Eigenfunction expansions
34 Asymptotic properties of eigenfunctions
36 Boundary-value problems, non-linear
40 Qualitative theory of first- and second-order equations and systems; singular points and limit cycles
41 Periodic and almost periodic solutions
42 Non-linear oscillations
43 Stability of first- and second-order equations and systems
50 Qualitative theory of equations and systems of order greater than 2; asymptotic behavior of solutions
51 Stability; periodic and almost periodic solutions
53 Perturbations
54 Singular perturbations; small parameter with highest derivative
60 Stochastic differential equations
65 Dynamical systems, systems on manifolds
70 Equations and systems of infinite order
75 Difference-differential equations
76 Solution by finite differences
80 Differential equations in Banach and other spaces
85 Optimal control problems

35. PARTIAL DIFFERENTIAL EQUATIONS
05 General properties of solutions, Cauchy-Kowalewski theorems, etc.
10 First-order equations and systems
12 Classification of equations and systems
13 Exact equations and integrable systems. Pfaffian equations
15 Characteristics
20 Separation of variables, elementary properties of the equations of mathematical physics
30 Elliptic equations and systems of first and second order
31 Strongly elliptic systems
32 Quasi-linear elliptic equations and systems
33 Schrödinger equation, wave equation
34 Maxwell's equations, Helmholtz equation
35 Spectral analysis, eigenfunctions, operators
36 Elliptic systems and equations of higher order
37 Biharmonic equation
40 Hyperbolic equations and systems of second order
41 Quasi-linear hyperbolic systems and equations of second order
45 Hyperbolic equations and systems of higher order
Parabolic equations of second order
55 Equations of mixed type and second order
Mixed type, order higher than two
Non-linear equations and systems
62 Approximation of solutions
Solution by finite differences
65 Special equations: Navier-Stokes, etc.
70 Asymptotic properties
75 Infinite systems, equations of infinite order
Equations in Banach spaces and other spaces

39. FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS
10 Finite differences, general
12 Solution of ordinary differential equations by finite differences
13 Solution of partial differential equations by finite differences
20 Difference equations
30 Functional equations

40. SEQUENCES, SERIES, SUMMABILITY
10 Series and sequences of numbers, general
11 Infinite products
12 Continued fractions
13 Convergence
15 Multiple series and sequences
20 Sequences and series of functions, convergence
30 Summability, general methods
31 Summability, matrix methods
32 Cesàro summability

33 Abel, Poisson summability
35 Summability of double series
40 Convergence and summability of integrals
42 Tauberian theorems

41. APPROXIMATIONS AND EXPANSIONS
10 Interpolation
15 Polynomial approximation
17 Approximation by rational functions
20 Approximate quadratures
40 Rate of convergence, degree of approximation, best approximation
41 Constructive theory of functions
45 Approximation by smoothing integrals
50 Asymptotic approximations
55 Other methods of approximation

42. FOURIER ANALYSIS
05 Trigonometric polynomials
06 Best approximation by trigonometric polynomials
08 Trigonometric interpolation
10 Fourier coefficients
11 Convergence of Fourier series
15 Orthogonal functions
16 Expansions in orthogonal functions
17 Completeness of sets of orthogonal functions
18 Completeness, closure, spectral synthesis
20 Summability of Fourier and generalized Fourier expansions
25 Fourier transform
26 Other Fourier-type transforms
27 Trigonometric moment problems
30 Almost periodic functions
35 Positive definite functions
40 Multiple Fourier series and integrals
50 Abstract harmonic analysis

44. INTEGRAL TRANSFORMS, OPERATIONAL CALCULUS
10 Laplace transform
25 Convolution
28 General transforms
30 Special transforms: Legendre, Hilbert, etc.
32 Multiple transforms
40 Operational calculus
42 Operational calculus in several variables
50 Fractional derivatives and integrals
60 Moment problems

45. INTEGRAL EQUATIONS
10 Linear integral equations, general
11 Fredholm equations
12 Eigenvalue problems
13 Volterra equations
15 Singular equations
20 Systems of linear equations
30 Non-linear integral equations
31 Singular non-linear equations
40 Integro-differential equations

46. FUNCTIONAL ANALYSIS
10 Topological linear spaces (including normed spaces and Hilbert spaces)
20 Partially ordered linear spaces
30 Special function spaces
40 Distributions
50 Single linear operators in linear or partially ordered spaces
60 Groups or semigroups of linear operators, representations of topological groups
65 Rings of operators, group algebras, abstract topological algebras, and their representations
70 Banach algebras of analytic functions
80 Differential and integral operators
90 Applications of functional analysis

49. CALCULUS OF VARIATIONS

50. GEOMETRY
05 Foundations
10 Euclidean
11 Triangles, tetrahedra
12 Circles, spheres
14 Constructions
15 Descriptive geometry
17 Analytic geometry
20 Affine geometry, general
25 Transformations
30 Projective geometry, general
35 Transformations
38 Second-order loci
40 Non-Euclidean geometry (i.e., hyperbolic or elliptic)
46 Minkowski geometry
48 Special geometries
50 Classical algebraic geometry
51 Foundations
52 Transformations
55 Curves
56 Surfaces
57 Abelian varieties, genera, integrals
60 Finite geometries
70 Projective planes
80 Configurations
90 Regular figures, divisions of space
95 Paradoxical decompositions

52. CONVEX SETS AND GEOMETRIC INEQUALITIES
10 Convex polyhedra
25 Convex curves
30 Convex regions
34 Helly-type theorems
40 Extremum problems and geometric inequalities
45 Packing and covering
50 Distance geometries

53. DIFFERENTIAL GEOMETRY
01 Curves and surfaces in Euclidean spaces
04 Minimal surfaces
10 Affine differential geometry
20 Projective differential geometry
25 Conformal differential geometry
30 Non-Euclidean differential geometry
32 Other special differential geometries
35 Vector and tensor analysis
38 Spinor analysis
40 Differentiable manifolds (general theory), e.g., definitions, jets, tangent bundles
42 Differential invariants (local theory), geometric objects
45 Differential calculus (global theory); forms, currents, integration, tensor fields, etc.
50 Connections (general theory), including G-structures
55 Affine connections
60 Projective connections
65 Conformal connections
70 Local Riemannian geometry
72 Riemannian manifolds
75 Global surface theory (convex surfaces)
78 Lorentz metrics and generalizations
80 Kählerian connections and generalizations
85 Finsler spaces and generalizations (areal metrics)
90 Integral geometry
99 Other generalizations (G-spaces of Busemann, etc.)

54. GENERAL TOPOLOGY
10 Axiomatics; generalizations of topological spaces
20 General topological spaces
21 Convergence notions

22 Compactness and generalizations
23 Compactifications, other extensions
24 Proximity spaces
25 Uniform spaces
26 Metric spaces, metrizability
27 Dimension theory
28 Mappings
29 Many-valued mappings
30 Continua
35 Topological semigroups, lattices, etc.
40 Topology of E_2, 2-manifolds
50 Topology of E_n, n-manifolds ($n > 2$)
60 Transformation groups
70 Topological dynamics
80 Fixed-point theorems, coincidence theorems, etc.

55. ALGEBRAIC TOPOLOGY
10 Graphs, map-coloring
20 Knots, links
30 Homology and cohomology theory
32 Sheaves
34 Cohomology (and homology) operations, Steenrod algebra
36 Periodic transformations, fixed points, coincidences (Smith theory)
38 Dimension theory
40 Homotopy theory, general
42 Algebra of mappings (including cohomotopy groups)
45 Computation of homotopy groups
50 Fiber spaces
52 Spectral sequences
60 Manifolds
62 Classification
64 Mappings
66 Generalized manifolds, local homology
68 Homotopy manifolds
70 Imbedding, immersion

57. TOPOLOGY AND GEOMETRY OF DIFFERENTIABLE MANIFOLDS
10 Differentiable structures, cobordism
20 Differentiable mappings, imbeddings, immersions, diffeomorphisms
30 Fiber bundles, tangent bundles
32 Characteristic classes
34 Topology of vector and tensor fields
36 Foliations
40 Topology of Lie groups
45 Topology of homogeneous spaces
50 Analysis on differentiable manifolds
60 Complex manifolds (also in several complex variables)
70 Infinite Lie groups, deformations

60. PROBABILITY
05 Foundations
10 Combinatorial probability theory
15 Geometric probability theory
20 Distributions
30 Limit theorems
40 Stochastic processes, general theory
45 Markov processes
48 Branching processes
50 Stationary processes
55 Random walks, Brownian motion
60 Queueing theory, telephone traffic, storage
62 Renewal theory
65 Other special processes
70 Random noise
90 Other applications

62. STATISTICS
05 Elementary descriptive statistics
10 Distributions of statistical functions
20 Estimation theory: parametric case
25 Testing of hypotheses: parametric case
30 Multivariate analysis, regression analysis
35 Order statistics
40 Non-parametric methods
50 Analysis of experiments
55 Design of experiments
60 Decision theory
65 Multistage decision procedures, sequential analysis
70 Statistical engineering, quality control
75 Life testing
80 Sampling surveys
85 Statistical analysis of stochastic processes, time series analysis
90 Applications

65. NUMERICAL METHODS
05 General mathematical methods, iteration, tables
15 Monte Carlo methods
20 Interpolation, smoothing, least squares, curve fitting, approximation of functions
25 Computation of special functions, series, integrals
30 Mathematical programming
35 Linear equations, determinants, matrices
40 Eigenvalues, eigenvectors, Rayleigh-Ritz method
50 Solution of algebraic and transcendental equations and systems
55 Numerical differentiation and integration, mechanical quadrature
60 Ordinary differential equations
65 Partial differential equations
70 Difference and functional equations
75 Integral and integro-differential equations
80 Error analysis
85 Graphical methods, nomography
90 Harmonic analysis and synthesis

68. COMPUTING MACHINES

69. GENERAL APPLIED MATHEMATICS

70. MECHANICS OF PARTICLES AND SYSTEMS
10 Foundations
20 Statics
30 Kinematics, mechanisms, linkages
40 Dynamics
50 Oscillations, stability
55 Non-linear oscillation
60 Exterior ballistics
70 Variable mass, rockets

73. ELASTICITY, PLASTICITY
05 Foundations of mechanics of deformable solids
06 Elasticity: general theorems
07 Finite deformation
10 Plane stress and strain
15 Three-dimensional problems
25 Beams and rods
30 Plates
32 Shells and membranes
35 Anisotropic bodies
37 Dislocation theory
45 Vibrations, structural dynamics
47 Aeroelasticity
50 Stability, buckling, failure
55 Wave propagation, impact
65 Visco elasticity
70 Plasticity, creep
80 Soil mechanics
90 Thermo-mechanics

76. FLUID MECHANICS, ACOUSTICS
- 05 Foundations
- 10 Incompressible fluids: general theory
- 15 Incompressible fluids with special boundaries
- 17 Airfoil theory
- 20 Free surface flows, water waves, jets, wakes
- 25 Viscous fluids
- 30 Boundary layer theory
- 35 Stability of flow
- 37 Convection
- 40 Turbulence
- 45 Rarefied gas flow
- 50 Compressible fluids: general theory
- 55 Compressible fluids: subsonic flow
- 60 Compressible fluids: transonic flow
- 65 Compressible fluids: supersonic and hypersonic flow
- 70 Shock waves
- 75 Aero- and hydrodynamic sound
- 80 Acoustics
- 85 Non-Newtonian fluids
- 90 Magnetohydro- and aerodynamics, ionized gas flow (see also 78.30 ; 82.25 ; 85.50)
- 92 Quantum hydrodynamics
- 93 Relativistic hydrodynamics
- 95 Diffusion, filtration

78. OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS
- 05 Geometric optics
- 10 Physical optics
- 20 Electron optics
- 30 Space charge waves
- 40 Electromagnetic theory
- 45 Electro- and magnetostatics
- 50 Waves and radiation
- 60 Diffraction, scattering
- 70 Antennas, wave-guides
- 80 Circuits, networks (for switching theory, see 94.70)
- 90 Technical applications

80. CLASSICAL THERMODYNAMICS, HEAT TRANSFER
- 10 Classical thermodynamics
- 20 Heat and mass transfer
- 30 Combustion, interior ballistics
- 40 Chemical kinetics

81. QUANTUM MECHANICS
- 10 General theory
- 15 Scattering theory
- 20 Mathematical field theory
- 22 Dispersion relations
- 24 Weak interactions
- 26 Elementary particles
- 28 Nuclear physics
- 30 Atomic physics
- 38 Molecular physics
- 50 Quantum mechanics of many-body systems
- 60 Superconductivity and superfluidity
- 70 Quantum statistical mechanics

82. STATISTICAL PHYSICS, STRUCTURE OF MATTER
- 05 General statistical mechanics
- 10 Quantum statistical mechanics
- 15 Statistical thermodynamics
- 17 Irreversible thermodynamics
- 20 Kinetic theory of gases
- 25 Plasma (see also 76.90)
- 30 Liquids
- 40 Solids
- 45 Crystals
- 50 Metals
- 60 Transport processes
- 65 Nuclear reactor theory

83. RELATIVITY
- 10 Special relativity
- 20 General relativity
- 30 Unified field theories
- 40 Other relativistic theories

85. ASTRONOMY
- 10 Celestial mechanics
- 15 Galactic and stellar dynamics
- 20 3- and n-body problems
- 27 Astronautics
- 30 Stellar structure
- 40 Stellar atmospheres, radiative transfer
- 50 Hydrodynamic and hydromagnetic problems
- 60 Statistical astronomy
- 70 Cosmology
- 80 Special problems
- 90 Radio astronomy

86. GEOPHYSICS
- 10 Hydrology, hydrography, oceanography
- 20 Meteorology
- 30 Seismology
- 40 Potentials, prospecting
- 50 Geo-electricity and magnetism
- 60 Geodesy, mapping problems
- 70 Atmospheric physics

90. ECONOMICS, OPERATIONS RESEARCH, GAMES
- 05 Decisions, utility
- 10 Economic models
- 15 Economic time-series analysis
- 17 Miscellaneous applications to economics
- 30 Management science, operations research
- 35 Highway traffic
- 40 Actuarial theory
- 50 Linear programming
- 52 Transportation problems
- 54 Flows in networks
- 56 Integer programming
- 58 Non-linear programming
- 60 Logistics, inventory, storage
- 70 Games

92. BIOLOGY AND BEHAVIORAL SCIENCES
- 10 Biology
- 20 Genetics
- 30 Population dynamics, epidemiology
- 40 Sociology
- 50 Psychology
- 55 Nervous networks

94. INFORMATION, COMMUNICATION, CONTROL
- 05 Foundations, coding theorem
- 10 Statistical theory of communication channels, filters
- 13 Detection
- 15 Codes, decoding
- 20 Data processing
- 30 Linguistics, machine translation
- 55 Control systems
- 57 Stability of control systems
- 60 Optimal control
- 65 Random control
- 70 Switching theory, relays
- 80 Servomechanisms
- 90 Automata (see also 02.95)

ABBREVIATIONS OF NAMES OF JOURNALS

This list gives the form of references used in MATHEMATICAL REVIEWS and the complete title; the place of publication and other pertinent information are given in parentheses when desirable for clarity.

Abh. Akad. Wiss. Göttingen Math.-Phys. Kl. Abhandlungen der Akademie der Wissenschaften in Göttingen. Mathematisch-Physikalische Klasse. (Göttingen)

Abh. Deutsch. Akad. Wiss. Berlin Kl. Math. Phys. Tech. Abhandlungen der Deutschen Akademie der Wissenschaften zu Berlin. Klasse für Mathematik, Physik und Technik. (Berlin)

Abh. Dokumentationszentrums Technik Wirtschaft. Abhandlungen des Dokumentationszentrums der Technik und Wirtschaft. (Vienna)

Abh. Math. Sem. Univ. Hamburg. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg. (Hamburg)

Abh. Sächs. Akad. Wiss. Leipzig Math.-Natur. Kl. Abhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig. Mathematisch-Naturwissenschaftliche Klasse. (Leipzig)

Acad. R. P. Romîne An. Romîno-Soviet. Ser. Mat.-Fiz. Academia Republicii Populare Romîne. Institutul de Studii Romîno-Sovietic. Analele Romîno-Sovietice. Seria Matematică-Fizică. Rumyno-Sovetskie Zapiski. Fiziko-Matematičeskaja Serija. (Bucharest)

Acad. R. P. Romîne Baza Cerc. Şti. Timişoara Stud. Cerc. Şti. Tehn. Academia Republicii Populare Romîne. Baza de Cercetări Ştiinţifice Timişoara. Studii şi Cercetări Ştiinţe Tehnice. (Timişoara)

Acad. R. P. Romîne Fil. Cluj Stud. Cerc. Mat. Academia Republicii Populare Romîne. Institutul de Calcul. Filiala Cluj. Studii şi Cercetări de Matematică. (Cluj)

Acad. R. P. Romîne Fil. Iaşi Stud. Cerc. Şti. Fiz. Şti. Tehn. Academia Republicii Populare Romîne. Filiala Iaşi. Studii şi Cercetări Stiinţifice. Fizică şi Ştiinţe Tehnice. (Iaşi)

Acad. R. P. Romîne Fil. Iaşi Stud. Cerc. Şti. Mat. Academia Republicii Populare Romîne. Filiala Iaşi. Studii şi Cercetări Ştiinţifice. Matematică. (Iaşi)

Acad. R. P. Romîne Stud. Cerc. Mat. Academia Republicii Populare Romîne. Institutul de Matematică. Studii şi Cercetări Matematice. Matematičeskie Trudy i Issledovanija. Études et Recherches Mathématiques. (Bucharest)

Acad. R. P. Romîne Stud. Cerc. Mec. Apl. Academia Republicii Populare Romîne. Institutul de Mecanică Aplicată "Traian Vuia". Studii şi Cercetări de Mecanică Aplicată. (Bucharest)

Acad. Roy. Belg. Bull. Cl. Sci. Académie Royale de Belgique. Bulletin de la Classe des Sciences. Koninklijke Belgische Academie. Mededelingen van de Klasse der Wetenschappen. (Brussels)

Acad. Roy. Belg. Cl. Sci. Mém. Coll. in-8°. Académie Royale de Belgique. Classe des Sciences. Mémoires. Collection in-8°. Koninklijke Belgische Academie. Klasse der Wetenschappen. Verhandelingen. Verzameling in-8°. (Brussels)

Acad. Serbe Sci. Arts Glas Cl. Sci. Math. Natur. Académie Serbe des Sciences et des Arts. Glas. Classe des Sciences Mathématiques et Naturelles. Nouvelle Série. (Belgrade)

Accad. Naz. Sci. Lett. Arti Modena Atti Mem. Accademia Nazionale di Scienze Lettere e Arti. Modena. Atti e Memorie. (Modena)

Acta Acad. Abo. Math. Phys. Acta Academiae Aboensis. Mathematica et Physica. (Åbo = Turku)

Acta Arith. Polska Akademia Nauk. Instytut Matematyczny. Acta Arithmetica. (Warsaw)

Acta Astronom. Sinica. Acta Astronomica Sinica. (Peking)

Acta Cryst. Acta Crystallographica. (Copenhagen)

Acta Fac. Natur. Univ. Comenian. Acta Facultatis Rerum Naturalium Universitatis Comenianae. (Bratislava)

Acta Math. Acta Mathematica. (Uppsala)

Acta Math. Acad. Sci. Hungar. Acta Mathematica Academiae Scientiarum Hungaricae. (Budapest)

Acta Math. Sinica. Acta Mathematica Sinica. (Peking) (Translated as: *Chinese Math.*)

Acta Mech. Sinica. Acta Mechanica Sinica. (Peking)

Acta Philos. Fenn. Acta Philosophica Fennica. (Helsinki)

Acta Phys. Acad. Sci. Hungar. Acta Physica Academiae Scientiarum Hungaricae. (Budapest)

Acta Phys. Austriaca. Acta Physica Austriaca. (Vienna)

Acta Phys. Polon. Polska Akademia Nauk. Instytut Fizyki. Acta Physica Polonica. (Warsaw)

Acta Polytech. Scand. Math. and Comput. Mach. Ser. Acta Polytechnica Scandinavica. Mathematics and Computing Machinery Series. (Stockholm) (Formerly: *Acta Polytech. Scandinav.*)

Acta Polytech. Scandinav. Acta Polytechnica Scandinavica. (Stockholm) (Continued as: *Acta Polytech. Scand. Math. and Comput. Mach. Ser.*)

Acta Salmant. Ci. Acta Salmanticensia. Ciencias. Neuva Serie. (Salamanca)

Acta Sci. Math. (Szeged). Acta Universitatis Szegediensis. Acta Scientiarum Mathematicarum. (Szeged)

Acta Tech. Acad. Sci. Hungar. Acta Technica Academiae Scientiarum Hungaricae. (Budapest)

Acta Univ. Debrecen. Acta Universitatis Debreceniensis de Ludovico Kossuth Nominatae. (Debrecen)

Acta Univ. Lund. Sect. II. Acta Universitatis Lundensis. Sectio II. Medica, Mathematica, Scientiae rerum naturalium. (Lund) (Formerly: *Lunds Univ. Årsskr. Avd. 2*)

Actes Soc. Helv. Sci. Natur. Actes de la Société Helvétique des Sciences Naturelles. Verhandlungen der Schweizerischen Naturforschenden Gesellschaft. Atti della Società Elvetica di Scienze Naturali. (Basel)

Actualités Sci. Indust. Actualités Scientifiques et Industrielles. (Paris)

Acustica. Acustica. (Zurich)

Advances in Math. Advances in Mathematics. (New York)

Advances in Phys. Advances in Physics. A Quarterly Supplement of the Philosophical Magazine. (London)

Aero. Quart. The Aeronautical Quarterly. (London)

AIAA J. AIAA Journal. (Easton, Pa.)

Akad. Nauk Armjan. SSR Dokl. Akademija Nauk Armjanskoĭ SSR. Doklady. (Erevan)

Akad. Nauk Azerbaĭdžan. SSR Dokl. Akademija Nauk Azerbaĭdžanskoĭ SSR. Doklady. (Baku)

Akad. Nauk Azerbaĭdžan. SSR Trudy Inst. Mat. Meh. Akademija Nauk Azerbaĭdžanskoĭ SSR. Trudy Instituta Matematiki i Mehaniki. (Baku)

Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze. Akademija Nauk Gruzinskoĭ SSR. Trudy Tbilisskogo Matematičeskogo Instituta im. A. M. Razmadze. (Tiflis)

Akad. Nauk Kazah. SSR Trudy Astrofiz. Inst. Akademija Nauk Kazahskoĭ SSR. Trudy Astrofizičeskogo Instituta. (Alma-Ata)

Akad. Nauk Kazah. SSR Trudy Sekt. Mat. Meh. Akademija Nauk Kazahskoĭ SSR. Trudy Sektora Matematiki i Mehaniki. (Alma-Ata)

Akad. Nauk Latv. SSR Trudy Inst. Fiz. Akademija Nauk Latviĭskoĭ SSR. Trudy Instituta Fiziki. (Riga)

Akad. Nauk SSSR Izv. Sibirsk. Otd. Akademija Nauk SSSR. Izvestija Sibirkskogo Otdelenija Akademii Nauk SSSR. (Novosibirsk) (Continued as: *Izv. Sibirsk. Otd. Akad. Nauk SSSR Ser. Tehn. Nauk*)

Akad. Nauk Uzbek. SSR Trudy Inst. Mat. Akademija Nauk Uzbekskoĭ SSR. Trudy Instituta Matematiki im. V. I. Romanovskogo. (Tashkent)

Akad. Wiss. Lit. Mainz Abh. Math.-Natur. Kl. Akademie der Wissenschaften und der Literatur in Mainz. Abhandlungen der Mathematisch-Naturwissenschaftlichen Klasse. (Wiesbaden)

Akust. Beihefte. Akustische Beihefte. (Stuttgart)

Akust. Ž. Akademija Nauk SSSR. Akustičeskiĭ Žurnal. (Moscow) (Translated as: *Soviet Physics Acoust.*)

Algebra i Logika Sem. Akademija Nauk SSSR. Sibirskoe Otdelenie. Institut Matematiki. Algebra i Logika. Seminar. (Novosibirsk)

Algorytmy. Algorytmy. Prace. Instytutu Maszyn Matematycznych. Polskiej Akademii Nauk. (Warsaw)

Allgemein. Statist. Arch. Allgemeines statistisches Archiv. Organ der Deutschen Statistischen Gesellschaft. (Munich)

Amer. J. Math. American Journal of Mathematics. (Baltimore, Md.)

Amer. J. Phys. American Journal of Physics. (New York)

Amer. Math. Monthly. The American Mathematical Monthly. (Buffalo, N.Y.)

Amer. Math. Soc. Notices. American Mathematical Society Notices. (Providence, R.I.) (Continued as: *Notices Amer. Math. Soc.*)

Amer. Math. Soc. Transl. American Mathematical Society Translations. (Providence, R.I.)

AMR. Applied Mechanics Reviews. (New York)

An. Acad. Brasil. Ci. Anais da Academia Brasileira de Ciencias. (Rio de Janeiro)

An. Inst. Mat. Univ. Nac. Autónoma México. Anales del Instituto de Matemáticas. Universidad Nacional Autónoma de México. (Mexico City)

An. Soc. Ci. Argentina. Anales de la Sociedad Cientifica Argentina. (Buenos Aires)

An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I. Analele Ştiinţifice ale Universităţii "Al. I. Cuza" din Iaşi. (Seria Nouă). Secţiunea I. (Matematică, Fizică, Chimie). (Iaşi)

An. Univ. Bucureşti Ser. Acta Logica. Analele Universităţii Bucureşti. Seria Acta Logica. (Bucharest)

An. Univ. Bucureşti Ser. Şti. Natur. Mat.-Fiz. Analele Universităţii Bucureşti. Seria Ştiinţele Naturii. Matematică-Fizică. (Bucharest) (Formerly: *Ann. Univ. C. I. Parhon Ser. Şti. Natur. Mat.-Fiz.*)

An. Univ. C. I. Parhon Ser. Şti. Natur. Mat.-Fiz. Analele Universităţii C. I. Parhon. Seria Ştiinţele Naturii. Matematică-Fizică. (Bucharest) (Continued as: *An. Univ. Bucureşti Ser. Şti. Natur. Mat.-Fiz.*)

Ann. Acad. Sci. Fenn. Ser. A I. Suomalaisen Tiedeakatemian Toimituksia. Sarja A. Annales Academiae Scientiarum Fennicae. Series A. I. Mathematica. (Helsinki)

Ann. Acad. Sci. Fenn. Ser. A VI. Suomalaisen Tiedeakatemian Toimituksia. Sarja A. Annales Academiae Scientiarum Fennicae. Series A. VI. Physica. (Helsinki)

Ann. Assoc. Internat. Calcul Anal. Annales de l'Association Internationale pour le Calcul Analogique. Proceedings of the International Association for Analog Computation. (Brussels)

Ann. Astrophys. Annales d'Astrophysique. (Paris)

Ann. Fac. Sci. Univ. Toulouse. Annales de la Faculté des Sciences de l'Université de Toulouse pour les Sciences Mathématiques et les Sciences Physiques. (Toulouse)

Ann. Geofis. Annali di Geofisica. Rivista dell'Istituto Nazionale di Geofisica. (Rome)

Ann. Inst. Fourier (Grenoble). Université de Grenoble. Annales de l'Institut Fourier. (Grenoble)

Ann. Inst. H. Poincaré. Annales de l'Institut Henri Poincaré. (Paris)

Ann. Inst. Statist. Math. Annals of the Institute of Statistical Mathematics. (Tokyo)

Ann. Japan Assoc. Philos. Sci. Annals of the Japan Association for Philosophy of Science. (Tokyo)

Ann. Mat. Pura Appl. Annali di Matematica Pura ed Applicata. (Bologna)

Ann. Math. Statist. The Annals of Mathematical Statistics. (Baltimore, Md.)

Ann. New York Acad. Sci. Annals of the New York Academy of Sciences. (New York)

Ann. of Math. Annals of Mathematics. (Princeton, N.J.)

Ann. of Sci. Annals of Science. (London)

Ann. Physics. Annals of Physics. (New York)

Ann. Physik. Annalen der Physik. (Leipzig)

Ann. Physique. Annales de Physique. (Paris)

Ann. Polon. Math. Polska Akademia Nauk. Annales Polonici Mathematici. (Warsaw)

Ann. Ponts Chaussées. Annales de Ponts et Chaussées. (Paris)

Ann. Radioélec. Annales de Radioélectricité. (Paris)

Ann. Sci. École Norm. Sup. Annales Scientifiques de l'École Normale Supérieure. (Paris)

Ann. Scuola Norm. Sup. Pisa. Annali della Scuola Normale Superiore de Pisa. Scienze Fisiche e Matematiche. (Pisa)

Ann. Soc. Sci. Bruxelles Sér. I. Annales de la Société Scientifique de Bruxelles. Série I. Sciences Mathématiques, Astronomiques et Physiques. (Brussels)

Ann. Télécommun. Annales des Télécommunications. (Paris)

Ann. Univ. Lyon Sect. A. Annales de l'Université de Lyon. Section A. Sciences Mathématiques et Astronomie. (Lyon)

Ann. Univ. Lyon Sect. B. Annales de l'Université de Lyon. Section B. Sciences Physiques et Chimiques. (Lyon)

Ann. Univ. Mariae Curie-Sklodowska Sect. A. Annales Universitatis Mariae Curie-Sklodowska. Section A. Mathematics. (Lublin)

Ann. Univ. Mariae Curie-Sklodowska Sect. AA. Annales Universitatis Mariae Curie-Sklodowska. Section AA. Physics and Chemistry. (Lublin)

Ann. Univ. Sarav. Annales Universitatis Saraviensis. (Saarbrücken) (Continued as: *Ann. Univ. Sarav. Math.-Natur. Fak.*)

Ann. Univ. Sarav. Math.-Natur. Fak. Annales Universitatis Saraviensis. Mathematisch-Naturwissenschaftliche Fakultät. (Berlin) (Formerly: *Ann. Univ. Sarav.*)

Ann. Univ. Sci. Budapest. Eötvös Sect. Math. Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae. Sectio Mathematica. (Budapest)

Ann. Univ. Turku Ser. A I. Annales Universitatis Turkuensis. Series A. I. Astronomica-Chemica-Physica-Mathematica. (Turku)

Annuaire Acad. Roy. Belg. Annuaire de l'Académie Royale de Belgique. Jaarboek van de Koninklijke Belgische Akademie. (Brussels)

Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 1 Math. Godišnik na Sofiĭskija Universitet Fiziko-Matematičeski Fakultet. Kniga 1—Matematika. Annuaire de l'Université de Sofia. Faculté des Sciences Physiques et Mathématiques. Livre 1—Mathématiques. (Sofia)

Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 2 Phys. Godišnik na Sofiĭskija Universitet Fiziko-Matematičeski Fakultet. Kniga 2—Fizika. Annuaire de l'Université de Sofia. Faculté des Sciences Physiques et Mathématiques. Livre 2—Physique. (Sofia)

Anuário Obs. São Paulo. Universidade de São Paulo. Instituto Astrônomico e Geofísico. Anuário do Observatório de S. Paulo. (São Paulo)

Apl. Mat. Československá Akademie Věd. Aplikace Matematiky. (Prague)

Appl. Mech. Rev. Applied Mechanics Reviews. (New York) (Sometimes abbreviated as *AMR* in the text of *Math. Reviews*)

Appl. Statist. Applied Statistics. A Journal of the Royal Statistical Society. (London)

Årbok Univ. Bergen Mat.-Natur. Ser. Årbok for Universitetet i Bergen. Mat.-Naturv. Serie. (Bergen)

Arch. Elektrotech. Archiv für Elektrotechnik. (Berlin-Göttingen-Heidelberg)

Arch. History Exact Sci. Archive for History of Exact Sciences. (Berlin)

Arch. Math. Archiv der Mathematik. Archives of Mathematics. Archives Mathématiques. (Basel-Stuttgart)

Arch. Math. Logik Grundlagenforsch. Archiv für mathematische Logik und Grundlagenforschung. (Stuttgart)

Arch. Mech. Stos. Polska Akademia Nauk. Instytut Podstawowych Problemów Techniki. Archiwum Mechaniki Stosowanej. (Warsaw)

Arch. Rational Mech. Anal. Archive for Rational Mechanics and Analysis. (Berlin)

Arch. Sci. Soc. Phys. Histoire Natur. Genève. Archives des Sciences. Société de Physique et d'Histoire Naturelle de Genève. (Geneva)

Archimede. Archimede. Rivista per gli Insegnanti e i Cultori di Matematiche Pure e Applicate. (Florence)

Ark. Astronom. Arkiv för Astronomi. (Stockholm)

Ark. Fys. Arkiv för Fysik. (Stockholm)

Ark. Mat. Arkiv för Matematik. (Stockholm)

Arkhimedes. Arkhimedes. Suomen Fyysikkoseura—Finlands Fysikerförening r.y. Suomen Matemaattinen Yhdistys—Finlands Matematiska Förening r.y. (Helsinki)

Arquivo Inst. Gulbenkian Ci. Sec. A Estud. Mat. Fís.-Mat. Arquivo do Instituto Gulbenkian de Ciência. Secção A. Estudos Matemáticos e Físico-Matemáticos. (Lisbon)

Assoc. Roy. Actuair. Belges Bull. Association Royale des Actuaires Belges. Bulletin. (Brussels)

Astronaut. Acta. Astronautica Acta. (Vienna)

Astronom. J. The Astronomical Journal. (New Haven, Conn.)

Astronom. Jber. Astronomischer Jahresbericht. (Berlin)

Astronom. Nachr. Astronomische Nachrichten. (Berlin)

Astronom. Ž. Akademija Nauk Sojuza SSR. Astronomičeskiĭ Žurnal. (Moscow) (Translated as: *Soviet Astronom. AJ*)

Astrophys. J. The Astrophysical Journal. (Chicago, Ill.)

Astrophys. Norvegica. Det Norske Videnskaps-Akademi i Oslo. Astrophysical Norvegica. (Oslo)

Atatürk Üniv. Yayınları Ser.-Mat. Atatürk Üniversitesi Yayınları. Araştırmalar Serisi—Matematik. Publications of Atatürk University. Research Series—Mathematics. (Erzurum)

Atti Accad. Gioenia Catania. Atti della Accademia Gioenia di Scienze Naturali in Catania. (Catania)

Atti Accad. Ligure. Atti della Accademia Ligure di Scienze e Lettere. (Genoa)

Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. I. Atti della Accademia Nazionale dei Lincei. Memorie. Classe di Scienze Fisiche, Matematiche e Naturali. Sezione Iª. (Matematica, Meccanica, Astronomia, Geodesia e Geofisica). (Rome)

Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Natur. Sez. II. Atti della Accademia Nazionale dei Lincei. Memorie. Classe di Scienze Fisiche, Matematiche e Naturali. Sezione IIª. (Fisica, Chimica, Geologia, Paleontologia e Mineralogia). (Rome)

Atti Accad. Naz. Lincei Rend. Adunanze Solenni. Atti della Accademia Nazionale dei Lincei. Rendiconti delle Adunanze Solenni. (Rome)

Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali. (Rome)

Atti Accad. Sci. Fis. Mat. Napoli. Atti dell'Accademia delle Scienze Fisiche e Matematiche di Napoli. (Naples)

Atti Accad. Sci. Ist. Bologna Cl. Sci. Fis. Mem. Atti della Accademia delle Scienze dell'Istituto di Bologna. Classe di Scienze Fisiche. Memorie. (Bologna)

Atti Accad. Sci. Ist. Bologna Cl. Sci. Fis. Rend. Atti della Accademia delle Scienze dell'Istituto di Bologna. Classe di Scienze Fisiche. Rendiconti. (Bologna)

Atti Accad. Sci. Lett. Arti Palermo Parte I. Atti della Accademia di Scienze Lettere e Arti di Palermo. Parte Prima: Scienze. (Palermo)

Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. Atti della Accademia delle Scienze di Torino. Classe di Scienze Fisiche, Matematiche e Naturali. (Turin)

Atti Sem. Mat. Fis. Univ. Modena. Atti del Seminario Matematico e Fisico dell'Università di Modena. (Modena)

Austral. J. Appl. Sci. Australian Journal of Applied Science. (Melbourne)

Austral. J. Phys. Australian Journal of Physics. (Melbourne)

Austral. J. Sci. The Australian Journal of Science. (Sydney)

Austral. J. Statist. The Australian Journal of Statistics. (Sydney)

Avh. Norske Vid.-Akad. Oslo I. Avhandlinger Utgitt av det Norske Videnskaps-Akademi i Oslo. I. Mat.-Naturv. Klasse. (Oslo)

Avtomat. i Telemeh. Akademija Nauk Sojuza SSR. Avtomatika i Telemehanika. (Moscow) (Translated as: *Automat. Remote Control*)

Avtomatika. Akademija Nauk Ukraïns'koï RSR. Institut Elektrotehniki. Avtomatika. (Kiev)

Azerbaĭdžan. Gos. Univ. Učen. Zap. Ser. Fiz.-Mat. i Him. Nauk. Azerbaĭdžanskiĭ Gosudarstvennyĭ Universitet im. S. M. Kirova. Učenye Zapiski. Serija Fiziko-Matematičeskih i Himičeskih Nauk. (Baku)

Bayer. Akad. Wiss. Math.-Natur. Kl. Abh. Bayerische Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse. Abhandlungen. (Munich)

Bayer. Akad. Wiss. Math.-Natur. Kl. S.-B. Bayerische Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse. Sitzungsberichte. (Munich)

Behavioral Sci. Behavioral Science. (Baltimore, Md.-Ann Arbor, Mich.)

Bell System Tech. J. The Bell System Technical Journal. (New York)

Biometrics. Biometrics. Journal of the Biometric Society. (Blacksburg, Va.)

Biometrika. Biometrika. (London)

BIT = Nordisk Tidskr. Informations-Behandling.

Bjull. Inst. Teoret. Astronom. Akademija Nauk Sojuza Sovetskih Socialističeskih Respublik. Bjulleten' Instituta Teoretičeskoĭ Astronomii. (Moscow)

Bol. Fac. Ingen. Agrimens. Montevideo. Boletín de la Facultad de Ingeniería y Agrimensura de Montevideo. (Montevideo)

Bol. Soc. Mat. Mexicana. Boletín de la Sociedad Matemática Mexicana. (Mexico, D.F.)

Bol. Soc. Mat. São Paulo. Boletim da Sociedade de Matemática de São Paulo. (São Paulo)

Bol. Univ. Paraná Fís. Teórica. Boletim da Universidade do Paraná. Conselho de Pesquisas. Instituto de Física. Física Teórica. (Curitiba)

Boll. Un. Mat. Ital. Bolletino della Unione Matematica Italiana. (Bologna)

Bonn. Math. Schr. Bonner Mathematische Schriften. (Bonn)

British J. Philos. Sci. The British Journal for the Philosophy of Science. (Edinburgh)

Bul. Inst. Politehn. București. Buletinul Institutului Politehnic București. (Bucharest)

Bul. Inst. Politehn. Iași. Buletinul Institutului Politehnic din Iași. Serie Nouă. (Iași)

Bul. Ști. Inst. Politehn. Cluj. Buletinul Științific al Institutului Politehnic din Cluj. (Cluj)

Bŭlgar. Akad. Nauk. Izv. Mat. Inst. Bŭlgarska Akademija na Naukite. Otdelenie za Fiziko-Matematičeski i Tehničeski Nauki. Izvestija na Matematičeskija Institut. (Sofia)

Bŭlgar. Akad. Nauk. Otd. Fiz.-Mat. Nauk. Izv. Fiz. Inst. s Aneb. Bŭlgarska Akademija na Naukite. Otdelenie za Fiziko-Matematičeski Nauki. Izvestija na Fizičeskija Institut s Aneb. (Sofia)

Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques. (Warsaw)

Bull. Acad. Serbe Sci. Arts Cl. Sci. Math. Natur. Sci. Math. Bulletin de l'Académie Serbe des Sciences et des Arts. Classe des Sciences Mathématiques et Naturelles Sciences. Mathématiques. (Belgrade) (Formerly: *Bull. Acad. Serbe Sci. Cl. Sci. Math. Nat. Sci. Math.*)

Bull. Acad. Serbe Sci. Cl. Sci. Math. Natur. Sci. Math. Bulletin de l'Académie Serbe des Sciences. Classe des Sciences Mathématiques et Naturelles Sciences. Mathématiques. (Belgrade) (Continued as: *Bull. Acad. Serbe Sci. Arts. Cl. Sci. Math. Natur. Sci. Math.*)

Bull. Amer. Math. Soc. Bulletin of the American Mathematical Society. (Providence, R.I.)

Bull. Astronom. Bulletin Astronomique. (Paris)

Bull. Astronom. Inst. Netherlands. Bulletin of the Astronomical Institutes of the Netherlands. (Haarlem)

Bull. Calcutta Math. Soc. Bulletin of the Calcutta Mathematical Society. (Calcutta)

Bull. College Sci. (Baghdad). Bulletin of the College of Science. (Baghdad)

Bull. Earthquake Res. Inst. Tokyo. Bulletin of the Earthquake Research Institute, University of Tokyo. (Tokyo) See *Bull. Earthquake Res. Inst. Univ. Tokyo.*

Bull. Earthquake Res. Inst. Univ. Tokyo. Bulletin of the Earthquake Research Institute, University of Tokyo. (Tokyo) (Formerly listed as: *Bull. Earthquake Res. Inst. Tokyo*)

Bull. Fukuoka Gakugei Univ. III. Bulletin of Fukuoka Gakugei University. III. Natural Sciences. (Fukuoka)

Bull. Géodésique. Bulletin Géodésique. (Paris)

Bull. Inst. Internat. Statist. Bulletin de l'Institut International de Statistique. (The Hague)

Bull. (Izv.) Acad. Sci. USSR Geophys. Ser. Bulletin (Izvestiya) Academy of Sciences, USSR. Geophysics Series. (Washington, D.C.) (Translation of *Izv. Akad. Nauk SSSR Ser. Geofiz.*)

Bull. JSME. Bulletin of JSME. (Japan Society of Mechanical Engineers). (Tokyo)

Bull. Kyoto Gakugei Univ. Ser. B. Bulletin of the Kyoto Gakugei University. Ser. B. Mathematics and Natural Science. (Kyoto)

Bull. Kyushu Inst. Tech. Math. Natur. Sci. Bulletin of the Kyushu Institute of Technology. Mathematics, Natural Science. (Tobata)

Bull. Math. Biophys. The Bulletin of Mathematical Biophysics. (Chicago, Ill.)

Bull. Math. Soc. Nanyang Univ. Majallah Tahunan 'Ilmu Pasti. Bulletin of Mathematical Society. Nanyang University. (Singapore)

Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine. Bulletin Mathématique de la Société des Sciences Mathématiques et Physiques de la République Populaire Roumaine. (Bucharest)

Bull. Math. Statist. Bulletin of Mathematical Statistics. (Fukuoka)

Bull. Res. Council Israel Sect. F. Bulletin of the Research Council of Israel. Section F. Mathematics and Physics. (Jerusalem) (Continued as: *Israel J. Math.*)

Bull. Sci. Conseil Acad. RSF Yougoslav. Bulletin Scientifique. Conseil des Académies de la RSF de Yougoslavie. (Zagreb)

Bull. Sci. Math. Bulletin des Sciences Mathématiques. (Paris)

Bull. Signal. 1 *Math. Pures Appl.* Ministère de l'Éducation Nationale. Centre National de la Recherche Scientifique. Bulletin Signalétique. 1. Mathématiques Pures et Appliquées. (Paris)

Bull. Soc. Amis Sci. Lettres Poznań Sér. B. Bulletin de la Société des Amis des Sciences et des Lettres de Poznań. Série B: Sciences Mathématiques et Naturelles. (Poznań)

Bull. Soc. Math. Belg. Bulletin de la Société Mathématique de Belgique. (Brussels)

Bull. Soc. Math. France. Bulletin de la Société Mathématique de France. (Paris)

Bull. Soc. Math. Grèce. Bulletin de la Société Mathématique de Grèce. (Athens)

Bull. Soc. Math. Phys. Macédoine. Bulletin de la Société des Mathématiciens et des Physiciens de la République Populaire de Macédoine. Bilten na Društvoto na Mathematičarite i Fizičarite od Narodna Republika Makedonija. (Skopje)

Bull. Soc. Math. Phys. Serbie. Bulletin de la Société des Mathématiciens et Physiciens de la R. P. de Serbie. Vesnik Društva Matematičara i Fizičara Narodne Republike Srbije. (Belgrade) (Continued as: *Mat. Vesnik*)

Bull. Soc. Roy. Sci. Liège. Bulletin de la Société Royale des Sciences de Liège. (Liège)

Bull. Soc. Sci. Lettres Łódź. Bulletin de la Société des Sciences et des Lettres de Łódź. (Łódź)

Bull. Tokyo Gakugei Univ. Bulletin of Tokyo Gakugei University. (Tokyo)

Bull. Trimest. Inst. Actuaires Franç. Bulletin Trimestriel de l'Institut des Actuaires Français. (Paris)

Bull. Univ. Osaka Prefecture Ser. A. Bulletin of University of Osaka Prefecture. Series A. Engineering and Natural Sciences. (Osaka)

C. R. Acad. Bulgare Sci. Doklady Bolgarskoĭ Akademii Nauk. Comptes Rendus de l'Académie Bulgare des Sciences. (Sofia)

C. R. Acad. Sci. Paris. Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. (Paris)

Cahiers Centre Études Rech. Opér. Cahiers du Centre d'Études de Recherche Opérationnelle. (Brussels) See *Cahiers Centre Études Recherche Opér.*

Cahiers Centre Études Recherche Opér. Cahiers du Centre d'Études de Recherche Opérationnelle. (Brussels) (Formerly listed as: *Cahiers Centre Études Rech. Opér.*)

Cahiers Centre Math. Statist. Appl. Sci. Social. Cahiers du Centre de Mathématique et de Statistique Appliquées aux Sciences Sociales de l'Institut de Sociologie Solvay. (Brussels)

Cahiers de Phys. Cahiers de Physique. (Paris)

Cahiers Rhodan. Cahiers Rhodaniens. (Lyon)

Calc. Automat. y Cibernet. Calculo Automatico y Cibernetica. (Madrid)

Calcutta Statist. Assoc. Bull. Calcutta Statistical Association Bulletin. (Calcutta)

Canad. J. Math. Canadian Journal of Mathematics. Journal Canadien de Mathématiques. (Toronto, Ont.)

Canad. J. Phys. Canadian Journal of Physics. (Ottawa, Ont.)

Canad. Math. Bull. Canadian Mathematical Bulletin. Bulletin Canadien de Mathématiques. (Toronto, Ont.)

Časopis Pěst. Mat. Československá Akademie Věd. Časopis pro Pěstování Matematiky. (Prague)

Centaurus. Centaurus. (Copenhagen)

Chalmers Tekn. Högsk. Handl. Chalmers Tekniska Högskolans Handlingar. Transactions of Chalmers University of Technology, Gothenburg, Sweden. (Gothenburg)

Chinese Math. Chinese Mathematics. Translation of Acta Mathematica Sinica. (Providence, R.I.) (Translation of *Acta Math. Sinica*)

Ciência (Lisboa). Ciência. Revista de Cultura Científica. (Lisbon)

Ciencias (Madrid). Las Ciencias. (Madrid)

Collect. Math. Consejo Superior de Investigaciones Científicas. Universidad de Barcelona. Collectanea Mathematica. (Barcelona)

Colloq. Math. Colloquium Mathematicum. (Warsaw)

Com. Acad. R. P. Romîne. Comunicările Academiei Republicii Populare Romîne. (Bucharest)

Comm. ACM. Communications of the Association for Computing Machinery. (New York)

Comm. and Electronics. Communication and Electronics. (New York) (Continued as: *IEEE Trans. Comm. and Electronics*)

Comm. Dublin Inst. Adv. Studies Ser. A. Communications of the Dublin Institute for Advanced Studies. Series A. (Dublin)

Comm. Fac. Sci. Univ. Ankara Sér. A. Communications de la Faculté des Sciences de l'Université d'Ankara. Série A. Mathématiques-Physique-Astronomie. (Ankara)

Comm. Pure Appl. Math. Communications on Pure and Applied Mathematics. (New York)

Comment. Math. Helv. Commentarii Mathematici Helvetici. (Zürich)

Comment. Math. Univ. Carolinae. Commentationes Mathematicae Universitatis Carolinae. (Prague)

Comment. Math. Univ. St. Paul. Commentarii Mathematici Universitatis Sancti Pauli. Rikkyô Daigaku Sûgaku Zassi. (Tokyo)

Compositio Math. Compositio Mathematica. (Groningen)

Comput. Bull. The Computer Bulletin. (London)

Comput. J. The Computer Journal. (London)

Comput. Rev. Computing Reviews. (New York) (Usually abbreviated *CR* in the text of *Math. Reviews*)

Confer. Sem. Mat. Univ. Bari. Conference del Seminario di Matematica dell'Università di Bari. (Bari)

Contributions to Differential Equations. Contributions to Differential Equations. (New York)

CORS J. CORS J. The Canadian Operation Research Society Journal. (Toronto, Ont.)

CR. Computing Reviews. (New York)

Cybernetica. Cybernetica. (Namur)

Czechoslovak J. Phys. Čehoslovackaja Akademija Nauk. Čehoslovackiĭ Fizičeskiĭ Žurnal. Czechoslovak Academy of Sciences. Czechoslovak Journal of Physics. (Prague)

Czechoslovak Math. J. Čehoslovackaja Akademija Nauk. Čehoslovackiĭ Matematičeskiĭ Žurnal. Czechoslovak Mathematical Journal. (Prague)

Deutsche Geodät. Kommiss. Bayer. Akad. Wiss. Reihe B Angew. Geodäsie. Deutsche Geodätische Kommission bei der Bayerischen Akademie der Wissenschaften. Reihe B. Angewandte Geodäsie. (Munich)

Dialectica. Dialectica. (Neuchâtel)

Discussions Faraday Soc. Discussions of the Faraday Society. (London)

Diskret. Analiz. Akademija Nauk SSSR. Sibirskoe Otdelenie. Institut Matematiki. Diskretnyĭ Analiz. Sbornik Trudov. (Novosibirsk)

Dokl. Akad. Nauk BSSR. Doklady Akademii Nauk BSSR. (Minsk)

Dokl. Akad. Nauk SSSR. Doklady Akademii Nauk SSSR. (Moscow) (Mathematics section translated as: *Soviet Math. Dokl.*; Physics section translated as: *Soviet Physics Dokl.*)

Dokl. Akad. Nauk Tadžik. SSR. Doklady Akademii Nauk Tadžikskoĭ SSR. (Dushanbe)

Dopovidi Akad. Nauk Ukraïn. RSR. Dopovidi Akademiï Nauk Ukraïns'koï Radjans'koï Socialističnoï Respubliki. (Kiev)

Duke Math. J. Duke Mathematical Journal. (Durham, N.C.)

Econometrica. Econometrica. Journal of the Econometric Society. (Chicago, Ill.)

Eesti NSV Tead. Akad. Toimetised Füüs.-Mat. Tehn. Tead. Seer. Eesti NSV Teaduste Akadeemia Toimetised. Füüsikalis-Matemaatiliste ja Tehniliste Teaduste Seeria. Izvestija Akademii Nauk Èstonskoĭ SSR. Serija Fiziko-Matematičeskih i Tehničeskih Nauk. (Tallin) (Continued as: *Eesti NSV Tead. Akad. Toimetised Füüs.-Mat. Tehn.-tead. Seer.*)

Eesti NSV Tead. Akad. Toimetised Füüs.-Mat. Tehn.-tead. Seer. Eesti NSV Teaduste Akadeemia Toimetised. Füüsika-Matemaatika-ja Tehnikateaduste Seeria. Izvestija Akademii Nauk Èstonskoĭ SSR. Serija Fiziko-Matematičeskih i Tehničeskih Nauk. (Tallin) (Formerly: *Eesti NSV Tead. Akad. Toimetised Füüs-Mat. Tehn. Tead. Seer.*)

Elektron. Datenverarbeitung. Elektronische Datenverarbeitung. Fachberichte über programmgesteuerte Maschinen und ihre Anwendung. (Braunschweig)

Elem. Math. Elemente der Mathematik. Revue de Mathématiques Élémentaires. Rivista de Matematica Elementare. (Basel)

English Abstracts Selected Articles Soviet Bloc and Mainland China Tech. J. Ser. I. English Abstracts of Selected Articles from Soviet Bloc and Mainland China Technical Journals. Series I—Physics and Mathematics. (Washington, D.C.)

Enseignement Math. L'Enseignement Mathématique. (Geneva)

Ericsson Technics. Ericsson Technics. (Stockholm)

Estadíst. Española. Estadística Española. (Madrid)

Estadística. Estadística. (Washington, D.C.)

Euclides (Groningen). Euclides. Tijdschrift voor de Didactiek der Exacte Vakken. (Groningen)

Euclides (Madrid). Euclides. Revista Mensual de Ciencias Exactas, Físicas, Químicas, Naturales y Applicaciones Técnicas. (Madrid)

Fac. Ingen. Agrimens. Montevideo Publ. Didact. Inst. Mat. Estadíst. Facultad de Ingeniería y Agrimensura. Montevideo. Publicaciones Didacticas del Instituto de Matemática y Estadística. (Montevideo)

Fac. Sci. Univ. Skopje Annuaire. Faculté des Sciences de l'Université de Skopje. Annuaire. (Skopje)

Fibonacci Quart. The Fibonacci Quarterly. Official Organ of The Fibonacci Association. A journal devoted to the study of integers with special properties. (St. Mary's College, Calif.)

Fiz.-Mat. Spis. Bŭlgar. Akad. Nauk. Bŭlgarska Akademija na Naukite Fizičeski Institut. Matematičeski Institut. Fiziko-Matematičesko Spisanie. (Sofia)

Fiz. Tverd. Tela. Fizika Tverdogo Tela. (Moscow-Leningrad) (Translated as: *Soviet Physics Solid State*)

Forsch. Gebiete Ingenieurwesens. Forschung auf dem Gebiete des Ingenieurwesens. (Berlin)

Fortschr. Physik. Fortschritte der Physik. (Berlin)

Fund. Math. Polska Akademia Nauk. Fundamenta Mathematicae. (Warsaw)

Funkcial. Ekvac. Fako de l'Funkcialaj Ekvacioj Japana Matematika Societo. Funkcialaj Ekvacioj. (Serio Internacia). (Kobe)

Gac. Mat. (Madrid). Gaceta Matemática. (Madrid)

Gaṇita. Gaṇita. (Lucknow)

Gaz. Mat. Fiz. Ser. A. Societatea de Ştiinţe Matematice şi Fizice din R. P. R. Gazeta Matematică şi Fizică. Publicaţie Pentru Studiul şi Răspîndirea Ştiinţelor Matematice şi Fizice. Seria A. (Bucharest)

Gaz. Mat. (Lisboa). Gazeta de Matemática. (Lisbon)

Genet. Res. Genetical Research. (New York)

Geodaet. Inst. Medd. Geodaetisk Institut. Meddelelse. (Copenhagen)

Geofys. Publ. Norske Vid.-Akad. Oslo. Geofysiske Publikasjoner Utgitt av det Norske Videnskaps-Akademi i Oslo. (Oslo)

Giorn. Ist. Ital. Attuari. Giornale dell'Istituto Italiano degli Attuari. (Rome)

Giorn. Mat. Battaglini. Giornale di Matematiche di Battaglini. (Naples)

Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II. Glasnik Matematičko-Fizički i Astronomski. Periodicum Mathematico-Physicum et Astronomicum. Serija II. Društvo Matematičara i Fizičara Hrvatske. Societas Mathematicorum et Physicorum Croatiae. (Zagreb)

Gruzin. Politehn. Inst. Trudy. Gruzinskiĭ Politehničeskiĭ Institut im. V. I. Lenina. Trudy. (Tiflis)

Helv. Phys. Acta. Helvetica Physica Acta. Societatis Physicae Helveticae Commentaria Publica. (Basel)

Hochfrequenztech. Elektroakust. Hochfrequenztechnik und Elektroakustik. (Leipzig)

Houille Blanche. La Houille Blanche. Revue de l'Ingénieur Hydraulicien. (Grenoble)

IBM J. Res. Develop. IBM Journal of Research and Development. (New York)

Icarus. Icarus. International Journal of the Solar System. (New York)

ICC Bull. ICC Bulletin. (Rome)

ICSU Rev. World Sci. ICSU Review of World Science. (Amsterdam)

IEEE Trans. Antennas and Propagation. IEEE Transactions on Antennas and Propagation. (New York)

IEEE Trans. Automatic Control. IEEE Transactions on Automatic Control. (New York)

IEEE Trans. Circuit Theory. IEEE Transactions on Circuit Theory. (New York)

IEEE Trans. Comm. and Electronics. IEEE Transactions on Communication and Electronics. (New York) (Formerly: *Comm. and Electronics*)

IEEE Trans. Comm. Systems. IEEE Transactions on Communication Systems. (New York)

IEEE Trans. Electronic Computers. IEEE Transactions on Electronic Computers. (New York)

IEEE Trans. Information Theory. IEEE Transactions on Information Theory. (New York)

IEEE Trans. Nuclear Science. IEEE Transactions on Nuclear Science. (New York)

IEEE Trans. Reliability. IEEE Transactions on Reliability. (New York)

IEEE Trans. Space Electronics and Telemetry. IEEE Transactions on Space Electronics and Telemetry. (New York)

Illinois J. Math. Illinois Journal of Mathematics. (Urbana, Ill.)

Indian J. Math. Indian Journal of Mathematics. (Allahabad)

Indian J. Mech. Math. Indian Journal of Mechanics and Mathematics. (Calcutta)

Indian J. Phys. Indian Journal of Physics and Proceedings of the Indian Association for the Cultivation of Science. (Calcutta)

Indian J. Theoret. Phys. Indian Journal of Theoretical Physics. (Calcutta)

Information and Control. Information and Control. (New York)

Information Processing in Japan. Information Processing in Japan. (Tokyo)

Information Storage and Retrieval. Information Storage and Retrieval. (Oxford-New York)

Ing.-Arch. Ingenieur-Archiv. (Berlin-Göttingen-Heidelberg)

Inst. Hautes Études Sci. Publ. Math. Institut des Hautes Études Scientifiques. Publications Mathématiques. (Paris)

Internat. J. Abstracts Statist. Theory Method. International Journal of Abstracts. Statistical Theory and Method. (London) (Continued as: *Statist. Theory Method Abstracts*)

Internat. J. Comput. Math. International Journal of Computer Mathematics. (New York)

Internat. J. Engrg. Sci. International Journal of Engineering Science. (Oxford-New York)

Internat. J. Mech. Sci. International Journal of Mechanical Sciences. (Oxford)

Internat. Math. Nachr. Internationale Mathematische Nachrichten. (Vienna)

Inž. Ž. Inženernyĭ Žurnal. Organ Otdelenija Tehničeskih Nauk i Instituta Mehaniki Akademiĭ Nauk SSSR. (Moscow)

Isis. Isis. An International Review Devoted to the History of Science and Civilization. (Cambridge, Mass.)

Israel J. Math. Israel Journal of Mathematics. (Jerusalem) (Formerly: *Bull. Res. Council Israel Sect. F*)

Ist. Lombardo Accad. Sci. Lett. Rend. A. Istituto Lombardo. Accademia di Scienze e Lettere. Rendiconti. Scienze Matematiche, Fisiche, Chimiche e Geologiche. A. (Milan)

Ist. Lombardo Accad. Sci. Lett. Rend. Parte Gen. e Atti Uff. Istituto Lombardo Accademia di Scienze e Lettere. Rendiconti. Parte Generale e Atti Ufficiali. (Milan) (Formerly: *Ist. Lombardo Sci. Lett. Rend. Parte Gen. e Atti Uff.*)

Ist. Lombardo Sci. Lett. Rend. Parte Gen. e Atti Uff. Istituto Lombardo di Scienze e Lettere. Rendiconti. Parte Generale e Atti Ufficiali. (Milan) (Continued as: *Ist. Lombardo Accad. Sci. Lett. Rend. Parte Gen. e Atti Uff.*)

Ist. Veneto Sci. Lett. Arti Atti Cl. Sci. Mat. Natur. Istituto Veneto di Scienze, Lettere ed Arti. Venezia. Atti. Classe di Scienze Matematiche e Naturali. (Venice)

İstanbul Tek. Üniv. Bül. İstanbul Teknik Üniveritesi Bülteni. Bulletin of the Technical University of Istanbul. (Istanbul)

İstanbul Üniv. Fen Fak. Mec. Ser. A. İstanbul Üniversitesi Fen Fakültesi Mecmuasi. Seri A: Sirfî ve Tatbikî Matematik. Revue de la Faculté des Sciences de l'Université d'Istanbul. Série A: Mathématiques Pures et Appliquées. (Istanbul)

İstanbul Üniv. Fen Fak. Mec. Ser. C. İstanbul Üniversitesi Fen Fakültesi Mecmuasi. Seri C: Astronomi-Fizik-Kimya. Revue de la Faculté des Sciences de l'Université d'Istanbul. Série C: Astronomie-Physique-Chimie. (Istanbul)

Istor.-Astronom. Issled. Istoriko-Astronomičeskie Issledovanija. (Moscow)

Istor.-Mat. Issled. Istoriko-Matematičeskie Issledovanija. (Moscow-Leningrad)

Istoriko-Mat. Zbirnik. Akademija Nauk Ukraïns'koï RSR. Institut Matematiki. Istoriko-Matematičnii Zbirnik. (Kiev)

Izv. Akad. Nauk Armjan. SSR Ser. Fiz.-Mat. Nauk. Izvestija Akademii Nauk Armjanskoï SSR. Serija Fiziko-Matematičeskih Nauk. (Erevan)

Izv. Akad. Nauk Armjan. SSR Ser. Tehn. Nauk. Izvestija Akademii Nauk Armjanskoï SSR. Serija Tehničeskih Nauk. (Erevan)

Izv. Akad. Nauk Azerbaĭdžan. SSR. Ser. Fiz.-Mat. Tehn. Nauk. Izvestija Akademii Nauk Azerbaĭdžanskoï SSR. Serija Fiziko-Matematičeskih i Tehničeskih Nauk. (Baku)

Izv. Akad. Nauk Kazah. SSR Ser. Fiz.-Mat. Nauk. Izvestija Akademii Nauk Kazahskoï SSR. Serija Fiziko-Matematičeskih Nauk. (Alma-Ata)

Izv. Akad. Nauk SSSR Ser. Fiz. Izvestija Akademii Nauk SSSR. Serija Fizičeskaja. (Moscow)

Izv. Akad. Nauk SSSR Ser. Geofiz. Izvestija Akademii Nauk SSSR. Serija Geofizičeskaja. (Moscow) (Translated as: *Bull. (Izv.) Acad. Sci. USSR Geophys. Ser.*)

Izv. Akad. Nauk SSSR Ser. Mat. Izvestija Akademii Nauk SSSR. Serija Matematičeskaja. (Moscow)

Izv. Akad. Nauk SSSR Tehn. Kibernet. Izvestija Akademii Nauk SSSR. Tehničeskaja Kibernetika. (Moscow)

Izv. Akad. Nauk Turkmen. SSR Ser. Fiz.-Tehn. Him. Geol. Nauk. Izvestija Akademii Nauk Turkmenskoï SSR. Serija Fiziko-Tehničeskih, Himičeskih i Geologičeskih Nauk. (Ashkhabad)

Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk. Izvestija Akademii Nauk UzSSR. Serija Fiziko-Matematičeskih Nauk. UzSSR Fanlar Akademijasining Ahboroti. Fizika-Matematika Fanlari Serijasi. (Tashkent)

Izv. Kazan. Fil. Akad. Nauk SSSR Ser. Fiz.-Mat. i Tehn. Nauk. Izvestija Kazanskogo Filiala Akademii Nauk SSSR. Serija Fiziko-Matematičeskih i Tehničeskih Nauk. (Kazan)

Izv. Krymsk. Ped. Inst. Izvestija Krymskogo Pedagogičeskogo Instituta im. M. V. Frunze. (Simferopol)

Izv. Sibirsk. Otd. Akad. Nauk SSSR Ser. Tehn. Nauk. Izvestija Sibirskogo Otdelenija. Akademija Nauk SSSR. Serija Tehničeskih Nauk. (Novosibirsk) (Formerly: *Akad. Nauk SSSR Izv. Sibirsk. Otd.*)

Izv. Vuzov = Izv. Vysš. Učebn. Zaved. Matematika.

Izv. Vysš. Učebn. Zaved. Matematika. Ministerstvo Vysšego Obrazovanija SSSR. Izvestija Vysših Učebnyh Zavedenii. Matematika. (Kazan)

J. Acoust. Soc. Amer. The Journal of the Acoustical Society of America. (New York)

J. Algebra. Journal of Algebra. (New York)

J. Amer. Statist. Assoc. Journal of the American Statistical Association. (Washington, D.C.)

J. Analyse Math. Journal d'Analyse Mathématique. (Jerusalem)

J. Annamalai Univ. Part B Sci. Journal of the Annamalai University. Part B. Sciences. (Annamalainagar)

J. Appl. Math. Mech. Journal of Applied Mathematics and Mechanics. (Translation of the Soviet journal *Prikladnaja Matematika i Mehanika*). (New York) (Translation of *Prikl. Mat. Meh.*)

J. Appl. Phys. Journal of Applied Physics. (New York)

J. Appl. Probability. Journal of Applied Probability. (East Lansing, Mich.)

J. Assoc. Comput. Mach. Journal of the Association for Computing Machinery. (New York)

J. Atmospheric Sci. Journal of the Atmospheric Sciences. (Boston, Mass.)

J. Austral. Math. Soc. The Journal of the Australian Mathematical Society. (Sydney)

J. Chem. Phys. The Journal of Chemical Physics. (New York)

J. College Arts Sci. Chiba Univ. Journal of the College of Arts and Sciences, Chiba University. (Chiba)

J. Electronics Control. Journal of Electronics and Control. (London)

J. Fac. Sci. Hokkaido Univ. Ser. I. Journal of the Faculty of Science. Hokkaido University. Series I. Mathematics. (Sapporo)

J. Fac. Sci. Niigata Univ. Ser. I. Journal of the Faculty of Science. Niigata University. Series I. Mathematics, Physics and Chemistry. (Niigata) (Continued as: *Sci. Rep. Niigata Univ. Ser. A*)

J. Fac. Sci. Univ. Tokyo Sect. I. Journal of the Faculty of Science. University of Tokyo. Section I. Mathematics, Astronomy, Physics, Chemistry. (Tokyo)

J. Fluid Mech. Journal of Fluid Mechanics. (London)

J. Franklin Inst. Journal of the Franklin Institute. (Philadelphia, Pa.)

J. Gakugei Tokushima Univ. Journal of the Gakugei College. Tokushima University. (Natural Science). (Tokushima)

J. Geophys. Res. Journal of Geophysical Research. (Washington, D.C.)

J. Indian Inst. Sci. Journal of the Indian Institute of Science. (Bangalore)

J. Indian Math. Soc. The Journal of the Indian Mathematical Society. (Madras)

J. Indian Soc. Agric. Statist. Journal of the Indian Society of Agricultural Statistics. (New Delhi)

J. Indian Statist. Assoc. Journal of the Indian Statistical Association. (Bombay)

J. Information Processing Soc. Japan. Journal of Information Processing Society of Japan. (Tokyo)

J. Inst. Actuar. Journal of the Institute of Actuaries. (London)

J. Karnatak Univ. Journal of the Karnatak University. (Dharwar) (Continued as: *J. Karnatak Univ. Sci.*)

J. Karnatak Univ. Sci. Journal of the Karnatak University. Science. (Dharwar) (Formerly: *J. Karnatak Univ.*)

J. London Math. Soc. The Journal of the London Mathematical Society. (London)

J. Madras Univ. B. Journal of the Madras University. B. Contributions in Mathematics, Physical and Biological Sciences. (Madras) (Formerly: *J. Madras Univ. Sect. B*)

J. Madras Univ. Sect. B. Journal of the Madras University. Section B. (Madras) (Continued as: *J. Madras Univ. B*)

J. Math. Anal. Appl. Journal of Mathematical Analysis and Applications. (New York)

J. Math. and Phys. Journal of Mathematics and Physics. (Cambridge, Mass.)

J. Math. Kyoto Univ. Journal of Mathematics of Kyoto University. (Kyoto)

J. Math. Mech. Journal of Mathematics and Mechanics. (Bloomington, Ind.)

J. Math. Osaka City Univ. Journal of Mathematics. Osaka City University. (Osaka) (Continued as: *Osaka J. Math.*)

J. Math. Pures Appl. Journal de Mathématiques Pures et Appliquées. (Paris)

J. Math. Soc. Japan. Journal of the Mathematical Society of Japan. (Tokyo)

J. Mathematical Phys. Journal of Mathematical Physics. (New York)

J. Mathematical Psychology. Journal of Mathematical Psychology. (New York)

J. Méc. Phys. Atmos. Journal de Mécanique et de Physique de l'Atmosphère. (Paris)

J. Mécanique. Journal de Mécanique. (Paris)

J. Mech. Phys. Solids. Journal of the Mechanics and Physics of Solids. (London)

J. Nara Gakugei Univ. The Journal of Nara Gakugei University. (Nara) (Continued as: *J. Nara Gakugei Univ. Natur. Sci.*)

J. Nara Gakugei Univ. Natur. Sci. The Journal of Nara Gakugei University. Natural Science. (Nara) (Formerly: *J. Nara Gakugei Univ.*)

J. Natur. Sci. and Math. The Journal of Natural Sciences and Mathematics. (Lahore)

J. Operations Res. Soc. Japan. Journal of the Operations Research Society of Japan. (Tokyo)

J. Opt. Soc. Amer. Journal of the Optical Society of America. (New York)

J. Osaka Inst. Sci. Tech. Part I. Journal of the Osaka Institute of Science and Technology. (Kinki University). Part I. Mathematics and Physics. (Osaka)

J. Phys. Soc. Japan. Journal of the Physical Society of Japan. (Tokyo)

J. Proc. Roy. Soc. New South Wales. Journal and Proceedings of the Royal Society of New South Wales. (Sydney)

J. Reine Angew. Math. Journal für die reine und angewandte Mathematik. (Berlin)

J. Res. Nat. Bur. Standards Sect. B. Journal of Research of the National Bureau of Standards. B. Mathematics and Mathematical Physics. (Washington, D.C.)

J. Res. Nat. Bur. Standards Sect. D. Journal of Research of the National Bureau of Standards. Section D. Radio Science. (Washington, D.C.)

J. Roy. Aero. Soc. The Journal of the Royal Aeronautical Society. (London) See *J. Roy. Aeronaut. Soc.*

J. Roy. Aeronaut. Soc. The Journal of the Royal Aeronautical Society. (London) (Formerly listed as: *J. Roy. Aero. Soc.*)

J. Roy. Astronom. Soc. Canada. The Journal of the Royal Astronomical Society of Canada. (Toronto, Ont.)

J. Roy. Statist. Soc. Ser. A. Journal of the Royal Statistical Society. Series A. (General). (London)

J. Roy. Statist. Soc. Ser. B. Journal of the Royal Statistical Society. Series B. (Methodological). (London)

J. Sci. Engrg. Res. Journal of Science and Engineering Research. (Kharagpur)

J. Sci. Hiroshima Univ. Ser. A-I Math. Journal of Science of the Hiroshima University. Series A-I (Mathematics). (Hiroshima)

J. Sci. Res. Banaras Hindu Univ. The Journal of Scientific Research of the Banaras Hindu University. (Banaras)

J. Ship Res. Journal of Ship Research. (New York)

J. SIAM Control Ser. A. Journal of the Society for Industrial and Applied Mathematics on Control. Series A. (Philadelphia, Pa.) See *J. Soc. Indust. Appl. Math. Ser. A Control.*

J. Soc. Indust. Appl. Math. Journal of the Society for Industrial and Applied Mathematics. (Philadelphia, Pa.)

J. Soc. Indust. Appl. Math. Ser. A Control. Journal of the Society for Industrial and Applied Mathematics. Series A: Control. (Philadelphia, Pa.) (Formerly listed as: *J. SIAM Control Ser. A*)

J. Symbolic Logic. The Journal of Symbolic Logic. (New Brunswick, N.J.)

J. Univ. Bombay. Journal of the University of Bombay. (Bombay)

J. Washington Acad. Sci. Journal of the Washington Academy of Sciences. (Washington, D.C.)

J. Zosen Kiokai. Journal of Zosen Kiokai. (Tokyo)

Janus. Janus. Revue Internationale de l'Histoire des Sciences, de la Médecine, de la Pharmacie et de la Technique. (Leiden)

Japan. J. Math. Science Council of Japan. Japanese Journal of Mathematics. (Tokyo)

Jber. Deutsch. Math.-Verein. Jahresbericht der Deutschen Mathematiker-Vereinigung. (Stuttgart)

Kabardino-Balkarsk. Gos. Univ. Učen. Zap. Ser. Fiz.-Mat. Kabardino-Balkarskiĭ Gosudarstvennyĭ Universitet. Učenye Zapiski. Serija Fiziko-Matematičeskaja. (Nalchik)

Kalinin. Gos. Ped. Inst. Učen. Zap. Kalininskiĭ Gosudarstvennyĭ Pedagogičeskiĭ Institut im. M. I. Kalinina. Učenye Zapiski. (Kalinin)

Kazan. Gos. Univ. Učen. Zap. Kazanskiĭ Ordena Trudovogo Krasnogo Znameni Gosudarstvennyĭ Universitet im. V. I. Ul'janova-Lenina. Učenye Zapiski. (Kazan)

Kišinev. Gos. Univ. Učen. Zap. Komitet Vysšego i Srednego Special'nogo Obrazovanija Soveta Ministrov Moldavskoĭ SSR. Kišinevskiĭ Gosudarstvennyĭ Universitet. Učenye Zapiski. (Kishinev)

Kōdai Math. Sem. Rep. Kōdai Mathematical Seminar Reports. (Tokyo)

Köz. Mat. Lapok. Középiskolai Matematikai Lapok. (Budapest)

Kristallografija. Akademija Nauk SSSR. Kristallografija. (Moscow) (Translated as: *Soviet Physics Cryst.*)

Kumamoto J. Sci. Ser. A. Kumamoto Journal of Science. Series A. (Mathematics, Physics and Chemistry). (Kumamoto)

Kungl. Fysiogr. Sällsk. i Lund Handl. Acta Regiae Societatis Physiographicae Lundensis. Kungl. Fysiografiska Sällskapets i Lund Handlingar. (Lund) (Same as: *Lunds Univ. Årsskr. Avd.* 2)

Kungl. Svenska Vetenskapsakad. Handl. Kungl. Svenska Vetenskapsakademiens Handlingar. (Stockholm)

Kungl. Tekn. Högsk. Handl. Stockholm. Kungl. Tekniska Högskolans Handlingar. Transactions of the Royal Institute of Technology, Stockholm, Sweden. (Stockholm)

Kybernetik. Kybernetik. (Berlin)

Kyungpook Math. J. Kyungpook Mathematical Journal. (Taegu)

Latvijas PSR Zinātņu Akad. Vēstis. Latvijas PSR Zinātņu Akademijas Vēstis. Izvestija Akademii Nauk Latviĭskoĭ SSR. (Riga)

Latvijas PSR Zinātņu Akad. Vēstis Fiz. Tehn. Zinātņu Ser. Latvijas PSR Zinātņu Akadēmijas Vēstis. Fizikas un Tehnisko Zinātņu Sērija. Izvestija Akademii Nauk Latviĭskoĭ SSR. Serija Fizičeskih i Tehničeskih Nauk. (Riga)

Latvijas Valsts Univ. Zinātn. Raksti. PSRS Augstākās Izglītības Ministrija. Pētera Stučkas Latvijas Valsts Universitāte. Zinātniskie Raksti. Ministerstvo Vysšego Obrazovanija. Latviĭskiĭ Gosudarstvennyĭ Universitet im. Petra Stučki. Učenye Zapiski. (Riga)

Leningrad. Gos. Ped. Inst. Učen. Zap. Leningradskiĭ Gosudarstvennyĭ Pedagogičeskiĭ Institut im. A. I. Gercena. Učenye Zapiski. (Leningrad)

Leningrad. Gos. Univ. Učen. Zap. Ser. Mat. Nauk. Leningradskiĭ Gosudarstvennyĭ Ordena Lenina Universitet im. A. A. Ždanova. Učenye Zapiski. Serija Matematičeskih Nauk. (Leningrad)

Litovsk. Mat. Sb. Vysšie Učebnye Zavedenija Litovskoĭ SSR Litovskiĭ Matematičeskiĭ Sbornik. (Vilna)

Logique et Analyse. Logique et Analyse. (Louvain)

Lucrăr. Şti. Inst. Ped. Timişoara Mat.-Fiz. Ministerul Învăţământului şi Culturii. Lucrările Ştiinţifice ale Institutului Pedagogic Timişoara. Matematică-Fizică. (Timişoara)

Lunds Univ. Årsskr. Avd. 2. Acta Universitatis Lundensis. Lunds Universitets Årsskrift. Andra Avdelningen. (Lund) (Same as: *Kungl. Fysiogr. Sällsk. i Lund. Handl.*) (Continued as: *Acta Univ. Lund. Sect. II*)

L'vov. Politehn. Inst. Naučn. Zap. Ser. Fiz.-Mat. MVO USSR. L'vovskiĭ Politehničeskiĭ Institut. Naučnye Zapiski. Serija Fiziko-Matematičeskaja. (Lvov)

Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. A Magyar Tudományos Akadémia Matematikai és Fizikai Tudományok Osztályának Közleményei. (Budapest)

Magyar Tud. Akad. Mat. Kutató Int. Közl. A Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei. (Budapest)

Management Sci. Management Science. Journal of the Institute of Management Science. (Philadelphia, Pa.)

Mat.-Fys. Medd. Danske Vid. Selsk. Matematiskfysiske Meddelelser udgivet af Det Kongelige Danske Videnskabernes Selskab. (Copenhagen)

Mat.-Fys. Skr. Danske Vid. Selsk. Matematiskfysiske Skrifter udgivet af Det Kongelige Danske Videnskabernes Selskab. (Copenhagen)

Mat.-Fyz. Časopis Sloven. Akad. Vied. Matematicko-Fyzikálny Časopis. Slovenská Akadémia Vied. (Bratislava)

Mat. Lapok. Matematikai Lapok. Bolyai János Matematikai Társulat. (Budapest)

Mat. Sb. Matematičeskiĭ Sbornik. Novaja Serija. (Moscow)

Mat. Vesnik. Matematički Vesnik. Nova Serija. (Belgrade) (Formerly: *Bull. Soc. Math. Phys. Serbie*)

Matematiche (Catania). Le Matematiche. (Catania)

Math. Ann. Mathematische Annalen. (Berlin-Göttingen-Heidelberg)

Math. Centrum Amsterdam Afd. Toegepaste Wisk. Mathematisch Centrum Amsterdam. Afd. Toegepaste Wiskunde. (Amsterdam)

Math. Centrum Amsterdam Afd. Zuivere Wisk. Mathematisch Centrum Amsterdam. Afdeling Zuivere Wiskunde. (Amsterdam) (Formerly: *Math. Centrum Amsterdam Rap.*)

Math. Centrum Amsterdam Rap. Mathematisch Centrum Amsterdam. Rapport. (Amsterdam) (Continued as: *Math. Centrum Amsterdam Afd. Zuivere Wisk.*)

Math. Centrum Amsterdam Rekenafdeling. Mathematisch Centrum Amsterdam. Rekenafdeling. (Amsterdam)

Math. Comp. Mathematics of Computation. (Washington, D.C.)

Math. Gaz. The Mathematical Gazette. (London)

Math. J. Okayama Univ. Mathematical Journal of Okayama University. (Okayama)

Math. Japon. Mathematica Japonicae. (Kobe)

Math. Mag. Mathematics Magazine. (Buffalo, N.Y.)

Math. Nachr. Mathematische Nachrichten. (Berlin)

Math. Naturwiss. Unterricht. Der mathematische und naturwissenschaftliche Unterricht. (Bonn)

Math. Notae. Mathematicae Notae. Boletín del Instituto de Matemática. (Rosario)

Math.-Phys. Semesterber. Mathematisch-Physikalische Semesterberichte. (Göttingen)

Math. Reviews. Mathematical Reviews. (Providence, R.I.) (Usually abbreviated as *MR* in the text of *Math. Reviews*)

Math. Scand. Mathematica Scandinavica. (Copenhagen)

Math. Student. The Mathematics Student. (Madras)

Math.-Tech.-Wirtschaft. Mathematik-Technik-Wirtschaft. Zeitschrift für moderne Rechentechnik und Automation. (Vienna)

Math. und Wirtschaft. Mathematik und Wirtschaft. (Berlin)

Math. Z. Mathematische Zeitschrift. (Berlin-Göttingen-Heidelberg)

Mathematica (Cluj). Societatea de Ştiinţe Matematice şi Fizice din R.P.R. Filiala Cluj. Mathematica. (Cluj)

Mathematika. Mathematika. A Journal of Pure and Applied Mathematics. (London)

Mathematikunterricht. Der Mathematikunterricht. Beiträge zu seiner wissenschaftlichen und methodischen Gestaltung. (Stuttgart)

Mathesis. Mathesis. Recueil Mathématique à l'Usage des Écoles Spéciales et des Établissements d'Instruction Moyenne. (Mons)

Matrix Tensor Quart. The Tensor Club of Great Britain. The Matrix and Tensor Quarterly. (London)

Medd. Lunds Astronom. Obs. Ser. I. Meddelanden från Lunds Astronomiska Observatorium. Series I. (Lund)

Medd. Lunds Astronom. Obs. Ser. II. Meddelanden från Lunds Astronomiska Observatorium. Series II. (Lund)

Medd. Lunds Univ. Mat. Sem. Meddelanden från Lunds Universitets Matematiska Seminarium. Communications du Séminaire Mathématique de l'Université de Lund. (Lund)

Mem. Acad. Ci. Madrid. Memorias de la Real Academia de Ciencias Exactas, Físicas y Naturales de Madrid. Serie de Ciencias Exactas. (Madrid)

Mem. Amer. Math. Soc. Memoirs of the American Mathematical Society. (Providence, R.I.)

Mem. Defense Acad. Memoirs of the Defense Academy (Mathematics, Physics, Chemistry and Engineering) (Yokosuka)

Mem. Fac. Ci. Univ. Habana Ser. Mat. Memorias de la Facultad de Ciencias. Universidad de la Habana. Serie Matemática. (Havana)

Mem. Fac. Ed. Kumamoto Univ. Memoirs of the Faculty of Education. Kumamoto University. (Kumamoto). (Continued as: *Mem. Fac. Ed. Kumamoto Univ. Sect.* 1)

Mem. Fac. Ed. Kumamoto Univ. Sect. 1. Memoirs of the Faculty of Education. Kumamoto University. Section I. (Natural Science). (Kumamoto) (Formerly: *Mem. Fac. Ed. Kumamoto Univ.*)

Mem. Fac. Engrg. Hiroshima Univ. Memoirs of the Faculty of Engineering. Hiroshima University. (Hiroshima)

Mem. Fac. Engrg. Miyazaki Univ. Memoirs of the Faculty of Engineering, Miyazaki University. (Miyazaki)

Mem. Fac. Sci. Kyushu Univ. Ser. A. Memoirs of the Faculty of Science. Kyushu University. Series A. Mathematics. (Fukuoka)

Mem. Ist. Lombardo Accad. Sci. Lett. Cl. Sci. Mat. Natur. Memorie dell'Istituto Lombardo. Accademia di Scienze e Lettere. Classe di Scienze Matematiche e Naturali. (Milan)

Mem. Kitami College Tech. Memoirs of the Kitami College of Technology. (Kitami)

Mem. Mat. Inst. "Jorge Juan". Consejo Superior de Investigaciones Científicas. Memorias de Matemática del Instituto "Jorge Juan". (Madrid)

Mem. Muroran Inst. Tech. Memoirs of the Muroran Institute of Technology. (Muroran)

Mem. Osaka Univ. Lib. Arts Ed. Ser. B. Memoirs of the Osaka University of the Liberal Arts and Education. Series B. Natural Science. (Osaka)

Mem. Proc. Manchester Lit. Philos. Soc. Memoirs and Proceedings of the Manchester Literary & Philosophical Society. (Manchester)

Mem. Real Acad. Ci. Art. Barcelona. Memorias de la Real Academia de Ciencas y Artes de Barcelona. (Barcelona)

Mem. Roy. Astronom. Soc. Memoirs of the Royal Astronomical Society. (London)

Mem. School Sci. Engrg. Waseda Univ. Tokyo. Memoirs of the School of Science and Engineering. Waseda University, Tokyo. (Tokyo)

Mem. Soc. Astronom. Ital. Memorie della Società Astronomica Italiana. (Milan)

Mém. Soc. Roy. Sci. Liège Coll. in-4°. Mémoires de la Société Royale des Sciences de Liège. Collection in-4°. (Liège)

Mém. Soc. Roy. Sci. Liège Coll. in-8°. Mémoires de la Société Royale des Sciences de Liège. Collection in-8°. (Liège)

Mémor. Sci. Math. Mémorial des Sciences Mathématiques. (Paris)

Mémor. Sci. Phys. Mémorial des Sciences Physiques. (Paris)

Météorol. La Météorologie. (Paris)

Metrika. Metrika. (Würzburg)

Metroecon. Metroeconomica. Rivista Internazionale di Economica. (Bologna)

Metron. Metron. (Rome)

Michigan Math. J. The Michigan Mathematical Journal. (Ann Arbor, Mich.)

Min. Proc. Roy. Irish Acad. Minutes of Proceedings. Royal Irish Academy. (Dublin)

Mind. Mind. A Quarterly Review of Psychology and Philosophy. (London)

Mitt. Inst. Angew. Math. Zürich. Mitteilungen aus dem Institut für angewandte Mathematik an der Eidgenössischen Technischen Hochschule in Zürich. (Zürich)

Mitt. Inst. Theoret. Phys. Geophys. Bergakad. Freiberg. Mitteilungen aus dem Institut für Theoretische Physik und Geophysik der Bergakademie Freiberg. (Freiberg)

Mitt. Math. Ges. Hamburg. Mitteilungen der mathematischen Gesellschaft in Hamburg. (Hamburg)

Mitt. Math. Sem. Giessen. Mitteilungen aus dem Mathematischen Seminar Giessen. (Giessen) (Formerly: *Mitt. Math. Sem. Univ. Giessen*)

Mitt. Math. Sem. Univ. Giessen. Mitteilungen des Mathematischen Seminars der Universität Giessen. (Giessen) (Continued as: *Mitt. Math. Sem. Giessen*)

Mitt. Max-Planck-Inst. Strömungsforsch. Mitteilungen aus dem Max-Planck-Institut für Strömungsforschung. (Göttingen)

Mitt. Naturforsch. Ges. Bern. Mitteilungen der Naturforschenden Gesellschaft Bern. (Bern)

Mitt. Verein. Schweiz. Versich.-Math. Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker. Bulletin de l'Association des Actuaires Suisses. (Bern)

Molecular Phys. Molecular Physics. (London)

Monatsb. Deutsch. Akad. Wiss. Berlin. Monatsberichte der Deutschen Akademie der Wissenschaften zu Berlin. (Berlin)

Monatsh. Math. Monatshefte für Mathematik. (Vienna)

Monthly Notices Roy. Astronom. Soc. Monthly Notices of the Royal Astronomical Society. (London)

Moskov. Gos. Univ. Soobšč. Gos. Astronom. Inst. Šternberg. Moskovskiĭ Gosudarstvennyĭ Universitet im. M. V. Lomonosova. Soobščenija Gosudarstvennogo Astronomičeskogo Instituta im. P. K. Šternberga. (Moscow)

Moskov. Gos. Univ. Trudy Gos. Astronom. Inst. Šternberg. Moskovskiĭ Gosudarstvennyĭ Universitet im. M. V. Lomonosova. Trudy Gosudarstvennogo Astronomičeskogo Instituta im. P. K. Šternberga. (Moscow)

Moskov. Oblast. Ped. Inst. Učen. Zap. Moskovskiĭ Oblastnoĭ Pedagogičeskiĭ Institut. Učenye Zapiski. (Moscow)

MR. Mathematical Reviews. (Providence, R.I.)

Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II. Nachrichten der Akademie der Wissenschaften in Göttingen. II. Mathematisch-Physikalische Klasse. (Göttingen)

Nagoya Math. J. Nagoya Mathematical Journal. (Nagoya)

Nat. Bur. Standards Appl. Math. Ser. National Bureau of Standards. Applied Mathematics Series. (Washington, D.C.)

Nat. Luchtvaartlab. Amsterdam Rep. Nationaal Luchtvaartlaboratorium. National Aeronautical Research Institute, Amsterdam. Report. (Amsterdam)

Natur. Sci. Rep. Ochanomizu Univ. Natural Science Report c Ochanomizu University. (Tokyo)

Naturwissenschaften. Die Naturwissenschaften. (Berlin-Gött Heidelberg)

Naval Res. Logist. Quart. Naval Research Logistics Quar (Washington, D.C.)

Navigation. Navigation. Journal of the Institute of Navig (Los Angeles, Calif.)

Nederl. Akad. Wetensch. Indag. Math. Koninklijke Nederl Akademie van Wetenschappen. Indagationes Mathematica Actis Quibus Titulus. Proceedings of the Section of Scie (Amsterdam) (Same as: *Nederl. Akad. Wetensch. Proc. Ser. A*)

Nederl. Akad. Wetensch. Proc. Ser. A. Koninklijke Nederl Akademie van Wetenschappen. Proceedings. Series A. M matical Sciences. (Amsterdam) (Same as: *Nederl. Akad. Wete Indag. Math.*)

Nederl. Akad. Wetensch. Proc. Ser. B. Koninklijke Nederl Akademie van Wetenschappen. Proceedings. Series B. Ph Sciences. (Amsterdam)

Nederl. Akad. Wetensch. Verslag Afd. Natuurk. Koninklijke N landse Akademie van Wetenschappen. Verslag van de Ge Vergadering van de Afdeling Natuurkunde. (Amsterdam)

Nederl. Tijdschr. Natuurk. Nederlands Tijdschrift voor Natuurk (Utrecht)

New Zealand J. Sci. New Zealand Journal of Science. (Wellin

Nieuw Arch. Wisk. Nieuw Archief voor Wiskunde. (Groningen

Nieuw Tijdschr. Wisk. Nieuw Tijdschrift voor Wiskunde. (Gro gen-Djakarta)

Nordisk Mat. Tidskr. Nordisk Matematisk Tidskrift. (Oslo)

Nordisk Tidskr. Informations-Behandling. Nordisk Tidskrift Informations-Behandling. (Copenhagen)

Norske Vid. Selsk. Forh. (Trondheim). Det Kongelige Norske V skabers Selskabs Forhandlinger. (Trondheim)

Norske Vid. Selsk. Skr. (Trondheim). Det Kongelige Norske V skabers Selskabs Skrifter. (Trondheim)

Notas Mat. Notas de Matemática. (Rio de Janeiro)

Notices Amer. Math. Soc. Notices of the American Mathem Society. (Providence, R.I.) (Formerly: *Amer. Math. Soc. N*

Notre Dame J. Formal Logic. Notre Dame Journal of Formal] (Notre Dame, Ind.)

Nova Acta Soc. Sci. Upsal. Nova Acta Regiae Societatis Scienti Upsaliensis. (Uppsala)

Nuclear Fusion. Nuclear Fusion. (Vienna)

Nuclear Phys. Nuclear Physics. (Amsterdam)

Nuclear Sci. Abstracts. Nuclear Science Abstracts. (Oak I Tenn.)

Numer. Math. Numerische Mathematik. (Berlin-Göttingen-H berg)

Nuovo Cimento. Il Nuovo Cimento. (Bologna)

Nuovo Cimento Suppl. Supplemento al Nuovo Cimento. (Bolog

O.N.E.R.A. Publ. Office National d'Études et de Recherches nautiques. Publication. (Paris)

Observatory. The Observatory. A Review of Astronomy. (Hail

Operations Res. Operations Research. The Journal of the Oper Research Society of America. (Baltimore, Md.) (Sometimes a viated as *OR* in *Math. Reviews*)

Optica Acta. Optica Acta. (London)

Optik. Optik. Zeitschrift für das gesamte Gebiet der Licht- Elektronenoptik. (Stuttgart)

Optika i Spektrosk. Akademija Nauk SSSR. Optika i Spektroskopija. (Moscow-Leningrad)

Osaka J. Math. Osaka Journal of Mathematics. (Osaka) (Formerly: *J. Math. Osaka City Univ.* and *Osaka Math. J.*)

Osaka Math. J. Osaka Mathematical Journal. (Osaka) (Continued as: *Osaka J. Math.*)

Österreich. Akad. Wiss. Math.-Natur. Kl. S.-B. II. Österreichische Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse. Sitzungsberichte. Abteilung II. Mathematik, Astronomie, Physik, Meteorologie und Technik. (Vienna)

Österreich. Ing.-Arch. Österreichisches Ingenieur-Archiv. (Vienna)

Pacific J. Math. Pacific Journal of Mathematics. (Berkeley, Calif.)

Papers and Proc. Roy. Soc. Tasmania. Papers and Proceedings of the Royal Society of Tasmania. (Hobart)

Period. Mat. Periodico di Matematiche. (Bologna)

Perm. Gos. Univ. Učen. Zap. Mat. Ministerstvo Vysšego i Srednego Special'nogo Obrazovanija RSFSR. Permskiĭ Gosudarstvennyĭ Universitet im. A. M. Gor'kogo. Učenye Zapiski. Matematika. (Perm)

Philos. Mag. The Philosophical Magazine. A Journal of Theoretical, Experimental and Applied Science. (London)

Philos. Rev. The Philosophical Review. (Ithaca, N.Y.)

Philos. Sci. Philosophy of Science. (Baltimore, Md.)

Philos. Trans. Roy. Soc. London Ser. A. Philosophical Transactions of the Royal Society of London. Series A. Mathematical and Physical Sciences. (London)

Phys. Fluids. The Physics of Fluids. (New York)

Phys. Lett. Physics Letters. (Amsterdam)

Phys. Norveg. Physica Norvegica. (Oslo)

Phys. Rev. The Physical Review. (New York)

Phys. Rev. Lett. Physical Review Letters. (New York)

Physica. Physica. (Amsterdam)

PMTF Ž. Prikl. Meh. i Tehn. Fiz. PMTF. Žurnal Prikladnoĭ Mehaniki i Tehničeskoĭ Fiziki. (Moscow)

Polska Biblio. Analit. Mech. Polska Akademia Nauk. Instytut Podstawowych Problemów Techniki. Polska Bibliografia Analityczna. Mechanika. Polish Scientific Abstracts. Mechanics. (Warsaw)

Portugal. Math. Portugaliae Mathematica. (Lisbon)

Portugal. Phys. Portugaliae Physica. (Lisbon)

Práce Brn. Českoslov. Akad. Věd. Práce Brněnské Základny Československé Akademie Věd. Acta Academiae Scientiarum Čechoslovenicae Basis Brunensis. (Brno)

Prace Mat. Roczniki Polskiego Towarzystwa Matematycznego. Seria I. Prace Matematyczne. (Warsaw)

Prakt. Akad. Athēnōn. Πρακτικὰ τῆς Ἀκαδημίας Ἀθηνῶν. (Athens)

Prikl. Mat. Meh. Akademija Nauk SSSR. Otdelenie Tehničeskih Nauk. Institut Mehaniki. Prikladnaja Matematika i Mehanika. (Moscow) (Translated as: *J. Appl. Math. Mech.*)

Prikladna Meh. Akademija Nauk Ukraïns'koï RSR. Institut Mehaniki. Prikladna Mehanika. (Kiev)

Problemy Kibernet. Problemy Kibernetiki. (Moscow)

Problemy Peredači Informacii. Akademija Nauk SSSR. Laboratorija Sistem Peredači Informacii. Problemy Peredači Informacii. (Moscow)

Proc. Amer. Math. Soc. Proceedings of the American Mathematical Society. (Providence, R.I.)

Proc. Cambridge Philos. Soc. Proceedings of the Cambridge Philosophical Society. (Cambridge, England)

Proc. Edinburgh Math. Soc. Proceedings of the Edinburgh Mathematical Society. (Edinburgh)

Proc. Edinburgh Math. Soc. Edinburgh Math. Notes. Proceedings of the Edinburgh Mathematical Society. Edinburgh Mathematical Notes. (Edinburgh)

Proc. Egyptian Acad. Sci. Proceedings of the Egyptian Academy of Sciences. (Cairo)

Proc. Fac. Engrg. Keio Univ. Proceedings of the Fujihara Memorial Faculty of Engineering. Keio University. (Tokyo)

Proc. Glasgow Math. Assoc. Proceedings of the Glasgow Mathematical Association. (Glasgow)

Proc. IEEE. Proceedings of the IEEE. (New York)

Proc. Indian Acad. Sci. Sect. A. Proceedings of the Indian Academy of Sciences. Section A. (Bangalore)

Proc. Indiana Acad. Sci. Proceedings of the Indiana Academy of Science. (Indianapolis, Ind.)

Proc. Inst. Elec. Engrs. B. The Proceedings of the Institution of Electrical Engineers. Part B. Radio and Electronic Engineering (including Communication Engineering). (London)

Proc. Inst. Elec. Engrs. C. The Proceedings of the Institution of Electrical Engineers. Part C. Monographs. (London)

Proc. Inst. Mech. Engrs. Proceedings of the Institution of Mechanical Engineers. (London)

Proc. Inst. Statist. Math. The Proceedings of the Institute of Statistical Mathematics. (Tokyo)

Proc. Iowa Acad. Sci. Proceedings of the Iowa Academy of Science. (Des Moines, Iowa)

Proc. Iraqi Sci. Soc. Proceedings of the Iraqi Scientific Societies. (Baghdad)

Proc. Japan Acad. Proceedings of the Japan Academy. (Tokyo)

Proc. London Math. Soc. Proceedings of the London Mathematical Society. (London)

Proc. Math. Phys. Soc. U.A.R. Proceedings of the Mathematical and Physical Society of U.A.R. (Cairo) (Continued as: *Proc. Math. Phys. Soc. U.A.R. (Egypt)*)

Proc. Math. Phys. Soc. U.A.R. (Egypt). Proceedings of the Mathematical and Physical Society of U.A.R. (Egypt). (Cairo) (Formerly: *Proc. Math. Phys. Soc. U.A.R.*)

Proc. Nat. Acad. Sci. India Sect. A. Proceedings of the National Academy of Sciences, India. Section—A. (Allahabad)

Proc. Nat. Acad. Sci. U.S.A. Proceedings of the National Academy of Sciences of the United States of America. (Washington, D.C.)

Proc. Nat. Inst. Sci. India Part A. Proceedings of the National Institute of Sciences of India. Part A. Physical Sciences. (New Delhi)

Proc. Pakistan Statist. Assoc. Proceedings of the Pakistan Statistical Association. (Lahore)

Proc. Phys. Soc. Proceedings of the Physical Society. (London)

Proc. Rajasthan Acad. Sci. The Proceedings of The Rajasthan Academy of Sciences. (Pilani)

Proc. Roy. Irish Acad. Sect. A. Proceedings of the Royal Irish Academy. Section A. (Dublin)

Proc. Roy. Soc. Edinburgh Sect. A. Proceedings of the Royal Society of Edinburgh. Section A. (Mathematical and Physical Sciences). (Edinburgh)

Proc. Roy. Soc. New Zealand. Proceedings of the Royal Society of New Zealand. (Wellington)

Proc. Roy. Soc. Ser. A. Proceedings of the Royal Society. Series A. Mathematical and Physical Sciences. (London)

Proc. Roy. Soc. Victoria. Proceedings of the Royal Society of Victoria. New Series. (Melbourne)

Proc. Vibration Problems. Polish Academy of Sciences. Institute of Basic Technical Problems. Proceedings of Vibration Problems. (Warsaw)

Progr. Theoret. Phys. Progress of Theoretical Physics. (Kyoto)

Psychometrika. Psychometrika. A Journal Devoted to the Development of Psychology as a Quantitative Rational Science. (Chapel Hill, N.C.)

Publ. Astronom. Soc. Pacific. Publications of the Astronomical Society of the Pacific. (San Francisco, Calif.)

Publ. Inst. Math. (Beograd). Institut Mathématique. Publications de l'Institut Mathématique. Nouvelle Série. (Belgrade)

Publ. Inst. Statist. Univ. Paris. Publications de l'Institut de Statistique de l'Université de Paris. (Paris)

Publ. Math. Debrecen. Publicationes Mathematicae. (Debrecen)

Publ. Sci. Tech. Ministère de l'Air (Paris). Publications Scientifiques et Techniques du Ministère de l'Air. (Paris)

Publ. Sci. Tech. Ministère de l'Air (Paris) Notes Tech. Publications Scientifiques et Techniques du Ministère de l'Air. Notes Techniques. (Paris)

Publ. Sci. Univ. Alger Sér. A. Publications Scientifiques de l'Université d'Alger. Série A. Sciences Mathématiques. (Algiers)

Publ. Sém. Géom. Univ. Neuchâtel. Publications du Séminaire de Géométrie de l'Université de Neuchâtel. (Neuchâtel)

Quart. Appl. Math. Quarterly of Applied Mathematics. (Providence, R.I.)

Quart. J. Math. Oxford Ser. The Quarterly Journal of Mathematics, Oxford Second Series. (Oxford)

Quart. J. Mech. Appl. Math. The Quarterly Journal of Mechanics and Applied Mathematics. (Oxford)

Quart. J. Roy. Astronom. Soc. The Quarterly Journal of the Royal Astronomical Society. (London)

Quart. Rev. Sci. Publ. Polish Academy of Sciences. Distribution Centre for Scientific Publications. Quarterly Review of Scientific Publications. (Warsaw)

RAAG Newsletter. RAAG Newsletter. (Tokyo)

RAAG Res. Notes. RAAG Research Notes. (Tokyo)

Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke. Rad Jugoslavenske Akademije Znanosti i Umjetnosti. Odjel za Matematičke, Fizičke i Tehničke Nauke. (Zagreb)

Radio Engrg. and Electronics. Radio Engineering and Electronics. (Translation of *Radiotehnika i Èlektronika*, a U.S.S.R. Academy of Sciences Publication). (New York) (Translation of *Radiotehn. i Èlektron.*)

Radiotehn. i Èlektron. Akademija Nauk SSSR. Radiotehnika i Èlektronika. (Moscow) (Translated as: *Radio Engrg. and Electronics*)

Ratio. Ratio. (Oxford)

RCA Rev. RCA Review. (New York)

Rech. Aéro. La Recherche Aéronautique. (Paris) (Continued as: *Recherche Aérospat.*)

Recherche Aérospat. La Recherche Aérospatiale. (Paris) (Formerly: *Rech. Aéro.*)

Rend. Accad. Naz. dei XL. Rendiconti, Accademia Nazionale dei XL. (Rome)

Rend. Accad. Sci. Fis. Mat. Napoli. Società Nazionale di Napoli. Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche. (Naples)

Rend. Circ. Mat. Palermo. Rendiconti del Circolo Matematico di Palermo. (Palermo)

Rend. Mat. e Appl. Università di Roma. Istituto Nazionale di Alta Matematica. Rendiconti di Matematica e delle sue Applicazioni. (Rome)

Rend. Sem. Fac. Sci. Univ. Cagliari. Rendiconti del Seminario della Facoltà di Scienze della Università di Cagliari. (Cagliari)

Rend. Sem. Mat. Fis. Milano. Rendiconti del Seminario Matematico e Fisico di Milano. (Milan)

Rend. Sem. Mat. Univ. Padova. Rendiconti del Seminario Matematico dell'Università di Padova. (Padova)

Rep. Inst. Sci. Tech. Univ. Tokyo. The Reports of the Institute of Science and Technology, University of Tokyo. (Tokyo)

Rep. Lib. Arts Sci. Fac. Shizuoka Univ. Sect. Natur. Sci. Reports of Liberal Arts and Science Faculty. Shizuoka University. Section Natural Science. (Shizuoka)

Rep. Res. Inst. Appl. Mech. Kyushu Univ. Reports of Research Institute for Applied Mechanics. Kyushu University. (Fukuoka)

Rep. Statist. Appl. Res. Un. Japan. Sci. Engrs. Reports of Statistical Application Research, Union of Japanese Scientists and Engineers. (Tokyo)

Repúb. Venezuela Bol. Acad. Ci. Fís. Mat. Natur. República de Venezuela. Boletín de la Academia de Ciencias Físicas, Matemáticas y Naturales. (Caracas)

Res. Bull. Panjab Univ. Research Bulletin of the Panjab University. (Hoshiarpur)

Res. Rep. Nagaoka Tech. College. Research Reports of the Nagaoka Technical College. (Nagaoka)

Rev. Acad. Ci. Madrid. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales de Madrid. (Madrid)

Rev. Acad. Ci. Zaragoza. Revista de la Academia de Ciencias Exactas, Físico-Químicas y Naturales de Zaragoza. (Zaragoza)

Rev. Acad. Colombiana Ci. Exact. Fís. Natur. Revista de la Academia Colombiana de Ciencias Exactas, Físicas y Naturales. (Bogotá)

Rev. Ci. (Lima). Revista de Ciencias. (Lima)

Rev. Fac. Ci. Univ. Coimbra. Revista de Faculdade de Ciências da Universidade de Coimbra. (Coimbra)

Rev. Française Traitement Information [Chiffres]. Revue Française de Traitement de l'Information. Chiffres. (Paris)

Rev. Gén. Sci. Pures Appl. Revue Générale des Sciences Pures et Appliquées et Bulletin de l'Association Française pour l'Avancement des Sciences. (Paris)

Rev. Histoire Sci. Appl. Revue d'Histoire des Sciences et de leurs Applications. (Paris)

Rev. Inst. Internat. Statist. Revue de l'Institut International de Statistique. Review of the International Statistical Institute. (The Hague)

Rev. Mat. Elem. Revista de Matemáticas Elementales. (Bogotá)

Rev. Mat. Hisp.-Amer. Revista Matemática Hispano-Americana. (Madrid)

Rev. Math. Pures Appl. Académie de la République Populaire Roumaine. Revue de Mathématiques Pures et Appliquées. (Bucharest)

Rev. Méc. Appl. Académie de la République Populaire Roumaine. Revue de Mécanique Appliquée. (Bucharest)

Rev. Mexicana Fís. Revista Mexicana de Física. (Mexico, D.F.)

Rev. Modern Phys. Reviews of Modern Physics. (New York)

Rev. Optique. Revue d'Optique Théorique et Instrumentale. (Paris)

Rev. Un. Mat. Argentina. Revista de la Unión Matemática Argentina. (Buenos Aires)

Rice Univ. Studies. Rice University Studies. (Houston, Tex.)

Ricerca (Napoli). La Ricerca. Rivista di Matematiche Pure ed Applicate. (Naples)

Ricerche Mat. Ricerche di Matematica. (Naples)

Riv. Mat. Univ. Parma. Rivista di Matematica della Università di Parma. (Parma)

Rjazansk. Gos. Ped. Inst. Učen. Zap. Rjazanskiĭ Gosudarstvennyĭ Pedagogičeskiĭ Institut. Učenye Zapiski. (Ryazan)

Rostov.-na-Donu Gos. Ped. Inst. Fiz.-Mat. Fak. Učen. Zap. Rostovskiĭ-na-Donu Gosudarstvennyĭ Pedagogičeskiĭ Institut. Fiziko-Matematičeskiĭ Fakul'tet. Učenye Zapiski. (Rostov-on-Don)

Rozpravy Československé Akad. Věd. Rozpravy Československé Akademie Věd. (Prague)

Rozprawy Inż. Polska Akademia Nauk. Instytut Podstawowych Problemów Techniki. Rozprawy Inżynierskie. (Warsaw)

Rozprawy Mat. Polska Akademia Nauk. Instytut Matematyczny. Rozprawy Matematyczne. (Warsaw)

Russian Math. Surveys. Russian Mathematical Surveys. (A translation of the survey articles and of selected biographical articles in *Uspehi Matematičeskih Nauk*). (London) (Translation of *Uspehi Mat. Nauk*)

RŽ Astronom. Akademija Nauk SSSR. Institut Naučnoĭ Informacii. Referativnyĭ Žurnal. Otdel'nyĭ Vypusk 51. Astronomija. (Moscow) (Formerly: *RŽ Astronom. Geod.*)

RŽ Astronom. Geod. Akademija Nauk SSSR. Institut Naučnoĭ Informacii. Referativnyĭ Žurnal. Astronomija Geodezija Referaty. (Moscow) (Continued as: *RŽ Astronom. and RŽ Geodez.*)

RŽ Geodez. Akademija Nauk SSSR. Institut Naučnoĭ Informacii. Referativnyĭ Žurnal. Otdel'nyĭ Vypusk 52. Geodezija. (Moscow) (Formerly: *RŽ Astronom. Geod.*)

RŽMat. Akademija Nauk SSSR. Institut Naučnoĭ Informacii. Referativnyĭ Žurnal. Matematika Referaty. (Moscow)

RŽMeh. Akademija Nauk SSSR. Institut Naučnoĭ Informacii. Referativnyĭ Žurnal. Mehanika Referaty. (Moscow)

SA. Science Abstracts. Section A. Physics Abstracts. (London)

Sankhyā Ser. A. Sankhyā. The Indian Journal of Statistics. Series A. (Calcutta)

Sankhyā Ser. B. Sankhyā. The Indian Journal of Statistics. Series B. (Calcutta)

S.-B. Berlin. Math. Ges. Sitzungsberichte der Berliner Mathematischen Gesellschaft. (Berlin)

S.-B. Deutsch. Akad. Wiss. Berlin Kl. Math. Phys. Tech. Sitzungsberichte der Deutschen Akademie der Wissenschaften zu Berlin. Klasse für Mathematik, Physik und Technik. (Berlin)

S.-B. Finn. Akad. Wiss. Sitzungsberichte der Finnischen Akademie der Wissenschaften. Proceedings of the Finnish Academy of Science and Letters. (Helsinki)

S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl. Sitzungsberichte der Heidelberger Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse. (Heidelberg)

S.-B. Phys.-Med. Soz. Erlangen. Sitzungsberichte der physikalisch-medizinischen Sozietät zu Erlangen. (Erlangen)

S.-B. Sächs. Akad. Wiss. Leipzig Math.-Natur. Kl. Sitzungsberichte der Sächsischen Akademie der Wissenschaften zu Leipzig. Mathematisch-naturwissenschaftliche Klasse. (Berlin)

Sb. Vysoké. Učení Tech. Brno. Sbornik Vysokého Učení Technického v Brně. (Brno)

Schr. Math. Inst. Univ. Münster. Schriftenreihe des Mathematischen Instituts der Universität Münster. (Münster)

Schweiz. Z. Vermessg. Kulturtech. Photogr. Schweizerische Zeitschrift für Vermessung, Kulturtechnik und Photogrammetrie. Revue Technique Suisse des Mensurations, du Génie Rural et de Photogrammétrie. (Winterthur)

Sci. Abstracts Sect. A. Science Abstracts. Section A. Physics Abstracts. (London) (Usually abbreviated *SA* in the text of *Math. Reviews*)

Sci. Abstracts Sect. B. Science Abstracts. Section B. Electrical Engineering Abstracts. (London)

Sci. Amer. Scientific American. (New York)

Sci. Information Notes. Scientific Information Notes. (Washington, D.C.)

Sci. Papers College Gen. Ed. Univ. Tokyo. Scientific Papers of the College of General Education. University of Tokyo. (Tokyo)

Sci. Papers Inst. Phys. Chem. Res. Scientific Papers of the Institute of Physical and Chemical Research. (Tokyo)

Sci. Proc. Roy. Dublin Soc. Ser. A. The Scientific Proceedings of the Royal Dublin Society. Series A. (Dublin)

Sci. Proc. Roy. Dublin Soc. Ser. B. The Scientific Proceedings of the Royal Dublin Society. Series B. (Dublin)

Sci. Rep. Fac. Lib. Arts Ed. Gifu Univ. Natur. Sci. Science Report of the Faculty of Liberal Arts and Education, Gifu University. (Natural Science). (Gifu)

Sci. Rep. Fac. Lit. Sci. Hirosaki Univ. Science Reports of the Faculty of Literature and Science. Hirosaki University. (Hirosaki)

Sci. Rep. Kagoshima Univ. Science Reports of the Kagoshima University. (Kagoshima)

Sci. Rep. Kanazawa Univ. The Science Reports of the Kanazawa University. (Kanazawa)

Sci. Rep. Niigata Univ. Ser. A. Science Reports of Niigata University. Series A (Mathematics). (Niigata) (Formerly: *J. Fac. Sci. Niigata Univ. Ser. I*)

Sci. Rep. Res. Inst. Theoret. Phys. Hiroshima Univ. Scientific Reports of the Research Institute for Theoretical Physics, Hiroshima University. (Hiroshima)

Sci. Rep. Res. Inst. Tôhoku Univ. Ser. A. The Science Reports of the Research Institutes. Tôhoku University. Series A. (Physics, Chemistry and Metallurgy). (Sendai)

Sci. Rep. Saitama Univ. Ser. A. The Science Reports of the Saitama University. Series A. Mathematics, Physics and Chemistry. (Urawa)

Sci. Rep. Tôhoku Univ. Ser. 1. The Science Reports of the Tôhoku University. First Series. (Physics, Chemistry, Astronomy). (Sendai)

Sci. Rep. Tokyo Kyoiku Daigaku Sect. A. Science Reports of the Tokyo Kyoiku Daigaku. Section A. (Tokyo)

Sci. Rep. Yokohama Nat. Univ. Sect. I. Science Reports of the Yokohama National University. Section I. Mathematics, Physics. (Yokohama)

Sci. Sinica. Scientia Sinica. (Peking)

Scripta Math. Scripta Mathematica. A Quarterly Journal Devoted to the Philosophy, History, and Expository Treatment of Mathematics. (New York)

Shuxue Jinzhan. Shuxue Jinzhan. (Shanghai)

SIAM Rev. SIAM Review. A Publication of the Society for Industrial and Applied Mathematics. (Philadelphia, Pa.)

Sibirsk. Mat. Ž. Sibirskiĭ Matematičeskiĭ Žurnal. (Moscow)

Simon Stevin. Simon Stevin. Wis-en Natuurkundig Tijdschrift. (Groningen-Djakarta)

Skand. Aktuarietidskr. Skandinavisk Aktuarietidskrift. (Uppsala)

Skr. Norske Vid.-Akad. Oslo I. Skrifter Utgitt av det Norske Videnskaps-Akademi i Oslo. I. Mat.-Naturv. Klasse. (Oslo)

Smithsonian Contrib. to Astrophys. Smithsonian Contributions to Astrophysics. (Washington, D.C.)

Soc. Actuar. Trans. Society of Actuaries. Transactions. (Chicago, Ill.)

Soc. Parana. Mat. Anuário. Anuário da Sociedade Paranaense de Matemática. (Curitiba)

Soc. Sci. Fenn. Comment. Phys.-Math. Societas Scientiarum Fennica. Commentationes Physico-Mathematicae. (Helsinki)

Soobšč. Akad. Nauk Gruzin. SSR. Soobščenija Akademii Nauk Gruzinskoĭ SSR. (Tiflis)

Soviet Astronom. AJ. Soviet Astronomy. AJ. (A translation of *Astronomičeskiĭ Žurnal* of the Academy of Sciences of the USSR). (New York) (Translation of *Astronom. Ž.*)

Soviet Math. Dokl. Soviet Mathematics. Doklady. (A translation of the mathematics section of *Doklady Akademii Nauk SSSR*). (Providence, R.I.) (Translation of mathematics section of *Dokl. Akad. Nauk SSSR*)

Soviet Physics Acoust. Soviet Physics. Acoustics. (A translation of *Akustičeskiĭ Žurnal* of the Academy of Sciences of the USSR) (New York) (Translation of *Akust. Ž.*)

Soviet Physics Cryst. Soviet Physics. Crystallography. (A translation of the journal *Kristallografija* of the Academy of Sciences of the USSR). (New York) (Translation of *Kristallografija*)

Soviet Physics Dokl. Soviet Physics. Doklady. (A translation of the physics sections of *Doklady Akademii Nauk SSSR*). (New York) (Translation of physics sections of *Dokl. Akad. Nauk SSSR*)

Soviet Physics JETP. Soviet Physics. JETP. (A translation of *Ž. Èksperimental'noĭ i Teoretičeskoĭ Fiziki* of the USSR). (New York) (Translation of *Ž. Èksper. Teoret. Fiz.*)

Soviet Physics Solid State. Soviet Physics. Solid State. (A translation of the journal *Fizika Tverdogo Tela* of the Academy of Sciences of the USSR). (New York) (Translation of *Fiz. Tverd. Tela*)

Soviet Physics Tech. Phys. Soviet Physics. Technical Physics. (A translation of *Žurnal Tehničeskoĭ Fiziki* of the Academy of Sciences of the USSR). (New York) (Translation of *Ž. Tehn. Fiz.*)

Soviet Physics Uspekhi. Soviet Physics. Uspekhi. (A translation of *Uspehi Fizičeskih Nauk* (Advances in the Physical Sciences) of the Academy of Sciences, U.S.S.R.). (New York) (Translation of *Uspehi Fiz. Nauk*)

Spisy Přírod. Fak. Univ. Brno. Spisy Přírodovědecké Fakulty University v Brně. Trudy Estestvenno-Istoričeskogo Fakul'teta Universiteta v g. Brno. Publications de la Faculté des Sciences de l'Université à Brno. (Brno)

SRI J. SRI Journal. (Menlo Park, Calif.)

Statist. Theory Method Abstracts. Statistical Theory and Method Abstracts. (London) (Formerly: *Internat. J. Abstracts Statist. Theory Method*)

Statistica (Bologna). Statistica. (Bologna)

Statistica Neerlandica. Statistica Neerlandica. Orgaan van de Vereniging voor Statistiek. (The Hague)

Stroje na Zpracování Informací. Stroje na Zpracování Informací. (Prague)

Studia Logica. Polska Akademia Nauk. Komitet Filozoficzny. Studia Logica. (Warsaw)

Studia Math. Polska Akademia Nauk. Studia Mathematica. (Warsaw)

Studia Univ. Babeş-Bolyai Ser. I Math. Phys. Studia Universitatis Babeş-Bolyai. Series I. Mathematica Physica. (Cluj) (Continued as: *Studia Univ. Babeş-Bolyai Ser. Math.-Phys.*)

Studia Univ. Babeş-Bolyai Ser. Math.-Phys. Studia Universitatis Babeş-Bolyai. Series Mathematica-Physica. (Cluj) (Formerly: *Studia Univ. Babeş-Bolyai Ser. I Math. Phys.*)

Sudhoffs Arch. Sudhoffs Archiv für Geschichte der Medizin und der Naturwissenschaften. (Wiesbaden)

Sûgaku. Sûgaku. (Tokyo)

Summa Brasil. Math. Summa Brasiliensis Mathematicae. (Rio de Janeiro)

Svenska Aeroplan A. B. Tech. Notes. SAAB Aircraft Company. Svenska Aeroplan Aktiebolaget. Technical Notes. (Linköping)

Tartu Riikl. Ül. Toimetised. Tartu Riikliku Ülikooli Toimetised. Učenye Zapiski Tartuskogo Gosudarstvennogo Universiteta. Matemaatika-Loodusteaduskonna Töid. Trudy Estestvenno-Matematičeskogo Fakulteta. (Tallin)

Tashkent. Gos. Ped. Inst. Učen. Zap. Ministerstvo Prosveščenija Uzbekskoĭ SSR. Tashkentskiĭ Gosudarstvennyĭ Pedagogičeskiĭ Institut im. Nizami. Učenye Zapiski. Fiziko-Matematičeskiĭ. (Tashkent)

Tbiliss. Gos. Univ. Trudy Ser. Meh.-Mat. Nauk. Tbilisskiĭ Gosudarstvennyĭ Universitet. Trudy. Serija Mehaniko-Matematičeskih Nauk. (Tiflis)

Tech. Moderne. La Technique Moderne. (Paris)

Tech. Rep. Engrg. Res. Inst. Kyoto Univ. Technical Reports of the Engineering Research Institute. Kyoto University. (Kyoto)

Tech. Transl. Technical Translations. (Washington, D.C.)

Technometrics. Technometrics. A Journal of Statistics for the Physical, Chemical and Engineering Sciences. (Princeton, N.J.)

Tensor. Tensor. New Series. (Sapporo)

Teor. Verojatnost. i Primenen. Akademija Nauk SSSR. Teorija Verojatnosteĭ i ee Primenenija. (Moscow) (Translated as: *Theor. Probability Appl.*)

Thalès. Thalès. Recueil Annuel des Travaux de l'Institut d'Histoire des Sciences et des Techniques de l'Université de Paris. (Paris)

Theor. Probability Appl. Theory of Probability and its Applications. (An English translation of the Soviet journal *Teorija Verojatnosteĭ i ee Primenenija*). (Philadelphia, Pa.) (Translation of *Teor. Verojatnost. i Primenen.*)

Theoria (Lund). Theoria. A Swedish Journal of Philosophy and Psychology. (Lund)

Tôhoku Math. J. The Tôhoku Mathematical Journal. (Sendai)

Tomsk. Gos. Univ. Učen. Zap. Tomskiĭ Gosudarstvennyĭ Universitet im. V. V. Kuĭbyševa. Učenye Zapiski. (Tomsk)

Topology. Topology. An International Journal of Mathematics. (Oxford)

Trabajos Estadist. Trabajos de Estadística. (Madrid)

Trans. Amer. Inst. Elec. Engrs. Transactions of the American Institute of Electrical Engineers. (New York)

Trans. Amer. Math. Soc. Transactions of the American Mathematical Society. (Providence, R.I.)

Trans. Amer. Philos. Soc. Transactions of the American Philosophical Society Held at Philadelphia for Promoting Useful Knowledge. (Philadelphia, Pa.)

Trans. ASME Ser. B. J. Engrg. Indust. Transactions of the ASME. Series B. Journal of Engineering for Industry. (New York)

Trans. ASME Ser. E. J. Appl. Mech. Transactions of the ASME. Series E. Journal of Applied Mechanics. (New York)

Trans. Faraday Soc. Transactions of the Faraday Society. (London)

Trans. New York Acad. Sci. Transactions of the New York Academy of Sciences. (New York)

Trans. Roy. Soc. Edinburgh. Transactions of the Royal Society of Edinburgh. (Edinburgh)

Trans. Roy. Soc. New Zealand General. Transactions of the Royal Society of New Zealand. General. (Wellington)

Trans. Roy. Soc. South Africa. Transactions of the Royal Society of South Africa. (Cape Town)

Trav. Soc. Sci. Lett. Wroclaw Ser. B. Travaux de la Société des Sciences et des Lettres de Wroclaw. Seria B. (Wroclaw)

Trudy Akad. Nauk Litov. SSR Ser. B. Lietuvos TSR Mokslų Akademijos. Darbai. Serija B. Akademii Nauk Litovskoĭ SSR. Trudy. Serija B. (Vilna)

Trudy Inst. Istor. Estest. Tehn. Akademija Nauk SSSR. Trudy Instituta Istorii Estestvoznanija i Tehniki. (Moscow)

Trudy Inst. Teoret. Astronom. Akademija Nauk SSSR. Trudy Instituta Teoretičeskoĭ Astronomii. (Moscow)

Trudy Leningrad. Tehn. Inst. Trudy Kafedr. Meh. Fak. Ministerstvo Vysšego Obrazovanija SSSR. Trudy Leningradskogo Tehnologičeskogo Instituta im. Lensoveta. Trudy Kafedr Mehaničeskogo Fakul'teta. (Leningrad)

Trudy Mat. Inst. Steklov. Akademija Nauk Sojuza Sovetskih Socialističeskih Respublik. Trudy Matematičeskogo Instituta im. V. A. Steklova. (Moscow-Leningrad)

Trudy Moskov. Mat. Obšč. Trudy Moskovskogo Matematičeskogo Obščestva. (Moscow)

Trudy Samarkand. Gos. Univ. Mat. Ministerstvo Vysšego i Srednego Special'nogo Obrazovanija UzSSR. Trudy Samarkandskogo Gosudarstvennogo Universiteta im. A. Navoi. Matematika. (Samarkand)

Trudy Sem. Vektor. Tenzor. Anal. Trudy Seminara po Vektornomu i Tenzornomu Analizu s ih Priloženijami k Geometrii, Mehaniki i Fiziki. (Moscow)

Trudy Tashkent. Gos. Univ. Trudy Tashkentskogo Gosudarstvennogo Universiteta im. V. I. Lenina. Matematika. (Tashkent)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Trudy Tomskogo Gosudarstvennogo Universiteta im. V. V. Kuĭbyševa. Serija Mehaniko-Matematičeskaja. (Tomsk) (Formerly: *Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb.*)

Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat. Geom. Sb. Trudy Tomskogo Gosudarstvennogo Universiteta im. V. V. Kuĭbyševa. Serija Mehaniko-Matematičeskaja. Geometričeskiĭ Sbornik. (Tomsk) (Continued as: *Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat.*)

Trudy Vyčisl. Centra Akad. Nauk Gruzin. SSR. Akademija Nauk Gruzinskoĭ SSR. Trudy Vyčislitel'nogo Centra. (Tiflis)

Tul'sk. Gos. Ped. Inst. Učen. Zap. Fiz.-Mat. Nauk. Tul'skiĭ Gosudarstvennyĭ Pedagogičeskiĭ Institut im. L. N. Tolstogo. Učenye Zapiski. Fiziko-Matematičeskie Nauki. (Tula)

U.S.S.R. Comput. Math. and Math. Phys. U.S.S.R. Computational Mathematics and Mathematical Physics. (Oxford) (Translation of *Ž. Vyčisl. Mat. i Mat. Fiz.*)

Učen. Zap. Borisoglebsk. Gos. Ped. Inst. Ministerstvo Prosveščenija RSFSR. Učenye Zapiski. Borisoglebskogo Gosudarstvennogo Pedagogičeskogo Instituta. (Borisoglebsk)

Učen. Zap. Karel. Ped. Inst. Ser. Fiz.-Mat. Nauk. Ministerstvo Prosveščenija RSFSR. Karel'skiĭ Pedagogičeskiĭ Institut. Učenye Zapiski. Serija Fiziko-Matematičeskih Nauk. (Petrozavodsk)

Učen. Zap. Ural. Gos. Univ. Učenye Zapiski Ural'skogo Gosudarstvennogo Universiteta im. A. M. Gor'kogo. (Pri Učastii Ural'skogo Matematičeskogo Obščestva.) (Sverdlovsk)

Ukrain. Mat. Ž. Akademija Nauk Ukrainskoĭ SSR. Institut Matematiki. Ukrainskiĭ Matematičeskiĭ Žurnal. (Kiev)

Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. Univerzitet u Beogradu. Publikacije Elektrotehničkog Fakulteta. Serija Matematika i Fizika. (Belgrade)

Univ. Buenos Aires Contrib. Ci. Ser. Fis. Universidad de Buenos Aires. Facultad de Ciencias Exactas, Físicas y Naturales. Contribuciones Científicas. Serie Física. (Buenos Aires)

Univ. Buenos Aires Contrib. Ci. Ser. Mat. Universidad de Buenos Aires. Facultad de Ciencias Exactas y Naturales. Contribuciones Científicas. Serie Matematica. (Buenos Aires)

Univ. California Publ. Math. University of California. Publications in Mathematics. (Berkeley-Los Angeles, Calif.)

Univ. California Publ. Statist. University of California. Publications in Statistics. (Berkeley-Los Angeles, Calif.)

Univ. Central Venezuela Bol. Fac. Ing. Universidad Central de Venezuela. Boletín de la Facultad de Ingeniería. (Caracas)

Univ. e Politec. Torino Rend. Sem. Mat. Università e Politecnico di Torino. Rendiconti del Seminario Matematico (già "Conferenze di Fisica e di Matematica"). (Turin)

Univ. Illinois Bull. Engrg. Exper. Station Bull. University of Illinois. Bulletin. Engineering Experiment Station Bulletin. (Urbana, Ill.)

Univ. Lisboa Revista Fac. Ci. A. Universidade de Lisboa. Revista da Faculdade de Ciências. 2.ª Série. A. Ciências Matemáticas. (Lisbon)

Univ. Madrid Publ. Sec. Mat. Fac. Ci. Universidad de Madrid. Publicaciones de la Sección de Matemática de la Facultad de Ciencias. (Madrid)

Univ. Nac. La Plata Publ. Fac. Ci. Fisicomat. Serie Segunda Rev. (or *Contrib.*) Universidad Nacional de La Plata. Publicaciones de la Facultad de Ciencias Fisicomatemáticas. Serie Segunda. Revista (or Contribuciones). (La Plata)

Univ. Nac. La Plata Publ. Fac. Ci. Fisicomat. Serie Tercera Publ. Esp. Universidad Nacional de La Plata. Publicaciones de la Facultad de Ciencias Fisicomatemáticas. Serie Tercera. Publicaciones Especiales. (La Plata)

Univ. Nac. Tucumán Publ. Universidad Nacional de Tucumán. Publicación. (Tucumán)

Univ. Nac. Tucumán Rev. Ser. A. Universidad Nacional de Tucumán. Facultad de Ciencias Exactas y Tecnologia. Revista. Serie A. Matemáticas y Física Teórica. (Tucumán)

Univ. Repúb. Fac. Ingen. Agrimens. Montevideo Publ. Inst. Mat. Estadíst. Universidad de la República. Facultad de Ingeniería y Agrimensura. Montevideo-Uruguay. Publicaciones del Instituto de Matemática y Estadística. (Montevideo)

Univ. Adama Mickiewicza w Poznan. Prace Wydz. Mat. Fiz. Chem. Ser. Mat. Uniwersytet im. Adama Mickiewicza w Poznaniu. Prace Wydziału Matematyki, Fiziki i Chemii. Seria Matematyka. (Poznań)

Ural. Gos. Univ. Mat. Zap. Ministerstvo Vysšego i Srednego Special'nogo Obrazovanija RSFSR. Ural'skiĭ Gosudarstvennyĭ Universitet im. A. M. Gor'kogo. Ural'skoe Matematičeskoe Obščestvo. Matematičeskie Zapiski. (Sverdlovsk)

Ural. Politehn. Inst. Ural'skiĭ Politehničeskiĭ Institut im. S. M. Kirova. (Sverdlovsk)

Uspehi Fiz. Nauk. Akademija Nauk SSSR. Uspehi Fizičeskih Nauk. (Moscow-Leningrad) (Translated as: *Soviet Physics Uspekhi*)

Uspehi Mat. Nauk. Akademija Nauk SSSR i Moskovskoe Matematičeskoe Obščestvo. Uspehi Matematičeskih Nauk. (Moscow-Leningrad) (Selected articles translated as: *Russian Math. Surveys*)

Užgorod. Gos. Univ. Naučn. Zap. MVO USSR. Užgorodskiĭ Gosudarstvennyĭ Universitet. Naučnye Zapiski. (Lvov)

Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. I. Verhandelingen der Koninklijke Nederlandse Akademie van Wetenschappen, Afdeling Natuurkunde. Eerste Sectie. (Amsterdam)

Vesci Akad. Navuk BSSR Ser. Fiz.-Tèhn. Navuk. Vesci Akadèmii Navuk BSSR. Serija Fizika-Tèhničnyh Navuk. (Minsk)

Vestnik Akad. Nauk Kazah. SSR. Vestnik Akademii Nauk Kazahskoĭ SSR. (Alma-Ata)

Vestnik Leningrad. Univ. Ser. Fiz. Him. Vestnik Leningradskogo Universiteta. Serija Fiziki, Himii. (Leningrad)

Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. Vestnik Leningradskogo Universiteta. Serija Matematiki, Mehaniki i Astronomii. (Leningrad)

Vestnik Moskov. Univ. Ser. I Mat. Meh. Vestnik Moskovskogo Universiteta. Serija I. Matematika, Mehanika. (Moscow)

Vestnik Moskov. Univ. Ser. III Fiz. Astronom. Vestnik Moskovskogo Universiteta. Serija III. Fizika, Astronomija. (Moscow)

Vestnik Statist. Vestnik Statistiki. Organ Central'nogo Statističeskogo Upravlenija pri Sovete Ministrov SSSR. (Moscow)

Vierteljschr. Naturforsch. Ges. Zürich. Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich. (Zürich)

Vijnana Parishad Anusandhan Patrika. Vijnana Parishad Anusandhan Patrika [The Research Journal of the Hindi Science Academy]. (Allahabad)

Vilniaus Valst. Univ. Mokslo Darbai Mat. Fiz. Chem. Mokslų Ser. TSRS Aukštojo Mokslo Ministerija. Vilniaus Valstybinis Universitetas. Mokslo Darbai. Matematikos, Fizikos ir Chemijos Mokslų Serija. Ministerstvo Vysšego Obrazovanija SSSR. Vil'njusskiĭ Gosudarstvennyĭ Universitet. Učenye Trudy. Serija Matematičeskih, Fizičeskih i Himičeskih Nauk. (Vilna)

Virginia J. Sci. The Virginia Journal of Science. A Journal issued Quarterly by the Virginia Academy of Science. (Blacksburg, Va.)

Voprosy Istor. Estest. i Tehn. Akademija Nauk SSSR. Voprosy Istorii Estestvoznanija i Tehniki. (Moscow)

Voprosy Kosmog. Akademija Nauk SSSR. Voprosy Kosmogonii. (Moscow)

Vsesojuz. Zaočn. Inst. Inž. Železn. Transport. Učen. Zap. Trudy Kafedr Vysš. Mat. i Teoret. Meh. SSSR. Ministerstvo Puteĭ Soobščenija Vsesojuznyĭ Zaočnyi Institut Inženerov Železnodorožnogo Transporta. Učenye Zapiski. Trudy Kafedr Vysšeĭ Matematiki i Teoretičeskoĭ Mehaniki. (Moscow)

Vyčisl. Sistemy. Akademija Nauk SSSR. Sibirskoe Otdelenie. Institut Matematiki. Vyčislitel'nye Sistemy. Sbornik Trudov. (Novosibirsk)

Wiadom. Mat. Roczniki Polskiego Towarzystwa Matematycznego. Ser. II. Wiadomości Matematyczne. (Warsaw)

Wiss. Z. Ernst-Moritz-Arndt-Univ. Greifswald Math.-Natur. Reihe. Wissenschaftliche Zeitschrift der Ernst-Moritz-Arndt-Universität Greifswald. Mathematisch-Naturwissenschaftliche Reihe. (Greifswald)

Wiss. Z. Friedrich-Schiller-Univ. Jena/Thüringen. Wissenschaftliche Zeitschrift der Friedrich-Schiller-Universität Jena/Thüringen. (Jena/Thüringen)

Wiss. Z. Hochsch. Elektrotech. Ilmenau. Wissenschaftliche Zeitschrift der Hochschule für Elektrotechnik. Ilmenau. (Ilmenau) (Continued as: *Wiss. Z. Techn. Hochsch. Ilmenau*)

Wiss. Z. Hochsch. Verkehrswesen "Friedrich List" Dresden. Wissenschaftliche Zeitschrift der Hochschule für Verkehrswesen "Friedrich List" in Dresden. Die Anwendung mathematischer Methoden im Transport- und Nachrichtenwesen. (Dresden)

Wiss. Z. Humboldt-Univ. Berlin Math.-Natur. Reihe. Wissenschaftliche Zeitschrift der Humboldt-Universität Berlin. Mathematisch-Naturwissenschaftliche Reihe. (Berlin)

Wiss. Z. Karl-Marx-Univ. Leipzig Math.-Natur. Reihe. Wissenschaftliche Zeitschrift der Karl-Marx-Universität Leipzig. Mathematisch-Naturwissenschaftliche Reihe. (Leipzig)

Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe. Wissenschaftliche Zeitschrift der Martin-Luther-Universität Halle-Wittenberg. Mathematisch-Naturwissenschaftliche Reihe. (Halle-Wittenberg)

Wiss. Z. Pädagog. Hochsch. Potsdam Math.-Natur. Reihe. Wissenschaftliche Zeitschrift der Pädagogischen Hochschule Potsdam. Mathematisch-Naturwissenschaftliche Reihe. (Potsdam)

Wiss. Z. Techn. Hochsch. Chem. Leuna-Merseburg. Wissenschaftliche Zeitschrift der Technischen Hochschule für Chemie Leuna-Merseburg. (Leuna-Merseburg)

Wiss. Z. Techn. Hochsch. Dresden. Wissenschaftliche Zeitschrift der Technischen Hochschule Dresden. (Dresden) (Continued as: *Wiss. Z. Techn. Univ. Dresden*)

Wiss. Z. Techn. Hochsch. Ilmenau. Wissenschaftliche Zeitschrift der Technischen Hochschule Ilmenau. (Ilmenau) (Formerly: *Wiss. Z. Hochsch. Elektrotech. Ilmenau*)

Wiss. Z. Techn. Univ. Dresden. Wissenschaftliche Zeitschrift der Technischen Universität Dresden. (Dresden) (Formerly: *Wiss. Z Techn. Hochsch. Dresden*)

Yokohama Math. J. Yokohama Mathematical Journal. (Yokohama)

Z. Angew. Math. Mech. Zeitschrift für Angewandte Mathematik und Mechanik. Ingenieurwissenschaftliche Forschungsarbeiten. (Berlin)

Z. Angew. Math. Phys. Zeitschrift für Angewandte Mathematik und Physik. ZAMP. Journal of Applied Mathematics and Physics. Journal de Mathématiques et de Physiques Appliquées. (Basel)

Z. Angew. Phys. Zeitschrift für Angewandte Physik. (Berlin)

Z. Astrophys. Zeitschrift für Astrophysik. (Berlin-Göttingen-Heidelberg)

Ž. Èksper. Teoret. Fiz. Akademija Nauk SSSR. Žurnal Èksperimentalnoĭ i Teoretičeskoĭ Fiziki. (Moscow) (Translated as: *Soviet Physics JETP*)

Z. Flugwiss. Zeitschrift für Flugwissenschaften. (Braunschweig)

Z. Math. Logik Grundlagen Math. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik. (Berlin)

Z. Meteorol. Zeitschrift für Meteorologie. (Berlin)

Z. Naturforsch. Zeitschrift für Naturforschung. (Tübingen)

Z. Phonetik Sprachwiss. Kommunikat. Zeitschrift für Phonetik Sprachwissenschaft und Kommunikationsforschung. (Berlin)

Z. Physik. Zeitschrift für Physik. (Berlin-Göttingen-Heidelberg)

Ž. Tehn. Fiz. Akademija Nauk SSSR. Žurnal Tehničeskoĭ Fiziki. (Moscow-Leningrad) (Translated as: *Soviet Physics Tech. Phys.*)

Ž. Vyčisl. Mat. i Mat. Fiz. Akademija Nauk SSSR. Žurnal Vyčislitel'noĭ Matematiki i Matematičeskoĭ Fiziki. (Moscow) (Translated as: *U.S.S.R. Comput. Math. and Math. Phys.*)

Z. Wahrscheinlichkeitstheorie und Verw. Gebiete. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete. (Berlin)

Zastos. Mat. Polska Akademia Nauk. Instytut Matematyczny. Zastosowania Matematyki. (Warsaw)

Zbirnik Prac' z Občis. Mat. i Tehn. Akademija Nauk Ukraïns'koï RSR. Občisljuval'niĭ Centr. Zbirnik Prac' z Občisljuval'noĭ Matematiki i Tehniki. (Kiev)

Zbl. Zentralblatt für Mathematik und ihre Grenzgebiete. (Berlin-Göttingen-Heidelberg)

Zbl. Math. Zentralblatt für Mathematik und ihre Grenzgebiete. (Berlin-Göttingen-Heidelberg) (Usually abbreviated *Zbl* in the text of *Math. Reviews*)

Zeszyty Nauk. Uniw. Jagiello. Prace Mat. Zeszyty Naukowe Uniwersytetu Jagiellońskiego. Prace Matematyczne. (Matematyka, Fizyka, Chemia) (Kraków)

JOURNALS IN TRANSLATION

Automat. Remote Control. Automation and Remote Control. (A translation of ***Avtomatika i Telemehanika***, a publication of the Academy of Sciences of the USSR). (Pittsburgh, Pa.) (Translation of ***Avtomat. i Telemeh.***)

Bull. (Izv.) Acad. Sci. USSR Geophys. Ser. Bulletin (Izvestiya) Academy of Sciences, USSR. Geophysics Series. (Washington, D.C.) (Translation of *Izv. Akad. Nauk SSSR Ser. Geofiz.*)

Chinese Math. Chinese Mathematics. Translation of Acta Mathematica Sinica. (Providence, R.I.) (Translation of *Acta Math. Sinica*)

J. Appl. Math. Mech. Journal of Applied Mathematics and Mechanics. (Translation of the Soviet journal *Prikladnaja Matematika i Mehanika*). (New York) (Translation of *Prikl. Mat. Meh.*)

Probleme der Kybernetik. Probleme der Kybernetik. (Berlin) (Translation of *Problemy Kibernet.*)

Radio Engrg. and Electronics. Radio Engineering and Electronics. (Translation of *Radiotehnika i Èlektronika*, a U.S.S.R. Academy of Sciences Publication). (New York) (Translation of *Radiotehn. i Èlektron.*)

Russian Math. Surveys. Russian Mathematical Surveys. (A translation of the survey articles and of selected biographical articles in *Uspehi Matematičeskih Nauk*). (London) (Translation of *Uspehi Mat. Nauk*)

Soviet Astronom. AJ. Soviet Astronomy. AJ. (A translation of *Astronomičeskiĭ Žurnal* of the Academy of Sciences of the USSR). (New York) (Translation of *Astronom. Ž.*)

Soviet Math. Dokl. Soviet Mathematics. Doklady. (A translation of the mathematics section of *Doklady Akademii Nauk SSSR*). (Providence, R.I.) (Translation of mathematics section of *Dokl. Akad. Nauk SSSR*)

Soviet Physics Acoust. Soviet Physics. Acoustics. (A translation of *Akustičeskiĭ Žurnal* of the Academy of Sciences of the USSR). (New York) (Translation of *Akust. Ž.*)

Soviet Physics Cryst. Soviet Physics. Crystallography. (A translation of the journal *Kristallografija* of the Academy of Sciences of the USSR). (New York) (Translation of *Kristallografija*)

Soviet Physics Dokl. Soviet Physics. Doklady. (A translation of the physics sections of *Doklady Akademii Nauk SSSR*). (New York) (Translation of physics sections of *Dokl. Akad. Nauk SSSR*)

Soviet Physics JETP. Soviet Physics. JETP. (A translation of *Ž. Èksperimental'noĭ i Teoretičeskoĭ Fiziki* of the USSR). (New York) (Translation of *Ž. Èksper. Teoret. Fiz.*)

Soviet Physics Solid State. Soviet Physics. Solid State. (A translation of the journal *Fizika Tverdogo Tela* of the Academy of Sciences of the USSR). (New York) (Translation of *Fiz. Tverd. Tela*)

Soviet Physics Tech. Phys. Soviet Physics. Technical Physics. (A translation of *Žurnal Tehničeskoĭ Fiziki* of the Academy of Sciences of the USSR). (New York) (Translation of *Ž. Tehn. Fiz.*)

Soviet Physics Uspekhi. Soviet Physics. Uspekhi. (A translation of *Uspehi Fizičeskih Nauk* (Advances in the Physical Sciences) of the Academy of Sciences, U.S.S.R.). (New York) (Translation of *Uspehi Fiz. Nauk*)

Theor. Probability Appl. Theory of Probability and its Applications. (An English translation of the Soviet journal *Teorija Verojatnostei i ee Primenenija*). (Philadelphia, Pa.) (Translation of *Teor. Verojatnost. i Primenen.*)

U.S.S.R. Comput. Math. and Math. Phys. U.S.S.R. Computational Mathematics and Mathematical Physics. (Oxford) (Translation of *Ž. Vyčisl. Mat. i Mat. Fiz.*)

ERRATA AND ADDENDA

VOLUME 19

p. 395: **Krasner, Marc**

In line 12 of the first review, read "If the field k is not ..." instead of "If the residue class field of k is not ...".

To the second review, add "If an analytic function is given by a Taylor series, then either it cannot be prolonged outside its circle of convergence or the circle of convergence does not include its circumference, and (in a suitable extension field) there is a singularity (i.e., a point to which the function cannot be prolonged) on the circumference".

VOLUME 26

#2537: **Šain, B. M.**

The reviewer would like to add the following to his review.

Here also the structure of generalized heaps is fully described. The special case of symmetric heaps was treated by the author in a previous paper [Naučn. Dokl. Vysš. Skoly Fiz.-Mat. Nauki **1959**, no. 1, 88–99; Zbl **94**, 10–11]. A detailed review can be found in RŽMat **1962** #8 A164.

#2870: **Semenov, E. M.**

The reviewer remarks that the upper limits of the integrals on lines -6 and -8 should be h, not 1; the integral in the first displayed formula is correct, however.

#3820: **Furstenberg, Harry**

The reviewer submits the following correction to his review. "As mentioned in the review, the author's result that a bounded solution of Laplace's equation on G/K is also annihilated by the other invariant differential operators on G/K has also been announced by F. A. Berezin [Dokl. Akad. Nauk SSSR **125** (1959), 1187–1189; MR **21** #3014]. However, Berezin's proof, as outlined there, seems faulty, being based on an incorrect form of the asymptotic formula for the spherical functions on G/K."

#3903: **Sato, Daihachiro**

The second displayed formula should read

$$k = k_{(q)} = \limsup \log^{[q-1]} M(r)/r^{\lambda}.$$

#4936: **Perkins, Peter**

The reviewer wishes, as a result of a recent discovery, to append the following significant priority references concerning the 'Wielandt bound' to Leopold Löwenheim [S.-B. Berlin. Math. Ges. **12** (1913), 65–71; Math. Ann. **73** (1913), 245–272].

Beurling, Arne; Livingston, A. E.

On p. 1424 of the index issue the review number associated with Beurling should be 2851 and not 2551; the listing is correctly given under Livingston.

VOLUME 27

#720: **Arhangel'skiĭ, A.**

The reviewer points out that the second metrization theorem in the paper is not new; indeed, it was proved earlier by F. Burton Jones [Proc. Amer. Math. Soc. **9** (1958), 487; MR **20** #278].

#2102: **Hsu, L. C.**

The remainder term in the displayed formula should be $O(N^{-p/m}\omega_p(N^{-1/m}))$.

#2103: **Hsu, L. C.**

On the right-hand side of the formula, read $(-1)^k/m!$ for $(-1)^k/m$.

#3480: **Brown, William M.**

Lines 10–11 of the penultimate paragraph should read "... Z-transform (sampled-data theory as developed by Zadeh and Ragazzini [see Ragazzini and Franklin, *Sampled-data control systems*, McGraw-Hill, ...".

#5240: **Church, P. T.; Hemmingsen, E.**

In line 5 of paragraph 4 of the review, replace the word "exceptionally" by "exceptionality".

#5299: **Hanisch, Herman; Hirsch, Warren M.**

In line 9 of the review, replace "$\mu_G \int_0^\infty t\, dG(t) < \infty$" by "$\mu_G = \int_0^\infty t\, dG(t) < \infty$".

#5861: **Macbeath, A. M.; Świerczkowski, S.**

In line 17 of the review the words "m, and almost" should be replaced by "m-almost".

#5964: **Lukaševič, N. A.**

The reference to Kukles in the review should be Dokl. Akad. Nauk SSSR and not Dokl. Akad. Nauk BSSR (as is implied by the use of "ibid.").

Volume 28

60: Skubenko, B. F.

In the last displayed expression, replace $\sum_{d|n}$ by $\sum_{d|N}$.

00: Walsh, J. L.

In line 22 of the review, replace "the norm is as in (1)" by "the norm is $[\max|f(z)-r_{jk}(z)|, z$ on $E]$".

171: Baker, A.

The author remarks that, beginning on p. 119 of the paper, the misprint $N=2A^k$ in place of $N=2^{A^k}$ appears worthy of attention.

340: Lorch, L.; Szego, P.

In line 4 of the review, replace "$0<f(x)\leqq\infty$" by "$0<f(\infty)\leqq\infty$".

773: Tihonov, A. N.; Gorbunov, A. D.

Interchange the text of this review with that of the following [#1774].

#1774: Tihonov, A. N.; Gorbunov, A. D.

Interchange the text of this review with that of the preceding [#1773].

#3138: Jacobs, K.

A list of errata is available from the Matematisk Institut, Aarhus Universitet, Aarhus.

#3988: Fröhlich, A.

That part of the review following the words "d'une" in line 31 of the first column on p. 772 and ending with the words "groupe d'inertie" in line 14 of the second column should be inserted between the words "l'idèle" and "que" in line 42, column 1, of p. 773.

#5164a-b: Sinaĭ, Ja. G.

The reviewer states that the preliminary lemmas, which the author asserts are used to prove the principal result (Theorem 2), are used nowhere in the proof of the main theorem.

TRANSLITERATION OF RUSSIAN

Russian Letter Cap. Ital.	Mathematical Reviews	Zentralblatt für *Mathematik*	Bulletin Signalétique	Applied Mechanics Reviews	Science Abstracts	U.S. Library of Congress	Amer. Slavic & E. European Review	Journal of Symboli[c] Lo[gic]
А *а*	a	a	a	a	a	a	a	a
Б *б*	b	b	b	b	b	b	b	b
В *в*	v	v	v	v	v	v	v	v
Г *г*	g	g	g	g	g	g	g	g
Д *д*	d	d	d	d	d	d	d	d
Е *е*	e	e	e	e	e	e	e	é
Ё *ё*	e	e	ë	ë	ë	ë	ë	é
Ж *ж*	ž	ž	zh	zh	zh	z͡h	ž	ž
З *з*	z	z	z	z	z	z	z	z
И *и*	i	i	i	i	i	i	i	i
Й *й*	ĭ	j	j	ĭ	ĭ	ĭ	j	j
К *к*	k	k	k	k	k	k	k	k
Л *л*	l	l	l	l	l	l	l	l
М *м*	m	m	m	m	m	m	m	m
Н *н*	n	n	n	n	n	n	n	n
О *о*	o	o	o	o	o	o	o	o
П *п*	p	p	p	p	p	p	p	p
Р *р*	r	r	r	r	r	r	r	r
С *с*	s	s	s	s	s	s	s	s
Т *т*	t	t	t	t	t	t	t	t
У *у*	u	u	u	u	u	u	u	u
Ф *ф*	f	f	f	f	f	f	f	f
Х *х*	h	ch	kh	kh	kh	k͡h	kh	h
Ц *ц*	c	c	c	ts	ts	t͡s	c	c
Ч *ч*	č	č	ch	ch	ch	c͡h	č	č
Ш *ш*	š	š	sh	sh	sh	s͡h	š	š
Щ *щ*	šč	šč	shch	shch	shch	shch	šč	šč
Ъ *ъ*	"			"	'	"	'	
Ы *ы*	y	y	y	í	ȳ	y	y	y
Ь *ь*	'	'	'	'	'	'	'	'
Э *э*	è	ė	e	é	é	é	è	e
Ю *ю*	ju*	ju	ju	yu	yu	i͡u	ju	ú
Я *я*	ja†	ja	ja	ya	ya	i͡a	ja	á

* formerly yu
† formerly ya